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THE NATURAL PARTIAL ORDER ON LINEAR TRANSFORMATION SEMIGROUPS WITH RESTRICTIONS ON NULLITY OR CO-RANK

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อันดับบางส่วนธรรมชาติ $\leq$ บนกึ่งกรุป S คืออันดับบางส่วนที่นิยามโดย

$a \leq b$ ก็ต่อเมื่อ $a = xb = by$ และ $a = ay$ สำหรับบาง $x, y \in S^1$

เมื่อ S^1 เป็นกึ่งกรุปที่ได้จาก S โดยที่ $S^1 = S$ ถ้า S มีเอกลักษณ์ และถ้า S ไม่มีเอกลักษณ์ ให้ S^1 คือ S ผนวกเอกลักษณ์ 1 เข้าไปใน S เป็นที่รู้กันว่าอันดับบางส่วนธรรมชาติบนกึ่งกรุปและกึ่งกรุปย่อยปรกติของกึ่งกรุปนั้นพ้องกัน ดังนั้นการศึกษาอันดับบางส่วนธรรมชาติบนกึ่งกรุปไม่ปรกติจึงเป็นที่สนใจ

ในวิทยานิพนธ์ฉบับนี้เราให้เงื่อนไขที่จำเป็นและเพียงพอสำหรับการที่สมาชิกในกึ่งกรุปการแปลงเชิงเส้นไม่ปรกติที่มีข้อจำกัดบนศูนยภาพหรือค่าลำดับชั้นร่วมจะมีความสัมพันธ์กันภายใต้อันดับบางส่วนธรรมชาติ นอกจากนี้ เราให้เงื่อนไขที่จำเป็นและเพียงพอสำหรับสมาชิกในกึ่งกรุปการแปลงเชิงเส้นเหล่านั้นที่จะเป็นสมาชิกใช้แทนกันได้ทางซ้ายและทางขวา สมาชิกเล็กสุดเฉพาะกลุ่มและสมาชิกใหญ่สุดเฉพาะกลุ่ม สมาชิกปกล่างและสมาชิกปกบน

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The natural partial order $\leq$ on a semigroup S is a partial order defined by

$$a \leq b \text{ if and only if } a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1$$

where S^1 is the semigroup obtained from S such that $S^1 = S$ if S has an identity and if S has no identity, let S^1 be S with the identity 1 adjoined. It is known that the natural partial orders on a semigroup and its regular subsemigroups coincide. Therefore, the study of the natural partial order on nonregular semigroups are of interest.

In this thesis, we give necessary and sufficient conditions for elements in nonregular linear transformation semigroups with restrictions on nullity or co-rank are related under the natural partial order. Furthermore, we provide necessary and sufficient conditions for elements in those linear transformation semigroups to be left and right compatible elements, minimal and maximal elements, lower and upper covers.

Department : Mathematics and Computer Science .... Student's Signature ....
Advisor's Signature ....
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CONTENTS

CHAPTER I

INTRODUCTION

In semigroup theory, the problem of defining a partial order on a semigroup has been studied for a long time. For a semigroup S, the set $E(S)$ of all idempotents in S can be ordered in the following manner: for any $e, f \in E(S)$,

$$e \leq_{E(S)} f \text{ if and only if } e = ef = fe.$$

From [8], this is a partial order on $E(S)$. There are many attempts to extend this order to any semigroup. In 1952, V. Wagner [16] introduced the notion of inverse semigroups and the partial order $\leq_{inv}$, that is, for an inverse semigroup S and $a, b \in S$,

$$a \leq_{inv} b \text{ if and only if } a = eb \text{ for some } e \in E(S).$$

The intersection of the partial order $\leq_{inv}$ and $E(S) \times E(S)$ is $\leq_{E(S)}$ where S is an inverse semigroup. Next, in 1980, according to [3] and [12], R. Hartwig and K. Nambooripad independently defined partial orders on a regular semigroup, which have different forms but equal, say $\leq_{reg}$. An interesting form of $\leq_{reg}$ is defined by H. Mitsch [11] in 1986 as follows. For a regular semigroup S and $a, b \in S$,

$$a \leq_{reg} b \text{ if and only if } a = xb = by \text{ and } a = xa \text{ for some } x, y \in S.$$

Unfortunately, this relation on a general semigroup may not be a partial order if the semigroup has no identity. To solve this problem, an element $1 \notin S$ is added to be the identity of S, that is, for a semigroup S, let S^1 be a semigroup with the identity 1 adjoined if S has no identity; otherwise, $S^1 = S$. Therefore, H. Mitsch defined in [11] that, for a semigroup S and $a, b \in S$,

$$a \leq b \text{ if and only if } a = xb = by \text{ and } a = xa \text{ for some } x, y \in S^1.$$

This order is a partial order on S called the *natural partial order*. In particular, the orders $\leq$ and $\leq_{reg}$ are equal on an arbitrary regular semigroup. Furthermore, $\leq$, $\leq_{reg}$ and $\leq_{inv}$ are equal on each inverse semigroup. In addition, H. Mitsch proved in [11] that the natural partial order $\leq$ has several equivalent forms, that is, for a semigroup S with the natural partial order $\leq$ and $a, b \in S$, the following are equivalent:

(i) $a \leq b$,

(ii) $a = xb = by$ and $a = ay$ for some $x, y \in S^1$,

(iii) $a = xb = by$ and $a = xa = ay$ for some $x, y \in S^1$.

Nevertheless, it is sometimes not convenient to verify related elements in a semigroup by using the definition of $\leq$. Therefore, the problem of finding necessary and sufficient conditions for elements in various semigroups to be related is of interest. Notice that the regularity of a semigroup is significant in studying the natural partial order on a semigroup. In 1994, P. M. Higgins [4] proved that regular subsemigroups of a semigroup inherit the natural partial order from the semigroup. Therefore the natural partial orders on nonregular subsemigroups of some semigroups are focused. From [2] and [5], they determined the natural partial order on nonregular semigroups.

Note that every semigroup can be embedded into a certain transformation semigroup, see [6]. This embedding additionally preserves the natural partial order since the natural partial order is defined via multiplication of a semigroup. There are many researches about the natural partial order on transformation semigroups; for example, [5], [7] and [9] provide characterizations for elements in some transformation semigroups to be related under the natural partial order.

For a vector space V, denote by $L(V)$ the set of all linear transformations on V. It is a semigroup under composition. In 2005, R. P. Sullivan [15] studied the natural partial order on the semigroup $L(V)$. Our attention is now on various subsemigroups of $L(V)$. In this thesis, certain nonregular subsemigroups of $L(V)$ are considered.

We consider several subsemigroups of $L(V)$, namely, $AM(V)$, $AE(V)$, $OM(V)$,

$OE(V)$, $K(V,\kappa)$, $CI(V,\kappa)$. They are linear transformation semigroups with restrictions on nullity or co-rank and their definitions can be found in Chapter II. Since the regularity of a semigroup is important in studying the natural partial order, we focus on the regularity of those subsemigroups of $L(V)$ listed above. The regularity of the semigroups $AM(V)$, $AE(V)$, $OM(V)$ and $OE(V)$ are described in [6]. However, the regularity of $K(V,\kappa)$ and $CI(V,\kappa)$ are still not investigated. We shall give necessary and sufficient conditions for $K(V,\kappa)$ and $CI(V,\kappa)$ to be nonregular.

We organize this thesis as follows: Chapter II contains notations, definitions and quoted results that will be used in this thesis. Then we provide necessary and sufficient conditions for elements in the nonregular semigroup $S(V)$ to be related under the natural partial order $\leq$ where $S(V)$ is defined as follows:

In Chapter III, $S(V)$ is either $AM(V)$ or $AE(V)$.

In Chapter IV, $S(V)$ is $OM(V)$, $OE(V)$, $K(V,\kappa)$ or $CI(V,\kappa)$.

Furthermore, we characterize left and right compatibility, minimality, maximality and the existence of lower and upper covers of elements in $(S(V),\leq)$ with or without the regularity of $S(V)$. At last, we give many examples associated with our results.

CHAPTER II
PRELIMINARIES

2.1 Notation and Definitions

For a nonempty set X, a *partition* P of X is a family of sets such that $\varnothing \notin P$, $\bigcup_{A \in P} A = P$ and $A \cap B = \varnothing$ for all distinct $A, B \in P$. A binary relation $\preccurlyeq$ on a nonempty set X is called a *partial order* on X if the following properties hold.

(i) Reflexivity: $x \preccurlyeq x$ for all $x \in X$.

(ii) Antisymmetry: for any $x, y \in X$, $x \preccurlyeq y$ and $y \preccurlyeq x$ imply $x = y$.

(iii) Transitivity: for any $x, y, z \in X$, $x \preccurlyeq y$ and $y \preccurlyeq z$ imply $x \preccurlyeq z$.

If $\preccurlyeq$ is a partial order on a set X, the pair $(X, \preccurlyeq)$ is said to be a *partially ordered set* or a *poset*. In the sequel, if there is no ambiguity about the partial order, we may write X instead of $(X, \preccurlyeq)$.

For a poset $(X, \preccurlyeq)$, an element x in X is called a *minimal element* in X if for any $y \in X$, $y \preccurlyeq x$ implies $y = x$. A *maximal element* x in X is defined by for any $y \in X$, $x \preccurlyeq y$ implies $x = y$. An element x in X is said to be the *minimum element* in X if $x \preccurlyeq y$ for all y in X. The *maximum element* x in X is the element such that $y \preccurlyeq x$ for all $y \in X$.

For any distinct elements x and y in a poset $(X \preccurlyeq)$, x is said to be a *lower cover* of y in X if $x \preccurlyeq y$ and there is no $z \in X \smallsetminus \{x, y\}$ such that $x \preccurlyeq z$ and $z \preccurlyeq y$. From this definition, y is said to be an *upper cover* of x in X. It is clear that there is no lower [upper] cover of minimal [maximal] elements.

Example 2.1.1. (i) Let X be a nonempty set and $a \in X$. Consider $\mathcal{P}(X)$ with the inclusion $\subseteq$. Obviously, $\varnothing$ has no lower cover in $\mathcal{P}(X)$ and $\varnothing$ is a lower cover of $\{a\}$ in $\mathcal{P}(X)$. Let $A, B \subseteq X$ be such that $A \subseteq B$. Then A is a lower cover of B in $\mathcal{P}(X)$ if and only if $B \smallsetminus A$ is a singleton.

(ii) Every element in the set of real numbers $\mathbb{R}$ has no lower and upper cover with respect to the relation "less than or equal to".

In this thesis, we illustrate many figures, so the notations are needed to introduce. Consider a poset $(X, \preccurlyeq)$, an element in X will be drawn as a vertex. For distinct $x, y \in X$, we draw a straight line from x upward to y if x is a lower cover of y in $(X, \preccurlyeq)$. If $x \preccurlyeq y$ on X, we use a dotted line from x upward to y; see the following notations.

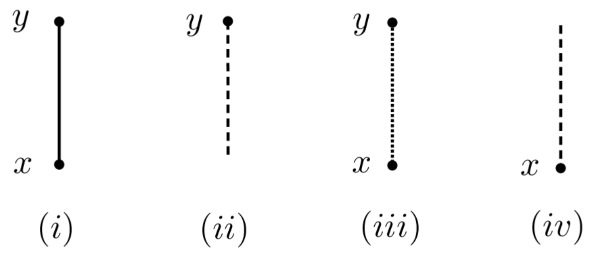

Figure 2.1: (i) This means that x is a lower cover of y. The meaning of (ii) is that y has a lower cover. (iii) means $x \preccurlyeq y$. (iv) x has an upper cover.

For example, see the figure below.

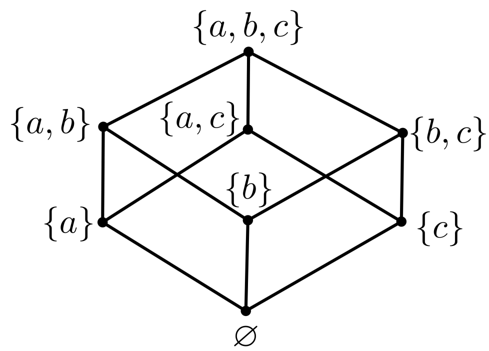

Figure 2.2: A diagram of $(\mathcal{P}(X), \subseteq)$ where $X = \{a, b, c\}$.

Let $\mathbb{N}$ be the set of all natural numbers with the standard relation "less than or equal to"$\leqslant$. Consider $\mathbb{N} \times \mathbb{N}$ with the partial order $\leqslant'$ defined by, for any $a, b, c, d \in \mathbb{N}$, $(a, b) \leqslant' (c, d)$ if and only if $a \leqslant c$ and $b \leqslant d$. Then $(\mathbb{N} \times \mathbb{N}, \leqslant')$ is a partially ordered set and it can be illustrated as in Figure 2.3.

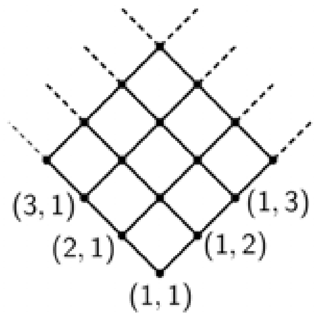

Figure 2.3: A diagram of $(\mathbb{N} \times \mathbb{N}, \leqslant')$.

Next, we provide definitions and notations in semigroup theory that will be used. A nonempty set S with a binary operation $\cdot$ on S is called a *semigroup* if $\cdot$ is associative. For any elements x, y in a semigroup S, we write $x \cdot y$ as xy. Next, for a partial order $\preccurlyeq$ on a semigroup S, an element $c \in S$ is said to be *left* [*right*] *compatible* on S if for any elements $a, b \in S$, $a \preccurlyeq b$ implies $ca \preccurlyeq cb$ [$ac \preccurlyeq bc$]. In particular, c is called *compatible* on S if it is both left and right compatible. If every element in S is compatible, then $\preccurlyeq$ is called *compatible* on S.

Example 2.1.2. (i) For a set X, consider the power set $\mathcal{P}(X)$ with the intersection $\cap$ and the union $\cup$. Then $(\mathcal{P}(X), \cap)$ and $(\mathcal{P}(X), \cup)$ are semigroups. Let $A, B, C \in \mathcal{P}(X)$ be such that $A \subseteq B$ on $(\mathcal{P}(X), \cap)$. Thus $C \cap A \subseteq C \cap B$ and $A \cap C \subseteq B \cap C$. Therefore, $\subseteq$ is compatible on $(\mathcal{P}(X), \cap)$. Similarly, $\subseteq$ is also compatible on $(\mathcal{P}(X), \cup)$.
(ii) Let $(\mathbb{R}^+, \cdot)$ denote the poset of all positive real numbers endowed with usual multiplication $\cdot$. Consider the order "less than or equal to"$\leqslant$ on $(\mathbb{R}^+, \cdot)$. Let $a, b, c \in \mathbb{R}^+$ be such that $a \leqslant b$ on $\mathbb{R}^+$. Then $ca \leqslant cb$ and $ac \leqslant bc$ on $\mathbb{R}^+$. Hence $\leqslant$ is compatible on $(\mathbb{R}^+, \cdot)$. Since $1 \leqslant 2$ and $1(-2) = (-2)1 = -2 \nleqslant -4 = (-2)2 = 2(-2)$, we have $\leqslant$ is neither left nor right compatible on $(\mathbb{R}, \cdot)$.

Let S be a semigroup. An element e in S is said to be an *idempotent* if $e^2 = e$. For an x in S, if $ax = x = xa$ for all $a \in S$, then x is called the *zero*. If e is an element in S with the property that $ae = a = ea$ for all $a \in S$, then e is said to be the *identity*. An element a in S is called *regular* if $a = axa$ for some $x \in S$. If every element in S is regular, then S is said to be a *regular semigroup*. For any

$A, B \subseteq S$, let

$$AB = \{ab \mid a \in A \text{ and } b \in B\}.$$

A nonempty subset A of a semigroup S is called a *left* [*right*] *ideal* of S if $SA \subseteq A$ [$AS \subseteq A$].

If S is a semigroup without identity, we can adjoin an element $1 \notin S$ and define a binary operation $*$ on $S \cup \{1\}$ by

$$\begin{aligned} a * b = ab & \quad \text{for all } a, b \in S, \\ a * 1 = a = 1 * a & \quad \text{for all } a \in S \cup \{1\}. \end{aligned}$$

For a semigroup S, we let

$$S^1 = \begin{cases} S & \text{if } S \text{ has the identity;} \\ S \cup \{1\} & \text{if } S \text{ has no identity.} \end{cases}$$

Then S^1 is a semigroup with identity.

Example 2.1.3. (i) Let X be a set. Note that X and $\varnothing$ are identities of $(\mathcal{P}(X), \cap)$ and $(\mathcal{P}(X), \cup)$, respectively. Hence $\mathcal{P}(X)^1 = \mathcal{P}(X)$ under the operations $\cap$ and $\cup$.
(ii) Let S be a semigroup defined by $ab = b$ for all $a, b \in S$. If S has more than one element, then S has no identity. Therefore, $S^1 \neq S$.

The *natural partial order* $\leq$ on a semigroup S is defined by Mitsch in [11] as follows, for any $a, b \in S$,

$$a \leq b \text{ if and only if } a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1.$$

This order is a partial order on S. For any $a, b \in S$, the relation $a < b$ stands for $a \leq b$ and $a \neq b$.

Example 2.1.4. Let X be a set. Consider the semigroups $(\mathcal{P}(X), \cap)$ and $(\mathcal{P}(X), \cup)$. Let $A, B \in \mathcal{P}(X)$ be such that $A \subseteq B$. Since $A = A \cap B = B \cap A$ and $A = A \cap A$, we obtain $A \leq B$ on $(\mathcal{P}(X), \cap)$. Since $B = B \cup A = A \cup B$ and $B = B \cup B$, we have $B \leq A$ on $(\mathcal{P}(X), \cup)$.

Example 2.1.5. Let S be a semigroup defined by $ab = b$ for all $a, b \in S$. Suppose that $a \leq b$ on S. Then $a = xb = by$ and $a = ay$ for some $x, y \in S^1$. This implies that $a = xb = b$. Hence $a \leq b$ on S if and only if $a = b$.

Consider S with the natural partial order $\leq$. If S has the zero element 0, then 0 is the minimum element by [2] and $x \in S$ is called a *minimal nonzero element* in S if x is an upper cover of 0.

In this thesis, we are interested in studying the natural partial order on linear transformation semigroups. We next introduce notations and definitions in linear algebra and the theory of linear transformation semigroups.

In this section, let V be a vector space over a field and let

$$L(V) = \{\alpha \mid \alpha \text{ is a linear transformation on } V\}.$$

From [6], $L(V)$ is a regular semigroup under composition. Let $\alpha \in L(V)$. Throughout this thesis, all functions act on the right-hand side of the argument. The *kernel of* α is the set of $v \in V$ such that $v\alpha = 0$ where 0 is the zero vector. The *image of* α means $V\alpha$. The kernel and the image of α are denoted by $\ker\alpha$ and $\operatorname{im}\alpha$, respectively. For a subspace U of V, let $\dim U$ represent the dimension of U. The notations $\dim(\ker\alpha)$ and $\dim(\operatorname{im}\alpha)$ are called the *nullity of* α and the *rank of* α, respectively, denoted by $\operatorname{nullity}\alpha$ and $\operatorname{rank}\alpha$. For a subset A of V, let $\langle A\rangle$ stand for the subspace spanned by A and $\langle v\rangle$ means $\langle\{v\}\rangle$ where $v \in V$. Denote by 0_V and 1_V the zero map on V and the identity map on V, respectively.

For a subspace W of V and a vector $v \in V$, the set $\{v+w \mid w \in W\}$ is denoted by $v+W$, called a *coset* of W in V. The set of all cosets of W in V is represented by

$$V/W = \{v + W \mid v \in V\}.$$

Then the set V/W is called the *quotient space of* V *modulo* W. In fact, V/W is a vector space under the following operations: for $u, v \in V$ and a scalar a,

$$(u + W) \oplus (v + W) = (u + v) + W$$

and

$$a \odot (u + W) = (au) + W.$$

Clearly, the zero vector in V/W is $0+W = W$. For each $\alpha \in L(V)$, $\dim(V/\operatorname{im}\alpha)$ is called the *corank of* α denoted by $\operatorname{corank}\alpha$. Note that if $\dim V$ is finite, then $\operatorname{corank}\alpha = \dim(V/\operatorname{im}\alpha) = \dim V - \dim(\operatorname{im}\alpha) = \dim(\ker\alpha) = \operatorname{nullity}\alpha$.

For any sets A, B, $A_0 \subseteq A$, $B_0 \subseteq B$ and any function $\phi : A \to B$, denote by $A_0\phi$ and $B_0\phi^{-1}$ the image of A_0 under ϕ and the inverse image of B_0 under ϕ, respectively. Moreover, ϕ^{-1} is the inverse relation of ϕ. For any $\alpha, \beta \in L(V)$, $V\alpha\beta^{-1} = \{v \in V \mid v\beta \in \operatorname{im}\alpha\}$ and let

$$E(\alpha, \beta) = \{v \in V \mid v\alpha = v\beta\}.$$

It is easy to see that $E(\alpha, \beta) \subseteq V\alpha\beta^{-1}$ for all $\alpha, \beta \in L(V)$, but it is not necessary true that $E(\alpha, \beta) = V\alpha\beta^{-1}$. Recall that any linear transformation $\alpha \in L(V)$ can be defined on a basis B of V as it can be extended linearly from a basis, that is,

$$(a_1v_1 + \cdots + a_nv_n)\alpha = a_1v_1\alpha + \cdots + a_nv_n\alpha$$

where $v_1, \ldots, v_n \in B$, $a_1, \ldots, a_n$ are scalars and n is a natural number.

Example 2.1.6. Let $\dim V > 1$ and let B be a basis of V and $u \in B$. Define $\beta \in L(V)$ by $v\beta = u$ for all $v \in B$. It is routine to verify that $V1_V\beta^{-1} = V$ and $E(1_V, \beta) = \langle u\rangle$. Hence $E(1_V, \beta) \subsetneq V1_V\beta^{-1}$.

Many linear transformations will be defined in this thesis. We shall use a bracket notation to represent many of them. Let B be a basis of V. If $\alpha \in L(V)$ is defined by, for each $v \in B$, $v\alpha = w_v$ where $w_v \in V$, then we write

$$\alpha = \begin{pmatrix} v \\ w_v \end{pmatrix}_{v \in B}.$$

For a natural number $n > 1$, assume B has a partition $\{B_1, B_2, \ldots, B_n\}$, $u_i \in V$ for all $i = 1, 2, \ldots, n-1$, and $\alpha \in L(V)$ is defined by

$$v\alpha = \begin{cases} u_i & \text{if } v \in B_i, i = 1, 2, \ldots, n-1; \\ w_v & \text{if } v \in B_n \end{cases}$$

where $w_v \in V$ for all $v \in V$. Then α will be written as

$$\alpha = \begin{pmatrix} B_1 & B_2 & \dots & B_{n-1} & v \\ u_1 & u_2 & \dots & u_{n-1} & w_v \end{pmatrix}_{v \in B_n}.$$

Next, let

$$AM(V) = \{\alpha \in L(V) \mid \operatorname{nullity} \alpha \text{ is finite}\},$$
$$AE(V) = \{\alpha \in L(V) \mid \operatorname{corank} \alpha \text{ is finite}\}.$$

Observe that 1_V is contained in $AM(V) \cap AE(V)$. Y. Kemprasit showed in [6] that these sets are subsemigroups of $L(V)$. Moreover, both $AM(V)$ and $AE(V)$ are nonregular if and only if $\dim V$ is infinite. Notice that both $AM(V)$ and $AE(V)$ do not contain the zero map 0_V whenever $\dim V$ is infinite.

In [6], Y. Kemprasit also proved that the following are nonregular subsemigroups of $L(V)$. For an infinite dimensional vector space V, let

$$OM(V) = \{\alpha \in L(V) \mid \operatorname{nullity} \alpha \text{ is infinite}\},$$
$$OE(V) = \{\alpha \in L(V) \mid \operatorname{corank} \alpha \text{ is infinite}\}.$$

It can be seen that both $OM(V)$ and $OE(V)$ contain 0_V, but 1_V is not an element in these semigroups.

For a cardinal number κ with $\kappa \leq \dim V$, let

$$K(V, \kappa) = \{\alpha \in L(V) \mid \operatorname{nullity} \alpha \geq \kappa\},$$
$$CI(V, \kappa) = \{\alpha \in L(V) \mid \operatorname{corank} \alpha \geq \kappa\}.$$

In 2005, S. Chaopraknoi and Y. Kemprasit proved in [1] that $K(V, \kappa)$ and $CI(V, \kappa)$ are subsemigroups of $L(V)$. Note that both $K(V, \kappa)$ and $CI(V, \kappa)$ contain the identity map 1_V if and only if $\kappa = 0$. Moreover, the zero map 0_V is always contained in $K(V, \kappa)$ and $CI(V, \kappa)$. Observe that $K(V, \aleph_0) = OM(V)$, $CI(V, \aleph_0) = OE(V)$ and $K(V, 0) = L(V) = CI(V, 0)$ where $\aleph_0$ is the aleph-null. If $\dim V$ is finite, then $K(V, \kappa) = CI(V, \kappa)$ since $\operatorname{corank} \alpha = \operatorname{nullity} \alpha$. Furthermore, significant properties of these semigroups are provided.

Proposition 2.1.7. [1] *If* $\dim V$ *be infinite, then* $K(V, \kappa) \neq CI(V, \iota)$ *for any nonzero cardinal numbers* $\kappa, \iota \leq \dim V$.

Therefore, we obtain the following proposition.

Proposition 2.1.8. *$K(V,\kappa) = CI(V,\kappa)$ if and only if $\dim V$ is finite or $\kappa = 0$.*

In [10], the authors proved that $K(V,\kappa) \cap CI(V,\kappa)$ is a regular subsemigroup of $L(V)$ when $\dim V$ is infinite.

The next proposition will be used in Section 4.2.

Proposition 2.1.9. [13] *(i) $K(V,\kappa)$ is a right ideal of $L(V)$.*
(ii) $CI(V,\kappa)$ is a left ideal of $L(V)$.

2.2 Elementary Results

This section contains basic results in set theory and linear algebra which are needed in this thesis. The following is a well-known fact in set theory.

Proposition 2.2.1. [14] *Let κ and λ be cardinal numbers such that at least one of them is infinite. Then $\kappa + \lambda = \max\{\kappa, \lambda\}$.*

For a set X, the cardinality of X is denoted by $|X|$. Next, we show that any infinite set can be partitioned into finitely many infinite sets as follows.

Proposition 2.2.2. *Let X be an infinite set and let n be a natural number. Then there exists a partition $\{X_1, X_2, \ldots, X_n\}$ of X such that $|X| = |X_i|$ for all $i = 1, 2, \ldots, n$.*

Proof. Let $x_1, x_2, \ldots, x_n \in X$ be distinct. Then

$$|X| = |X \times \{x_1\}| = |X \times \{x_2\}| = \cdots = |X \times \{x_n\}|.$$

Since $X \times \{x_1\}, X \times \{x_2\}, \ldots, X \times \{x_{n-1}\}$ and $X \times \{x_n\}$ are disjoint, by Proposition 2.2.1,

$$\left|\bigcup_{i=1}^{n} (X \times \{x_i\})\right| = |X \times \{x_1\}| + |X \times \{x_2\}| + \cdots + |X \times \{x_n\}| = |X|.$$

Thus there exists a bijection $\phi : \bigcup_{i=1}^{n}(X \times \{x_i\}) \to X$. For any $i = 1, 2, \ldots, n$, choose $X_i = [X \times \{x_i\}]\phi$. Hence $|X_i| = |X \times \{x_i\}| = |X|$ and

$$\bigcup_{i=1}^{n} X_i = \bigcup_{i=1}^{n} [X \times \{x_i\}]\phi = \left[\bigcup_{i=1}^{n} (X \times \{x_i\})\right] \phi = X.$$

Furthermore, it is easy to see that $X_i \cap X_j = \varnothing$ for all distinct $i, j \in \{1, 2, \ldots, n\}$. Therefore $\{X_1, X_2, \ldots, X_n\}$ is a partition of X. □

Now we let V be a vector space. Here are some facts about linear maps, bases and quotient spaces.

Proposition 2.2.3. *Let B be a basis of V, $A \subseteq B$ and let $\varphi : B \smallsetminus A \to V$ be such that $(B \smallsetminus A)\varphi$ is a linearly independent subset of V. Let $\alpha \in L(V)$ be defined by*

$$\alpha = \begin{pmatrix} A & v \\ 0 & v\varphi \end{pmatrix}_{v \in B \smallsetminus A}.$$

Then the following statements hold.
(i) If φ is an injection, then $\ker \alpha = \langle A \rangle$.
(ii) $\operatorname{im} \alpha = \langle (B \smallsetminus A)\varphi \rangle$.

Proof. (i) Assume that φ is injective. Clearly, $\langle A \rangle \subseteq \ker \alpha$. Let $u \in \ker \alpha$. Then

$$u = \sum_i a_i u_i + \sum_j b_j v_j$$

where $u_i \in A$, $v_j \in B \smallsetminus A$, a_i, b_j are scalars and both summations are over finite index sets I and J. Hence

$$0 = u\alpha = \sum_j b_j v_j \alpha = \sum_j b_j v_j \varphi.$$

Since φ is injective, $v_j\varphi \neq v_{j'}\varphi$ for all distinct $j, j' \in J$. Since $(B \smallsetminus A)\varphi$ is linearly independent, we have $b_j = 0$ for all $j \in J$. This implies that $u = \sum_i a_i u_i \in \langle A \rangle$. Therefore $\ker \alpha = \langle A \rangle$.
(ii) Obviously, $\langle (B \smallsetminus A)\varphi \rangle \subseteq \operatorname{im} \alpha$. Let $v \in \operatorname{im} \alpha$. Then $v = u\alpha$ for some $u \in V$. We write

$$u = \sum_i a_i u_i + \sum_j b_j v_j$$

for some $u_i \in A$, $v_j \in B \smallsetminus A$ and scalars a_i, b_j where $i \in I$, $j \in J$ and I, J are finite index sets. It follows that

$$v = u\alpha = \sum_j b_j v_j \alpha = \sum_j b_j v_j \varphi \in \langle (B \smallsetminus A)\varphi \rangle .$$

Hence $\operatorname{im}\alpha = \langle (B \smallsetminus A)\varphi \rangle$, as desired. $\square$

The next proposition shows that a basis of a vector space V and a basis of $\ker\alpha$ where $\alpha \in L(V)$ can be used to construct a basis of $\operatorname{im}\alpha$.

Proposition 2.2.4. *Let $\alpha \in L(V)$, B_1 be a basis of $\ker\alpha$ and B a basis of V containing B_1. Then the following statements hold.*
(i) For any $v_1, v_2 \in B \smallsetminus B_1$, $v_1 = v_2$ if and only if $v_1\alpha = v_2\alpha$.
(ii) $(B \smallsetminus B_1)\alpha$ is a basis of $\operatorname{im}\alpha$.

Proof. (i) Let $v_1, v_2 \in B \smallsetminus B_1$. The necessity is clear. Suppose that $v_1\alpha = v_2\alpha$. Then $v_1 - v_2 \in \ker\alpha = \langle B_1 \rangle$. Since $v_1, v_2 \in B \smallsetminus B_1$, we obtain $v_1 - v_2 \in \langle B \smallsetminus B_1 \rangle \cap \langle B_1 \rangle = \{0\}$. Hence $v_1 = v_2$.
(ii) Assume that

$$\sum_i a_i v_i = 0$$

where $v_i \in (B \smallsetminus B_1)\alpha$, a_i is a scalar and this summation is over finite index set I. Then for each $i \in I$, $v_i = u_i\alpha$ for some $u_i \in B \smallsetminus B_1$. Hence

$$0 = \sum_i a_i v_i = \sum_i a_i u_i \alpha = \left(\sum_i a_i u_i \right) \alpha.$$

Thus $\sum_i a_i u_i \in \langle B \smallsetminus B_1 \rangle \cap \langle B_1 \rangle = \{0\}$ and so $a_i = 0$ for all $i \in I$. Therefore $(B \smallsetminus B_1)\alpha$ is linearly independent. By Proposition 2.2.3 (ii), we have $\operatorname{im}\alpha$ is spanned by $(B \smallsetminus B_1)\alpha$. Hence $(B \smallsetminus B_1)\alpha$ is a basis of $\operatorname{im}\alpha$. $\square$

Proposition 2.2.5. *Let B be a basis of V and $A \subseteq B$. Then the following statements hold.*
(i) $\{v + \langle A \rangle \mid v \in B \smallsetminus A\}$ is a basis of the quotient space $V/\langle A \rangle$.
(ii) $\dim(V/\langle A \rangle) = |B \smallsetminus A|$.

Proof. (i) Assume that

$$\sum_i a_i(v_i + \langle A \rangle) = \langle A \rangle$$

where $v_i \in B \smallsetminus A$, a_i is a scalar, $i \in I$ and I is a finite index set. Then

$$\sum_i a_i v_i + \langle A \rangle = \langle A \rangle .$$

It follows that $\sum\limits_i a_i v_i \in \langle A \rangle$. If $A = \varnothing$, then $a_i = 0$ for all $i \in I$. Suppose that $A \neq \varnothing$. Hence we write $\sum\limits_i a_i v_i = \sum\limits_j b_j u_j$ where $u_j \in A$, b_j is a scalar, $j \in J$ and J is a finite index set. Since B is linearly independent, we have $a_i = 0$ for all $i \in I$. Therefore $\{v + \langle A \rangle \mid v \in B \smallsetminus A\}$ is linearly independent. Next, let $v + \langle A \rangle \in V/\langle A \rangle$. We write

$$v = \sum_i a_i u_i + \sum_j b_j v_j$$

where $u_i \in A$, $v_j \in B \smallsetminus A$, a_i, b_j are scalars, $i \in I$, $j \in J$ and I, J are finite index sets. Then

$$\begin{aligned} v + \langle A \rangle &= \sum_i (a_i u_i + \langle A \rangle) + \sum_j (b_j v_j + \langle A \rangle) \\ &= \langle A \rangle + \sum_j (b_j v_j + \langle A \rangle) \\ &= \sum_j b_j (v_j + \langle A \rangle) \in \langle \{v + \langle A \rangle \mid v \in B \smallsetminus A\} \rangle . \end{aligned}$$

Therefore $\{v + \langle A \rangle \mid v \in B \smallsetminus A\}$ is a basis of $V/\langle A \rangle$.
(ii) By (i), $\dim(V/\langle A \rangle) = |\{v + \langle A \rangle \mid v \in B \smallsetminus A\}| = |B \smallsetminus A|$. □

Recall that $E(\alpha, \beta) = \{v \in V \mid v\alpha = v\beta\}$. The following are elementary results of linear transformation semigroups.

Proposition 2.2.6. *Let $\alpha, \beta \in L(V)$ be such that $V\alpha\beta^{-1} = E(\alpha, \beta)$. Then $\ker \beta \subseteq \ker \alpha$.*

Proof. Let $v \in \ker \beta$. Then $v\beta = 0 \in V\alpha$, and thus $v \in V\alpha\beta^{-1}$. Since $V\alpha\beta^{-1} = E(\alpha, \beta)$, we have $v \in E(\alpha, \beta)$. Hence $v\alpha = v\beta = 0$. Therefore $v \in \ker \alpha$. □

Proposition 2.2.7. *Let $\alpha, \beta \in L(V)$ be such that $V\alpha\beta^{-1} = E(\alpha, \beta)$. Then the following are equivalent.*
(i) $\alpha = \beta$,
(ii) $\operatorname{im}\beta \subseteq \operatorname{im}\alpha$,
(iii) $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ *and* $\ker\alpha \subseteq \ker\beta$.

Proof. It is easy to see that (i) implies (ii) and (iii).
$(ii) \Rightarrow (i)$: Assume that $\operatorname{im}\beta \subseteq \operatorname{im}\alpha$. Let $v \in V$. Then $v\beta \in \operatorname{im}\beta \subseteq \operatorname{im}\alpha$, so $v\beta = u\alpha$ for some $u \in V$. It follows that $v \in V\alpha\beta^{-1} = E(\alpha, \beta)$. Hence $v\alpha = v\beta$, so $\alpha = \beta$.
$(iii) \Rightarrow (i)$: Suppose that $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $\ker\alpha \subseteq \ker\beta$. Let $v \in V$. Then $v\alpha \in \operatorname{im}\alpha$. Since $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$, we get $v\alpha = u\beta$ for some $u \in V$. Thus $u \in V\alpha\beta^{-1} = E(\alpha, \beta)$, so $u\alpha = u\beta = v\alpha$. Hence $v - u \in \ker\alpha \subseteq \ker\beta$. Therefore $v\beta = u\beta = v\alpha$, and that $\alpha = \beta$. □

Remark 2.2.8. Let $\alpha, \beta \in L(V)$ be distinct and $V\alpha\beta^{-1} = E(\alpha, \beta)$. Then $\ker\beta \subsetneq \ker\alpha$ and $\operatorname{im}\alpha \neq \operatorname{im}\beta$ by Propositions 2.2.6 and 2.2.7.

For convenience, we write the set $\{x_i \in V \mid i \in I\}$ in short by $\{x_i\}_{i\in I}$ where I is an index set. Next, we give a result extracted from the proof of Theorem 2.5 in [15].

Proposition 2.2.9. *Let $\alpha, \beta \in L(V)$ be such that $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha, \beta)$. Then*

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \quad \textit{and} \quad \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & v_j & u_k \end{pmatrix}_{j\in J, k\in K}$$

where $\{x_i\}_{i\in I}$ is a basis of $\ker\beta$, $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}$ is a basis of $\ker\alpha$, $\{u_k\}_{k\in K}$ is a basis of $\operatorname{im}\alpha$, $\{v_j\}_{j\in J} \cup \{u_k\}_{k\in K}$ is a basis of $\operatorname{im}\beta$ and $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ is a basis of V.

Proof. Let $\{x_i\}_{i\in I}$ be a basis of $\ker\beta$. By Proposition 2.2.6, $\ker\beta \subseteq \ker\alpha$. Then extend $\{x_i\}_{i\in I}$ to a basis $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}$ of $\ker\alpha$. Let $\{u_k\}_{k\in K}$ be a basis of $\operatorname{im}\alpha$. Now we let $k \in K$. Since $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$, we have $u_k = z_k\beta$ for some $z_k \in V$. Thus

$z_k \in V\alpha\beta^{-1} = E(\alpha,\beta)$, so $z_k\alpha = z_k\beta = u_k$. Assume that $y_j\beta = v_j$ for all $j \in J$. Next, we show that $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ is a basis of V. Suppose that

$$\sum_i a_i x_i + \sum_j b_j y_j + \sum_k c_k z_k = 0$$

where a_i, b_j, c_k are scalars, $i \in I'$, $j \in J'$, $k \in K'$ and I', J', K' are finite subsets of I, J, K, respectively. Then

$$0 = 0\alpha = \sum_k c_k z_k \alpha = \sum_k c_k u_k,$$

and hence $c_k = 0$ for all $k \in K'$. Since $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}$ is a basis of $\ker\alpha$, we have $a_i = b_j = 0$ for all $i \in I'$ and $j \in J'$. Hence $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ is linearly independent. Let $v \in V$. Since $v\alpha \in \operatorname{im}\alpha$ and $\{u_k\}_{k\in K}$ is a basis of $\operatorname{im}\alpha$, we have

$$v\alpha = \sum_k c_k u_k = \sum_k c_k z_k \alpha$$

where c_k is a scalar, $k \in K'$ and K' is a finite subset of K. This implies that $v - \sum_k c_k z_k \in \ker\alpha$. Hence we can write

$$v - \sum_k c_k z_k = \sum_i a_i x_i + \sum_j b_j y_j$$

where a_i, b_j are scalars, $i \in I'$, $j \in J'$, and I', J' are finite subsets of I, J, respectively. Then $v \in \langle \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}\rangle$. So, $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ is a basis of V. Note that $\{v_j\}_{j\in J} \cup \{u_k\}_{k\in K} = (\{y_j\}_{j\in J} \cup \{z_k\}_{k\in K})\beta$ is a basis of $\operatorname{im}\beta$ by Proposition 2.2.4 (ii). Therefore, α and β can be written as desired. □

The following is a useful tool to verify when the condition $V\alpha\beta^{-1} = E(\alpha,\beta)$ holds, where $\alpha, \beta \in L(V)$.

Lemma 2.2.10. *Let* $\alpha, \beta \in L(V)$ *be such that* $\ker\beta \subseteq \ker\alpha$ *and let* A_1, A_2, A_3 *be disjoint linearly independent sets such that* A_1, $A_1 \cup A_2$ *and* $A_1 \cup A_2 \cup A_3$ *are bases of* $\ker\beta$, $\ker\alpha$ *and* V, *respectively. If* $v\alpha = v\beta$ *for all* $v \in A_3$, *then* $V\alpha\beta^{-1} = E(\alpha,\beta)$.

Proof. Assume that $v\alpha = v\beta$ for all $v \in A_3$. Since $E(\alpha, \beta) \subseteq V\alpha\beta^{-1}$, it remains to prove that $V\alpha\beta^{-1} \subseteq E(\alpha, \beta)$. Let $v \in V\alpha\beta^{-1}$. Then $v\beta = v'\alpha$ for some $v' \in V$. Hence we write

$$\begin{aligned} v &= \sum_i a_i x_i + \sum_j b_j y_j + \sum_k c_k z_k, \\ v' &= \sum_i a'_i x_i + \sum_j b'_j y_j + \sum_k c'_k z_k \end{aligned}$$

for some $x_i \in A_1$, $y_j \in A_2$, $z_k \in A_3$ and some scalars $a_i, a'_i, b_j, b'_j, c_k, c'_k$ where these summations are over finite index sets. Since $A_1 \subseteq \ker \beta$ and $A_1 \cup A_2 \subseteq \ker \alpha$, we obtain

$$\begin{aligned} v\beta &= \sum_j b_j y_j \beta + \sum_k c_k z_k \beta, \\ v'\alpha &= \sum_k c'_k z_k \alpha = \sum_k c'_k z_k \beta. \end{aligned}$$

By Proposition 2.2.4 (ii), $(A_2 \cup A_3)\beta$ is linearly independent. Since $v\beta = v'\alpha$, we have $b_j = 0$ for all $j \in J$. It follows that $v = \sum_i a_i x_i + \sum_k c_k z_k$. Thus $v\alpha = \sum_k c_k z_k \alpha = \sum_k c_k z_k \beta = v\beta$ since $z_k \in A_3$ for all $k \in K$. Hence $v \in E(\alpha, \beta)$. Therefore, $V\alpha\beta^{-1} = E(\alpha, \beta)$. □

The converse of this lemma is not true and the counterexample is provided in Remark 4.2.3. We end this section by an observation on the semigroups $AM(V)$ and $AE(V)$. Y. Kemprasit showed in [6] that if $\dim V$ is finite, then $AM(V) = AE(V) = L(V)$. Now we prove that the converse is also true.

Proposition 2.2.11. *$AM(V) = AE(V) = L(V)$ if and only if $\dim V$ is finite.*

Proof. We shall prove the sufficiency by contrapositive. Assume that $\dim V$ is infinite. Let B be a basis of V. Then there is a partition $\{B_1, B_2\}$ of B such that $|B| = |B_1| = |B_2|$. Let $u \in B_1$ and let $\phi : B \setminus \{u\} \to B_2$ be a bijection. Now define $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} u & v \\ 0 & v\phi \end{pmatrix}_{v \in B \setminus \{u\}} \quad \text{and} \quad \beta = \begin{pmatrix} B_1 & v \\ 0 & v\phi^{-1} \end{pmatrix}_{v \in B_2}.$$

Then nullity $\alpha = 1$ and corank $\alpha = |B \setminus B_2| = |B_1|$ by Propositions 2.2.3 (i) and 2.2.5 (ii), respectively. Thus $\alpha \in AM(V) \setminus AE(V)$. This implies that $AM(V) \neq AE(V)$ and $AE(V) \neq L(V)$. Moreover, by Propositions 2.2.3 (i) and 2.2.5 (ii), nullity $\beta = |B_1|$ and corank $\beta = |B \setminus (B \setminus \{u\})| = 1$, respectively. Hence $\beta \in AE(V) \setminus AM(V)$ and so $AM(V) \neq L(V)$.

The necessity is followed from [6]. □

2.3 A Glance on the Natural Partial Order

In this section, we provide some known results of the natural partial order on a semigroup. Now let S be a semigroup with the natural partial order $\leq$.

Proposition 2.3.1. [2] (i) *If S has the zero element, then it is the minimum element in S.*
(ii) *For any $s \in S$ and the identity 1 in S, $s \leq 1$ on S if and only if s is an idempotent in S.*
(iii) *For any subsemigroup T of S and $a, b \in T$, $a \leq b$ on T implies $a \leq b$ on S.*

The following propositions are very important in studying the natural partial order on a semigroup. P. M. Higgins showed in [4] that the natural partial orders on a semigroup and its regular subsemigroup coincide. Hence the study of nonregular semigroups is of interest.

Proposition 2.3.2. [4] *Let T be a regular subsemigroup of S and $a, b \in T$. Then $a \leq b$ on T if and only if $a \leq b$ on S.*

Proposition 2.3.3. *Let T be a regular subsemigroup of S and $x \in T$. If x is left [right] compatible on S, then x is left [right] compatible on T.*

Proof. Suppose that x is left [right] compatible on S. Let $a, b \in T$ be such that $a \leq b$ on T. Then $a \leq b$ on S. By assumption, $xa \leq xb$ $[ax \leq bx]$ on S. Since T is a regular subsemigroup of S, by Proposition 2.3.2, $xa \leq xb$ $[ax \leq bx]$ on T. Hence x is left [right] compatible on T. □

The converse of this proposition is not true in general as we will show in Example 4.2.1.

Proposition 2.3.4. *Let T be a subsemigroup of S and $a, b \in T$. If a is a lower cover of b in S, then a is a lower cover of b in T. In other words, if b is an upper cover of a in S, then b is an upper cover of a in T.*

Proof. Assume that a is a lower cover of b in S. Let $c \in T$ be such that $a \leq c \leq b$ on T. Then, by Proposition 2.3.1 (iii), $a \leq c \leq b$ on S. By assumption, $a = c$ or $c = b$. □

The converse of this proposition is also not true in general, see Example 4.4.8.

Let V be a vector space. In this thesis, for any subsemigroup $S(V)$ of $L(V)$, we consider $S(V)$ with the natural partial order $\leq$ and then we may write $(S(V), \leq)$ in short by $S(V)$. In [15], R. P. Sullivan gave a characterization of the natural partial order on $L(V)$. The proof of the converse of this theorem will be used, so we recall the proof of the sufficiency and omit the forward implication.

Theorem 2.3.5. [15] *Let $\alpha, \beta \in L(V)$. Then $\alpha \leq \beta$ on $L(V)$ if and only if $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha, \beta)$.*

Proof. Assume that $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha, \beta)$. From Proposition 2.2.9, α and β can be written as

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \quad \text{and} \quad \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & v_j & u_k \end{pmatrix}_{j\in J, k\in K}$$

with $\{x_i\}_{i\in I}$, $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}$, $\{u_k\}_{k\in K}$, $\{v_j\}_{j\in J} \cup \{u_k\}_{k\in K}$ and $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ are bases of $\ker\beta$, $\ker\alpha$, $\operatorname{im}\alpha$, $\operatorname{im}\beta$ and V, respectively. Then we define $\lambda, \mu \in L(V)$ by

$$\lambda = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & z_k \end{pmatrix}_{k\in K} \quad \text{and} \quad \mu = \begin{pmatrix} \{v_j\}_{j\in J} \cup \{w_l\}_{l\in L} & u_k \\ 0 & u_k \end{pmatrix}_{k\in K}$$

where $\{v_j\}_{j\in J} \cup \{u_k\}_{k\in K} \cup \{w_l\}_{l\in L}$ is a basis of V. For any $i \in I$, $j \in J$, $k \in K$,

$$0 = x_i\alpha = x_i\lambda\beta = x_i\beta\mu = x_i\alpha\mu,$$

$$0 = y_j\alpha = y_j\lambda\beta = y_j\beta\mu = y_j\alpha\mu,$$

$$u_k = z_k\alpha = z_k\lambda\beta = z_k\beta\mu = z_k\alpha\mu.$$

It follows that $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$. Hence $\alpha \leq \beta$ on $L(V)$. □

Let $\alpha, \beta \in L(V)$ be such that $\alpha \leq \beta$ on $L(V)$. Then, by Theorem 2.3.5, $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha, \beta)$. By Proposition 2.2.9, we can write α and β as

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \quad \text{and} \quad \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & v_j & u_k \end{pmatrix}_{j\in J, k\in K}$$

where $\{x_i\}_{i\in I}$ is a basis of $\ker\beta$, $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}$ is a basis of $\ker\alpha$, $\{u_k\}_{k\in K}$ is a basis of $\operatorname{im}\alpha$ and $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ is a basis of V. That is, β is different from α by sending $\{y_j\}_{j\in J}$ to $\{v_j\}_{j\in J}$. Hence we illustrate α and β as follows.

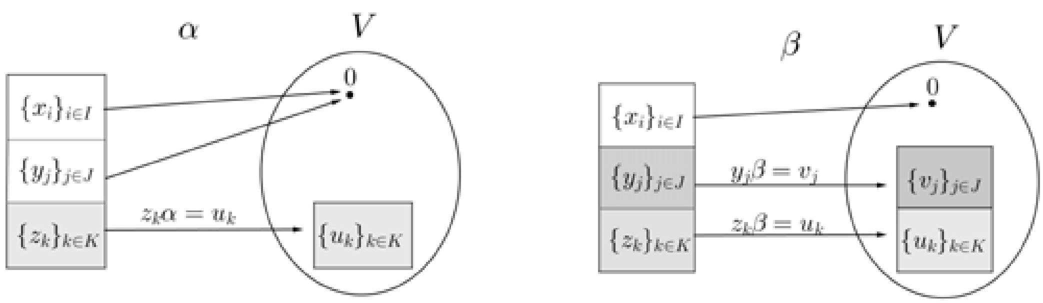

Figure 2.4: $\alpha \leq \beta$ on $L(V)$.

Furthermore, R. P. Sullivan [15] described left and right compatible elements in $L(V)$.

Theorem 2.3.6. [15] *Let* $\dim V \geq 2$ *and let* $\gamma \in L(V)$ *be nonzero. Then the following statements hold.*

(i) γ *is left compatible on* $L(V)$ *if and only if* γ *is an epimorphism.*

(ii) γ *is right compatible on* $L(V)$ *if and only if* γ *is a monomorphism.*

Remark 2.3.7. $(i) \leq$ is not compatible on $L(V)$ where $\dim V \geq 2$ since an element in $L(V)$ which is neither injective nor surjective is not compatible on $L(V)$.
(ii) If $\dim V = 0$, then $L(V) = \{0_V\}$ and it is clear that 0_V is compatible on $L(V)$.
(iii) Suppose that $\dim V = 1$. Claim that, for any $\alpha, \beta \in L(V)$, if $\alpha \leq \beta$ on $L(V)$, then either $\alpha = \beta$ or $\alpha = 0_V$ and $\operatorname{rank}\beta = 1$. Let $\alpha, \beta \in L(V)$ be such that $\alpha \leq \beta$ on $L(V)$ and $\operatorname{rank}\alpha = \operatorname{rank}\beta = 1$. Then $\operatorname{im}\alpha = \operatorname{im}\beta$. By Theorem 2.3.5 and Proposition 2.2.7, $\alpha = \beta$. Hence the claim is proven. Therefore, $\leq$ is compatible on $L(V)$.

Note that 0_V is the minimum element in $L(V)$ by Proposition 2.3.1 (i). Hence minimal nonzero elements in $L(V)$ is of interest by R. P. Sullivan.

Theorem 2.3.8. [15] *Let $\alpha \in L(V)$. Then*
(i) *α is a minimal nonzero element in $L(V)$ if and only if* $\operatorname{rank}\alpha = 1$.
(ii) *α is a maximal element in $L(V)$ if and only if α is a monomorphism or an epimorphism.*

Theorems 2.3.5, 2.3.6 and 2.3.8 will be used to compare to our main results in the next chapters. Notice that many proofs in the remaining chapters always refer to Propositions 2.2.1–2.2.4. For convenience, we sometimes omit these details.

CHAPTER III

THE SEMIGROUPS $AM(V)$ AND $AE(V)$

Recall that, for a vector space V,

$$AM(V) = \{\alpha \in L(V) \mid \operatorname{nullity} \alpha \text{ is finite}\},$$

$$AE(V) = \{\alpha \in L(V) \mid \operatorname{corank} \alpha \text{ is finite}\}.$$

By Proposition 2.2.11, we know that $AM(V) = AE(V) = L(V)$ if and only if $\dim V$ is finite. Then we study the semigroups $AM(V)$ and $AE(V)$ when $\dim V$ is infinite, which also implies that $AM(V)$ and $AE(V)$ are nonregular semigroups. Since $1_V \in AM(V) \cap AE(V)$, we have $AM(V)^1 = AM(V)$ and $AE(V)^1 = AE(V)$.

Our main purpose in this chapter is to give necessary and sufficient conditions for elements in the semigroups $AM(V)$ and $AE(V)$ to be comparable under the natural partial order $\leq$. Then we provide characterizations for elements in $AM(V)$ and $AE(V)$ to be left and right compatible elements, minimal and maximal elements. In addition, lower and upper covers of elements in $L(V)$, $AM(V)$ and $AE(V)$ are also described. Throughout this chapter, unless stated otherwise, let V be an infinite dimensional vector space. Furthermore, let $S(V)$ stand for $AM(V)$ or $AE(V)$.

3.1 The Natural Partial Orders on $AM(V)$ and $AE(V)$

We first provide the necessary and sufficient conditions for elements in $S(V)$ to be related under the natural partial order.

Theorem 3.1.1. *Let $\alpha, \beta \in S(V)$. Then $\alpha \leq \beta$ on $S(V)$ if and only if* $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$ *and $V\alpha\beta^{-1} = E(\alpha, \beta)$.*

Proof. Suppose that $\alpha \leq \beta$ on $S(V)$. Then, by Proposition 2.3.1 (iii), $\alpha \leq \beta$ on $L(V)$ since $S(V)$ is a subsemigroup of $L(V)$. By Theorem 2.3.5, $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$

and $V\alpha\beta^{-1} = E(\alpha, \beta)$.

Conversely, assume the conditions hold. It follows from Proposition 2.2.9 that

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \text{ and } \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & v_j & u_k \end{pmatrix}_{j\in J, k\in K}$$

where $\{x_i\}_{i\in I}$, $\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}$, $\{u_k\}_{k\in K}$, $\{v_j\}_{j\in J}\cup\{u_k\}_{k\in K}$ and $\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}\cup\{z_k\}_{k\in K}$ are bases of $\ker\beta$, $\ker\alpha$, $\operatorname{im}\alpha$, $\operatorname{im}\beta$ and V, respectively. Let $\{v_j\}_{j\in J} \cup \{u_k\}_{k\in K} \cup \{w_l\}_{l\in L}$ be a basis of V. Next, define $\lambda, \mu \in L(V)$ by

$$\lambda = \begin{pmatrix} \{y_j\}_{j\in J} & v \\ 0 & v \end{pmatrix}_{v\in\{x_i\}_{i\in I}\cup\{z_k\}_{k\in K}} \text{ and } \mu = \begin{pmatrix} \{v_j\}_{j\in J} & v \\ 0 & v \end{pmatrix}_{v\in\{u_k\}_{k\in K}\cup\{w_l\}_{l\in L}}.$$

For any $i \in I$, $j \in J$, $k \in K$,

$$0 = x_i\alpha = x_i\lambda\beta = x_i\beta\mu = x_i\alpha\mu,$$

$$0 = y_j\alpha = y_j\lambda\beta = y_j\beta\mu = y_j\alpha\mu,$$

$$u_k = z_k\alpha = z_k\lambda\beta = z_k\beta\mu = z_k\alpha\mu.$$

Therefore, $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$. We claim that $\lambda, \mu \in S(V)$. From the definition of β, $|\{y_j\}_{j\in J}| = |\{v_j\}_{j\in J}|$ by Proposition 2.2.4 (i). If $S(V) = AM(V)$, then

$$\operatorname{nullity}\lambda = |\{y_j\}_{j\in J}| \leq |\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}| = \operatorname{nullity}\alpha < \infty$$

and $\operatorname{nullity}\mu = |\{v_j\}_{j\in J}| = |\{y_j\}_{j\in J}| < \infty$. Thus $\lambda, \mu \in AM(V)$. If $S(V) = AE(V)$, then

$$\begin{aligned} \operatorname{corank}\lambda &= |\{y_j\}_{j\in J}| = |\{v_j\}_{j\in J}| \\ &\leq |\{v_j\}_{j\in J} \cup \{w_l\}_{l\in L}| \\ &= \operatorname{corank}\alpha < \infty \end{aligned}$$

and $\operatorname{corank}\mu = |\{v_j\}_{j\in J}| < \infty$, so $\lambda, \mu \in AE(V)$. Hence $\alpha \leq \beta$ on $S(V)$. $\square$

Next, we show that Theorem 3.1.1 cannot be directly proven by the proof of Theorem 2.3.5 although the conditions of these theorems are the same.

Example 3.1.2. Let B be a basis of V and $u \in B$. Since $|B \smallsetminus \{u\}| = |B|$, there is a partition $\{B_1, B_2\}$ of $B \smallsetminus \{u\}$ such that the cardinalities of $B \smallsetminus \{u\}$, B_1 and B_2 are equal. Then there exists a bijection $\phi : B \smallsetminus \{u\} \to B_2$.

(i) Define $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} u & v \\ 0 & v\phi \end{pmatrix}_{v \in B \smallsetminus \{u\}} \quad \text{and} \quad \beta = \begin{pmatrix} u & v \\ u & v\phi \end{pmatrix}_{v \in B \smallsetminus \{u\}}.$$

Hence, by Proposition 2.2.3 (i) and β is a monomorphism, $\operatorname{nullity} \alpha = 1$ and $\alpha, \beta \in AM(V)$. Since $\varnothing$, $\{u\}$ and $B \smallsetminus \{u\}$ satisfy Lemma 2.2.10, we have $V\alpha\beta^{-1} = E(\alpha, \beta)$. It is clear that $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$. Therefore $\alpha \leq \beta$ on $AM(V)$ by Theorem 3.1.1. According to the proof of Theorem 2.3.5, there exist $\lambda, \mu \in L(V)$ of the forms

$$\lambda = \begin{pmatrix} u & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus \{u\}} \quad \text{and} \quad \mu = \begin{pmatrix} B_1 \cup \{u\} & v \\ 0 & v \end{pmatrix}_{v \in B_2}$$

such that $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$. By Proposition 2.2.3 (i), we obtain $\operatorname{nullity} \mu = |B_1 \cup \{u\}|$ is infinite and so $\mu \notin AM(V)$.

(ii) Define $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 \cup \{u\} & v \\ 0 & v\phi^{-1} \end{pmatrix}_{v \in B_2} \quad \text{and} \quad \beta = \begin{pmatrix} B_1 & u & v \\ 0 & u & v\phi^{-1} \end{pmatrix}_{v \in B_2}.$$

Then β is an epimorphism and $\operatorname{corank} \alpha = |B \smallsetminus (B \smallsetminus \{u\})| = 1$ by Proposition 2.2.5 (ii). Hence $\alpha, \beta \in AE(V)$. The sets B_1, $\{u\}$ and B_2 fulfill Lemma 2.2.10, so $V\alpha\beta^{-1} = E(\alpha, \beta)$. Moreover, $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$. Thus $\alpha \leq \beta$ on $AE(V)$ by Theorem 3.1.1. Therefore, by the proof of Theorem 2.3.5, there exist $\lambda, \mu \in L(V)$ in the forms

$$\lambda = \begin{pmatrix} B_1 \cup \{u\} & v \\ 0 & v \end{pmatrix}_{v \in B_2} \quad \text{and} \quad \mu = \begin{pmatrix} u & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus \{u\}}$$

such that $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$. However, $\operatorname{corank} \lambda = |B \smallsetminus B_2|$ is infinite, so $\lambda \notin AE(V)$.

By Theorems 2.3.5 and 3.1.1, the natural partial order on $S(V)$ can be derived from the natural partial order on $L(V)$.

Corollary 3.1.3. *Let $\alpha, \beta \in S(V)$. Then $\alpha \leq \beta$ on $S(V)$ if and only if $\alpha \leq \beta$ on $L(V)$.*

Observe that the natural partial orders on $S(V)$ and $L(V)$ are the same even if $S(V)$ is nonregular.

Theorem 3.1.1 can be used to check easily whether elements in $S(V)$ are related under the natural partial order. In Example 3.1.2, we give $\alpha, \beta \in S(V)$ with $\alpha \leq \beta$ on $S(V)$. The following shows that there are infinitely many elements in $(S(V), \leq)$.

Example 3.1.4. Let B be a basis of V and let B_1, B_2 be disjoint finite subsets of B. Define $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 \cup B_2 & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus (B_1 \cup B_2)} \quad \text{and} \quad \beta = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus B_1}.$$

Then $\operatorname{nullity} \alpha = |B_1 \cup B_2| = \operatorname{corank} \alpha$ and $\operatorname{nullity} \beta = |B_1| = \operatorname{corank} \beta$. These cardinal numbers are finite, and hence $\alpha, \beta \in AM(V) \cap AE(V)$. Observe that $V\alpha\beta^{-1} = E(\alpha, \beta)$ since the sets B_1, B_2 and $B \smallsetminus (B_1 \cup B_2)$ satisfy Lemma 2.2.10. Moreover, $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$. Therefore, by Theorem 3.1.1, we have $\alpha \leq \beta$ on $S(V)$.

The following proposition shows that comparable elements in $AM(V)$ must be both in either $AM(V) \cap AE(V)$ or $AM(V) \smallsetminus AE(V)$.

Proposition 3.1.5. *Let $\alpha, \beta \in AM(V)$ be such that $\alpha \leq \beta$ on $AM(V)$. Then $\alpha \in AE(V)$ if and only if $\beta \in AE(V)$.*

Proof. For the forward implication, suppose that $\alpha \in AE(V)$. Since $\alpha \leq \beta$ on $AM(V)$, we obtain $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$ by Theorem 3.1.1. It follows that $\operatorname{corank} \beta \leq \operatorname{corank} \alpha < \infty$, so $\beta \in AE(V)$.

Conversely, assume that $\alpha \notin AE(V)$. Since $\alpha \leq \beta$ on $AM(V)$, by Theorem 3.1.1, $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$ and $V\alpha\beta^{-1} = E(\alpha, \beta)$. From Proposition 2.2.9, we have

$$\alpha = \begin{pmatrix} \{x_i\}_{i \in I} \cup \{y_j\}_{j \in J} & z_k \\ 0 & u_k \end{pmatrix}_{k \in K} \quad \text{and} \quad \beta = \begin{pmatrix} \{x_i\}_{i \in I} & y_j & z_k \\ 0 & v_j & u_k \end{pmatrix}_{j \in J, k \in K}$$

with $\{x_i\}_{i\in I}$, $\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}$, $\{u_k\}_{k\in K}$, $\{v_j\}_{j\in J}\cup\{u_k\}_{k\in K}$ and $\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}\cup\{z_k\}_{k\in K}$ are bases of $\ker\beta$, $\ker\alpha$, $\operatorname{im}\alpha$, $\operatorname{im}\beta$ and V, respectively. Let $\{v_j\}_{j\in J}\cup\{u_k\}_{k\in K}\cup\{w_l\}_{l\in L}$ be a basis of V. Since $\alpha\in AM(V)$, we get $\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}$ is finite. Then $\{y_j\}_{j\in J}$ is a finite set, and so is $\{v_j\}_{j\in J}$. Since $\alpha\notin AE(V)$, we have

$$|\{v_j\}_{j\in J}\cup\{w_l\}_{l\in L}| = |(\{v_j\}_{j\in J}\cup\{u_k\}_{k\in K}\cup\{w_l\}_{l\in L})\setminus\{u_k\}_{k\in K}| = \operatorname{corank}\alpha$$

is infinite. Hence $\operatorname{corank}\beta = |\{w_l\}_{l\in L}|$ is infinite, which implies $\beta\notin AE(V)$. □

Similarly, for any $\alpha,\beta\in AE(V)$ such that $\alpha\leq\beta$ on $AE(V)$, we have $\alpha,\beta\in AM(V)\cap AE(V)$ or $\alpha,\beta\in AE(V)\setminus AM(V)$.

Proposition 3.1.6. *Let $\alpha,\beta\in AE(V)$ be such that $\alpha\leq\beta$ on $AE(V)$. Then $\alpha\in AM(V)$ if and only if $\beta\in AM(V)$.*

Proof. To prove the forward implication, assume that $\alpha\in AM(V)$. By Theorem 3.1.1, $V\alpha\beta^{-1} = E(\alpha,\beta)$. Then $\ker\beta\subseteq\ker\alpha$ by Proposition 2.2.6. Hence $\operatorname{nullity}\beta\leq\operatorname{nullity}\alpha<\infty$ and so $\beta\in AM(V)$.

For the converse, suppose that $\alpha\notin AM(V)$. As $\alpha\leq\beta$ on $AE(V)$, by Theorem 3.1.1, $\operatorname{im}\alpha\subseteq\operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha,\beta)$. By Proposition 2.2.9,

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I}\cup\{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \quad\text{and}\quad \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & v_j & u_k \end{pmatrix}_{j\in J, k\in K}$$

where $\{x_i\}_{i\in I}$, $\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}$, $\{u_k\}_{k\in K}$, $\{v_j\}_{j\in J}\cup\{u_k\}_{k\in K}$ and $\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}\cup\{z_k\}_{k\in K}$ are bases of $\ker\beta$, $\ker\alpha$, $\operatorname{im}\alpha$, $\operatorname{im}\beta$ and V, respectively. Let $\{v_j\}_{j\in J}\cup\{u_k\}_{k\in K}\cup\{w_l\}_{l\in L}$ be a basis of V. Since $\alpha\in AE(V)\setminus AM(V)$, we get $\operatorname{corank}\alpha = |\{v_j\}_{j\in J}\cup\{w_l\}_{l\in L}|<\infty$ and $\operatorname{nullity}\alpha = |\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}|$ is infinite. By Proposition 2.2.4 (i), $|\{y_j\}_{j\in J}| = |\{v_j\}_{j\in J}|$ which is finite. Since $|\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}|$ is infinite and $|\{y_j\}_{j\in J}|$ is finite, $\operatorname{nullity}\beta = |\{x_i\}_{i\in I}|$ is infinite. Therefore, $\beta\notin AM(V)$. □

We provide Figures 3.1 and 3.2 to demonstrate the above propositions.

By Propositions 3.1.5 and 3.1.6, elements in $AM(V)\cap AE(V)$ force its related element to be in $AM(V)\cap AE(V)$. Hence the following corollary holds.

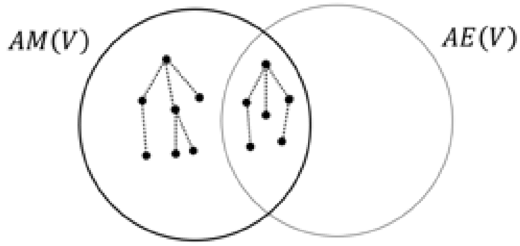

Figure 3.1: An example of elements in $(AM(V), \leq)$.

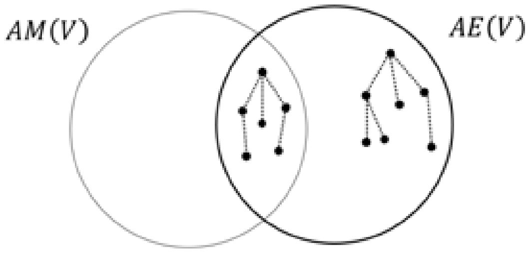

Figure 3.2: An example of elements in $(AE(V), \leq)$.

Corollary 3.1.7. *Let $\alpha, \beta \in AM(V) \cap AE(V)$. Then the following are equivalent.*
(i) $\alpha \leq \beta$ on $AM(V)$,
(ii) $\alpha \leq \beta$ on $AE(V)$,
(iii) $\alpha \leq \beta$ on $L(V)$.

Proposition 3.1.8. *Let $\alpha, \beta \in L(V)$ be such that $\alpha \leq \beta$ on $L(V)$.*
(i) If $\alpha \in AM(V)$, then $\beta \in AM(V)$.
(ii) If $\alpha \in AE(V)$, then $\beta \in AE(V)$.

Proof. Since $\alpha \leq \beta$ on $L(V)$, by Theorem 2.3.5, we have $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$ and $V\alpha\beta^{-1} = E(\alpha, \beta)$. From Proposition 2.2.6, we obtain $\ker \beta \subseteq \ker \alpha$.
(i) If $\alpha \in AM(V)$, then $\operatorname{nullity} \beta \leq \operatorname{nullity} \alpha < \infty$, so $\beta \in AM(V)$.
(ii) If $\alpha \in AE(V)$, then

$$\operatorname{corank} \beta = \dim(V/\operatorname{im} \beta) \leq \dim(V/\operatorname{im} \alpha) \leq \operatorname{corank} \alpha < \infty,$$

so $\beta \in AE(V)$. □

The converse of this proposition may not be true as in the below example.

Example 3.1.9. Let $\beta \in S(V)$. Since 0_V is the minimum element in $L(V)$ by Proposition 2.3.1 (i), we get $0_V \leq \beta$ on $L(V)$. However $0_V \notin S(V)$ as $\operatorname{nullity} 0_V = \operatorname{corank} 0_V = \dim V$ is infinite.

Furthermore, we obtain a result, followed from Corollary 3.1.7 and Proposition 3.1.8.

Corollary 3.1.10. *Let $\alpha \in AM(V) \cap AE(V)$ and $\beta \in L(V)$. Then the following are equivalent.*
(i) $\alpha \leq \beta$ on $AM(V)$,
(ii) $\alpha \leq \beta$ on $AE(V)$,
(iii) $\alpha \leq \beta$ on $L(V)$.

3.2 Left and Right Compatible Elements in $(AM(V), \leq)$ and $(AE(V), \leq)$

Left and right compatibility of elements in $L(V)$ are characterized in [15] as mentioned in Theorem 2.3.6 whenever $\dim V \geq 2$, that is, for a nonzero $\gamma \in L(V)$, γ is left [right] compatible on $L(V)$ if and only if γ is an epimorphism [monomorphism]. In this section, we show that characterizations of left and right compatible elements in $S(V)$ and $L(V)$ coincide. Recall that $1_V \in S(V)$.

Theorem 3.2.1. *Let $\gamma \in S(V)$. Then*
(i) γ is left compatible on $S(V)$ if and only if γ is an epimorphism.
(ii) γ is right compatible on $S(V)$ if and only if γ is a monomorphism.

Proof. (i) Suppose that γ is left compatible on $S(V)$. Choose $z \in \operatorname{im}\gamma \setminus \{0\}$. Let B be a basis of V containing z and let $u \in B \setminus \{z\}$. Define $\alpha \in S(V)$ by letting

$$\alpha = \begin{pmatrix} z & v \\ u & v \end{pmatrix}_{v \in B \setminus \{z\}}.$$

Since $z\alpha^2 = u\alpha = u = z\alpha$ and $v\alpha^2 = v\alpha$ for all $v \in B \setminus \{z\}$, we get α is an idempotent. Then $\alpha \leq 1_V$ on $S(V)$ by Proposition 2.3.1 (ii). Since γ is left compatible on $S(V)$, we have $\gamma\alpha \leq \gamma$ on $S(V)$. Thus $\operatorname{im}\gamma\alpha \subseteq \operatorname{im}\gamma$ by Theorem 3.1.1. Since $z \in \operatorname{im}\gamma$ and $u = z\alpha$, we obtain $u \in \operatorname{im}\gamma\alpha \subseteq \operatorname{im}\gamma$. As u is an arbitrary element in $B \setminus \{z\}$, we get $B \subseteq \operatorname{im}\gamma$. Therefore γ is an epimorphism.

Conversely, assume that γ is an epimorphism. Then γ is left compatible on $L(V)$ by Theorem 2.3.6 (i). Let $\alpha, \beta \in S(V)$ be such that $\alpha \leq \beta$ on $S(V)$. Then $\alpha \leq \beta$ on $L(V)$, so $\gamma\alpha \leq \gamma\beta$ on $L(V)$. Hence $\gamma\alpha \leq \gamma\beta$ on $S(V)$ by Corollary 3.1.3.

(ii) Assume that γ is right compatible on $S(V)$ and γ is not a monomorphism. Choose $z \in \ker\gamma \setminus \{0\}$. Let B be a basis of V containing z and let $u \in B \setminus \{z\}$. Define $\alpha \in S(V)$ by

$$\alpha = \begin{pmatrix} z & v \\ u & v \end{pmatrix}_{v \in B \setminus \{z\}}.$$

Then α is an idempotent. By Proposition 2.3.1 (ii), we have $\alpha \leq 1_V$ on $S(V)$. Thus $\alpha\gamma \leq \gamma$ on $S(V)$ by the right compatibility of γ. It follows that $\alpha\gamma = \gamma\mu$ for some $\mu \in S(V)$. Then

$$u\gamma = z\alpha\gamma = z\gamma\mu = 0,$$

so $u \in \ker\gamma$. Since $u \in \ker\gamma$ for all $u \in B \setminus \{z\}$, we have $B \subseteq \ker\gamma$. Hence $\gamma = 0_V \notin S(V)$, a contradiction. Therefore, γ is a monomorphism.

For the converse, suppose that γ is a monomorphism. By Theorem 2.3.6 (ii), γ is right compatible on $L(V)$. Let $\alpha, \beta \in S(V)$ be such that $\alpha \leq \beta$ on $S(V)$. Similar to (i), $\alpha\gamma \leq \beta\gamma$ on $S(V)$ by Corollary 3.1.3. □

Corollary 3.2.2. *Let $\gamma \in S(V)$. Then γ is left [right] compatible on $S(V)$ if and only if γ is left [right] compatible on $L(V)$.*

There are infinitely many left and right compatible elements in $S(V)$ as in the following example.

Example 3.2.3. Let B be a basis of V and B_1 a nonempty finite subset of B. Since $|B| = |B \setminus B_1|$, there exists a bijection $\phi : B \to B \setminus B_1$. Define $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v\phi^{-1} \end{pmatrix}_{v \in B \setminus B_1} \quad \text{and} \quad \beta = \begin{pmatrix} v \\ v\phi \end{pmatrix}_{v \in B}.$$

Then α is an epimorphism, β is a monomorphism, nullity $\alpha = |B_1| < \infty$ and

$$\operatorname{corank} \beta = |B \smallsetminus (B \smallsetminus B_1)| = |B_1| < \infty.$$

Hence $\alpha, \beta \in AM(V) \cap AE(V)$. By Theorem 3.2.1, α is left compatible on $S(V)$ and β is right compatible on $S(V)$.

If we have related elements in $S(V)$, we can construct a new one by using left and right compatible elements as follows.

Example 3.2.4. Let B be a basis of V and let $u_1, u_2 \in B$ be distinct. Define $\alpha, \beta \in AM(V) \cap AE(V)$ by

$$\alpha = \begin{pmatrix} \{u_1, u_2\} & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus \{u_1, u_2\}} \quad \text{and} \quad \beta = \begin{pmatrix} u_1 & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus \{u_1\}}.$$

Then $\alpha \leq \beta$ on $S(V)$ by Theorem 3.1.1. Since $|B| = |B \smallsetminus \{u_1, u_2\}|$, we let $\phi : B \to B \smallsetminus \{u_1, u_2\}$ be a bijection. Define $\gamma, \delta \in L(V)$ by

$$\gamma = \begin{pmatrix} \{u_1, u_2\} & v \\ 0 & v\phi^{-1} \end{pmatrix}_{v \in B \smallsetminus \{u_1, u_2\}} \quad \text{and} \quad \delta = \begin{pmatrix} v \\ v\phi \end{pmatrix}_{v \in B}.$$

Then $\gamma, \delta \in S(V)$, γ is left compatible on $S(V)$ and δ is right compatible on $S(V)$ by Theorem 3.2.1. Hence $\gamma\alpha \leq \gamma\beta$ and $\alpha\delta \leq \beta\delta$ on $S(V)$. Since ϕ^{-1} is surjective, there are $x_1, x_2 \in B \smallsetminus \{u_1, u_2\}$ such that $x_1\phi^{-1} = u_1$ and $x_2\phi^{-1} = u_2$. Note that

$$\gamma\alpha = \begin{pmatrix} \{u_1, u_2, x_1, x_2\} & v \\ 0 & v\phi^{-1} \end{pmatrix}_{v \in B \smallsetminus \{u_1, u_2, x_1, x_2\}}, \quad \gamma\beta = \begin{pmatrix} \{u_1, u_2, x_1\} & v \\ 0 & v\phi^{-1} \end{pmatrix}_{v \in B \smallsetminus \{u_1, u_2, x_1\}},$$

$$\alpha\delta = \begin{pmatrix} \{u_1, u_2\} & v \\ 0 & v\phi \end{pmatrix}_{v \in B \smallsetminus \{u_1, u_2\}} \quad \text{and} \quad \beta\delta = \begin{pmatrix} u_1 & v \\ 0 & v\phi \end{pmatrix}_{v \in B \smallsetminus \{u_1\}}.$$

It can be observed that $\gamma\alpha \neq \alpha$, $\alpha\delta \neq \alpha$, $\gamma\beta \neq \beta$, $\beta\delta \neq \beta$.

3.3 Minimal and Maximal Elements in $(AM(V), \leq)$ and $(AE(V), \leq)$

In this section, we focus on minimal, minimum, maximal and maximum elements in $S(V)$. The following theorem shows that there are no minimal elements in $S(V)$.

Theorem 3.3.1. *$S(V)$ has no minimal element.*

Proof. Let $\beta \in S(V)$ and B_1 a basis of $\ker\beta$. Extend B_1 to a basis B of V. Note that $B_1 \neq B$ and $(B \smallsetminus B_1)\beta$ is linearly independent, since $0_V \notin S(V)$ and by Proposition 2.2.4 (ii), respectively. Let C be a basis of V containing $(B \smallsetminus B_1)\beta$. Now we let $w \in (B \smallsetminus B_1)\beta$. Then $w = u\beta$ for some $u \in B \smallsetminus B_1$. Define $\alpha \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 \cup \{u\} & v \\ 0 & v\beta \end{pmatrix}_{v \in B \smallsetminus (B_1 \cup \{u\})}.$$

By using B_1, $\{u\}$ and $B \smallsetminus (B_1 \cup \{u\})$ in Lemma 2.2.10, we have $V\alpha\beta^{-1} = E(\alpha, \beta)$. If $u\beta \in \operatorname{im}\alpha$, then $u \in V\alpha\beta^{-1} = E(\alpha,\beta)$ and hence $u\beta = u\alpha = 0$, which is a contradiction. Thus $u\beta \notin \operatorname{im}\alpha$. This implies $\operatorname{im}\alpha \subsetneq \operatorname{im}\beta$, and that $\alpha \neq \beta$. If $S(V) = AM(V)$, then B_1 is finite and so is $|B_1 \cup \{u\}| = \operatorname{nullity}\alpha$. Thus $\alpha \in AM(V)$. If $S(V) = AE(V)$, then

$$|C \smallsetminus (B \smallsetminus B_1)\beta| = \operatorname{corank}\beta < \infty.$$

Hence

$$\operatorname{corank}\alpha = |C \smallsetminus (B \smallsetminus (B_1 \cup \{u\}))\beta| = \operatorname{corank}\beta + 1 < \infty,$$

so $\beta \in AE(V)$. Therefore $\alpha < \beta$ on $S(V)$ by Theorem 3.1.1. Hence $(S(V), \leq)$ has no minimal element. $\square$

From this theorem, $S(V)$ has no minimum element. Next, we give necessary and sufficient conditions for elements in $S(V)$ to be maximal.

Theorem 3.3.2. *Let $\alpha \in S(V)$. Then α is maximal in $S(V)$ if and only if α is a monomorphism or an epimorphism.*

Proof. To show the necessity, assume that α is neither a monomorphism nor an epimorphism. Let $w \in V \smallsetminus \operatorname{im}\alpha$, $u \in V \smallsetminus \{0\}$ and let B_1 be a basis of $\ker\alpha$ containing u. Extend B_1 to a basis B of V. Define $\beta \in L(V)$ by

$$\beta = \begin{pmatrix} B_1 \smallsetminus \{u\} & u & v \\ 0 & w & v\alpha \end{pmatrix}_{v \in B \smallsetminus B_1}.$$

Thus $\operatorname{im}\alpha \subsetneq \operatorname{im}\beta$ and hence $\alpha \neq \beta$. By using $B_1 \setminus \{u\}$, $\{u\}$ and $B \setminus B_1$ in Lemma 2.2.10, we obtain $V\alpha\beta^{-1} = E(\alpha, \beta)$. If $S(V) = AM(V)$, then

$$\operatorname{nullity}\beta = |B_1 \setminus \{u\}| \leq |B_1| = \operatorname{nullity}\alpha < \infty.$$

So, $\beta \in AM(V)$. If $S(V) = AE(V)$, then $\operatorname{corank}\beta \leq \operatorname{corank}\alpha < \infty$ and hence $\beta \in AE(V)$. By Theorem 3.1.1, $\alpha < \beta$ on $S(V)$. Therefore α is not maximal in $S(V)$.

For the sufficiency, suppose that α is a monomorphism or an epimorphism. By Theorem 2.3.8 (*ii*), α is maximal in $L(V)$. Hence α is also maximal in $S(V)$. □

Remark 3.3.3. Since there are many distinct monomorphisms and epimorphisms in $S(V)$, which are maximal in $S(V)$, we have $(S(V), \leq)$ has no maximum element.

The next corollary is obtained from Proposition 2.3.8 (*ii*) and Theorems 3.2.1 and 3.3.2.

Corollary 3.3.4. *Let $\alpha \in S(V)$. Then the following statements hold.*
(i) α is maximal in $S(V)$ if and only if α is maximal in $L(V)$.
(ii) α is maximal in $S(V)$ if and only if α is left compatible or right compatible on $S(V)$.

3.4 Lower and Upper Covers of Elements in $(L(V), \leq)$, $(AM(V), \leq)$ and $(AE(V), \leq)$

Lower and upper covers of elements can be used to illustrate a diagram of partially ordered sets. In this section, we first consider when an element in $L(V)$ has a lower cover and an upper cover and then consider the semigroup $S(V)$. Note that when dealing with the semigroup $L(V)$ we assume V is a general vector space, and when considering on $S(V)$ we suppose that $\dim V$ is infinite. The following proposition shows the necessary and sufficient conditions for α in $L(V)$ to be a lower cover of β in $L(V)$.

Lemma 3.4.1. *Let* $\alpha, \beta \in L(V)$ *such that* $\alpha < \beta$ *on* $L(V)$. *Then* α *is a lower cover of* β *in* $L(V)$ *if and only if* $\dim(\ker\alpha/\ker\beta) = 1$. *In other words,* β *is an upper cover of* α *in* $L(V)$ *if and only if* $\dim(\ker\alpha/\ker\beta) = 1$.

Proof. Assume that $\dim(\ker\alpha/\ker\beta) = 1$. Suppose that $\alpha < \gamma \leq \beta$ on $L(V)$ for some $\gamma \in L(V)$. Thus $V\alpha\gamma^{-1} = E(\alpha,\gamma)$ and $V\gamma\beta^{-1} = E(\gamma,\beta)$ by Theorem 2.3.5. It follows that $\ker\beta \subseteq \ker\gamma \subsetneq \ker\alpha$ by Remark 2.2.8. Let B_1 be a basis of $\ker\beta$. Extend it to a basis B_2 of $\ker\alpha$. Since $\dim(\ker\alpha/\ker\beta) = 1$, we have $|B_2 \smallsetminus B_1| = 1$. Let $u \in B_2 \smallsetminus B_1$. Thus $B_2 = B_1 \cup \{u\}$. Note that $u \notin \ker\gamma$ since $\ker\gamma \subsetneq \ker\alpha$. Let $w \in \ker\gamma$. Thus $w \in \ker\alpha$. We write $w = \sum\limits_i a_i x_i + bu$ where $x_i \in B_1$, a_i, b are scalars and i is an element in a finite index set I. Then $0 = w\gamma = 0 + bu\gamma$. Since $u\gamma \neq 0$, we get $b = 0$. Hence $w = \sum\limits_i a_i x_i \in \ker\beta$, and that $\ker\gamma = \ker\beta$. As $\gamma \leq \beta$ on $L(V)$, we obtain $\operatorname{im}\gamma \subseteq \operatorname{im}\beta$. Since $\ker\gamma = \ker\beta$, $\operatorname{im}\gamma \subseteq \operatorname{im}\beta$ and $V\gamma\beta^{-1} = E(\gamma,\beta)$, by Proposition 2.2.7, we have $\gamma = \beta$.

For the forward implication, suppose that $\dim(\ker\alpha/\ker\beta) \neq 1$. As $\alpha < \beta$ on $L(V)$, from Theorem 2.3.5, $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha,\beta)$. By Remark 2.2.8, we obtain that $\dim(\ker\alpha/\ker\beta) > 1$ since $\alpha \neq \beta$. By Proposition 2.2.9, α and β can be written as

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \quad \text{and} \quad \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & v_j & u_k \end{pmatrix}_{j\in J, k\in K}$$

where $\{x_i\}_{i\in I}$, $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}$, $\{u_k\}_{k\in K}$, $\{v_j\}_{j\in J} \cup \{u_k\}_{k\in K}$ and $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ are bases of $\ker\beta$, $\ker\alpha$, $\operatorname{im}\alpha$, $\operatorname{im}\beta$ and V, respectively. Since $|\{y_j\}_{j\in J}| = \dim(\ker\alpha/\ker\beta) > 1$, let $j_0 \in J$. Define $\gamma \in L(V)$ by

$$\gamma = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J\smallsetminus\{j_0\}} & y_{j_0} & z_k \\ 0 & v_{j_0} & u_k \end{pmatrix}_{k\in K}.$$

Then $\operatorname{im}\alpha \subsetneq \operatorname{im}\gamma \subsetneq \operatorname{im}\beta$. Moreover, $V\alpha\gamma^{-1} = E(\alpha,\gamma)$ and $V\gamma\beta^{-1} = E(\gamma,\beta)$ hold since the sets $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J\smallsetminus\{j_0\}}$, $\{y_{j_0}\}$ and $\{z_k\}_{k\in K}$, and the sets $\{x_i\}_{i\in I}$, $\{y_j\}_{j\in J\smallsetminus\{j_0\}}$ and $\{y_{j_0}\} \cup \{z_k\}_{k\in K}$ satisfying Lemma 2.2.10, respectively. Hence $\alpha < \gamma < \beta$ on $L(V)$ by Theorem 2.3.5. Therefore, α is not a lower cover of β in $L(V)$. $\square$

Hence we can describe the set of all lower covers of an element in $L(V)$.

Corollary 3.4.2. *Let $\beta \in L(V)$. Then*

$$\{\alpha \in L(V) \mid \alpha < \beta \text{ on } L(V) \text{ and } \dim(\ker\alpha/\ker\beta) = 1\}$$

is the set of all lower covers of β in $L(V)$.

For any $\alpha, \beta \in L(V)$ such that α is a lower cover of β in $L(V)$, we write α and β as in Proposition 2.2.9 and thus $|J| = 1$ since $\dim(\ker\alpha/\ker\beta) = 1$. Therefore the following corollary is obtained.

Corollary 3.4.3. *Let $\alpha, \beta \in L(V)$ be such that $\alpha < \beta$ on $L(V)$. Then α is a lower cover of β in $L(V)$ if and only if* $\dim(\operatorname{im}\beta/\operatorname{im}\alpha) = 1$.

Now, we characterize when elements in $L(V)$ have a lower cover and an upper cover. Note that 0_V is the minimum element in $L(V)$ so it has no lower cover.

Theorem 3.4.4. (i) *Every nonzero $\beta \in L(V)$ has a lower cover in $L(V)$.*
(ii) *For each $\alpha \in L(V)$, α has an upper cover in $L(V)$ if and only if α is not maximal in $L(V)$.*

Proof. (i) Let $\beta \in L(V)$ be nonzero and let B_1 be a basis of $\ker\beta$. Extend B_1 to a basis B of V. We let $w \in (B \setminus B_1)\beta$. Then there is $u \in B \setminus B_1$ such that $u\beta = w$. Define $\alpha \in L(V)$ as in Theorem 3.3.1 by

$$\alpha = \begin{pmatrix} B_1 \cup \{u\} & v \\ 0 & v\beta \end{pmatrix}_{v \in B \setminus (B_1 \cup \{u\})}.$$

Thus $\alpha < \beta$ on $L(V)$. By Lemma 3.4.1, α is a lower cover of β in $L(V)$ since $\dim(\ker\alpha/\ker\beta) = 1$.

(ii) Let $\alpha \in L(V)$. If α is a maximal element in $L(V)$, then there is no $\beta \in L(V)$ such that $\alpha < \beta$, so α has no upper cover in $L(V)$.

Conversely, suppose that α is not maximal in $L(V)$. Then, by Theorem 2.3.8 (ii), α is neither a monomorphism nor an epimorphism. Let $w \in V \setminus \operatorname{im}\alpha$ and

$u \in \ker\alpha \smallsetminus \{0\}$. Let B_1 be a basis of $\ker\alpha$ containing u. Extend B_1 to a basis B of V. Define $\beta \in L(V)$ as in Theorem 3.3.2 by

$$\beta = \begin{pmatrix} B_1 \smallsetminus \{u\} & u & v \\ 0 & w & v\alpha \end{pmatrix}_{v \in B \smallsetminus B_1}.$$

Then $\alpha < \beta$ on $L(V)$ and $\dim(\ker\alpha/\ker\beta) = 1$. Therefore β is an upper cover of α in $L(V)$ by Lemma 3.4.1. □

Next we study lower and upper covers of elements in $S(V)$ when $\dim V$ is infinite. Since $AM(V) = AE(V) = L(V)$ if and only if $\dim V$ is finite, we consider $S(V)$ when $\dim V$ is infinite.

Proposition 3.4.5. *Let* $\dim V$ *be infinite and let* $\alpha, \beta \in S(V)$. *Then* α *is a lower cover of* β *in* $S(V)$ *if and only if* α *is a lower cover of* β *in* $L(V)$. *In other words,* β *is an upper cover of* α *in* $S(V)$ *if and only if* β *is an upper cover of* α *in* $L(V)$.

Proof. Assume that α is a lower cover of β in $L(V)$. By Proposition 2.3.4, α is a lower cover of β in $S(V)$.

To show the forward implication, suppose that α is not a lower cover of β in $L(V)$. Then $\alpha < \gamma < \beta$ on $L(V)$ for some $\gamma \in L(V)$. Since $\alpha < \gamma$ on $L(V)$, by Proposition 3.1.8, $\gamma \in S(V)$. Hence $\alpha < \gamma < \beta$ on $S(V)$ by Corollary 3.1.3. Therefore, α is not a lower cover of β in $S(V)$. □

We next determine when an element in $S(V)$ has a lower cover and an upper cover where $\dim V$ is infinite. Recall that $0_V \notin S(V)$.

Theorem 3.4.6. (i) *Every* $\beta \in S(V)$ *has a lower cover in* $S(V)$.
(ii) *For each* $\alpha \in S(V)$, α *has an upper cover in* $S(V)$ *if and only if* α *is not a maximal element in* $S(V)$.

Proof. (i) Let $\beta \in S(V)$. Since β is nonzero, by Theorem 3.4.4, β has a lower cover in $L(V)$, say α. It follows from Proposition 3.1.8 that $\alpha \in S(V)$. Hence α is a lower cover of β in $S(V)$ by Proposition 3.4.5.
(ii) Let $\alpha \in S(V)$. Suppose that β is an upper cover of α in $S(V)$. Then β is

also an upper cover of α in $L(V)$ by Proposition 3.4.5. This implies that α is not a maximal element in $L(V)$. Hence, by Theorem 2.3.8 (ii), α is neither a monomorphism nor an epimorphism. Therefore, α is not a maximal element in $S(V)$ by Theorem 3.3.2.

The converse can be proven in a reverse way. □

Remark 3.4.7. Let $\beta \in S(V)$ and let B be a basis of V containing a basis B_1 of $\ker\beta$. Then $C_1 = (B \setminus B_1)\beta$ is a basis of $\operatorname{im}\beta$ by Proposition 2.2.4 (ii). Extend C_1 to a basis C of V. If $S(V) = AM(V)$, then $B \setminus B_1$ is infinite since B_1 is finite. If $S(V) = AE(V)$, then $|C \setminus C_1| = \operatorname{corank}\beta$ is finite, and that $(B \setminus B_1)\beta = C_1$ is infinite and so is $B \setminus B_1$. For each $u \in B \setminus B_1$, define $\alpha_u \in S(V)$ by

$$\alpha_u = \begin{pmatrix} B_1 \cup \{u\} & v \\ 0 & v\beta \end{pmatrix}_{v \in B \setminus (B_1 \cup \{u\})}.$$

Then α_u is a lower cover of β in $S(V)$ for all $u \in B \setminus B_1$ as $\dim(\ker\alpha_u/\ker\beta) = 1$ and by Proposition 3.4.5. Since $B \setminus B_1$ is infinite, β has infinitely many lower covers in $S(V)$.

To end this chapter, we provide a figure of some related elements in $AM(V)$ and $AE(V)$.

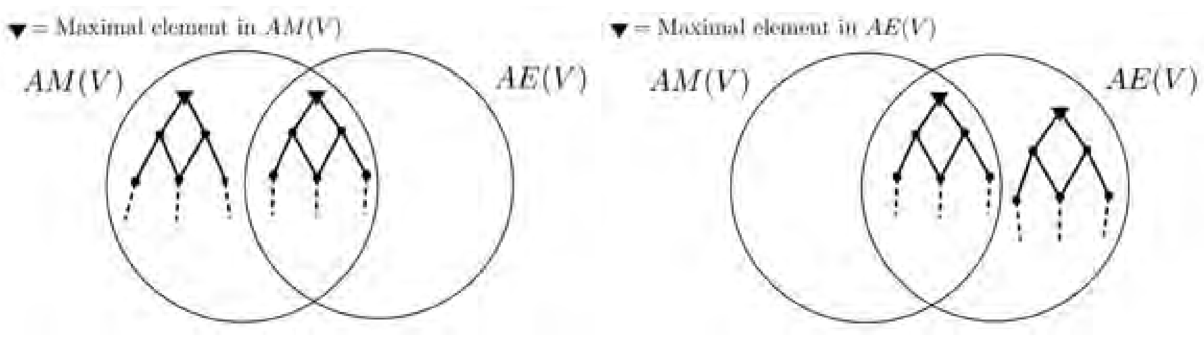

Figure 3.3: An example of comparable elements in $AM(V)$ and $AE(V)$.

CHAPTER IV

THE SEMIGROUPS $K(V, \kappa)$ AND $CI(V, \kappa)$

Unless stated otherwise, we suppose throughout this chapter that V is a vector space. Let us recall definitions of linear transformation semigroups given in Chapter II as follows. For a cardinal number κ with $\kappa \leq \dim V$,

$$K(V, \kappa) = \{\alpha \in L(V) \mid \text{nullity}\, \alpha \geq \kappa\},$$
$$CI(V, \kappa) = \{\alpha \in L(V) \mid \text{corank}\, \alpha \geq \kappa\}.$$

In particular, $OM(V) = K(V, \aleph_0)$ and $OE(V) = CI(V, \aleph_0)$ when $\dim V$ is infinite. It can be observed that 0_V is contained in $K(V, \kappa) \cap CI(V, \kappa)$. Since $K(V, 0) = CI(V, 0) = L(V)$, we suppose throughout that $0 < \kappa \leq \dim V$. Then $K(V, \kappa)$ and $CI(V, \kappa)$ do not contain any monomorphisms and epimorphisms, respectively. For convenience, let $S(V, \kappa)$ be $K(V, \kappa)$ or $CI(V, \kappa)$.

The main purpose of this chapter is to provide characterizations of the natural partial order on the semigroups $OM(V)$, $OE(V)$, $K(V, \kappa)$ and $CI(V, \kappa)$. Furthermore, left and right compatible elements, minimal and maximal elements, lower and upper covers of elements in these partially ordered sets are investigated.

4.1 The Natural Partial Orders on $K(V, \kappa)$ and $CI(V, \kappa)$

From Proposition 2.3.2, the regularity of a semigroup is important in studying the natural partial order on a semigroup. We first determine the regularity of the semigroups $K(V, \kappa)$ and $CI(V, \kappa)$. We recall that if $\dim V$ is finite, then $K(V, \kappa) = CI(V, \kappa)$.

Theorem 4.1.1. *$S(V, \kappa)$ is regular if and only if $\dim V$ is finite.*

Proof. For the necessity, suppose that $\dim V$ is infinite. Let B be a basis of V. Then there is a partition $\{B_1, B_2\}$ of B such that the cardinality of B_1, B_2 and

B are equal. There is a bijection $\phi : B_2 \to B$. Define $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v\phi \end{pmatrix}_{v \in B_2} \text{ and } \beta = \begin{pmatrix} v \\ v\phi^{-1} \end{pmatrix}_{v \in B}.$$

Thus $\alpha \in K(V, \kappa)$ and $\beta \in CI(V, \kappa)$ since $\operatorname{nullity} \alpha = |B_1| = |B| \geq \kappa$ and $\operatorname{corank} \beta = |B \smallsetminus B_2| = |B_1| = |B| \geq \kappa$. Let $\gamma \in L(V)$ be such that $\alpha = \alpha\gamma\alpha$. Since α is an epimorphism, we have $\gamma\alpha = 1_V$. This implies that γ is a monomorphism. Therefore $\gamma \notin K(V, \kappa)$, so $K(V, \kappa)$ is not regular. Next, let $\lambda \in L(V)$ be such that $\beta = \beta\lambda\beta$. Since β is a monomorphism, we have $\beta\lambda = 1_V$. Then λ is an epimorphism and so $\lambda \notin CI(V, \kappa)$. Therefore, $CI(V, \kappa)$ is not regular.

For the sufficiency, assume that $\dim V$ is finite. Then $K(V, \kappa) = CI(V, \kappa)$. Let $\alpha \in K(V, \kappa)$ and B_1 a basis of $\ker \alpha$. Extend B_1 to a basis B of V. Then $C_1 = (B \smallsetminus B_1)\alpha$ is a basis of $\operatorname{im} \alpha$ by Proposition 2.2.4 (*ii*). Extend C_1 to a basis C of V. Note that $\alpha|_{B \smallsetminus B_1} : B \smallsetminus B_1 \to C_1$ is bijective by Proposition 2.2.4 (*i*). Now define $\gamma \in L(V)$ by

$$\gamma = \begin{pmatrix} C \smallsetminus C_1 & v \\ 0 & v\alpha^{-1} \end{pmatrix}_{v \in C_1}.$$

It follows that

$$\begin{aligned} |C \smallsetminus C_1| &= \operatorname{nullity} \gamma = \dim V - \operatorname{rank} \gamma \\ &= |B| - |B \smallsetminus B_1| = |B_1| = \operatorname{nullity} \alpha \geq \kappa, \end{aligned}$$

so $\gamma \in K(V, \kappa)$. Observe that for any $v \in B_1$, $v\alpha = 0 = v\alpha\gamma\alpha$. For any $v \in B \smallsetminus B_1$, $v\alpha\gamma\alpha = (v\alpha\alpha^{-1})\alpha = v\alpha$. Hence $\alpha = \alpha\gamma\alpha$. Therefore $S(V, \kappa)$ is regular. □

By Proposition 2.3.2, we focus on studying the natural partial order on non-regular semigroups. Then, from the above proposition, we will consider the semigroups $K(V, \kappa)$ and $CI(V, \kappa)$ when V is an infinite dimensional vector space and $\kappa > 0$.

Remark 4.1.2. Note that 1_V is not contained in both $K(V, \kappa)$ and $CI(V, \kappa)$ since $\kappa > 0$. Suppose that γ is the identity in $K(V, \kappa)$. Consider α in the proof of Theorem 4.1.1. Since α is an epimorphism and $\alpha = \alpha\gamma$, we have $\gamma = 1_V$, a contradiction. Similarly, if γ is the identity in $CI(V,\kappa)$, then we use β in the proof of the above theorem and that $\gamma\beta = \beta$ implies $\gamma = 1_V$, which is a contradiction. Hence, both $K(V,\kappa)$ and $CI(V,\kappa)$ have no identity. Therefore $K(V,\kappa)^1 \neq K(V,\kappa)$ and $CI(V,\kappa)^1 \neq CI(V,\kappa)$.

The following example shows that there are $\alpha, \beta \in K(V,\kappa)$ such that $\alpha \leq \beta$ on $L(V)$ but $\alpha \nleq \beta$ on $K(V,\kappa)$. Therefore, the natural partial order on $K(V,\kappa)$ cannot be derived from the natural partial order on $L(V)$.

Example 4.1.3. Let $\dim V$ be infinite and κ a cardinal number such that $\kappa > 1$ and let B be a basis of V. Then there is a partition $\{B_1, B_2\}$ of B such that $|B| = |B_1| = |B_2|$. Let $u \in B_2$. There exists a bijection $\phi : B_2 \smallsetminus \{u\} \to B \smallsetminus \{u\}$. Define distinct $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 \cup \{u\} & v \\ 0 & v\phi \end{pmatrix}_{v \in B_2 \smallsetminus \{u\}} \quad \text{and} \quad \beta = \begin{pmatrix} B_1 & u & v \\ 0 & u & v\phi \end{pmatrix}_{v \in B_2 \smallsetminus \{u\}}.$$

Since $\operatorname{nullity} \alpha = |B_1 \cup \{u\}| = |B| \geq \kappa$ and $\operatorname{nullity} \beta = |B_1| = |B| \geq \kappa$, we have $\alpha, \beta \in K(V,\kappa)$. Moreover, $\operatorname{im} \alpha \subsetneq \operatorname{im} \beta$. The sets B_1, $\{u\}$ and $B_2 \smallsetminus \{u\}$ satisfy the conditions in Lemma 2.2.10, so $V\alpha\beta^{-1} = E(\alpha,\beta)$. Hence $\alpha \leq \beta$ on $L(V)$ by Theorem 2.3.5. Claim that $\alpha \nleq \beta$ on $K(V,\kappa)$. Let $\mu \in L(V)$ be such that $\alpha = \beta\mu$. We shall show that $\mu \notin K(V,\kappa)^1$. Note that $\mu \neq 1$ since $\alpha \neq \beta$. Then $0 = u\alpha = u\beta\mu = u\mu$. For each $v \in B_2 \smallsetminus \{u\}$, $v\phi = v\alpha = v\beta\mu = (v\phi)\mu$. Since $(B_2 \smallsetminus \{u\})\phi = B \smallsetminus \{u\}$, we have

$$\mu = \begin{pmatrix} u & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus \{u\}}.$$

It follows that $\operatorname{nullity} \mu = 1 < \kappa$, so $\mu \notin K(V,\kappa)^1$. Therefore, $\alpha \nleq \beta$ on $K(V,\kappa)$ but $\alpha \leq \beta$ on $L(V)$. Note that if $\kappa = 1$, it can be shown that $\alpha \leq \beta$ on $K(V,\kappa)$ but the prove is routine so we omit it.

Theorem 4.1.4. *Let* $\dim V$ *be infinite and* $\alpha, \beta \in K(V,\kappa)$. *Then* $\alpha \leq \beta$ *on* $K(V,\kappa)$ *if and only if* *(i)* $\alpha = \beta$ *or*

(ii) $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$, $V\alpha\beta^{-1} = E(\alpha,\beta)$ *and* $\alpha \in CI(V,\kappa)$.

Proof. Assume that $\alpha < \beta$ on $K(V,\kappa)$. Then $\alpha < \beta$ on $L(V)$, so $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha,\beta)$ by Theorem 2.3.5. Next, we shall prove that $\alpha \in CI(V,\kappa)$. Since $\alpha < \beta$ on $K(V,\kappa)$, we obtain $\alpha = \alpha\mu$ for some $\mu \in K(V,\kappa)$. Let B_1 be a basis of $\operatorname{im}\alpha$ and B_2 a basis of $\ker\mu$. To see that $B_1 \cap B_2 = \varnothing$, let $v \in B_1 \cap B_2$. Then $v\mu = 0$ and $v = u\alpha$ for some $u \in V$. Thus $v = u\alpha = u\alpha\mu = v\mu = 0$, a contradiction. Hence $B_1 \cap B_2 = \varnothing$. Now we claim that $B_1 \cup B_2$ is linearly independent. Assume that

$$\sum_i a_i v_i + \sum_j b_j w_j = 0$$

for some $v_i \in B_1$, $w_j \in B_2$ and scalars a_i, b_j where both summations are over finite index sets I and J. Note that for each $i \in I$, $v_i = u_i\alpha$ for some $u_i \in V$. Hence

$$\begin{aligned} 0 &= \sum_i a_i v_i + \sum_j b_j w_j = \sum_i a_i u_i \alpha + \sum_j b_j w_j, \text{so} \\ 0 &= \Big(\sum_i a_i u_i \alpha + \sum_j b_j w_j\Big)\mu = \sum_i a_i u_i \alpha\mu = \sum_i a_i u_i \alpha = \sum_i a_i v_i. \end{aligned}$$

Thus $a_i = 0$ for all $i \in I$, and that $b_j = 0$ for all $j \in J$. Hence the claim is proven. Extend $B_1 \cup B_2$ to a basis B of V. Since $\mu \in K(V,\kappa)$ and $B_1 \cap B_2 = \varnothing$, we get

$$\operatorname{corank}\alpha = |B \smallsetminus B_1| \geq |B_2| \geq \kappa.$$

Therefore, $\alpha \in CI(V,\kappa)$.

For the sufficiency, suppose that the condition *(ii)* holds. By Proposition 2.2.9, we write

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \text{ and } \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & v_j & u_k \end{pmatrix}_{j\in J, k\in K}$$

where $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ is a basis of V, $\{x_i\}_{i\in I}$, $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}$, $\{u_k\}_{k\in K}$ and $\{v_j\}_{j\in J} \cup \{u_k\}_{k\in K}$ are bases of $\ker\beta$, $\ker\alpha$, $\operatorname{im}\alpha$ and $\operatorname{im}\beta$, respectively. Let $\{v_j\}_{j\in J} \cup \{u_k\}_{k\in K} \cup \{w_l\}_{l\in L}$ be a basis of V. Define $\lambda, \mu \in L(V)$ by

$$\lambda = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & z_k \end{pmatrix}_{k\in K} \text{ and } \mu = \begin{pmatrix} \{v_j\}_{j\in J} \cup \{w_l\}_{l\in L} & u_k \\ 0 & u_k \end{pmatrix}_{k\in K}.$$

Then $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$. Observe that

$$\text{nullity}\,\lambda = |\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}| = \text{nullity}\,\alpha \geq \kappa.$$

Since $\alpha \in CI(V,\kappa)$, we get

$$\text{nullity}\,\mu = |\{v_j\}_{j\in J} \cup \{w_l\}_{l\in L}| = \text{corank}\,\alpha \geq \kappa.$$

Thus $\lambda, \mu \in K(V,\kappa)$. Hence $\alpha \leq \beta$ on $K(V,\kappa)$. □

The below example shows that there are infinitely many related element in $K(V,\kappa)$.

Example 4.1.5. Let $\dim V$ be infinite and B a basis of V. Then there exists a partition $\{B_1, B_2, B_3\}$ of B such that the cardinalities of B, B_1, B_2 and B_3 are equal. We let $\phi : B_2 \to B_1 \cup B_2$ be a bijection. Now define α and β in $L(V)$ by

$$\alpha = \begin{pmatrix} B_1 \cup B_2 & v \\ 0 & v \end{pmatrix}_{v\in B_3} \quad \text{and} \quad \beta = \begin{pmatrix} B_1 & w & v \\ 0 & w\phi & v \end{pmatrix}_{w\in B_2, v\in B_3}.$$

Obviously, $\text{nullity}\,\alpha = |B_1 \cup B_2| = |B| \geq \kappa$, $\text{nullity}\,\beta = |B_1| = |B| \geq \kappa$, $\text{corank}\,\alpha = |B \smallsetminus B_3| = |B| \geq \kappa$ and $\text{im}\,\alpha \subsetneq \text{im}\,\beta$. Then $\beta \in K(V,\kappa)$ and $\alpha \in K(V,\kappa) \cap CI(V,\kappa)$. Now we have B_1, B_2 and B_3 satisfying Lemma 2.2.10 and then $V\alpha\beta^{-1} = E(\alpha,\beta)$. Therefore, by Theorem 4.1.4, $\alpha < \beta$ on $K(V,\kappa)$.

By taking $\kappa = \aleph_0$ in Theorem 4.1.4, we have

Corollary 4.1.6. *Let $\alpha, \beta \in OM(V)$. Then $\alpha \leq \beta$ on $OM(V)$ if and only if*
(i) $\alpha = \beta$ or
(ii) $\text{im}\,\alpha \subseteq \text{im}\,\beta$, $V\alpha\beta^{-1} = E(\alpha,\beta)$ and $\alpha \in OE(V)$.

Similar to $K(V,\kappa)$, there are $\alpha, \beta \in CI(V,\kappa)$ such that $\alpha \leq \beta$ on $L(V)$ but $\alpha \not\leq \beta$ on $CI(V,\kappa)$, as shown in the example below.

Example 4.1.7. Let $\dim V$ be infinite, κ a cardinal number with $\kappa > 1$ and B a basis of V. Then there is a partition $\{B_1, B_2\}$ of B where B, B_1 and B_2 have the same cardinality. Let $u \in B_1$. There is a bijection $\phi : B \smallsetminus \{u\} \to B_2$. Define $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} u & v \\ 0 & v\phi \end{pmatrix}_{v\in B\smallsetminus\{u\}} \quad \text{and} \quad \beta = \begin{pmatrix} u & v \\ u & v\phi \end{pmatrix}_{v\in B\smallsetminus\{u\}}.$$

Then $\operatorname{im}\alpha \subsetneq \operatorname{im}\beta$ and $\alpha \neq \beta$. Note that

$$\operatorname{corank}\alpha = |B \smallsetminus B_2| = |B_1| = |B| \geq \kappa,$$

$$\operatorname{corank}\beta = |B \smallsetminus (B_2 \cup \{u\})| = |B_1 \smallsetminus \{u\}| = |B| \geq \kappa.$$

Thus $\alpha, \beta \in CI(V,\kappa)$. Moreover, $V\alpha\beta^{-1} = E(\alpha,\beta)$ since $\varnothing$, $\{u\}$ and $B \smallsetminus \{u\}$ satisfy Lemma 2.2.10. Hence, $\alpha < \beta$ on $L(V)$ by Theorem 2.3.5. Let $\lambda \in L(V)$ be such that $\alpha = \lambda\beta$ and let $v \in B \smallsetminus \{u\}$. Claim that $\lambda \notin CI(V,\kappa)^1$. Since $\alpha \neq \beta$, we have $\lambda \neq 1$. Consider $v\beta = v\phi = v\alpha = v\lambda\beta$. Since β is a monomorphism, $v\lambda = v$. Hence $B \smallsetminus \{u\} \subseteq \operatorname{im}\lambda$, so $\operatorname{corank}\lambda \leq 1 < \kappa$. Thus $\lambda \notin CI(V,\kappa)^1$. Therefore $\alpha \nleq \beta$ on $CI(V,\kappa)$. If $\kappa = 1$, it can be seen that $\alpha \leq \beta$ on $CI(V,\kappa)$ and we omit the proof.

Hence $CI(V,\kappa)$ does not inherit the natural partial order from $L(V)$.

Theorem 4.1.8. *Let* $\dim V$ *be infinite and let* $\alpha, \beta \in CI(V,\kappa)$. *Then* $\alpha \leq \beta$ *on* $CI(V,\kappa)$ *if and only if*
(i) $\alpha = \beta$ *or*
(ii) $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$, $V\alpha\beta^{-1} = E(\alpha,\beta)$ *and* $\alpha \in K(V,\kappa)$.

Proof. Suppose that $\alpha < \beta$ on $CI(V,\kappa)$. Then $\alpha < \beta$ on $L(V)$. By Theorem 2.3.5, $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha,\beta)$. It remains to prove that $\alpha \in K(V,\kappa)$. As $\alpha < \beta$ on $CI(V,\kappa)$, we obtain $\alpha = \lambda\beta$ for some $\lambda \in CI(V,\kappa)$. By Proposition 2.2.9, we have

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \quad \text{and} \quad \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & v_j & u_k \end{pmatrix}_{j\in J, k\in K}$$

where $\{x_i\}_{i\in I}$, $\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}$, $\{u_k\}_{k\in K}$, $\{v_j\}_{j\in J}\cup\{u_k\}_{k\in K}$ and $\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}\cup\{z_k\}_{k\in K}$ are bases of $\ker\beta$, $\ker\alpha$, $\operatorname{im}\alpha$, $\operatorname{im}\beta$ and V, respectively. Claim that for each $k \in K$, $z_k + w_k \in \operatorname{im}\lambda$ for some $w_k \in \ker\beta$. Let $k_0 \in K$. We write

$$z_{k_0}\lambda = \sum_i a_i x_i + \sum_j b_j y_j + \sum_k c_k z_k$$

where a_i, b_j and c_k are scalars, $i \in I'$, $j \in J'$, $k \in K'$ and I', J', K' are finite subsets of I, J, K, respectively. Then

$$u_{k_0} = z_{k_0}\alpha = z_{k_0}\lambda\beta = \sum_j b_j v_j + \sum_k c_k u_k.$$

Since $\{v_j\}_{j\in J} \cup \{u_k\}_{k\in K}$ is a basis of $\operatorname{im}\beta$, we have $b_j = 0$, $c_k = 0$ and $c_{k_0} = 1$ for all $j \in J'$ and $k \in K' \setminus \{k_0\}$. It follows that $z_{k_0} + \sum_i a_i x_i = z_{k_0}\lambda \in \operatorname{im}\lambda$ and $\sum_i a_i x_i \in \ker\beta$. By choosing $w_{k_0} = \sum_i a_i x_i$, the claim is proven. Next, we show that $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k + w_k\}_{k\in K}$ is a basis of V. Since $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ spans V and $w_k \in \ker\beta = \langle\{x_i\}_{i\in I}\rangle$ for all $k \in K$, we obtain that $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k + w_k\}_{k\in K}$ also spans V. Now assume that

$$\sum_i a_i x_i + \sum_j b_j y_j + \sum_k c_k(z_k + w_k) = 0$$

for some scalars a_i, b_j and c_k where $i \in I'$, $j \in J'$, $k \in K'$ and I', J', K' are finite subsets of I, J, K, respectively. Then

$$\sum_i a_i x_i + \sum_j b_j y_j + \sum_k c_k z_k + \sum_k c_k w_k = 0.$$

Since $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ is a basis of V and $w_k \in \ker\beta = \langle\{x_i\}_{i\in I}\rangle$ for all $k \in K'$, we get $b_j = 0 = c_k$ for all $j \in J'$, $k \in K'$, and hence $a_i = 0$ for all $i \in I'$. This implies $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k + w_k\}_{k\in K}$ is a basis of V. Since $\lambda \in CI(V,\kappa)$ and $\{z_k + w_k\}_{k\in K} \subseteq \operatorname{im}\lambda$, we have

$$\operatorname{nullity}\alpha = |\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}| \geq \operatorname{corank}\lambda \geq \kappa.$$

Therefore $\alpha \in K(V,\kappa)$, as desired.

For the converse, suppose that the condition (ii) holds. By Proposition 2.2.9, we get

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \quad \text{and} \quad \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & v_j & u_k \end{pmatrix}_{j\in J, k\in K}$$

where $\{x_i\}_{i\in I}$, $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}$, $\{u_k\}_{k\in K}$, $\{v_j\}_{j\in J} \cup \{u_k\}_{k\in K}$ and $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ are bases of $\ker\beta$, $\ker\alpha$, $\operatorname{im}\alpha$, $\operatorname{im}\beta$ and V, respectively. Let $\{v_j\}_{j\in J} \cup \{u_k\}_{k\in K} \cup \{w_l\}_{l\in L}$ be a basis of V. Define $\lambda, \mu \in L(V)$ as in the proof of Theorem 2.3.5 by

$$\lambda = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & z_k \end{pmatrix}_{k\in K} \quad \text{and} \quad \mu = \begin{pmatrix} \{v_j\}_{j\in J} \cup \{w_l\}_{l\in L} & u_k \\ 0 & u_k \end{pmatrix}_{k\in K}.$$

Then $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$. This implies that $\operatorname{corank}\lambda = |\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}| = \operatorname{nullity}\alpha \geq \kappa$ since $\alpha \in K(V,\kappa)$. As $\operatorname{im}\mu \subseteq \operatorname{im}\beta$ and $\beta \in CI(V,\kappa)$, we get $\operatorname{corank}\mu \geq \operatorname{corank}\beta \geq \kappa$. Thus $\lambda, \mu \in CI(V,\kappa)$. Therefore, $\alpha \leq \beta$ on $CI(V,\kappa)$. □

Below is a result observed from Theorems 4.1.4 and 4.1.8.

Corollary 4.1.9. *Let* $\dim V$ *be infinite. The following statements hold.*
(i) There are no $\alpha \in K(V,\kappa) \setminus CI(V,\kappa)$ *and* $\beta \in K(V,\kappa)$ *such that* $\alpha < \beta$ *on* $K(V,\kappa)$.
(ii) There are no $\alpha \in CI(V,\kappa) \setminus K(V,\kappa)$ *and* $\beta \in CI(V,\kappa)$ *such that* $\alpha < \beta$ *on* $CI(V,\kappa)$.

The previous theorem is useful to examine related elements in $CI(V,\kappa)$. By considering α and β in Example 4.1.7, we have $\operatorname{nullity}\alpha = 1 < \kappa$, so $\alpha \notin CI(V,\kappa)$. Hence $\alpha \not\leq \beta$ on $CI(V,\kappa)$ by Theorem 4.1.8.

Example 4.1.10. Let $\dim V$ be infinite and let $B = B_1 \cup B_2 \cup B_3$ be a basis of V such that the cardinality of B, B_1, B_2 and B_3 are the same and B_1, B_2, B_3 are disjoint. Let $\varphi : B_1 \to B_2$ and $\phi : B_2 \cup B_3 \to B_3$ be bijections. Define $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v\phi \end{pmatrix}_{v\in B_2\cup B_3} \quad \text{and} \quad \beta = \begin{pmatrix} w & v \\ w\varphi & v\phi \end{pmatrix}_{w\in B_1, v\in B_2\cup B_3}.$$

Since $\operatorname{corank}\alpha = |B_1 \cup B_2| = \operatorname{nullity}\alpha$ and $\operatorname{corank}\beta = |B_1|$, we have $\alpha, \beta \in CI(V,\kappa)$ and $\alpha \in K(V,\kappa)$. Observe that $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$. Then $V\alpha\beta^{-1} = E(\alpha,\beta)$ since the sets $\varnothing$, B_1 and $B_2 \cup B_3$ satisfy the conditions in Lemma 2.2.10. Therefore, by Theorem 4.1.8, $\alpha \leq \beta$ on $CI(V,\kappa)$.

The next corollary is a consequence of Theorem 4.1.8 by letting $\kappa = \aleph_0$.

Corollary 4.1.11. *Let* $\alpha, \beta \in OE(V)$. *Then* $\alpha \leq \beta$ *on* $OE(V)$ *if and only if*
(i) $\alpha = \beta$ *or*
(ii) $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$, $V\alpha\beta^{-1} = E(\alpha,\beta)$ *and* $\alpha \in OM(V)$.

Next, this corollary follows from Theorems 2.3.5, 4.1.4 and 4.1.8.

Corollary 4.1.12. *Let* $\dim V$ *be infinite. The following statemenets hold.*
(i) For any $\alpha, \beta \in K(V, \kappa)$, $\alpha < \beta$ *on* $K(V, \kappa)$ *if and only if* $\alpha < \beta$ *on* $L(V)$ *and* $\alpha \in CI(V, \kappa)$.
(ii) For any $\alpha, \beta \in CI(V, \kappa)$, $\alpha < \beta$ *on* $CI(V, \kappa)$ *if and only if* $\alpha < \beta$ *on* $L(V)$ *and* $\alpha \in K(V, \kappa)$.

The following figures are obtained from Theorems 4.1.4 and 4.1.8. Recall that we draw a dotted line from α upward to β to represent $\alpha \leq \beta$ and 0_V is the minimum element in $K(V, \kappa)$ and $CI(V, \kappa)$ by Proposition 2.3.1 (i).

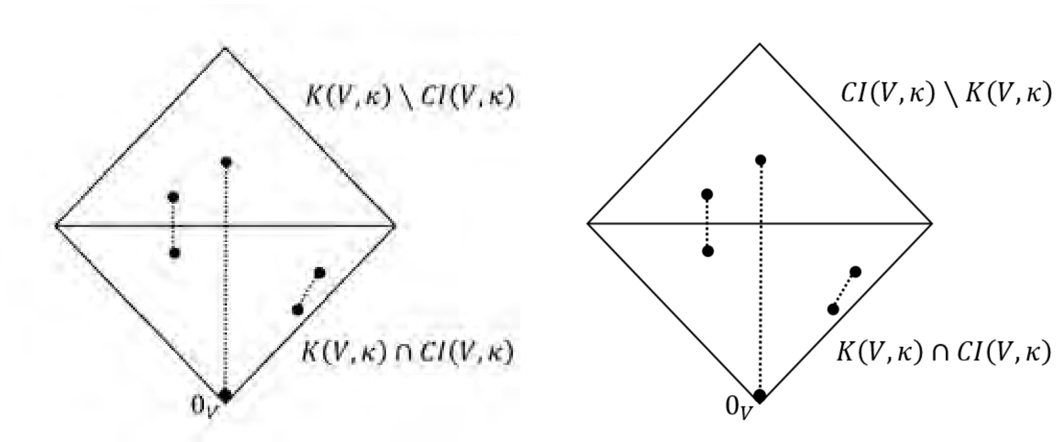

Figure 4.1: An example of related elements in $K(V, \kappa)$ and $CI(V, \kappa)$.

If $\alpha \leq \beta$ on $L(V)$ and $\beta \in K(V, \kappa)$ $[CI(V, \kappa)]$, we can conclude that $\alpha \in K(V, \kappa)$ $[CI(V, \kappa)]$; see the following proposition.

Proposition 4.1.13. *Let* $\alpha, \beta \in L(V)$ *be such that* $\alpha \leq \beta$ *on* $L(V)$.
(i) If $\beta \in K(V, \kappa)$, *then* $\alpha \in K(V, \kappa)$.
(ii) If $\beta \in CI(V, \kappa)$, *then* $\alpha \in CI(V, \kappa)$.

Proof. The condition $\alpha \leq \beta$ on $L(V)$ implies that $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha, \beta)$ by Theorem 2.3.5. Then $\ker\beta \subseteq \ker\alpha$ by Proposition 2.2.6.
(i) Assume that $\beta \in K(V, \kappa)$. Since $\operatorname{nullity}\alpha \geq \operatorname{nullity}\beta \geq \kappa$, we have $\alpha \in K(V, \kappa)$.
(ii) Suppose that $\beta \in CI(V, \kappa)$. Since $\operatorname{corank}\alpha \geq \operatorname{corank}\beta \geq \kappa$, we get $\alpha \in CI(V, \kappa)$. □

Remark 4.1.14. Let $\alpha \in L(V)$ and $\beta \in K(V,\kappa) \cap CI(V,\kappa)$ be such that $\alpha \leq \beta$ on $L(V)$. Then $\alpha \in K(V,\kappa) \cap CI(V,\kappa)$ by Proposition 4.1.13.

4.2 Left and Right Compatible Elements in $(\boldsymbol{K(V,\kappa)}, \leq)$ and $(\boldsymbol{CI(V,\kappa)}, \leq)$

This section is dedicated to the study of left and right compatible elements in $K(V,\kappa)$ and $CI(V,\kappa)$. Theorem 2.3.6 states that a nonzero $\gamma \in L(V)$, where $\dim V \geq 2$, is left [right] compatible on $L(V)$ if and only if γ is an epimorphism [a monomorphism]. However, the following example shows that, when $\dim V = 2$ and $\kappa = 1$, there exists $\gamma \in S(V,\kappa)$ which is a compatible element in $S(V,\kappa)$ but neither a monomorphism nor an epimorphism.

Example 4.2.1. Suppose that $\dim V = 2$ and $\kappa = 1$. Now we let $\{u, v\}$ be a basis of V. Define $\gamma \in L(V)$ by

$$\gamma = \begin{pmatrix} u & v \\ 0 & v \end{pmatrix}.$$

Then $\operatorname{nullity}\gamma = \operatorname{corank}\gamma = 1 \geq \kappa$ and hence $\gamma \in S(V,\kappa)$. Claim that γ is left and right compatible on $S(V,\kappa)$. Let $\alpha, \beta \in S(V,\kappa)$ be such that $\alpha \leq \beta$ on $S(V,\kappa)$. Then $\alpha \leq \beta$ on $L(V)$ by Proposition 2.3.1 (iii), so $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha,\beta)$ by Theorem 2.3.5. If $\alpha = \beta$, it is clear that $\gamma\alpha = \gamma\beta$ and $\alpha\gamma = \beta\gamma$. Suppose that $\alpha \neq \beta$. Then $\operatorname{im}\alpha \subsetneq \operatorname{im}\beta$ by Remark 2.2.8. Since $\kappa = 1$, we have $1 \leq \operatorname{corank}\beta < \operatorname{corank}\alpha \leq \dim V = 2$. It follows that $\operatorname{corank}\alpha = 2$. Thus $\alpha = 0_V$, so $\gamma\alpha = 0_V = \alpha\gamma$. Hence $\gamma\alpha \leq \gamma\beta$ and $\alpha\gamma \leq \beta\gamma$ on $S(V,\kappa)$. Therefore γ is left and right compatible on $S(V,\kappa)$. Note that γ is neither left nor right compatible on $L(V)$ by Theorem 2.3.6.

Observe that $0_V \in S(V,\kappa)$ and it is easy to see that 0_V is compatible, so nonzero left and right compatible elements will be determined.

Lemma 4.2.2. *Let $\gamma \in S(V,\kappa)$ be nonzero. The following statements hold.*
(i) Let $\dim V$ be infinite. If γ is left compatible on $S(V,\kappa)$, then γ is an epimorphism.

(ii) Let $\dim V$ *be infinite. If* γ *is right compatible on* $S(V,\kappa)$*, then* γ *is a monomorphism.*
(iii) Let $\dim V < \infty$*. If* γ *is left or right compatible on* $S(V,\kappa)$*, then* $\kappa = \dim V - 1$*.*

Proof. We use the following facts to show all of (i), (ii) and (iii). Let B be a basis of V containing a basis B_1 of $\ker\gamma$. By Proposition 2.2.4 (ii), we obtain $C_1 = (B \setminus B_1)\gamma$ is a basis of $\operatorname{im}\gamma$. Extend it to a basis C of V.

(i) Assume that γ is not an epimorphism. Let $u \in C \setminus C_1$. Since γ is nonzero, we have $w \in C_1$. Then $w = w_0\gamma$ for some $w_0 \in B \setminus B_1$. Define $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} C \setminus \{u,w\} & \{u,w\} \\ 0 & w \end{pmatrix} \text{ and } \beta = \begin{pmatrix} C \setminus \{u,w\} & u & w \\ 0 & w & u \end{pmatrix}.$$

Then $\operatorname{im}\alpha \subsetneq \operatorname{im}\beta$, and $V\alpha\beta^{-1} = E(\alpha,\beta)$ since $C \setminus \{u,w\}$, $\{u-w\}$ and $\{u\}$ satisfy Lemma 2.2.10. Moreover, $\alpha, \beta \in K(V,\kappa) \cap CI(V,\kappa)$ since $\operatorname{nullity}\alpha = \operatorname{corank}\alpha = \dim V - 1 = \dim V \geq \kappa$ and $\operatorname{nullity}\beta = \operatorname{corank}\beta = \dim V - 2 = \dim V \geq \kappa$. Hence $\alpha < \beta$ on $S(V,\kappa)$ by Theorems 4.1.4 and 4.1.8. Claim that $\operatorname{im}\gamma\alpha \nsubseteq \operatorname{im}\gamma\beta$ by showing that $w \in \operatorname{im}\gamma\alpha \setminus \operatorname{im}\gamma\beta$. Since $w = w\alpha = w_0\gamma\alpha$, we have $w \in \operatorname{im}\gamma\alpha$. Let $v \in B$. If $v \in B_1$, then $v\gamma\beta = 0\beta = 0$. Otherwise, $v\gamma\beta \in C_1\beta = \{0,u\}$. Hence $\operatorname{im}\gamma\beta = \langle u \rangle$ and so we have the claim. Therefore γ is not left compatible on $S(V,\kappa)$ by Theorems 4.1.4 and 4.1.8.

(ii) Assume that γ is not a monomorphism. Recall that B_1 is a basis of $\ker\gamma$. Let $u \in B_1$. Since γ is nonzero, let $w \in B \setminus B_1$. Define $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} B \setminus \{u,w\} & \{u,w\} \\ 0 & w \end{pmatrix} \text{ and } \beta = \begin{pmatrix} B \setminus \{u,w\} & u & w \\ 0 & w & u \end{pmatrix}.$$

Similar to (i), we have $\alpha, \beta \in K(V,\kappa) \cap CI(V,\kappa)$ and $\alpha < \beta$ on $S(V,\kappa)$. Then

$$\alpha\gamma = \begin{pmatrix} B \setminus \{u,w\} & \{u,w\} \\ 0 & w\gamma \end{pmatrix} \text{ and } \beta\gamma = \begin{pmatrix} B \setminus \{u\} & u \\ 0 & w\gamma \end{pmatrix}.$$

It follows that $\ker\beta\gamma \nsubseteq \ker\alpha\gamma$. Hence, by Proposition 2.2.6, $V(\alpha\gamma)(\beta\gamma)^{-1} \neq E(\alpha\gamma,\beta\gamma)$ and so $\alpha\gamma \nleq \beta\gamma$ on $S(V,\kappa)$ by Theorems 4.1.4 and 4.1.8. Hence γ is not right compatible on $S(V,\kappa)$.

(iii) Suppose that $\kappa \neq \dim V - 1$. We note that if $\kappa > \dim V - 1$, then $\kappa = \dim V$ and hence $\gamma = 0_V$ which is a contradiction. Then $\kappa < \dim V - 1$. It follows that $\dim V \geq 3$ since $\kappa > 0$. Note that $K(V, \kappa) = CI(V, \kappa)$. Since $\kappa > 0$, we get γ is neither a monomorphism nor an epimorphism. To show that γ is not left compatible, we define α and β as in (i). Observe that nullity $\alpha = \dim V - 1 > \kappa$ and nullity $\beta = \dim V - 2 \geq \kappa$. Thus $\alpha, \beta \in K(V, \kappa) = CI(V, \kappa)$. Observe from Theorem 4.1.1 that $S(V, \kappa)$ is regular. Hence, by Proposition 2.3.2 and Theorem 2.3.5, $\alpha < \beta$ on $S(V, \kappa)$, but $\gamma\alpha \not\leq \gamma\beta$ on $S(V, \kappa)$ since $\operatorname{im} \gamma\alpha \nsubseteq \operatorname{im} \gamma\beta$. This implies that γ is not left compatible on $S(V, \kappa)$. To see that γ is not right compatible, define α and β as in (ii). Since nullity $\alpha = \dim V - 1 > \kappa$ and nullity $\beta = \dim V - 2 \geq \kappa$, we obtain $\alpha, \beta \in S(V, \kappa)$. Moreover, $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$ and $V\alpha\beta^{-1} = E(\alpha, \beta)$. By Proposition 2.3.2 and Theorem 2.3.5, $\alpha < \beta$ on $S(V, \kappa)$. Since $\ker \beta\gamma \nsubseteq \ker \alpha\gamma$, we obtain $\alpha\gamma \not\leq \beta\gamma$ on $S(V, \kappa)$. Therefore γ is not left compatible on $S(V, \kappa)$. $\square$

Remark 4.2.3. By the definitions of α and β, we have $(C \smallsetminus \{u, w\}) \cup \{u - w\} \cup \{w\}$ is a basis of V but $w\alpha = w$ and $w\beta = u$. Hence the converse of Lemma 2.2.10 is not true.

We now determine left and right compatible elements in $S(V, \kappa)$ when $\dim V$ is finite.

Theorem 4.2.4. *Let* $\dim V < \infty$ *and let* $\gamma \in S(V, \kappa)$ *be nonzero. Then the following statements are equivalent.*
(i) *γ is left compatible on* $S(V, \kappa)$,
(ii) *γ is right compatible on* $S(V, \kappa)$,
(iii) $\kappa = \dim V - 1$.

Proof. Suppose that the condition (iii) holds. Let $\alpha, \beta \in S(V, \kappa)$ be such that $\alpha \leq \beta$ on $S(V, \kappa)$. If $\alpha = \beta$, then $\gamma\alpha = \gamma\beta$ and $\alpha\gamma = \beta\gamma$. Suppose that $\alpha < \beta$ on $S(V, \kappa)$. Then $\alpha < \beta$ on $L(V)$ by Proposition 2.3.1 (iii). This implies $V\alpha\beta^{-1} = E(\alpha, \beta)$ by Theorem 2.3.5. From Remark 2.2.8, $\ker \beta \subsetneq \ker \alpha$. Thus

$$\dim V - 1 = \kappa \leq \text{nullity}\, \beta < \text{nullity}\, \alpha \leq \dim V.$$

Since $\dim V$ is finite, we have nullity $\beta = \dim V - 1$ and nullity $\alpha = \dim V$. Hence $\alpha = 0_V$ since $\dim V$ is finite. Then $\gamma\alpha = 0_V = \alpha\gamma$, so $\gamma\alpha \leq \gamma\beta$ and $\alpha\gamma \leq \beta\gamma$ on $S(V,\kappa)$. Therefore, the conditions (i) and (ii) hold.

The conditions (i) and (ii) imply (iii) by Lemma 4.2.2 (iii). □

Remark 4.2.5. Let $\dim V = n$ be a natural number and $CK(V,\kappa)$ be the set of left compatible elements in $K(V,\kappa)$, which is also right compatible elements in $K(V,\kappa)$. By Theorem 4.2.4, if $\kappa \neq n-1$, we have

$$\{0_V\} = CK(V,1) = CK(V,2) = \cdots = CK(V,n-2) = CK(V,n) = K(V,n),$$

and if $\kappa = n-1$, then $CK(V,n-1) = K(V,n-1)$.

Theorem 4.2.6. *Let* $\dim V$ *be infinite and let* $\gamma \in K(V,\kappa)$ *be nonzero. Then the following hold.*
(i) *γ is left compatible on $K(V,\kappa)$ if and only if γ is an epimorphism.*
(ii) *γ is not right compatible on $K(V,\kappa)$.*

Proof. (i) The forward implication follows from Lemma 4.2.2 (i).

Conversely, suppose that γ is an epimorphism. By Theorem 2.3.6 (i), γ is left compatible on $L(V)$. Let $\alpha, \beta \in K(V,\kappa)$ be such that $\alpha \leq \beta$ on $K(V,\kappa)$. Note that the case $\alpha = \beta$ is obvious. Assume that $\alpha \neq \beta$. Then, by Corollary 4.1.12, $\alpha < \beta$ on $L(V)$ and $\alpha \in CI(V,\kappa)$. Hence $\gamma\alpha \leq \gamma\beta$ on $L(V)$ by the left compatibility of γ, and $\gamma\alpha \in CI(V,\kappa)$ since $CI(V,\kappa)$ is a left ideal of $L(V)$ by Proposition 2.1.9 (ii). Therefore, by Theorem 4.1.4, $\gamma\alpha \leq \gamma\beta$ on $K(V,\kappa)$.
(ii) Since every element in $K(V,\kappa)$ is not a monomorphism, by Lemma 4.2.2 (ii), we have γ is not right compatible on $K(V,\kappa)$. □

By taking $\kappa = \aleph_0$, we have the following corollary.

Corollary 4.2.7. *Let $\gamma \in OM(V)$ be nonzero. Then the following hold.*
(i) *γ is left compatible on $OM(V)$ if and only if γ is an epimorphism.*
(ii) *γ is not right compatible on $OM(V)$.*

The next corollary is obtained from Theorems 2.3.6 and 4.2.6.

Corollary 4.2.8. *Let* $\dim V$ *be infinite. Then the following hold.*
(i) $K(V,\kappa)$ *has no nonzero compatible elements.*
(ii) For each $\gamma \in K(V,\kappa)$*,* γ *is left compatible on* $K(V,\kappa)$ *if and only if* γ *is left compatible on* $L(V)$*.*

Next, left and right compatibility of elements in $CI(V,\kappa)$ are described.

Theorem 4.2.9. *Let* $\dim V$ *be infinite and let* $\gamma \in CI(V,\kappa)$ *be nonzero. Then the following hold.*
(i) γ *is not left compatible on* $CI(V,\kappa)$*.*
(ii) γ *is right compatible on* $CI(V,\kappa)$ *if and only if* γ *is a monomorphism.*

Proof. (i) Since $\gamma \in CI(V,\kappa)$ is not an epimorphism, we have γ is not left compatible on $CI(V,\kappa)$ by Lemma 4.2.2 (i).
(ii) To see the sufficiency, suppose that γ is a monomorphism. Theorem 2.3.6 (ii) implies that γ is right compatible on $L(V)$. Let $\alpha, \beta \in CI(V,\kappa)$ be such that $\alpha \leq \beta$ on $CI(V,\kappa)$. The case $\alpha = \beta$ is clear. Then we suppose that $\alpha \neq \beta$. Thus $\alpha < \beta$ on $L(V)$ and $\alpha \in K(V,\kappa)$ by Corollary 4.1.12. Since γ is right compatible on $L(V)$, we have $\alpha\gamma \leq \beta\gamma$ on $L(V)$. Furthermore, $\alpha\gamma \in K(V,\kappa)$ since $K(V,\kappa)$ is a right ideal of $L(V)$ by Proposition 2.1.9 (i). Therefore, by Theorem 4.1.8, $\alpha\gamma \leq \beta\gamma$ on $CI(V,\kappa)$.

The forward implication follows from Lemma 4.2.2 (ii). □

Corollary 4.2.10. *Let* $\gamma \in OE(V)$ *be nonzero. Then the following hold.*
(i) γ *is not left compatible on* $OE(V)$*.*
(ii) γ *is right compatible on* $OE(V)$ *if and only if* γ *is a monomorphism.*

Corollary 4.2.11. *Let* $\dim V$ *be infinite. The following hold.*
(i) $CI(V,\kappa)$ *has no nonzero compatible elements.*
(ii) For each $\gamma \in CI(V,\kappa)$*,* γ *is right compatible on* $CI(V,\kappa)$ *if and only if* γ *is right compatible on* $L(V)$*.*

In $S(V,\kappa)$, we can construct other elements in $\leq$ from one using left and right compatible elements.

Example 4.2.12. Let $\dim V$ be infinite and B a basis of V. Then there is a partition $\{B_1, B_2, B_3\}$ of B such that B, B_1, B_2 and B_3 have the same cardinality. Define $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 \cup B_2 & v \\ 0 & v \end{pmatrix}_{v \in B_3} \text{ and } \beta = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B_2 \cup B_3}.$$

Observe that $\operatorname{nullity} \alpha = \operatorname{corank} \alpha = |B_1 \cup B_2| \geq \kappa$ and $\operatorname{nullity} \beta = \operatorname{corank} \beta = |B_1| \geq \kappa$, so $\alpha, \beta \in K(V, \kappa) \cap CI(V, \kappa)$. It is easy to see that $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$. As B_1, B_2 and B_3 satisfy Lemma 2.2.10, we have $V\alpha\beta^{-1} = E(\alpha, \beta)$. Hence, by Theorems 4.1.4 and 4.1.8, $\alpha \leq \beta$ on $S(V, \kappa)$. Since $|B_3| = |B_1 \cup B_3|$ and $|B| = |B_3|$, let $\varphi : B_3 \to B_1 \cup B_3$ and $\phi : B \to B_3$ be bijections. Now define $\gamma, \delta \in L(V)$ as

$$\gamma = \begin{pmatrix} B_1 & w & v \\ 0 & w & v\varphi \end{pmatrix}_{w \in B_2, v \in B_3} \text{ and } \delta = \begin{pmatrix} v \\ v\phi \end{pmatrix}_{v \in B}.$$

It follows that $\gamma \in K(V, \kappa)$ and $\delta \in CI(V, \kappa)$ since $\operatorname{nullity} \gamma = |B_1|$ and $\operatorname{corank} \delta = |B \setminus B_3|$. Moreover, γ is an epimorphism and δ is a monomorphism. Thus γ is left compatible on $K(V, \kappa)$ and δ is right compatible on $CI(V, \kappa)$ by Theorems 4.2.6 and 4.2.9, respectively. Therefore, $\gamma\alpha \leq \gamma\beta$ on $K(V, \kappa)$ and $\alpha\delta \leq \beta\delta$ on $CI(V, \kappa)$. Since φ and ϕ are not identity functions, we have $\gamma\alpha \neq \alpha$, $\alpha\delta \neq \alpha$, $\gamma\beta \neq \beta$ and $\beta\delta \neq \beta$. Note that $\gamma\alpha$, $\gamma\beta$, $\alpha\delta$ and $\beta\delta$ are in the following forms.

$$\gamma\alpha = \begin{pmatrix} B_1 \cup B_2 & v \\ 0 & v\varphi \end{pmatrix}_{w \in B_3}, \ \gamma\beta = \begin{pmatrix} B_1 & w & v \\ 0 & w & v\varphi \end{pmatrix}_{w \in B_2, v \in B_3},$$

$$\alpha\delta = \begin{pmatrix} B_1 \cup B_2 & v \\ 0 & v\phi \end{pmatrix}_{v \in B_3} \text{ and } \beta\delta = \begin{pmatrix} B_1 & v \\ 0 & v\phi \end{pmatrix}_{v \in B_2 \cup B_3}.$$

4.3 Minimal and Maximal Elements in $(K(V, \kappa), \leq)$ and $(CI(V, \kappa), \leq)$

Recall that 0_V is the minimum element in $K(V, \kappa)$ and $CI(V, \kappa)$ by Proposition 2.3.1 (i). The main purpose of this section is to find necessary and sufficient conditions for elements in $K(V, \kappa)$ and $CI(V, \kappa)$ to be minimal nonzero elements and maximal elements where V is any vector space and $0 < \kappa \leq \dim V$. Recall that $S(V, \kappa)$ stands for $K(V, \kappa)$ or $CI(V, \kappa)$.

Theorem 4.3.1. *Let $\beta \in S(V, \kappa)$. Then β is a minimal nonzero element in $S(V, \kappa)$ if and only if* $\operatorname{rank} \beta = 1$.

Proof. Assume that β is a minimal nonzero element in $S(V, \kappa)$. Since $0 < \kappa \leq \dim V$, we have $\dim V \geq 1$. If $\dim V = 1$, it is clear that $\operatorname{rank} \beta = 1$. Suppose that $\dim V \geq 2$. Let B_1 be a basis of $\ker \beta$. Extend this to a basis B of V. As β is nonzero, there exists $u \in B \smallsetminus B_1$. Let C_1 be a basis of $\operatorname{im} \beta$ containing $u\beta$. Extend C_1 to a basis C of V. Define $\alpha \in L(V)$ by

$$\alpha = \begin{pmatrix} B \smallsetminus \{u\} & u \\ 0 & u\beta \end{pmatrix}.$$

Then $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$. Consider B_1, $B \smallsetminus (B_1 \cup \{u\})$ and $\{u\}$ in Lemma 2.2.10, we obtain $V\alpha\beta^{-1} = E(\alpha, \beta)$. Note that $|B| > |B_1|$ and $|C_1| \geq 1$ since β is nonzero.
Case 1: $\beta \in K(V, \kappa)$. Then $|B_1| \geq \kappa$. Hence $\operatorname{nullity} \alpha = |B \smallsetminus \{u\}| \geq |B_1| \geq \kappa$ and $\operatorname{corank} \alpha = |C \smallsetminus \{u\beta\}| = |B \smallsetminus \{u\}| \geq \kappa$, so $\alpha \in K(V, \kappa) \cap CI(V, \kappa)$.
Case 2: $\beta \in CI(V, \kappa)$. Then $|C \smallsetminus C_1| = \operatorname{corank} \beta \geq \kappa$. It follows that $\operatorname{corank} \alpha = |C \smallsetminus \{u\beta\}| \geq |C \smallsetminus C_1| \geq \kappa$ and $\operatorname{nullity} \alpha = |B \smallsetminus \{u\}| = |C \smallsetminus \{u\beta\}| \geq \kappa$. Thus $\alpha \in K(V, \kappa) \cap CI(V, \kappa)$.
In any case, by Theorems 4.1.4 and 4.1.8, $\alpha \leq \beta$ on $S(V, \kappa)$. Since β is minimal in $S(V, \kappa)$, we get $\alpha = \beta$. Therefore, $\operatorname{rank} \beta = 1$.

The converse is clear by Theorem 2.3.8 (*i*). □

Now we let $S(V)$ be $OM(V)$ or $OE(V)$.

Corollary 4.3.2. *Let $\beta \in S(V)$. Then β is a minimal nonzero element in $S(V)$ if and only if* $\operatorname{rank} \beta = 1$.

The next corollary is a consequence of Theorems 2.3.8 (*i*) and 4.3.1.

Corollary 4.3.3. *Let $\beta \in S(V, \kappa)$. Then β is a minimal nonzero element in $S(V, \kappa)$ if and only if β is a minimal nonzero element in $L(V)$.*

The following lemma is obtained from Theorems 4.1.4 and 4.1.8.

Lemma 4.3.4. *Let* $\dim V$ *be infinite. The following statements hold.*
(i) Every $\alpha \in K(V,\kappa) \smallsetminus CI(V,\kappa)$ *is maximal in* $K(V,\kappa)$.
(ii) Every $\alpha \in CI(V,\kappa) \smallsetminus K(V,\kappa)$ *is maximal in* $CI(V,\kappa)$.

Proof. (i) Let $\alpha \in K(V,\kappa) \smallsetminus CI(V,\kappa)$. Suppose that $\alpha \leq \beta$ on $K(V,\kappa)$ for some $\beta \in K(V,\kappa)$. Then $\alpha = \beta$ by Theorem 4.1.4.
(ii) It can be proven similar to (i). □

The below lemma shows more conditions for elements in $S(V,\kappa)$ to be maximal.

Lemma 4.3.5. (i) *Every* $\alpha \in K(V,\kappa)$ *with* $\operatorname{nullity}\alpha = \kappa < \infty$ *is maximal in* $K(V,\kappa)$.
(ii) *Every* $\alpha \in CI(V,\kappa)$ *with* $\operatorname{corank}\alpha = \kappa < \infty$ *is maximal in* $CI(V,\kappa)$.

Proof. (i) Let $\alpha \in K(V,\kappa)$ be such that $\operatorname{nullity}\alpha = \kappa < \infty$. Suppose that $\alpha \leq \beta$ on $K(V,\kappa)$ for some $\beta \in K(V,\kappa)$. By Proposition 2.3.1 (iii), we have $\alpha \leq \beta$ on $L(V)$. This implies $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha,\beta)$ by Theorem 2.3.5. Then $\ker\beta \subseteq \ker\alpha$ by Proposition 2.2.6. Thus

$$\kappa \leq \operatorname{nullity}\beta \leq \operatorname{nullity}\alpha = \kappa < \infty.$$

It follows that $\operatorname{nullity}\beta = \operatorname{nullity}\alpha = \kappa$. Since κ is finite, $\ker\alpha = \ker\beta$. Since $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $\ker\alpha = \ker\beta$, by Proposition 2.2.7, we have $\alpha = \beta$.
(ii) Let $\alpha \in CI(V,\kappa)$ be such that $\operatorname{corank}\alpha = \kappa < \infty$. Assume that $\alpha \leq \beta$ for some $\beta \in CI(V,\kappa)$. Then, by Proposition 2.3.1 (iii), $\alpha \leq \beta$ on $L(V)$. By Theorem 2.3.5, $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$. Hence

$$\kappa \leq \operatorname{corank}\beta \leq \operatorname{corank}\alpha = \kappa.$$

This implies $\operatorname{corank}\beta = \operatorname{corank}\alpha = \kappa$. Claim that $\operatorname{im}\alpha = \operatorname{im}\beta$. Let C_1 be a basis of $\operatorname{im}\alpha$. Since $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$, extend C_1 to a basis C_2 of $\operatorname{im}\beta$. Let C be a basis of V containing C_2. Then

$$|C \smallsetminus C_2| = \operatorname{corank}\beta = \kappa = \operatorname{corank}\alpha = |C \smallsetminus C_1|.$$

Since κ is finite, we have $C_1 = C_2$. Hence $\operatorname{im}\alpha = \operatorname{im}\beta$, the claim is proven. By Proposition 2.2.7, we get $\alpha = \beta$. □

Now, necessary and sufficient conditions for elements in $S(V,\kappa)$ to be maximal elements are provided.

Theorem 4.3.6. *(i) For each $\alpha \in K(V,\kappa)$, α is maximal in $K(V,\kappa)$ if and only if $\alpha \notin CI(V,\kappa)$ or* $\operatorname{nullity}\alpha = \kappa < \infty$.
(ii) For each $\alpha \in CI(V,\kappa)$, $\alpha \in CI(V,\kappa)$ is maximal in $CI(V,\kappa)$ if and only if $\alpha \notin K(V,\kappa)$ or $\operatorname{corank}\alpha = \kappa < \infty$.

Proof. The sufficient conditions of (i) and (ii) follow from Lemmas 4.3.4 and 4.3.5.

We shall prove the necessities of (i) and (ii) by contrapositive. Both (i) and (ii) can be shown by using the following facts. Let $\alpha \in K(V,\kappa) \cap CI(V,\kappa)$. Then α is neither a monomorphism nor an epimorphism. Choose $w \in V \smallsetminus \operatorname{im}\alpha$ and $u \in \ker\alpha \smallsetminus \{0\}$. Let B_1 be a basis of $\ker\alpha$ containing u. Extend B_1 to a basis B of V. Since $(B \smallsetminus B_1)\alpha$ is a basis of $\operatorname{im}\alpha$ by Proposition 2.2.4 (ii) and $w \notin \operatorname{im}\alpha$, let C be a basis of V containing $(B \smallsetminus B_1)\alpha \cup \{w\}$. Define $\beta \in L(V)$ by

$$\beta = \begin{pmatrix} B_1 \smallsetminus \{u\} & u & v \\ 0 & w & v\alpha \end{pmatrix}_{v \in B \smallsetminus B_1}.$$

Then $\operatorname{im}\alpha \subsetneq \operatorname{im}\beta$, and $V\alpha\beta^{-1} = E(\alpha,\beta)$ by using $B_1 \smallsetminus \{u\}$, $\{u\}$ and $B \smallsetminus B_1$ in Lemma 2.2.10.

(i) Assume that $|B_1| = \operatorname{nullity}\alpha > \kappa$ or κ is infinite.

Case 1: $|B_1| > \kappa$. Then $\dim V \geq 2$ since $\kappa > 0$, and $\operatorname{nullity}\beta = |B_1 \smallsetminus \{u\}| \geq \kappa$. It follows that $\beta \in K(V,\kappa)$.

Case 2: κ is infinite. Then $\operatorname{nullity}\beta = |B_1 \smallsetminus \{u\}| = |B_1| = \operatorname{nullity}\alpha \geq \kappa$, so $\beta \in K(V,\kappa)$.

In either case, we obtain $\alpha < \beta$ on $K(V,\kappa)$ by Theorem 4.1.4.

(ii) Assume that $\operatorname{corank}\alpha > \kappa$ or κ is infinite.

Case 1: $\operatorname{corank}\alpha > \kappa$. Then $\dim V \geq 2$ as $\kappa > 0$, and

$$\operatorname{corank}\beta = |C \smallsetminus ((B \smallsetminus B_1)\alpha \cup \{w\})| = \operatorname{corank}\alpha - 1 \geq \kappa.$$

Hence $\beta \in CI(V, \kappa)$.

Case 2: κ is infinite. Then

$$\operatorname{corank} \beta = |C \smallsetminus ((B \smallsetminus B_1)\alpha \cup \{w\})| = |C \smallsetminus ((B \smallsetminus B_1)\alpha| = \operatorname{corank} \alpha \geq \kappa.$$

This implies $\beta \in CI(V, \kappa)$.

In either case, $\alpha < \beta$ on $CI(V, \kappa)$ by Theorem 4.1.8. □

Consequently, we have the following corollary.

Corollary 4.3.7. *(i) $OM(V) \smallsetminus OE(V)$ is the set of all maximal elements in $OM(V)$.*
(ii) $OE(V) \smallsetminus OM(V)$ is the set of all maximal elements in $OE(V)$.

Lemma 4.3.4 says that the elements in $K(V, \kappa) \smallsetminus CI(V, \kappa)$ are maximal in $K(V, \kappa)$. Also, elements in $CI(V, \kappa) \smallsetminus K(V, \kappa)$ are maximal in $CI(V, \kappa)$. The following examples present that there are elements in $K(V, \kappa) \cap CI(V, \kappa)$ which are maximal in $S(V, \kappa)$ when $\dim V$ is infinite and κ is finite.

Example 4.3.8. Suppose that $\dim V$ is infinite. Let κ be a natural number and let B and C be bases of V. There exist $B_0 \subseteq B$ and $C_0 \subseteq C$ such that $|B_0| = \kappa = |C_0|$. Then $|B \smallsetminus B_0| = |C \smallsetminus C_0|$, so there is a bijection $\phi : B \smallsetminus B_0 \to C \smallsetminus C_0$. Moreover, let $\{B_1, B_2\}$ be a partition of B such that $|B| = |B_1| = |B_2|$. Let ψ be a bijection from $B \smallsetminus B_0$ to B_2.

(i) Define $\alpha \in L(V)$ by

$$\alpha = \begin{pmatrix} B_0 & v \\ 0 & v\psi \end{pmatrix}_{v \in B \smallsetminus B_0}.$$

Observe that $\operatorname{nullity} \alpha = |B_0| = \kappa$ and $\operatorname{corank} \alpha = |B_1| > \kappa$, so $\alpha \in K(V, \kappa) \cap CI(V, \kappa)$. By Theorem 4.3.6, α is maximal in $K(V, \kappa)$ but not maximal in $CI(V, \kappa)$.

(ii) Define $\alpha \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v\psi^{-1} \end{pmatrix}_{v \in B_2}.$$

Thus $\operatorname{nullity} \alpha > \kappa$ and $\operatorname{corank} \alpha = |B \smallsetminus (B \smallsetminus B_0)| = |B_0| = \kappa$, and that $\alpha \in K(V, \kappa) \cap CI(V, \kappa)$. Hence α is maximal in $CI(V, \kappa)$ but not maximal in $K(V, \kappa)$

by Theorem 4.3.6.

(iii) Define $\alpha \in L(V)$ by

$$\alpha = \begin{pmatrix} B_0 & v \\ 0 & v\phi \end{pmatrix}_{v \in B \setminus B_0}.$$

Then $\operatorname{nullity} \alpha = \operatorname{corank} \alpha = \kappa$, so $\alpha \in K(V,\kappa) \cap CI(V,\kappa)$. Hence α is maximal in $K(V,\kappa)$ and $CI(V,\kappa)$ by Theorem 4.3.6.

Observe that every epimorphism in $K(V,\kappa)$ is not contained in $CI(V,\kappa)$. Similarly, every monomorphism in $CI(V,\kappa)$ is not contained in $K(V,\kappa)$. Therefore, by Theorems 4.2.6 (i), 4.2.9 (ii) and 4.3.6 (i) and (ii), we have the following corollary.

Corollary 4.3.9. *Let* $\dim V$ *be infinite. The following statements hold.*
(i) *For each nonzero* $\alpha \in K(V,\kappa)$*, if* α *is left compatible on* $K(V,\kappa)$*, then* α *is maximal in* $K(V,\kappa)$*.*
(ii) *For each nonzero* $\alpha \in CI(V,\kappa)$*, if* α *is right compatible on* $CI(V,\kappa)$*, then* α *is maximal in* $CI(V,\kappa)$*.*

4.4 Lower and Upper Covers of Elements in $(K(V,\kappa),\leq)$ and $(CI(V,\kappa),\leq)$

In the last section, we give necessary and sufficient conditions for elements in $K(V,\kappa)$ and $CI(V,\kappa)$ to have lower and upper covers where V is a general vector space and $\kappa > 0$. Note that we let $S(V,\kappa)$ be $K(V,\kappa)$ or $CI(V,\kappa)$.

Lemma 4.4.1. *Let* $\alpha, \beta \in K(V,\kappa) \cap CI(V,\kappa)$*. Then* α *is a lower cover of* β *in* $S(V,\kappa)$ *if and only if* α *is a lower cover of* β *in* $L(V)$*.*

Proof. The sufficiency is obtained from Proposition 2.3.4.

To show the necessity, assume that α is a lower cover of β in $S(V,\kappa)$. Suppose that $\alpha < \gamma \leq \beta$ on $L(V)$. From Remark 4.1.14, $\gamma \in K(V,\kappa) \cap CI(V,\kappa)$.

Case 1: $\dim V$ is finite. Then $K(V,\kappa) = CI(V,\kappa)$ is regular by Theorem 4.1.1.

Hence $\alpha < \gamma \leq \beta$ on $S(V,\kappa)$ by Proposition 2.3.2.

Case 2: $\dim V$ is infinite. Then $\alpha < \gamma \leq \beta$ on $S(V,\kappa)$ by Theorems 4.1.4 and 4.1.8 and Remark 4.1.14.

In either case, we obtain $\gamma = \beta$. Therefore α is a lower cover of β in $L(V)$. $\square$

The next result will be used in our main theorem.

Lemma 4.4.2. *Let* $\dim V$ *be infinite and let* $\alpha \in K(V,\kappa)$ *and* $\beta \in K(V,\kappa) \smallsetminus CI(V,\kappa)$ *be such that* $\alpha < \beta$ *on* $K(V,\kappa)$*. Then* α *is a lower cover of* β *in* $K(V,\kappa)$ *if and only if* $\operatorname{corank}\alpha = \kappa < \infty$*.*

Proof. Assume that $\operatorname{corank}\alpha = \kappa < \infty$. Suppose that $\alpha \leq \gamma < \beta$ on $K(V,\kappa)$ for some $\gamma \in K(V,\kappa)$. Then, by Theorem 4.1.4, $\operatorname{im}\alpha \subseteq \operatorname{im}\gamma$, $V\alpha\gamma^{-1} = E(\alpha,\gamma)$ and $\gamma \in CI(V,\kappa)$, so

$$\kappa \leq \operatorname{corank}\gamma \leq \operatorname{corank}\alpha = \kappa < \infty.$$

Thus $\operatorname{corank}\gamma = \kappa = \operatorname{corank}\alpha$. Similar to the proof of Lemma 4.3.5 (ii), $\operatorname{im}\gamma = \operatorname{im}\alpha$. Since $\operatorname{im}\gamma = \operatorname{im}\alpha$ and $V\alpha\gamma^{-1} = E(\alpha,\gamma)$, by Proposition 2.2.7, $\gamma = \alpha$. Therefore α is a lower cover of β in $K(V,\kappa)$.

For the forward implication, suppose that $\operatorname{corank}\alpha \neq \kappa$ or κ is infinite. Since $\alpha < \beta$ on $K(V,\kappa)$, by Theorem 4.1.4, $\alpha \in CI(V,\kappa)$, $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha,\beta)$. By Proposition 2.2.9, we write α and β as

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \quad \text{and} \quad \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & v_j & u_k \end{pmatrix}_{j\in J, k\in K}$$

with $\{x_i\}_{i\in I}$, $\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}$, $\{u_k\}_{k\in K}$, $\{v_j\}_{j\in J}\cup\{u_k\}_{k\in K}$ and $\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}\cup\{z_k\}_{k\in K}$ are bases of $\ker\beta$, $\ker\alpha$, $\operatorname{im}\alpha$, $\operatorname{im}\beta$ and V, respectively. Let $\{v_j\}_{j\in J}\cup\{u_k\}_{k\in K}\cup\{w_l\}_{l\in L}$ be a basis of V. Since $\beta \notin CI(V,\kappa)$, we have $|\{w_l\}_{l\in L}| = \operatorname{corank}\beta < \kappa$. Thus $|L| < \kappa$. Since $\alpha \in CI(V,\kappa)$, we obtain $|\{v_j\}_{j\in J}\cup\{w_l\}_{l\in L}| = \operatorname{corank}\alpha \geq \kappa$. As $|L| < \kappa$ and we assume that $\operatorname{corank}\alpha > \kappa$ or κ is infinite, we obtain $|J| > 1$. Then we let $j_0 \in J$. Define $\gamma \in L(V)$ by

$$\gamma = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J\smallsetminus\{j_0\}} & y_{j_0} & z_k \\ 0 & v_{j_0} & u_k \end{pmatrix}_{k\in K}.$$

Case 1: $\operatorname{corank}\alpha > \kappa$. Then $\operatorname{corank}\gamma = \operatorname{corank}\alpha - 1 \geq \kappa$.

Case 2: κ is infinite. Since $\alpha \in CI(V,\kappa)$, we have $\operatorname{corank}\alpha$ is also infinite. Thus $\operatorname{corank}\gamma = \operatorname{corank}\alpha - 1 = \operatorname{corank}\alpha \geq \kappa$.

In any case, $\gamma \in CI(V,\kappa)$. It is obvious that $\gamma \in K(V,\kappa)$ since $\operatorname{nullity}\gamma = |\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J\smallsetminus\{j_0\}}| \geq |\{x_i\}_{i\in I}| = \operatorname{nullity}\beta \geq \kappa$. Note that $\operatorname{im}\alpha \subsetneq \operatorname{im}\gamma \subsetneq \operatorname{im}\beta$. The sets $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J\smallsetminus\{j_0\}}$, $\{y_{j_0}\}$ and $\{z_k\}_{k\in K}$ fulfill Lemma 2.2.10, so we get $V\alpha\gamma^{-1} = E(\alpha,\gamma)$. Next, the sets $\{x_i\}_{i\in I}$, $\{y_j\}_{j\in J\smallsetminus\{j_0\}}$ and $\{y_{j_0}\} \cup \{z_k\}_{k\in K}$ satisfy Lemma 2.2.10. Then $V\gamma\beta^{-1} = E(\gamma,\beta)$. By Theorem 4.1.4, $\alpha < \gamma < \beta$ on $K(V,\kappa)$. Therefore, α is not a lower cover of β in $K(V,\kappa)$. □

Therefore, we describe the set of all lower covers of an element in $K(V,\kappa) \smallsetminus CI(V,\kappa)$ where κ is finite.

Corollary 4.4.3. *Let* $\dim V$ *be infinite,* κ *a natural number and* $\beta \in K(V,\kappa) \smallsetminus CI(V,\kappa)$*. Then*

$$\{\alpha \in K(V,\kappa) \mid \alpha < \beta \text{ on } K(V,\kappa) \text{ and } \operatorname{corank}\alpha = \kappa\}$$

is the set of all lower covers of β *in* $K(V,\kappa)$*.*

Remark 4.4.4. Consider α and γ in the proof of the forward implication of Lemma 4.4.2. Note that α is a lower cover of γ in $L(V)$ by Lemma 3.4.1. Then, by Proposition 4.4.1, α is also a lower cover of γ in $K(V,\kappa)$. Suppose that κ is infinite. It follows that J is infinite and then let $j_1 \in J \smallsetminus \{j_0\}$. Next, we define $\gamma_1 \in L(V)$ by

$$\gamma_1 = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J\smallsetminus\{j_0,j_1\}} & y_{j_0} & y_{j_1} & z_k \\ 0 & v_{j_0} & v_{j_1} & u_k \end{pmatrix}_{k\in K}.$$

Then $\alpha < \gamma < \gamma_1 < \beta$ on $K(V,\kappa)$. It can be seen that γ is a lower cover of γ_1 by Lemma 3.4.1. Since J is infinite, we can construct infinitely many $\gamma_i \in K(V,\kappa)$ similar to γ_1 such that $\alpha < \gamma < \gamma_1 < \gamma_2 < \cdots < \gamma_i < \cdots < \beta$ on $K(V,\kappa)$ where i is a natural number. In particular, γ_i is a lower cover of γ_{i+1} for all natural numbers i.

Next, we provide a characterization for elements in $K(V,\kappa)$ to have a lower cover.

Theorem 4.4.5. (i) *Every nonzero* $\beta \in K(V,\kappa) \cap CI(V,\kappa)$ *has a lower cover in* $K(V,\kappa)$.
(ii) *Let* $\dim V$ *be infinite and let* $\beta \in K(V,\kappa) \smallsetminus CI(V,\kappa)$. *Then* β *has a lower cover in* $K(V,\kappa)$ *if and only if* κ *is finite.*

Proof. (i) Let $\beta \in K(V,\kappa) \cap CI(V,\kappa)$ be nonzero. By Theorem 3.4.4 (i), β has a lower cover in $L(V)$, say α. Since $\beta \in K(V,\kappa) \cap CI(V,\kappa)$, by Remark 4.1.14, we get $\alpha \in K(V,\kappa) \cap CI(V,\kappa)$. Let $\gamma \in K(V,\kappa)$ be such that $\alpha < \gamma \leq \beta$ on $K(V,\kappa)$. Then $\alpha < \gamma \leq \beta$ on $L(V)$. Since α is a lower cover of β in $L(V)$, we obtain $\gamma = \beta$. Hence α is a lower cover of β in $K(V,\kappa)$.

(ii) To show the sufficiency, suppose that κ is finite. Let B be a basis of V containing a basis B_1 of $\ker \beta$. Denote by $C_1 = (B \smallsetminus B_1)\beta$, a basis of $\operatorname{im} \beta$ by Proposition 2.2.4 (ii). Extend C_1 to a basis C of V. Since $|C \smallsetminus C_1| = \operatorname{corank} \beta < \kappa < \infty$, we get C_1 is infinite. Then there is $C_2 \subseteq C_1$ such that $|(C \smallsetminus C_1) \cup C_2| = \kappa$. Note that $\beta|_{B \smallsetminus B_1} : B \smallsetminus B_1 \to C_1$ is a bijection by Proposition 2.2.4 (i). Let $B_0 = C_2\beta^{-1}$. Define $\alpha \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 \cup B_0 & v \\ 0 & v\beta \end{pmatrix}_{v \in B \smallsetminus (B_1 \cup B_0)}.$$

Then $\operatorname{im} \alpha \subsetneq \operatorname{im} \beta$, and $V\alpha\beta^{-1} = E(\alpha,\beta)$ holds since B_1, B_0 and $B \smallsetminus (B_1 \cup B_0)$ fulfill Lemma 2.2.10. Hence $\alpha < \beta$ on $L(V)$ by Theorem 2.3.5. Since $\beta \in K(V,\kappa)$, by Proposition 4.1.13 (i), we have $\alpha \in K(V,\kappa)$. Since $B_0\beta = C_2$, we have

$$\dim(V/\operatorname{im} \alpha) = |C \smallsetminus [B \smallsetminus (B_1 \cup B_0)]\beta| = |C \smallsetminus (C_1 \smallsetminus C_2)| = |(C \smallsetminus C_1) \cup C_2| = \kappa.$$

Thus $\alpha \in CI(V,\kappa)$. Therefore, Theorem 4.1.4 implies that $\alpha < \beta$ on $K(V,\kappa)$. Since $\operatorname{corank} \alpha = \kappa < \infty$, by Lemma 4.4.2, α is a lower cover of β in $K(V,\kappa)$.

The forward implication is a consequence from Lemma 4.4.2. □

By letting $\kappa = \aleph_0$, we obtain the below corollary.

Corollary 4.4.6. (i) *Every nonzero* $\beta \in OM(V) \cap OE(V)$ *has a lower cover in* $OM(V)$.
(ii) *Every* $\beta \in OM(V) \setminus OE(V)$ *has no lower covers in* $OM(V)$.

We conclude the following remark from Theorems 4.3.6 (ii) and 4.4.5 (ii).

Remark 4.4.7. Suppose that κ is a natural number. Let $\beta \in K(V,\kappa) \setminus CI(V,\kappa)$. Every lower cover of β in $K(V,\kappa)$ is a maximal element in $CI(V,\kappa)$.

Next, we give an example of the existence of $\alpha, \beta \in K(V,\kappa)$ such that α is a lower cover of β in $K(V,\kappa)$ but α is not a lower cover of β in $L(V)$. Moreover, some elements in $K(V,\kappa) \setminus CI(V,\kappa)$ have distinct lower covers in $K(V,\kappa)$.

Example 4.4.8. Let κ be a natural number such that $\kappa > 1$. Suppose that $\dim V$ is infinite. Let B be a basis of V. Then there exists a partition $\{B_1, B_2\}$ of B such that $|B| = |B_1| = |B_2|$. Thus there is a bijection $\phi : B_2 \to B$. Let $B_0 \subseteq B_2$ be such that $|B_0| = \kappa$. Define $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 \cup B_0 & v \\ 0 & v\phi \end{pmatrix}_{v \in B_2 \setminus B_0} \quad \text{and} \quad \beta = \begin{pmatrix} B_1 & v \\ 0 & v\phi \end{pmatrix}_{v \in B_2}.$$

As $\operatorname{nullity} \alpha = |B_1 \cup B_0|$ and $\operatorname{nullity} \beta = |B_1|$, we have $\alpha, \beta \in K(V,\kappa)$. Since β is an epimorphism, $\beta \notin CI(V,\kappa)$. Note that

$$\operatorname{corank} \alpha = |B \setminus (B \setminus B_0\phi)| = |B_0\phi| = |B_0| = \kappa.$$

Thus $\alpha \in CI(V,\kappa)$. Moreover, $\operatorname{im} \alpha \subsetneq \operatorname{im} \beta$. The sets B_1, B_0 and $B_2 \setminus B_0$ satisfy Lemma 2.2.10. Thus $V\alpha\beta^{-1} = E(\alpha,\beta)$. Therefore $\alpha < \beta$ on $K(V,\kappa)$ by Theorem 4.1.4. Since $\operatorname{corank} \alpha = \kappa$, by Lemma 4.4.2, α is a lower cover of β in $K(V,\kappa)$. However, α is not a lower cover of β in $L(V)$ by Proposition 3.4.2. Since there are infinitely many subsets of B_2 which their cardinalities are κ, the number of lower covers of β in $K(V,\kappa)$ is also infinite. Note that if $\kappa = 1$, then α is a lower cover of β in $K(V,\kappa)$ and $L(V)$.

Now we pay attention on results of lower covers of elements in $CI(V,\kappa)$. The following result is similar to Lemma 4.4.2.

Lemma 4.4.9. *Let* $\dim V$ *be infinite and let* $\alpha \in CI(V,\kappa)$ *and* $\beta \in CI(V,\kappa) \setminus K(V,\kappa)$ *be such that* $\alpha < \beta$ *on* $CI(V,\kappa)$*. Then* α *is a lower cover of* β *in* $CI(V,\kappa)$ *if and only if* $\operatorname{nullity}\alpha = \kappa < \infty$.

Proof. Suppose that $\operatorname{nullity}\alpha = \kappa < \infty$ and $\alpha \leq \gamma < \beta$ on $CI(V,\kappa)$ for some $\gamma \in CI(V,\kappa)$. Then, by Theorem 4.1.8, $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\gamma^{-1} = E(\alpha,\gamma)$. By Proposition 2.2.6, $\ker\gamma \subseteq \ker\alpha$, so $\operatorname{nullity}\gamma \leq \operatorname{nullity}\alpha = \kappa$. Notice that $\gamma \in K(V,\kappa)$ by Theorem 4.1.8. It follows that

$$\kappa \leq \operatorname{nullity}\gamma \leq \operatorname{nullity}\alpha = \kappa < \infty.$$

Thus $\operatorname{nullity}\gamma = \kappa = \operatorname{nullity}\alpha$. Hence $\ker\gamma = \ker\alpha$. Since $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $\ker\gamma = \ker\alpha$, we have $\gamma = \alpha$ by Proposition 2.2.7. Therefore α is a lower cover of β in $CI(V,\kappa)$.

To prove the necessity, assume that $\operatorname{nullity}\alpha \neq \kappa$ or κ is infinite. As $\alpha < \beta$ on $CI(V,\kappa)$, we have $\alpha \in K(V,\kappa)$, $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha,\beta)$ by Theorem 4.1.8. Then $\operatorname{nullity}\alpha \geq \kappa$. By Proposition 2.2.9, we write α and β as follows.

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \quad \text{and} \quad \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & v_j & u_k \end{pmatrix}_{j\in J, k\in K}$$

where $\{x_i\}_{i\in I}$, $\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}$, $\{u_k\}_{k\in K}$, $\{v_j\}_{j\in J}\cup\{u_k\}_{k\in K}$ and $\{x_i\}_{i\in I}\cup\{y_j\}_{j\in J}\cup\{z_k\}_{k\in K}$ are bases of $\ker\beta$, $\ker\alpha$, $\operatorname{im}\alpha$, $\operatorname{im}\beta$ and V, respectively. Since $\alpha \in K(V,\kappa)$ and $\beta \notin K(V,\kappa)$, we have $|\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}| = \operatorname{nullity}\alpha > \kappa$ and $|\{x_i\}_{i\in I}| = \operatorname{nullity}\beta < \kappa$, respectively. Since $\operatorname{nullity}\alpha > \kappa$ or κ is infinite, $|J| > 1$ so we let $j_0 \in J$. Next define $\gamma \in L(V)$ by

$$\gamma = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J\setminus\{j_0\}} & y_{j_0} & z_k \\ 0 & v_{j_0} & u_k \end{pmatrix}_{k\in K}.$$

Observe that $\operatorname{im}\alpha \subsetneq \operatorname{im}\gamma \subsetneq \operatorname{im}\beta$.

Case 1: $\operatorname{nullity}\alpha > \kappa$. Then $\operatorname{nullity}\gamma = \operatorname{nullity}\alpha - 1 \geq \kappa$.

Case 2: κ is infinite. Since $\alpha \in K(V,\kappa)$, we have $\operatorname{nullity}\gamma = \operatorname{nullity}\alpha - 1 = \operatorname{nullity}\alpha \geq \kappa$.

In any case, $\gamma \in K(V, \kappa)$. Since $\operatorname{im} \gamma \subseteq \operatorname{im} \beta$, we have $\operatorname{corank} \gamma \geq \operatorname{corank} \beta \geq \kappa$. Thus $\gamma \in CI(V, \kappa)$. Similar to the proof of the necessity of Lemma 4.4.2, $V\alpha\gamma^{-1} = E(\alpha, \gamma)$ and $V\gamma\beta^{-1} = E(\gamma, \beta)$. Hence $\alpha < \gamma < \beta$ on $CI(V, \kappa)$ by Theorem 4.1.8. Therefore α is not a lower cover of β in $CI(V, \kappa)$. □

We now show the set of all lower covers of an element in $CI(V, \kappa) \setminus K(V, \kappa)$ when κ is finite.

Corollary 4.4.10. *Let* $\dim V$ *be infinite,* κ *a natural number and* $\beta \in CI(V, \kappa) \setminus K(V, \kappa)$. *Then*

$$\{\alpha \in CI(V, \kappa) \mid \alpha < \beta \textit{ on } CI(V, \kappa) \textit{ and } \operatorname{nullity} \alpha = \kappa\}$$

is the set of all lower covers of β *in* $CI(V, \kappa)$.

From the proof of the necessity of Lemma 4.4.9, the below remark is obtained.

Remark 4.4.11. Suppose that κ is infinite. Then J is infinite and let $j_1 \in J \setminus \{j_0\}$. By defining γ_1 as in Remark 4.4.4, we also obtain γ is a lower cover of γ_1 in $CI(V, \kappa)$. Then construct infinitely many $\gamma_i \in CI(V, \kappa)$ such that $\alpha < \gamma < \gamma_1 < \gamma_2 < \cdots < \gamma_i < \cdots < \beta$ on $CI(V, \kappa)$ where i is a natural number. Furthermore, γ_i is a lower cover of γ_{i+1} in $CI(V, \kappa)$ for all natural number i.

Next, we characterize when elements in $CI(V, \kappa)$ have lower covers.

Theorem 4.4.12. (i) *Every nonzero* $\beta \in K(V, \kappa) \cap CI(V, \kappa)$ *has a lower cover in* $CI(V, \kappa)$.
(ii) *Let* $\dim V$ *be infinite and let* $\beta \in CI(V, \kappa) \setminus K(V, \kappa)$. *Then* β *has a lower cover in* $CI(V, \kappa)$ *if and only if* κ *is finite.*

Proof. (i) Let $\beta \in K(V, \kappa) \cap CI(V, \kappa)$ be nonzero. Then, by Theorem 4.4.5 (i), β has a lower cover in $K(V, \kappa)$, say α. That is $\alpha < \beta$ on $K(V, \kappa)$. By Theorems 4.1.4 and 4.1.8, we obtain $\alpha \in K(V, \kappa) \cap CI(V, \kappa)$ and $\alpha < \beta$ on $CI(V, \kappa)$, respectively. To show that α is a lower cover of β in $CI(V, \kappa)$, let $\gamma \in CI(V, \kappa)$ be such that $\alpha < \gamma \leq \beta$ on $CI(V, \kappa)$. From Theorems 4.1.8 and 4.1.4, we have $\gamma \in K(V, \kappa)$ and $\alpha < \gamma \leq \beta$ on $K(V, \kappa)$, respectively. Since α is a lower cover of β in $K(V,\kappa)$, we can conclude that $\gamma = \beta$. Therefore α is a lower cover of β in $CI(V,\kappa)$.

(ii) To show the sufficiency, assume that κ is finite. Let B_1 be a basis of $\ker\beta$ and B a basis of V containing B_1. As $\beta \notin K(V,\kappa)$, we have $|B_1| = \operatorname{nullity}\beta < \kappa < \infty$. Then $B \setminus B_1$ is infinite, so there is a nonempty set $B_0 \subsetneq B \setminus B_1$ such that $|B_1 \cup B_0| = \kappa$. Define $\alpha \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 \cup B_0 & v \\ 0 & v\beta \end{pmatrix}_{v \in B \setminus (B_1 \cup B_0)}.$$

Thus $\operatorname{im}\alpha \subsetneq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha,\beta)$ holds by using B_1, B_0 and $B \setminus (B_1 \cup B_0)$ in Lemma 2.2.10. Hence $\alpha < \beta$ on $L(V)$ by Theorem 2.3.5. Since $\beta \in CI(V,\kappa)$, we obtain $\alpha \in CI(V,\kappa)$ by Lemma 4.1.13 (ii). Furthermore, $\operatorname{nullity}\alpha = |B_1 \cup B_0| = \kappa$. Hence $\alpha \in K(V,\kappa) \cap CI(V,\kappa)$. Therefore, Theorem 4.1.8 implies that $\alpha < \beta$ on $CI(V,\kappa)$. Since $\operatorname{nullity}\alpha = \kappa$, by Lemma 4.4.9, α is a lower cover of β in $CI(V,\kappa)$.

For the necessity, suppose that κ is infinite. By Lemma 4.4.9, we have β has no lower cover in $CI(V,\kappa)$. □

Corollary 4.4.13. (i) *Every nonzero* $\beta \in OM(V) \cap OE(V)$ *has a lower cover in* $OE(V)$.
(ii) *Every* $\beta \in OE(V) \setminus OM(V)$ *has no lower covers in* $OE(V)$.

By Theorems 4.3.6 (i) and 4.4.12 (ii), we have the following remark.

Remark 4.4.14. Let κ be a natural number and $\beta \in CI(V,\kappa) \setminus K(V,\kappa)$. Every lower cover of β in $CI(V,\kappa)$ is a maximal element in $K(V,\kappa)$.

We present the below example to demonstrate that there is an element in $CI(V,\kappa)$ whose lower covers in $L(V)$ and $CI(V,\kappa)$ are different. Moreover, it can be observed that a lower cover of an element in $CI(V,\kappa) \setminus K(V,\kappa)$ need not to be unique.

Example 4.4.15. Suppose that $\dim V$ is infinite. Let κ be a natural number such that $\kappa > 1$ and let B be a basis of V. Then there is a partition $\{B_1, B_2\}$

of V such that $|B| = |B_1| = |B_2|$. Thus there exists a bijection $\psi : B \to B_2$. Let $B_0 \subseteq B_2$ be such that $|B_0| = \kappa$. Define $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} B_0 & v \\ 0 & v\psi \end{pmatrix}_{v \in B \setminus B_0} \quad \text{and} \quad \beta = \begin{pmatrix} v \\ v\psi \end{pmatrix}_{v \in B}.$$

Observe that nullity $\alpha = |B_0| = \kappa$, hence that $\alpha \in K(V, \kappa)$. Since $\operatorname{im} \alpha \subsetneq \operatorname{im} \beta$, we have $\operatorname{corank} \alpha \geq \operatorname{corank} \beta = |B_1| \geq \kappa$. Then $\alpha, \beta \in CI(V, \kappa)$. Since β is a monomorphism, $\beta \notin K(V, \kappa)$. The sets $\varnothing$, B_0 and $B \setminus B_0$ satisfy Lemma 2.2.10 and hence $V\alpha\beta^{-1} = E(\alpha, \beta)$. Thus $\alpha < \beta$ on $CI(V, \kappa)$ by Theorem 4.1.8. Therefore α is a lower cover of β in $CI(V, \kappa)$ by Lemma 4.4.9. Since $\kappa > 1$, by Corollary 3.4.2, α is not a lower cover of β in $L(V)$. Note that B_2 has infinitely many subsets which their cardinalities equal to κ. Hence β has infinite lower covers in $CI(V, \kappa)$. If $\kappa = 1$, then α is a lower cover of β in $CI(V, \kappa)$ and $L(V)$.

Next, characterizations when elements in $K(V, \kappa)$ and $CI(V, \kappa)$ have upper covers in $K(V, \kappa)$ and $CI(V, \kappa)$, respectively, are investigated.

Theorem 4.4.16. *Let $\alpha \in S(V, \kappa)$. Then α has an upper cover in $S(V, \kappa)$ if and only if α is not maximal in $S(V, \kappa)$.*

Proof. Suppose that α is not maximal in $S(V, \kappa)$. Then, by Theorem 4.3.6, $\alpha \in K(V, \kappa) \cap CI(V, \kappa)$. Let $w \in V \setminus \operatorname{im} \alpha$ and $u \in \ker \alpha \setminus \{0\}$, and let B_1 be a basis of $\ker \alpha$ containing u. Extend B_1 to a basis B of V. Let C be a basis of V containing $(B \setminus B_1)\alpha \cup \{w\}$. Define $\beta \in L(V)$ by

$$\beta = \begin{pmatrix} B_1 \setminus \{u\} & u & v \\ 0 & w & v\alpha \end{pmatrix}_{v \in B \setminus B_1}.$$

Similar to the proof of Theorem 4.3.6, we get $\beta \in S(V, \kappa)$ and then $\alpha < \beta$ on $S(V, \kappa)$. By Lemma 3.4.1, β is an upper cover of α in $L(V)$. Hence, Proposition 2.3.4 implies that β is an upper cover of α in $S(V, \kappa)$.

The forward implication is obvious. □

By taking $\kappa = \aleph_0$, we obtain the below corollary.

Corollary 4.4.17. *Let $S(V)$ be $OM(V)$ or $OE(V)$, and let $\alpha \in S(V)$. Then α has an upper cover in $S(V)$ if and only if α is not maximal in $S(V)$.*

The following corollaries are obtained from Theorems 4.3.6 and 4.4.16.

Corollary 4.4.18. *Let $\alpha \in K(V,\kappa)$. Then the following are equivalent.*
(i) α is maximal in $K(V,\kappa)$.
(ii) $\alpha \notin CI(V,\kappa)$ or $\operatorname{nullity}\alpha = \kappa < \infty$.
(iii) α has no upper cover in $K(V,\kappa)$.

Corollary 4.4.19. *Let $\alpha \in CI(V,\kappa)$. Then the following are equivalent.*
(i) α is maximal in $CI(V,\kappa)$.
(ii) $\alpha \notin K(V,\kappa)$ or $\operatorname{corank}\alpha = \kappa < \infty$.
(iii) α has no upper cover in $CI(V,\kappa)$.

We illustrate examples of elements in $S(V,\kappa)$ where $\dim V$ is infinite and κ is finite in Figure 4.2. In Figure 4.3, we consider when $\dim V$ and κ are infinite.

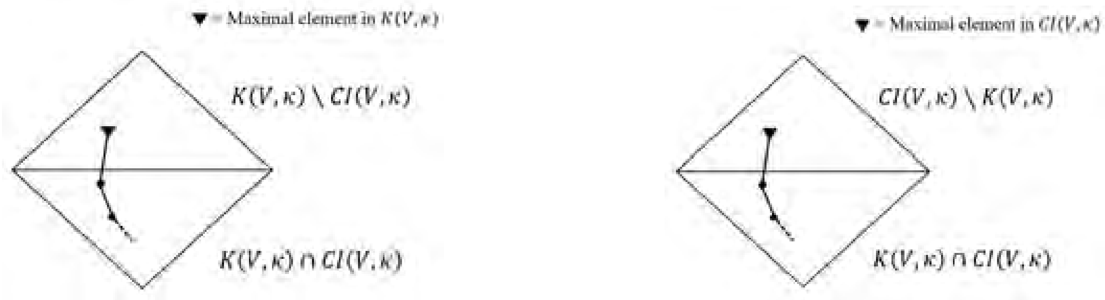

Figure 4.2: These are followed from Theorems 4.4.5 and 4.4.12.

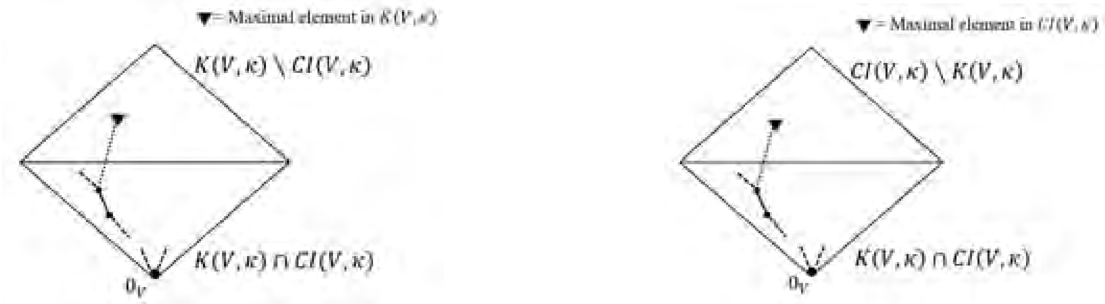

Figure 4.3: These are obtained by Theorems 4.4.5 (ii) and 4.4.12 (ii).

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