กฎอย่างเข้มของเลขจำนวนมากสำหรับตัวแปรสุ่มที่ไม่เป็นอิสระต่อกันเชิงจตุภาคลบเป็นคู่

นางสาวธนพร ธโนดมเดช

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STRONG LAW OF LARGE NUMBERS FOR PAIRWISE NEGATIVELY QUADRANT DEPENDENT RANDOM VARIABLES

Miss Thanaporn Thanodomdech

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By	Miss Thanaporn Thanodomdech
Field of Study	Mathematics
Thesis Advisor	Assistant Professor Nattakarn Chaidee, Ph.D.
Thesis Co-advisor	Jiraphan Suntornchost, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

. . . . Dean of the Faculty of Science

(Associate Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

. . . . Chairman

(Professor Kritsana Neammanee, Ph.D.)

. . . . Thesis Advisor

(Assistant Professor Nattakarn Chaidee, Ph.D.)

. . . . Thesis Co-advisor

(Jiraphan Suntornchost, Ph.D.)

. . . . Examiner

(Associate Professor Imchit Termwuttipong, Ph.D.)

. . . . External Examiner

(Assistant Professor Petcharat Rattanawong, Ph.D.)

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เราศึกษาความสัมพันธ์ของโครงสร้างความไม่เป็นอิสระของตัวแปรสุ่มและเสนอกฎอย่างเข้มของเลขจำนวนมากแบบมาร์ซินกีวิกซ์-ซิกมันสำหรับตัวแปรสุ่มที่ไม่เป็นอิสระต่อกันเชิงจตุภาคลบแบบขยายและมีการแจกแจงแบบเดียวกัน และเรายังได้กฎอย่างเข้มของเลขจำนวนมากแบบมาร์ซินกีวิกซ์-ซิกมันสำหรับตัวแปรสุ่มที่ไม่เป็นอิสระต่อกันเชิงจตุภาคลบเป็นคู่

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		ลายมือชื่อ อ.ที่ปรึกษาหลัก	....
สาขาวิชา	คณิตศาสตร์	ลายมือชื่อ อ.ที่ปรึกษาร่วม	....
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We study the relationship of dependence structures of random variables and propose the Marcinkiewicz-Zygmund strong law of large numbers for extended negatively dependent and identically distributed random variables and we also obtain the Marcinkiewicz-Zygmund strong law of large numbers for pairwise negatively quadrant dependent random variables.

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CHAPTER I

INTRODUCTION

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables and $S_n = \sum_{i=1}^{n} X_i$. One common objective of different versions of strong law of large numbers (SLLN) is to find sufficient conditions on the sequence of random variables $(X_n)_{n\in\mathbb{N}}$ that make $\frac{S_n - a_n}{b_n}$ converges to 0 almost surely ($a.s.$) where $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of real numbers such that $b_n > 0$ for all $n \in \mathbb{N}$. It is called the Kolmogorov SLLN when $a_n = ES_n$ and $b_n = n$ for each $n \in \mathbb{N}$ and it is called the Marcinkiewicz-Zygmund(M-Z) SLLN if $a_n = ES_n$ and $b_n = n^{\frac{1}{p}}$ for $n \in \mathbb{N}$ and $0 < p < 2$. Therefore the Kolmogorov SLLN is a special case of the Marcinkiewicz-Zygmund SLLN when $p = 1$.

For the classical SLLN, the random variables X_n's are assumed to be independent and identically distributed (i.i.d.) random variables with finite first moment which is stated as follows.

Theorem 1.1. *([8]) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables. If $E|X_1| < \infty$, then*

$$\frac{S_n - nEX_1}{n} \longrightarrow 0 \ a.s.$$

The classical SLLN has been extended into two different directions, relaxing the dependency assumption and removing the assumption of identical distribution. For example, Etemadi[7] proposed an alternative version of the SLLN by relaxing the dependency assumption of the sequence of random variables $(X_n)_{n\in\mathbb{N}}$ from independence to pairwise independence. Sancetta[14] replaced pairwise independence assumption studied in Etemadi[7] by pairwise positively quadrant dependence (pairwise PQD) assumption without assuming the identical distribution.

In Chapter 3, we study the relations of dependence structures and provide some examples which show their relations.

In Chapter 4, we consider the SLLN with the assumption of the identical distribution. For example, Kruglov[10] gave the SLLN for pairwise i.i.d. with infinite means and then Sung[17] proposed the SLLN for pairwise i.i.d. random variables with general moment conditions $\left(\sum_{n=1}^{\infty} P(|X_n| > b_n) < \infty\right)$ where $(b_n)_{n\in\mathbb{N}}$ is a sequence of positive constants. Later, Korchevsky[9] obtained the Marcinkiewicz-Zygmund SLLN for pairwise i.i.d. random variables. In 1992, Matula[13] relaxed the dependency assumption studied in Etemadi[7] by pairwise negatively quadrant dependence (pairwise NQD) assumption. Then the SLLN for pairwise NQD and identically distributed random variables with infinite means which relaxed the moment condition of Matula[13] and the dependency assumption of Kruglov[10] is proposed by Chaidee and Neammanee[2]. Later in 2014, Shen et al.[16] gave the SLLN for pairwise NQD and identically distributed random variables with general moment conditions by relaxing the moment condition of Chaidee and Neammanee[2] and the dependency assumption of Sung[17]. Moreover, Chen et al.[5] presented the SLLN for extended negatively dependent (END) random variables with assuming finite first moment.

In this study, we propose the Marcinkiewicz-Zygmund SLLN for END random variables which is a generalization of the Marcinkiewicz-Zygmund SLLN for pairwise i.i.d. random variables of Korchevsky[9]. Moreover, our result generalizes the Kolmogorov SLLN for END random variables in Chen et al.[5]. Our result is stated as follows.

Theorem 1.2. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of END and identically distributed random variables. For $0 < p < 2$, if $E|X_1|^p < \infty$,*

$$\frac{S_n - nE(X_1)}{n^{\frac{1}{p}}} \longrightarrow 0 \ a.s.$$

In chapter 5, we obtain the Marcinkiewicz-Zygmund SLLN for pairwise NQD random variables which is generalized from the SLLN of Chandra and Goswami[4] and it also relaxes the identical distribution and pairwise independence assumption in the result of Korchevsky[9]. Our result is stated as follows.

Theorem 1.3. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of pairwise NQD random variables. For $0 < p < 2$, if*

(i) $\sup_{n\in\mathbb{N}} \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} E|X_i| < \infty$,

(ii) $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{p}}} E|X_n| I(|X_n| > n^{\alpha}) < \infty$ *for some* $\alpha \in \left(0, \min\left(\frac{1}{2}, \frac{1}{p} - \frac{1}{2}\right)\right)$,

then $\frac{S_n - ES_n}{n^{\frac{1}{p}}} \longrightarrow 0$ *a.s.*

Tables 1.1 and 1.2 shown below explain series of studies in the context of SLLN and our new contributions for identically distributed random variables and non-identically distributed random variables, respectively.

	pairwise independent	pairwise NQD	END
Kolmogorov SLLN	Sung(2013) $\sum_{n=1}^{\infty} P(\lvert X_n\rvert > b_n) < \infty$ where $\frac{b_n}{n} \uparrow \infty$	Shen et al.(2014) $\sum_{n=1}^{\infty} P(\lvert X_n\rvert > b_n) < \infty$ where $\frac{b_n}{n} \uparrow \infty$	
	Kruglov(2008) $E\lvert X_1\rvert = \infty$	Chaidee and Neammanee (2009) $E\lvert X_1\rvert = \infty$	
	Etemadi(1981) $E\lvert X_1\rvert < \infty$	Matula(1992) $E\lvert X_1\rvert < \infty$	Chen et al.(2010) $E\lvert X_1\rvert < \infty$
M-Z SLLN	Korchevsky(2014) $E\lvert X_1\rvert^p < \infty$ for $0 < p < 2$	**Our work** $E\lvert X_1\rvert^p < \infty$ for $0 < p < 2$ **(Chapter 4)**	**Our work** $E\lvert X_1\rvert^p < \infty$ for $0 < p < 2$ **(Chapter 4)**

Table 1.1: SLLN for identically distributed random variables

pairwise independent	pairwise NQD
Kolmogorov SLLN	M-Z SLLN
Chandra and Goswami(2003) $\sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{i=1}^{n} E\lvert X_i\rvert < \infty,$ $\sum_{n=1}^{\infty} \frac{1}{n} E\lvert X_n\rvert I(\lvert X_n\rvert > n^{\alpha}) < \infty$ for some $\alpha \in \left(0, \frac{1}{2}\right)$	**Our work** $\sup_{n\in\mathbb{N}} \frac{1}{n^{\frac{1}{p}}}\sum_{i=1}^{n} E\lvert X_i\rvert < \infty,$ $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{p}}} E\lvert X_n\rvert I(\lvert X_n\rvert > n^{\alpha}) < \infty$ for some $\alpha \in \left(0, \min\left(\frac{1}{2}, \frac{1}{p} - \frac{1}{2}\right)\right)$ for $0 < p < 2$ **(Chapter 5)**

Table 1.2: SLLN for non-identically distributed random variables

CHAPTER II

PRELIMINARIES

In this chapter, we review some basic knowledge in probability and collect some definitions and theorems which will be used in this work. These can be found in Chung[6] and Grimmett and Stirzaker[8] .

2.1 Basic Knowledge in Probability

In this section, we review some basic knowledge in probability such as event, probability measure and random variable.

Definition 2.1. Let Ω be a set and $\mathcal{F}$ be a σ-algebra. Let $P : \mathcal{F} \to [0, 1]$ be a measure such that $P(\Omega) = 1$. Then $(\Omega, \mathcal{F}, P)$ is called a *probability space* and P is called a *probability measure*. The set Ω is the *sure event* and the elements of $\mathcal{F}$ are called *events*.

Definition 2.2. A *random variable* is a function $X : \Omega \to \mathbb{R}$ with the property that for every Borel set B in $\mathbb{R}$,

$$\{ \omega \in \Omega \mid X(\omega) \in B \} \in \mathcal{F}.$$

Note that the event $\{ \omega \in \Omega \mid X(\omega) \in B \}$ is always abbreviated by $(X \in B)$.

Definition 2.3. Let E be an event on Ω. A function $I_E : \Omega \to \mathbb{R}$ defined by

$$I_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E, \\ 0 & \text{if } \omega \notin E \end{cases}$$

is a random variable, called an *indicator random variable*.

Theorem 2.4. *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ and f be a Borel measurable function. Then $f(X)$ is a random variable on a probability space $(\Omega, \mathcal{F}, P)$.*

Corollary 2.5. *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$. Then X^+ and X^- are random variables on a probability space $(\Omega, \mathcal{F}, P)$ where $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$.*

Proposition 2.6. *Let X and Y be random variables on a probability space $(\Omega, \mathcal{F}, P)$. Then $aX + bY$ and XY are random variables on $(\Omega, \mathcal{F}, P)$ for any $a, b \in \mathbb{R}$.*

Theorem 2.7. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables on $(\Omega, \mathcal{F}, P)$. Then $\liminf\limits_{n\to\infty} X_n, \limsup\limits_{n\to\infty} X_n$ and $\lim\limits_{n\to\infty} X_n$ are random variables.*

2.2 Expectation, Variance and Covariance

In this section, we review definitions of expectation, variance and covariance and their properties.

Definition 2.8. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ and a measurable function $g : \mathbb{R} \to \mathbb{R}$. The *expectation or mean* of the random variable $g(X)$ is defined as

$$E\left(g(X)\right) = \int_\Omega g(X)dP.$$

Note that $E(I_E) = P(E)$.

Definition 2.9. If k is a positive integer, the k^{th} *moment* of X is defined as $E(X^k)$. The k^{th} *central moment* is $E[(X - E(X))^k]$.
The 2^{nd} central moment is called the *variance* of X is denoted by $Var(X)$.
The *covariance* of X and Y is $Cov(X, Y) = E[(X - E(X))(Y - E(Y))]$.

Remark 2.10. (i) $Var(X) = E\left[(X - E(X))^2\right] = E(X^2) - (E(X))^2$.

(ii) $Cov(X, Y) = E(XY) - E(X)E(Y)$.

Theorem 2.11. *Let X and Y be random variables and $a, b \in \mathbb{R}$. Then the followings are true.*

(i) If $E(X), E(Y) < \infty$, then $E(aX + bY) = aE(X) + bE(Y)$ for any $a, b \in \mathbb{R}$.

(ii) If $X \leq Y$, then $E(X) \leq E(Y)$.

(iii) $|E(X)| \leq E|X|$.

Theorem 2.12 (Chebyshev's Inequality)**.** *Let X be a random variable. Then for any $t > 0$,*

$$P(|X - E(X)| \geq t) \leq \frac{Var(X)}{t^2}.$$

Theorem 2.13 (Hölder's inequality)**.** *Let X and Y be random variables. If $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$E|XY| \leq (E|X|^p)^{\frac{1}{p}} (E|Y|^q)^{\frac{1}{q}}.$$

Remark 2.14. In counting measure, for all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}.$$

2.3 Type of Convergence

In this section, we review type of convergence and its useful properties which will be used in this work.

Definition 2.15. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$.

(i) We say that X_n *converges to X almost surely*, written $X_n \longrightarrow X$ *a.s.*, if $\{\omega \in \Omega \mid X_n(\omega) \to X(\omega) \text{ as } n \to \infty\}$ is an event whose probability is 1, i.e.,

$$P(\lim_{n \to \infty} X_n = X) = 1,$$

(ii) We say that X_n *converges to* X *in probability*, written $X_n \longrightarrow X$ *in probability*, if for every $\epsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0,$$

(iii) We say that X_n *converges to* X *in distribution*, written $X_n \longrightarrow X$ *in distribution*, if

$$\lim_{n\to\infty} P(X_n \le x) = P(X \le x)$$

for all points $x \in \mathbb{R}$ at which the function $F_X(x) = P(X \le x)$ is continuous.

Theorem 2.16. *Suppose that* $X_n \longrightarrow X$ *a.s. and* $Y_n \longrightarrow Y$ *a.s.*
Then for any $a, b \in \mathbb{R}$,

$$aX_n + bY_n \longrightarrow aX + bY \ a.s. \ and \ X_nY_n \longrightarrow XY \ a.s.$$

Theorem 2.17. *Suppose that* $X_n \longrightarrow X$ *a.s. and* f *is a continuous function. Then*

$$f(X_n) \longrightarrow f(X) \ a.s.$$

Remark 2.18. Theorem 2.16 and Theorem 2.17 are also true for the convergence in probability and the convergence in distribution.

2.4 Law of Large Numbers

In this section, we review the concept of law of large numbers.

Definition 2.19. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$ and $S_n = \sum_{i=1}^{n} X_i$. We say that $(X_n)_{n\in\mathbb{N}}$ satisfies the *strong law of large numbers (SLLN)* if

$$\frac{S_n - a_n}{b_n} \longrightarrow 0 \ a.s. \quad \text{as } n \to \infty,$$

where $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of real numbers such that $b_n > 0$ for all $n \in \mathbb{N}$ and we say that $(X_n)_{n\in\mathbb{N}}$ satisfies the *weak law of large numbers (WLLN)* if

$$\frac{S_n - a_n}{b_n} \longrightarrow 0 \text{ in probability} \quad \text{as } n \to \infty.$$

Remark 2.20. We say that a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ satisfies the *Kolmogorov SLLN* if for every $n \in \mathbb{N}, a_n = E(S_n)$ and $b_n = n$ and we say that $(X_n)_{n\in\mathbb{N}}$ satisfies the *Marcinkiewicz-Zygmund SLLN* if for every $n \in \mathbb{N}, a_n = E(S_n)$ and $b_n = n^{\frac{1}{p}}$ for $0 < p < 2$. Thus the Kolmogorov SLLN is a special case of the Marcinkiewicz-Zygmund SLLN when $p = 1$.

CHAPTER III
DEPENDENCE STRUCTURE

In this chapter, we study the relations of dependence structures which are studied in SLLN and give some examples.

Definition 3.1. The collection of random variables $X_1, X_2, \ldots, X_n$ is called *independent* if for all $x_i \in \mathbb{R}, i = 1, 2, \ldots, n$

$$P\left(\bigcap_{i=1}^{n}(X_i \leq x_i)\right) = \prod_{i=1}^{n} P(X_i \leq x_i).$$

An infinite sequence $(X_n)_{n\in\mathbb{N}}$ is call *independent* if for each positive integer n, the random variables $X_1, X_2, \ldots, X_n$ are *independent*.

A relaxation of independence was introduced which is called pairwise independence. The definition of pairwise independence is given as follows.

Definition 3.2. A sequence $(X_n)_{n\in\mathbb{N}}$ of random variables is called *pairwise independent* if for $i \neq j$, any real numbers x_i, x_j

$$P(X_i \leq x_i, X_j \leq x_j) = P(X_i \leq x_i)P(X_j \leq x_j).$$

In 1966, Lehmann[11] introduced the concept of negatively quadrant dependence (NQD). The definition of NQD is stated as follows.

Definition 3.3. The collection of random variables $X_1, X_2, \ldots, X_n$ is called *negatively quadrant dependent (NQD)* if for all $x_i \in \mathbb{R}, i = 1, 2, \ldots, n$

$$P\left(\bigcap_{i=1}^{n}(X_i \leq x_i)\right) \leq \prod_{i=1}^{n} P(X_i \leq x_i).$$

An infinite sequence $(X_n)_{n\in\mathbb{N}}$ is call *NQD* if for each positive integer n, the random variables $X_1, X_2, \ldots, X_n$ are NQD.

The dependency assumption used in some articles is pairwise independence which could be not applicable in some applications. Therefore relaxations of the condition were considered. For example, Lehmann[11] introduced a concept of pairwise negatively quadrant dependence (pairwise NQD) and pairwise positively quadrant dependence (pairwise PQD) which are more flexible than pairwise independence. The definition of pairwise NQD and pairwise PQD are given as follows.

Definition 3.4. A sequence $(X_n)_{n\in\mathbb{N}}$ of random variables is called *pairwise negatively quadrant dependent (pairwise NQD)* if for $i \neq j$, any real numbers x_i, x_j

$$P(X_i \leq x_i, X_j \leq x_j) \leq P(X_i \leq x_i)P(X_j \leq x_j)$$

equivalently

$$P(X_i > x_i, X_j > x_j) \leq P(X_i > x_i)P(X_j > x_j)$$

and it is called *pairwise positively quadrant dependent (pairwise PQD)* if for $i \neq j$, any real numbers x_i, x_j

$$P(X_i \leq x_i, X_j \leq x_j) \geq P(X_i \leq x_i)P(X_j \leq x_j)$$

equivalently

$$P(X_i > x_i, X_j > x_j) \geq P(X_i > x_i)P(X_j > x_j).$$

The following is an example of a sequence of random variables which are not pairwise independence but pairwise NQD.

Example 3.5. A box contains p balls of p different colors ($p \geq 3$). Choose two balls randomly (without replacement).
Let $\widetilde{X}_i, i = 1, 2, \ldots, p$ ($p \geq 3$) be a random variable indicating the presence of a ball of the i^{th} color such that

$$\widetilde{X}_i = \begin{cases} 1 & \text{if the } i^{th} \text{color is picked,} \\ 0 & \text{otherwise.} \end{cases}$$

For $i \in \mathbb{N}$, let X_i be a random variable defined by

$$X_i = \begin{cases} \widetilde{X}_i & \text{if } 1 \leq i \leq p, \\ 0 & \text{otherwise.} \end{cases}$$

For each $i, j = 1, 2, \ldots, p$ and $i \neq j$,

$$P(X_i = 1, X_j = 1) = \frac{2}{p(p-1)} < \left(\frac{2}{p}\right)^2 = P(X_i = 1)P(X_j = 1).$$

Therefore $(X_i)_{i\in\mathbb{N}}$ is not a sequence of pairwise independent random variables. For $1 \leq i, j \leq p,\ 0 \leq a < 1$ and $0 \leq b < 1$, we have

$$P(X_i > a) = P(X_i = 1) = \frac{2}{p} = P(X_j > b)$$

and for $i \neq j$

$$P(X_i > a, X_j > b) = P(X_i = 1, X_j = 1) = \frac{2}{p(p-1)}.$$

Then

$$P(X_i > a, X_j > b) = \frac{2}{p(p-1)} \leq \left(\frac{2}{p}\right)^2 = P(X_i > a)P(X_j > b).$$

Therefore $(X_i)_{i\in\mathbb{N}}$ is a sequence of pairwise NQD random variables.

The properties of a sequence of pairwise NQD random variables are presented in the following proposition.

Proposition 3.6. *([11]) Let (X_n) be a sequence of pairwise NQD random variables. Then the following results hold.*

(i) $Cov(X_i, X_j) \leq 0$ *for all* $i \neq j$*;*

(ii) $(f_n(X_n))$ *is a pairwise NQD sequence for any sequence of monotonically increasing functions* $(f_n)_{n\in\mathbb{N}}$*;*

(iii) $(g_n(X_n))$ *is a pairwise NQD sequence for any sequence of monotonically decreasing functions* $(g_n)_{n\in\mathbb{N}}$.

Remark 3.7. From Proposition 3.6(ii), if $(X_n)_{n\in\mathbb{N}}$ is a sequence of pairwise NQD, then $(X_n^+)_{n\in\mathbb{N}}$ and $(X_n^-)_{n\in\mathbb{N}}$ are pairwise NQD sequences.

The properties of a sequence of pairwise PQD random variables are presented in the following proposition.

Proposition 3.8. *([11]) Let (X_n) be a sequence of pairwise PQD random variables. Then the following results hold.*

(i) $Cov(X_i, X_j) \geq 0$ *for all* $i \neq j$*;*

(ii) $(f_n(X_n))$ *is a pairwise PQD sequence for any sequence of monotonically increasing functions* $(f_n)_{n\in\mathbb{N}}$*;*

(iii) $(g_n(X_n))$ *is a pairwise PQD sequence for any sequence of monotonically decreasing functions* $(g_n)_{n\in\mathbb{N}}$*.*

Remark 3.9. From Proposition 3.8(ii), if $(X_n)_{n\in\mathbb{N}}$ is a sequence of pairwise PQD, then $(X_n^+)_{n\in\mathbb{N}}$ and $(X_n^-)_{n\in\mathbb{N}}$ are pairwise PQD sequences.

Proposition 3.10. *([11]) Let X and Y are random variables. Then X and Y are pairwise NQD if and only if X and $-Y$ are pairwise PQD.*

The converse of Proposition 3.6(i) and Proposition 3.8(i) are not always true such as this following example.

Example 3.11. Let $X = \begin{cases} -1 & \text{with probability } 0.5, \\ 1 & \text{with probability } 0.5. \end{cases}$

If $X = -1$, let $Y = 0$ and if $X = 1$, let $Y = \begin{cases} -1 & \text{with probability } 0.5, \\ 1 & \text{with probability } 0.5. \end{cases}$

Then $EX = -1(0.5) + 1(0.5) = 0, EY = 0(0.5) + (-1)(0.25) + 1(0.25) = 0$

and $EXY = (1)(-1)(0.25) + (1)(1)(0.25) = 0$.

Therefore $Cov(X, Y) = EXY - (EX)(EY) = 0$.

Note that

$$\begin{aligned} P(X \le -1, Y \le 0) &= P(X = -1, Y = 0) \\ &= 0.5 \\ &> (0.5)(0.25) + (0.5)(0.5) \\ &= P(X = -1)P(Y = -1) + P(X = -1)P(Y = 0) \\ &= P(X \le -1)P(Y \le 0) \qquad (3.1) \end{aligned}$$

and

$$\begin{aligned} P(X \le -1, Y \le -1) &= 0 \\ &< (0.5)(0.25) \\ &= P(X = -1)P(Y = -1) \\ &= P(X \le -1)P(Y \le -1). \qquad (3.2) \end{aligned}$$

From (3.1) and (3.2), X and Y are neither pairwise NQD nor pairwise PQD random variables.

Further extensions of dependence structures in the context of SLLN have also been considered. For example, Liu[12] introduced the concept of extended negatively dependence (END), which is a generalization of NQD. The definition is stated as follows.

Definition 3.12. The collection of random variables $X_1, X_2, \ldots, X_n$ is called *lower extended negatively dependent (LEND)* if there is some $M > 0$ such that, for all $x_i \in \mathbb{R}, i = 1, 2, \ldots, n$

$$P\left(\bigcap_{i=1}^{n}(X_i \le x_i)\right) \le M \prod_{i=1}^{n} P(X_i \le x_i).$$

It is called *upper extended negatively dependent (UEND)* if there is some $M > 0$ such that, for all $x_i \in \mathbb{R}, i = 1, 2, \ldots, n$

$$P\left(\bigcap_{i=1}^{n}(X_i > x_i)\right) \le M \prod_{i=1}^{n} P(X_i > x_i).$$

It is called *extended negatively dependent (END)* if it is both LEND and UEND. An infinite sequence $(X_n)_{n\in\mathbb{N}}$ is call *LEND, UEND* or *END* if for each positive integer n, the random variables $X_1, X_2, \ldots, X_n$ are LEND, UEND or END, respectively.

When $M = 1$, the END sequence induced to NQD sequence. The following is an example of a sequence of random variables which are not pairwise NQD but END.

Example 3.13 (Pólya's Urn Process)**.** An urn contains a red balls and b green balls. A ball is randomly drawn from the urn and replaced back to the urn along with additional(fixed) c balls of the same color as the drawn ball.

Let X_i denote the color of the ball selected at time i such that

$$X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ picked ball is red,} \\ 0 & \text{if the } i^{th} \text{ picked ball is green.} \end{cases}$$

Let $n \in \mathbb{N}$. First, we will show that $X_1, X_2, \ldots, X_n$ are exchangeable, that is

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = P(X_1 = x_{\pi(1)}, X_2 = x_{\pi(2)}, \ldots, X_n = x_{\pi(n)})$$

for every permutation $\pi(\cdot)$ on $\{1, 2, \ldots, n\}$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}$.
Note that the joint probability depends only on number of events of observing red balls.
For each $n \in \mathbb{N}$, let $x_1, x_2, \ldots, x_n \in \{0, 1\}$ and $k = x_1 + x_2 + \cdots + x_n$ where $1 \le k \le n - 1$,

$$\begin{aligned} &P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) \\ &= \frac{a(a+c)\cdots(a+(k-1)c)b(b+c)\cdots(b+(n-k-1)c)}{(a+b)(a+b+c)(a+b+2c)\cdots(a+b+(n-1)c)} \\ &= P(X_1 = x_{\pi(1)}, X_2 = x_{\pi(2)}, \ldots, X_n = x_{\pi(n)}). \end{aligned}$$

That is the variables $X_1, X_2, \ldots, X_n$ are exchangeable.
Then for each $i = 1, 2, \ldots, n$,

$$P(X_i = 1) = P(X_1 = 1) = \frac{a}{a+b} \text{ and } P(X_i = 0) = P(X_1 = 0) = \frac{b}{a+b}. \quad (3.3)$$

and for $i \neq j$,

$$P(X_i = 1, X_j = 1) = P(X_1 = 1, X_2 = 1) = \left(\frac{a}{a+b}\right)\left(\frac{a+c}{a+b+c}\right).$$

Therefore for each $i \neq j$,

$$\begin{aligned}
Cov(X_i, X_j) &= E(X_iX_j) - EX_iEX_j \\
&= P(X_i = 1, X_j = 1) - P(X_i = 1)P(X_j = 1) \\
&= \left(\frac{a}{a+b}\right)\left(\frac{a+c}{a+b+c}\right) - \left(\frac{a}{a+b}\right)^2 \\
&= \frac{a}{a+b} \cdot \frac{bc}{(a+b)(a+b+c)} \\
&= \frac{abc}{(a+b)^2(a+b+c)} \\
&> 0.
\end{aligned}$$

By Proposition 3.6(i), $(X_n)_{n\in\mathbb{N}}$ is not a sequence of pairwise NQD random variables.

For arbitrary fixed $n \in \mathbb{N}$, we will show that $X_1, X_2, \ldots, X_n$ are LEND and UEND random variables. From (3.3), we have that for each $i = 1, 2, \ldots, n$,

$$P(X_i \leq x_i) = \begin{cases} 0 & \text{if } x_i \in (-\infty, 0), \\ \dfrac{b}{a+b} & \text{if } x_i \in [0, 1), \\ 1 & \text{if } x_i \in [1, \infty). \end{cases}$$

Then we consider $P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n)$.

Case 1 There exists $i \in \{1, 2, \ldots, n\}$ such that $x_i \in (-\infty, 0)$.

Then $P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) = 0 = \prod_{i=1}^{n} P(X_i \leq x_i)$.

Case 2 For all $i \in \{1, 2, \ldots, n\}, x_i \in [1, \infty)$.

Then $P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) = 1 = \prod_{i=1}^{n} P(X_i \leq x_i)$.

Case 3 There exists a nonempty subset A of $\{1, 2, \ldots, n\}$ such that for all $i \in A$, $x_i \in [0, 1)$ and for all $i \notin A, x_i \in [1, \infty)$.

Since the variables X_i and X_j are exchangeable, we can assume without loss of generality that $x_1, x_2, \ldots, x_k \in [0,1)$ and $x_{k+1}, x_{k+2}, \ldots, x_n \in [1,\infty)$. Then

$$
\begin{aligned}
&P(X_1 \le x_1, \ldots, X_k \le x_k, X_{k+1} \le x_{k+1}, \ldots, X_n \le x_n) \\
&= P(X_1 = 0, \ldots, X_k = 0, X_{k+1} \le 1, \ldots, X_n \le 1) \\
&= P(X_1 = 0, \ldots, X_k = 0) \\
&= \left(\frac{b}{a+b}\right)\left(\frac{b+c}{a+b+c}\right)\cdots\left(\frac{b+(k-1)c}{a+b+(k-1)c}\right) \\
&= \prod_{i=1}^{k} \frac{b+(i-1)c}{a+b+(i-1)c} \\
&= \frac{\left(\prod_{i=1}^{k} \frac{b+(i-1)c}{a+b+(i-1)c}\right)\prod_{i=1}^{k} P(X_i \le x_i)}{\left(\frac{b}{a+b}\right)^k} \\
&\le M_n \prod_{i=1}^{k} P(X_i \le x_i) \\
&= M_n \prod_{i=1}^{n} P(X_i \le x_i).
\end{aligned}
$$

where $M_n = \dfrac{\prod_{i=1}^{n} \frac{b+(i-1)c}{a+b+(i-1)c}}{\left(\frac{b}{a+b}\right)^n}$ and we use the fact that $\dfrac{b+kc}{a+b+kc} \ge \dfrac{b}{a+b}$ for all $k \in \mathbb{N}$ in the first inequality.

From the above 3 cases and the fact that $M_n \ge 1$, $X_1, X_2, \ldots, X_n$ are LEND random variables.

Similarly, we can show that $X_1, X_2, \ldots, X_n$ are UEND random variables by choosing

$$
M_n = \frac{\prod_{i=1}^{n} \frac{a+(i-1)c}{a+b+(i-1)c}}{\left(\frac{a}{a+b}\right)^n}.
$$

Hence $X_1, X_2, \ldots, X_n$ are END random variables where

$$M_n = \max\left(\frac{\prod_{i=1}^{n} \frac{a+(i-1)c}{a+b+(i-1)c}}{\left(\frac{a}{a+b}\right)^n}, \frac{\prod_{i=1}^{n} \frac{b+(i-1)c}{a+b+(i-1)c}}{\left(\frac{b}{a+b}\right)^n} \right).$$

The properties of a sequence of END random variables are presented in the following proposition.

Proposition 3.14. *([12]) Let (X_n) be a sequence of END random variables. Then the following results hold.*

(i) For each $n \in \mathbb{N}$, there exists a constant $M > 0$ such that

$$E\left(\prod_{i=1}^{n} X_i^+\right) \leq M \prod_{i=1}^{n} EX_i^+;$$

(ii) $(f_n(X_n))$ is an END sequence for any sequence of monotonically increasing functions $(f_n)_{n\in\mathbb{N}}$;

(iii) $(g_n(X_n))$ is an END sequence for any sequence of monotonically decreasing functions $(g_n)_{n\in\mathbb{N}}$.

Remark 3.15. From Proposition 3.14(ii), if $(X_n)_{n\in\mathbb{N}}$ is a sequence of END, then $(X_n^+)_{n\in\mathbb{N}}$ and $(X_n^-)_{n\in\mathbb{N}}$ are END sequences.

CHAPTER IV

STRONG LAW OF LARGE NUMBERS FOR IDENTICALLY DISTRIBUTED RANDOM VARIABLES

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. We define $S_n = \sum_{i=1}^{n} X_i$, $S_n^+ = \sum_{i=1}^{n} X_i^+$ and $S_n^- = \sum_{i=1}^{n} X_i^-$.

In 2014, Korchevsky[9] obtained the Marcinkiewicz-Zygmund SLLN for pairwise i.i.d. random variables. His result is presented as follows.

Theorem 4.1. *([9]) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of pairwise i.i.d. random variables. For $0 < p < 2$, if $E|X_1|^p < \infty$, then*

$$\frac{S_n - nE(X_1)}{n^{\frac{1}{p}}} \longrightarrow 0 \ a.s.$$

In this chapter, we propose a relaxation of dependency assumption in Korchevsky[9] and derive the Marcinkiewicz-Zygmund SLLN for END and identically distributed random variables. To obtain this result, we follow the techniques in Chen et al.[5].

We first restate a definition of infinitely often and a lemma of Chen[5] which are used in the proof of our result (Theorem 4.4).

Definition 4.2. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of events. Then

$$\{A_n \ i.o.\} = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

is the event that an infinite number of the events A_n occur. The *i.o.* stands for "infinitely often".

Lemma 4.3. *([5]) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of END and identically distributed random variables with mean 0 and a dominating constant $M > 0$. For arbitrarily* *fixed* $0 < v < 1$*, define for each* $i = 1, 2, \ldots$ *and* $x \in (0, \infty)$

$$\widetilde{X}_i = -vxI(X_i < -vx) + X_iI(-vx \le X_i \le vx) + vxI(X_i > vx).$$

Then for every $\gamma > 0, 0 < \delta \le 1$ *and* $0 < \theta < 1$*, there is some* $x_0 = x_0(v, \gamma, \delta, \theta) > 0$ *such that for all* $n \in \mathbb{N}$ *and* $x \ge \max\{\gamma n, x_0\}$*,*

$$P(|\widetilde{S}_n| > x) \le 2M \left(\frac{1}{(vx)^\delta} \int_{|y| \le vx} |y|^{1+\delta} F(dy) + vxP(|X_1| > vx) \right)^{\frac{1-\theta}{v}},$$

where $\widetilde{S}_n = \sum_{i=1}^{n} \widetilde{X}_i$ *and* F *denoted the distribution of* X_1*.*

Our result is stated as follows.

Theorem 4.4. *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of END and identically distributed random variables. For* $0 < p < 2$*, if* $E|X_1|^p < \infty$*,*

$$\frac{S_n - nE(X_1)}{n^{\frac{1}{p}}} \longrightarrow 0 \ a.s.$$

Proof. For $0 < p \le 1$, it was completed in Sawyer[15]. It remains to prove the case when $1 < p < 2$. For this case, it suffices to prove that both

$$\frac{S_n^+ - n\mu_+}{n^{\frac{1}{p}}} \longrightarrow 0 \ a.s. \text{ and } \frac{S_n^- - n\mu_-}{n^{\frac{1}{p}}} \longrightarrow 0 \ a.s.$$

hold where $\mu_+ = EX_1^+$ and $\mu_- = EX_1^-$.

For arbitrary fixed $v > 0$ and $n \in \mathbb{N}$, we define for each $1 < p < 2, i = 1, 2, \ldots, n$,

$$\begin{aligned} \widetilde{X}_{i,n}^+ = &- vn^{\frac{1}{p}} I(X_i^+ - \mu_+ < -vn^{\frac{1}{p}}) + (X_i^+ - \mu_+) I(-vn^{\frac{1}{p}} \le X_i^+ - \mu_+ \le vn^{\frac{1}{p}}) \\ &+ vn^{\frac{1}{p}} I(X_i^+ - \mu_+ > vn^{\frac{1}{p}}) \end{aligned}$$

and $\widetilde{S}_n^+ = \sum_{i=1}^{n} \widetilde{X}_{i,n}^+$.

Then we follow the proof of Theorem 1.1 in Chen[5] as follows.

Let $\epsilon > 0$ and $\alpha > 1$ be arbitrarily fixed.

By Lemma 4.3, with suitable chosen $0 < v < 1$ and $0 < \theta < 1$ such that $\dfrac{1-\theta}{v} = 1$, there is a positive integer $n_0 = n_0(v, \epsilon, \delta, \theta)$ such that for all $n \ge n_0$,

$$P\left(\left| \frac{\widetilde{S}_n^+}{n^{\frac{1}{p}}} \right| > \epsilon \right) \le 2M \left[\frac{1}{(v\epsilon n^{\frac{1}{p}})^\delta} \int_{|y| \le v\epsilon n^{\frac{1}{p}}} |y|^{1+\delta} \widetilde{F}^+(dy) + v\epsilon n^{\frac{1}{p}} P\left(|X_1^+ - \mu_+| > v\epsilon n^{\frac{1}{p}} \right) \right], \tag{4.1}$$

where $\widetilde{F}^+$ denotes the distribution of $X_1^+ - \mu_+$ and M is a dominating constant such that $M > 0$. Then

$$\sum_{n=1}^{\infty} P\left(\left|\frac{\widetilde{S}^+_{\lfloor\alpha^n\rfloor}}{\lfloor\alpha^n\rfloor^{\frac{1}{p}}}\right| > \epsilon\right) = \sum_{n=1}^{\max\{\lfloor\log_\alpha n_0\rfloor, -\lfloor\frac{\log(\alpha-1)}{\log\alpha}\rfloor\}} P\left(\left|\frac{\widetilde{S}^+_{\lfloor\alpha^n\rfloor}}{\lfloor\alpha^n\rfloor^{\frac{1}{p}}}\right| > \epsilon\right)$$
$$+ \sum_{n=\max\{\lfloor\log_\alpha n_0\rfloor+1, 1-\lfloor\frac{\log(\alpha-1)}{\log\alpha}\rfloor\}}^{\infty} P\left(\left|\frac{\widetilde{S}^+_{\lfloor\alpha^n\rfloor}}{\lfloor\alpha^n\rfloor^{\frac{1}{p}}}\right| > \epsilon\right)$$
$$\leq \max\{\lfloor\log_\alpha n_0\rfloor, -\lfloor\frac{\log(\alpha-1)}{\log\alpha}\rfloor\}$$
$$+ \sum_{n=\max\{\lfloor\log_\alpha n_0\rfloor+1, 1-\lfloor\frac{\log(\alpha-1)}{\log\alpha}\rfloor\}}^{\infty} P\left(\left|\frac{\widetilde{S}^+_{\lfloor\alpha^n\rfloor}}{\lfloor\alpha^n\rfloor^{\frac{1}{p}}}\right| > \epsilon\right). \quad (4.2)$$

From (4.1), for $n \geq \max\{\lfloor\log_\alpha n_0\rfloor + 1, 1 - \lfloor\frac{\log(\alpha-1)}{\log\alpha}\rfloor\}$

$$P\left(\left|\frac{\widetilde{S}^+_{\lfloor\alpha^n\rfloor}}{\lfloor\alpha^n\rfloor^{\frac{1}{p}}}\right| > \epsilon\right)$$
$$\leq 2M\left[\frac{1}{(v\epsilon\lfloor\alpha^n\rfloor^{\frac{1}{p}})^\delta}\int_{|y|\leq v\epsilon\lfloor\alpha^n\rfloor^{\frac{1}{p}}} |y|^{1+\delta}\widetilde{F}^+(dy) + v\epsilon\lfloor\alpha^n\rfloor^{\frac{1}{p}} P\left(|X_1^+ - \mu_+| > v\epsilon\lfloor\alpha^n\rfloor^{\frac{1}{p}}\right)\right].$$

Then

$$\sum_{n=\max\{\lfloor\log_\alpha n_0\rfloor+1, 1-\lfloor\frac{\log(\alpha-1)}{\log\alpha}\rfloor\}}^{\infty} P\left(\left|\frac{\widetilde{S}^+_{\lfloor\alpha^n\rfloor}}{\lfloor\alpha^n\rfloor^{\frac{1}{p}}}\right| > \epsilon\right)$$
$$\leq 2M\sum_{n=1}^{\infty}\frac{1}{(v\epsilon\lfloor\alpha^n\rfloor^{\frac{1}{p}})^\delta}\int_{|y|\leq v\epsilon\lfloor\alpha^n\rfloor^{\frac{1}{p}}} |y|^{1+\delta}\widetilde{F}^+(dy) + 2Mv\epsilon\sum_{n=1}^{\infty}\lfloor\alpha^n\rfloor^{\frac{1}{p}} P\left(|X_1^+ - \mu_+| > v\epsilon\lfloor\alpha^n\rfloor^{\frac{1}{p}}\right)$$
$$\leq 2M\left[\sum_{n=1}^{\infty}\frac{1}{(v\epsilon\alpha^{\frac{n-1}{p}})^\delta}\int_{|y|\leq v\epsilon\lfloor\alpha^n\rfloor^{\frac{1}{p}}} |y|^{1+\delta}\widetilde{F}^+(dy)\right] + 2Mv\epsilon\Sigma_2$$
$$\leq \frac{2M}{(v\epsilon\alpha^{\frac{-1}{p}})^\delta}\sum_{n=1}^{\infty}\frac{1}{\left(\alpha^{\frac{n}{p}}\right)^\delta}\int_{|y|\leq v\epsilon\alpha^{\frac{n}{p}}} |y|^{1+\delta}\widetilde{F}^+(dy) + 2Mv\epsilon\Sigma_2$$
$$= \frac{2M}{(v\epsilon\alpha^{\frac{-1}{p}})^\delta}\Sigma_1 + 2Mv\epsilon\Sigma_2, \quad (4.3)$$

where $\Sigma_1 = \sum_{n=1}^{\infty}\frac{1}{\left(\alpha^{\frac{n}{p}}\right)^\delta}\int_{|y|\leq v\epsilon\alpha^{\frac{n}{p}}} |y|^{1+\delta}\widetilde{F}^+(dy)$, $\Sigma_2 = \sum_{n=1}^{\infty}\lfloor\alpha^n\rfloor^{\frac{1}{p}} P\left(|X_1^+ - \mu_+| > v\epsilon\lfloor\alpha^n\rfloor^{\frac{1}{p}}\right)$ and we use the fact that $\alpha^{n-1} \leq \lfloor\alpha^n\rfloor \leq \alpha^n$ for $\alpha > 1$.

By (4.2) and (4.3),

$$\sum_{n=1}^{\infty} P\left(\left|\frac{\widetilde{S}^{+}_{\lfloor\alpha^n\rfloor}}{\lfloor\alpha^n\rfloor^{\frac{1}{p}}}\right| > \epsilon\right) \le \max\{\lfloor\log_\alpha n_0\rfloor, -\lfloor\frac{\log(\alpha-1)}{\log\alpha}\rfloor\} + \frac{2M}{(v\epsilon\alpha^{\frac{-1}{p}})^\delta}\Sigma_1 + 2Mv\epsilon\Sigma_2. \tag{4.4}$$

Note that

$$\begin{aligned}
\Sigma_1 &= \sum_{n=1}^{\infty} \frac{1}{\left(\alpha^{\frac{n}{p}}\right)^\delta} \int_{|y|\le v\epsilon\alpha^{\frac{n}{p}}} |y|^{1+\delta}\widetilde{F}^{+}(dy) \\
&= \int_{-\infty}^{\infty} \left(\sum_{n=\max\{1,\log_\alpha \frac{|y|^p}{(v\epsilon)^p}\}}^{\infty} \frac{1}{\alpha^{\frac{n\delta}{p}}}\right) |y|^{1+\delta}\widetilde{F}^{+}(dy) \\
&\le \int_{-\infty}^{\infty} \left(\sum_{n=\max\{1,\lfloor\log_\alpha \frac{|y|^p}{(v\epsilon)^p}\rfloor\}}^{\infty} \frac{1}{\alpha^{\frac{n\delta}{p}}}\right) |y|^{1+\delta}\widetilde{F}^{+}(dy) \\
&= \int_{-\infty}^{\infty} \frac{1}{\alpha^{\frac{\delta}{p}\max\{1,\lfloor\log_\alpha \frac{|y|^p}{(v\epsilon)^p}\rfloor\}}\left(1-\alpha^{-\frac{\delta}{p}}\right)} |y|^{1+\delta}\widetilde{F}^{+}(dy) \\
&\le \frac{(v\epsilon)^\delta}{1-\alpha^{-\frac{\delta}{p}}} \int_{-\infty}^{\infty} |y|\widetilde{F}^{+}(dy) \\
&= K_1 E|X_1^{+} - \mu_{+}| \\
&< \infty, \end{aligned} \tag{4.5}$$

where $K_1 = \dfrac{(v\epsilon)^\delta}{1-\alpha^{-\frac{\delta}{p}}} > 0$ and

$$\begin{aligned}
\Sigma_2 &= \sum_{n=1}^{\infty} \lfloor\alpha^n\rfloor^{\frac{1}{p}} P\left(|X_1^{+} - \mu_{+}| > v\epsilon\lfloor\alpha^n\rfloor^{\frac{1}{p}}\right) \\
&\le \sum_{n=1}^{\infty} \frac{\lfloor\alpha^n\rfloor^{\frac{1}{p}}}{\lfloor(\alpha-1)\alpha^{n-1}\rfloor} \sum_{i=1}^{\lfloor(\alpha-1)\alpha^{n-1}\rfloor} P\left(|X_1^{+} - \mu_{+}| > v\epsilon\left(\lfloor\alpha^{n-1}\rfloor + i\right)^{\frac{1}{p}}\right) \\
&= \sum_{n=1}^{\infty} \frac{\lfloor\alpha^n\rfloor^{\frac{1}{p}}}{\lfloor(\alpha-1)\alpha^{n-1}\rfloor} \sum_{i=1}^{\lfloor(\alpha-1)\alpha^{n-1}\rfloor} P\left(|X_1^{+} - \mu_{+}|^p > (v\epsilon)^p\left(\lfloor\alpha^{n-1}\rfloor + i\right)\right)
\end{aligned}$$

$$\leq \sum_{n=1}^{\infty} \frac{\lfloor \alpha^n \rfloor}{\lfloor (\alpha-1)\alpha^{n-1} \rfloor} \sum_{i=1}^{\lfloor (\alpha-1)\alpha^{n-1} \rfloor} P\left(|X_1^+ - \mu_+|^p > (v\epsilon)^p \left(\lfloor \alpha^{n-1} \rfloor + i\right)\right)$$
$$\leq K_2 \sum_{n=1}^{\infty} P\left(|X_1^+ - \mu_+|^p > (v\epsilon)^p n\right)$$
$$= K_2 E \left| \frac{X_1^+ - \mu_+}{v\epsilon} \right|^p$$
$$= \frac{K_2}{(v\epsilon)^p} E|X_1^+ - \mu_+|^p$$
$$< \infty, \tag{4.6}$$

for some a constant $K_2 > 0$.

From (4.4) – (4.6), we can show that

$$\sum_{n=1}^{\infty} P\left(\left| \frac{\widetilde{S}^+_{\lfloor \alpha^n \rfloor}}{\lfloor \alpha^n \rfloor^{\frac{1}{p}}} \right| > \epsilon \right) < \infty. \tag{4.7}$$

Hence

$$P\left(\left| \frac{S^+_{\lfloor \alpha^n \rfloor} - \lfloor \alpha^n \rfloor \mu_+}{\lfloor \alpha^n \rfloor^{\frac{1}{p}}} \right| > \epsilon \ i.o.\right)$$
$$\leq P\left(\left| \frac{\widetilde{S}^+_{\lfloor \alpha^n \rfloor}}{\lfloor \alpha^n \rfloor^{\frac{1}{p}}} \right| > \epsilon \ i.o.\right) + P\left(\frac{S^+_{\lfloor \alpha^n \rfloor} - \lfloor \alpha^n \rfloor \mu_+}{\lfloor \alpha^n \rfloor^{\frac{1}{p}}} \neq \frac{\widetilde{S}^+_{\lfloor \alpha^n \rfloor}}{\lfloor \alpha^n \rfloor^{\frac{1}{p}}} \ i.o.\right)$$
$$\leq \lim_{m\to\infty} P\left(\bigcup_{n=m}^{\infty} \left(\frac{\widetilde{S}^+_{\lfloor \alpha^n \rfloor}}{\lfloor \alpha^n \rfloor^{\frac{1}{p}}} > \epsilon \right)\right) + \lim_{m\to\infty} P\left(\bigcup_{n=m}^{\infty} \bigcup_{i=1}^{\lfloor \alpha^n \rfloor} \left(X_i^+ - \mu_+ \neq \widetilde{X}^+_{i,\lfloor \alpha^n \rfloor} \right)\right)$$
$$\leq \limsup_{m\to\infty} \sum_{n=m}^{\infty} P\left(\frac{\widetilde{S}^+_{\lfloor \alpha^n \rfloor}}{\lfloor \alpha^n \rfloor^{\frac{1}{p}}} > \epsilon \right) + \limsup_{m\to\infty} \sum_{n=m}^{\infty} \lfloor \alpha^n \rfloor P\left(X_i - \mu_+ > v \lfloor \alpha^n \rfloor^{\frac{1}{p}} \right)$$
$$= 0$$

where in the last step the convergence of the first series is from (4.7) and the convergence of the second series can be verified in the same way as (4.3).

Then

$$\lim_{n\to\infty} \frac{S^+_{\lfloor \alpha^n \rfloor} - \lfloor \alpha^n \rfloor \mu_+}{\lfloor \alpha^n \rfloor^{\frac{1}{p}}} = 0 \ a.s. \tag{4.8}$$

For every positive integer n, there is a unique positive integer k_n such that

$\lfloor\alpha^{k_n-1}\rfloor \le n < \lfloor\alpha^{k_n}\rfloor$. Thus for $1<p<2$, $\lfloor\alpha^{k_n-1}\rfloor^{\frac{1}{p}} \le n^{\frac{1}{p}} < \lfloor\alpha^{k_n}\rfloor^{\frac{1}{p}}$. Therefore

$$\begin{aligned}\frac{S_n^+ - n\mu_+}{n^{\frac{1}{p}}} &\le \frac{S^+_{\lfloor\alpha^{k_n}\rfloor} - \lfloor\alpha^{k_n-1}\rfloor\mu_+}{\lfloor\alpha^{k_n-1}\rfloor^{\frac{1}{p}}} \\ &= \frac{\lfloor\alpha^{k_n}\rfloor^{\frac{1}{p}}}{\lfloor\alpha^{k_n-1}\rfloor^{\frac{1}{p}}}\left[\frac{S^+_{\lfloor\alpha^{k_n}\rfloor} - \lfloor\alpha^{k_n}\rfloor\mu_+}{\lfloor\alpha^{k_n}\rfloor^{\frac{1}{p}}}\right] + \frac{\left(\lfloor\alpha^{k_n}\rfloor - \lfloor\alpha^{k_n-1}\rfloor\right)\mu_+}{\lfloor\alpha^{k_n-1}\rfloor^{\frac{1}{p}}} \\ &\le \frac{\lfloor\alpha^{k_n}\rfloor^{\frac{1}{p}}}{\lfloor\alpha^{k_n-1}\rfloor^{\frac{1}{p}}}\left[\frac{S^+_{\lfloor\alpha^{k_n}\rfloor} - \lfloor\alpha^{k_n}\rfloor\mu_+}{\lfloor\alpha^{k_n}\rfloor^{\frac{1}{p}}}\right] + \lfloor\alpha^{k_n}\rfloor^{1-\frac{1}{p}}\left(\frac{\lfloor\alpha^{k_n}\rfloor}{\lfloor\alpha^{k_n-1}\rfloor} - 1\right)\mu_+\end{aligned}$$

and

$$\begin{aligned}\frac{S_n^+ - n\mu_+}{n^{\frac{1}{p}}} &\ge \frac{S^+_{\lfloor\alpha^{k_n-1}\rfloor} - \lfloor\alpha^{k_n}\rfloor\mu_+}{\lfloor\alpha^{k_n}\rfloor^{\frac{1}{p}}} \\ &= \frac{\lfloor\alpha^{k_n-1}\rfloor^{\frac{1}{p}}}{\lfloor\alpha^{k_n}\rfloor^{\frac{1}{p}}}\left[\frac{S^+_{\lfloor\alpha^{k_n-1}\rfloor} - \lfloor\alpha^{k_n-1}\rfloor\mu_+}{\lfloor\alpha^{k_n-1}\rfloor^{\frac{1}{p}}}\right] + \frac{\left(\lfloor\alpha^{k_n-1}\rfloor - \lfloor\alpha^{k_n}\rfloor\right)\mu_+}{\lfloor\alpha^{k_n}\rfloor^{\frac{1}{p}}} \\ &= \frac{\lfloor\alpha^{k_n-1}\rfloor^{\frac{1}{p}}}{\lfloor\alpha^{k_n}\rfloor^{\frac{1}{p}}}\left[\frac{S^+_{\lfloor\alpha^{k_n-1}\rfloor} - \lfloor\alpha^{k_n-1}\rfloor\mu_+}{\lfloor\alpha^{k_n-1}\rfloor^{\frac{1}{p}}}\right] + \lfloor\alpha^{k_n}\rfloor^{1-\frac{1}{p}}\left(\frac{\lfloor\alpha^{k_n-1}\rfloor}{\lfloor\alpha^{k_n}\rfloor} - 1\right)\mu_+\end{aligned}$$

It follows from (4.8) that

$$\frac{\lfloor\alpha^{k_n}\rfloor^{1-\frac{1}{p}}(1-\alpha)\mu_+}{\alpha} \le \liminf_{n\to\infty}\frac{S_n^+ - n\mu_+}{n^{\frac{1}{p}}} \le \limsup_{n\to\infty}\frac{S_n^+ - n\mu_+}{n^{\frac{1}{p}}} \le \lfloor\alpha^{k_n}\rfloor^{1-\frac{1}{p}}(\alpha-1)\mu_+$$

Since α is arbitrary, we have that $\lim_{n\to\infty}\frac{S_n^+ - n\mu_+}{n^{\frac{1}{p}}} = 0$ *a.s.*

By the same technique, we can show that $\lim_{n\to\infty}\frac{S_n^- - n\mu_-}{n^{\frac{1}{p}}} = 0$ *a.s.*

Then the proof is completed. □

CHAPTER V

STRONG LAW OF LARGE NUMBERS FOR ARBITRARY PAIRWISE NQD RANDOM VARIABLES

There are some extensions of SLLN considering a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ which is not necessary to assume identically distributed random variables. For example, Chandra and Goswami[4] proposed the SLLN for pairwise independent random variables without assuming identical distribution. In this chapter, we consider the SLLN for non-identically distributed random variables.

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. We define $S_n = \sum_{i=1}^{n} X_i$, $S_n^+ = \sum_{i=1}^{n} X_i^+$ and $S_n^- = \sum_{i=1}^{n} X_i^-$.

First, we recall a Kronecker Lemma which is used in this chapter.

Lemma 5.1 (Kronecker Lemma)**.** *([6]) Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers such that* $\sum_{n=1}^{\infty} x_n$ *converges. Then we have for all* $0 \le b_1 \le b_2 \le b_3 \le \dots$ *and* $b_n \to \infty$ *that*

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} b_i x_i = 0.$$

5.1 Marcinkiewicz-Zygmund SLLN for Pairwise NQD Random Variables

In the begin of this section, we obtain the Marcinkiewicz-Zygmund SLLN for non-negative random variables with $EX_iX_j \le EX_iEX_j$ for all $i \ne j$ by using the technique of Chandra and Goswami[4]. Our result relaxes the identical distribution and pairwise independence assumption studied in Korchevsky[9] and it also generalizes the Kolmogorov SLLN of Chandra and Goswami[4] by considering the SLLN when $b_n = n^{\frac{1}{p}}$ for any $0 < p < 2$.

For proving our main theorem, we first recall a theorem and introduce an inequality which are used in the proof of our result.

Theorem 5.2. *([3]) Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of positive constants with $b_n \uparrow \infty$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of non-negative random variables with finite $Var(X_n)$. Assume that*

(i) $\sup_{n\in\mathbb{N}} \frac{1}{b_n} \sum_{i=1}^{n} EX_i < \infty$,

(ii) there is a double sequence (ρ_{ij}) of non-negative real numbers such that $Var(S_n) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_{ij}$,

(iii) $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\rho_{ij}}{(\max(b_i, b_j))^2} < \infty.$

Then $\frac{S_n - ES_n}{b_n} \longrightarrow 0$ *a.s.*

Lemma 5.3. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers. For each $j \in \mathbb{N}$ and $p > 0$,*

$$\sum_{k=j}^{\infty} \frac{a_k}{k^{\frac{2}{p}}} \leq \frac{2}{j^{\frac{1}{p}}} \sup_{m\in\mathbb{N}} \frac{1}{m^{\frac{1}{p}}} \sum_{k=1}^{m} a_k.$$

Proof. For each $n \in \mathbb{N}$, let $A_n = \sum_{k=1}^{n} a_k$ and $S = \sup_{m\in\mathbb{N}} \frac{1}{m^{\frac{1}{p}}} \sum_{k=1}^{m} a_k$.
By Abel's summation formula([1]), for any positive sequence $(b_k)_{k\in\mathbb{N}}$ such that

$b_k \geq b_{k+1}$ for all $k \in \mathbb{N}$

$$\begin{aligned}
\sum_{k=j}^{n} a_k b_k &= b_{n+1} A_n + \sum_{k=j}^{n} A_k (b_k - b_{k+1}) \\
&\leq n^{\frac{1}{p}} b_{n+1} S + \sum_{k=j}^{n} k^{\frac{1}{p}} S (b_k - b_{k+1}) \\
&= n^{\frac{1}{p}} b_{n+1} S + \big[j^{\frac{1}{p}} (b_j - b_{j+1}) + (j+1)^{\frac{1}{p}} (b_{j+1} - b_{j+2}) + \cdots \\
&\quad + (n-1)^{\frac{1}{p}} (b_{n-1} - b_n) + n^{\frac{1}{p}} (b_n - b_{n+1}) \big] S \\
&= \big[j^{\frac{1}{p}} b_j + ((j+1)^{\frac{1}{p}} - j^{\frac{1}{p}}) b_{j+1} + ((j+2)^{\frac{1}{p}} - (j+1)^{\frac{1}{p}}) b_{j+2} \\
&\quad + \cdots + (n^{\frac{1}{p}} - (n-1)^{\frac{1}{p}}) b_n \big] S.
\end{aligned}$$

Choose $b_k = \dfrac{1}{k^{\frac{2}{p}}}$ for all $k \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{k=j}^{n} \frac{a_k}{k^{\frac{2}{p}}} &\leq \left[\frac{1}{j^{\frac{1}{p}}} + \frac{(j+1)^{\frac{1}{p}} - j^{\frac{1}{p}}}{(j+1)^{\frac{2}{p}}} + \cdots + \frac{n^{\frac{1}{p}} - (n-1)^{\frac{1}{p}}}{n^{\frac{2}{p}}} \right] S \\
&\leq \frac{1}{j^{\frac{1}{p}}} S + \left[\frac{(j+1)^{\frac{1}{p}} - j^{\frac{1}{p}}}{j^{\frac{1}{p}} (j+1)^{\frac{1}{p}}} + \cdots + \frac{n^{\frac{1}{p}} - (n-1)^{\frac{1}{p}}}{(n-1)^{\frac{1}{p}} n^{\frac{1}{p}}} \right] S \\
&= \frac{1}{j^{\frac{1}{p}}} S + \left[\left(\frac{1}{j^{\frac{1}{p}}} - \frac{1}{(j+1)^{\frac{1}{p}}} \right) + \cdots + \left(\frac{1}{(n-1)^{\frac{1}{p}}} - \frac{1}{n^{\frac{1}{p}}} \right) \right] S \\
&= \left(\frac{2}{j^{\frac{1}{p}}} - \frac{1}{n^{\frac{1}{p}}} \right) S \\
&\leq \frac{2}{j^{\frac{1}{p}}} S.
\end{aligned}$$

□

Theorem 5.4. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of non-negative random variables satisfying $EX_iX_j \leq EX_iEX_j$ for all $i \neq j$. For $0 < p < 2$, if*

(i) $\sup_{n \in \mathbb{N}} \dfrac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} EX_i < \infty$,

(ii) $\sum_{n=1}^{\infty} \dfrac{1}{n^{\frac{1}{p}}} EX_n I(X_n > n^{\alpha}) < \infty$ *for some* $\alpha \in \left(0, \min\left(\frac{1}{2}, \frac{1}{p} - \frac{1}{2}\right)\right)$.

Then $\dfrac{S_n - ES_n}{n^{\frac{1}{p}}} \longrightarrow 0$ *a.s.*

Proof. The argument proceeds essentially along the same steps as in the proof of Theorem 2.1(b) in Chandra and Goswami[4].

For each $i \in \mathbb{N}$, define $Y_i = X_i I(X_i \leq i^{\frac{1}{p}})$ and $T_n = \sum_{i=1}^{n} Y_i$. Then

$$\frac{S_n - ES_n}{n^{\frac{1}{p}}} = \frac{T_n - ET_n}{n^{\frac{1}{p}}} + \frac{ET_n - ES_n}{n^{\frac{1}{p}}} + \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} X_i I(X_i > i^{\frac{1}{p}}). \tag{5.1}$$

To prove the main result, we will show that each of three terms on the right side of (5.1) converges to zero almost surely.

The convergence of the second term follows immediately from the assumption (ii) and the Kronecker Lemma(Lemma 5.1), that is

$$\frac{1}{n^{\frac{1}{p}}}(ET_n - ES_n) = -\frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} EX_i I(X_i > i^{\frac{1}{p}}) \longrightarrow 0 \quad \text{as } n \to \infty. \tag{5.2}$$

To prove the convergence of the third term, we notice that

$$\begin{aligned} \sum_{n=1}^{\infty} P(X_n \neq Y_n) &= \sum_{n=1}^{\infty} P(X_n > n^{\frac{1}{p}}) \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{p}}} EX_n I(X_n > n^{\frac{1}{p}}) \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{p}}} EX_n I(X_n > n^{\alpha}) \\ &< \infty. \end{aligned}$$

This implies that $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are tail equivalent.

Hence $\sum_{n=1}^{\infty} X_n I(X_n > n^{\frac{1}{p}})$ converges almost surely.

By Kronecker Lemma(Lemma 5.1), $\frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} i^{\frac{1}{p}} X_i I(X_i > i^{\frac{1}{p}}) \longrightarrow 0$ $a.s.$ This implies

$$\frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} X_i I(X_i > i^{\frac{1}{p}}) \longrightarrow 0 \ a.s. \tag{5.3}$$

The last step in proving this theorem is to show that

$$\frac{1}{n^{\frac{1}{p}}}(T_n - ET_n) \longrightarrow 0 \ a.s.$$

To this end, we will show that $(Y_n)_{n\in\mathbb{N}}$ satisfies the three conditions of Theorem 5.2 stated below :

(a) $\sup_{n\in\mathbb{N}} \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} EY_i < \infty$,

(b) $Var(T_n) \le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij}$,

(c) $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\rho_{ij}}{(\max(i^{\frac{1}{p}}, j^{\frac{1}{p}}))^2} < \infty$,

where $\rho_{ij} = [Cov(Y_i, Y_j)]^+$.

First, we see that (a) and (b) are clearly proved as the followings.

$$\sup_{n\in\mathbb{N}} \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} EY_i = \sup_{n\in\mathbb{N}} \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} EX_i I(X_i \le i^{\frac{1}{p}}) \le \sup_{n\in\mathbb{N}} \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} EX_i < \infty$$

and

$$Var(T_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(Y_i, Y_j) \le \sum_{i=1}^{n}\sum_{j=1}^{n} [Cov(Y_i, Y_j)]^+.$$

Then it remains to prove (c).

Note that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\rho_{ij}}{(\max(i^{\frac{1}{p}}, j^{\frac{1}{p}}))^2} = \sum_{i=1}^{\infty} \frac{\rho_{ii}}{i^{\frac{2}{p}}} + 2\sum_{i=2}^{\infty}\sum_{j=1}^{i-1} \frac{\rho_{ij}}{i^{\frac{2}{p}}}. \tag{5.4}$$

To prove that the first term on the right of (5.4) converges, we notice that

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{\rho_{ii}}{i^{\frac{2}{p}}} &\le \sum_{i=1}^{\infty} \frac{1}{i^{\frac{2}{p}}} EY_i^2 \\
&= \sum_{i=1}^{\infty} \frac{1}{i^{\frac{2}{p}}} EX_i^2 I(X_i \le i^{\alpha}) + \sum_{i=1}^{\infty} \frac{1}{i^{\frac{2}{p}}} EX_i^2 I(i^{\alpha} < X_i \le i^{\frac{1}{p}}) \\
&\le \sum_{i=1}^{\infty} \frac{1}{i^{2(\frac{1}{p}-\alpha)}} P(X_i \le i^{\alpha}) + \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{p}}} EX_i I(i^{\alpha} < X_i \le i^{\frac{1}{p}}) \\
&\le \sum_{i=1}^{\infty} \frac{1}{i^{2(\frac{1}{p}-\alpha)}} + \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{p}}} EX_i I(X_i > i^{\alpha}) \\
&< \infty.
\end{aligned} \tag{5.5}$$

For the second term on the right of (5.4), since X_n's are non-negative with $EX_iX_j \leq EX_iEX_j$ for all $i \neq j$, we can show that

$$\begin{aligned}
\rho_{ij} &= [Cov(Y_i, Y_j)]^+ \\
&= [E(Y_iY_j) - EY_iEY_j]^+ \\
&\leq [E(X_iX_j) - EY_iEY_j]^+ \\
&\leq [EX_iEX_j - EY_iEY_j]^+ \\
&= EX_iEX_j - EY_iEY_j \\
&= (EX_i - EY_i)EX_j + EY_i(EX_j - EY_j) \\
&= EX_jEX_iI(X_i > i^{\frac{1}{p}}) + EY_iEX_jI(X_j > j^{\frac{1}{p}}) \\
&\leq EX_jEX_iI(X_i > i^{\frac{1}{p}}) + EX_iEX_jI(X_j > j^{\frac{1}{p}}).
\end{aligned}$$

Thus,

$$\sum_{i=2}^{\infty}\sum_{j=1}^{i-1}\frac{\rho_{ij}}{i^{\frac{2}{p}}} \leq \sum_{i=2}^{\infty}\sum_{j=1}^{i-1}\frac{1}{i^{\frac{2}{p}}}EX_jEX_iI(X_i > i^{\frac{1}{p}}) + \sum_{i=2}^{\infty}\sum_{j=1}^{i-1}\frac{1}{i^{\frac{2}{p}}}EX_iEX_jI(X_j > j^{\frac{1}{p}}). \tag{5.6}$$

Finally it remains to show the convergence of the two terms on the right of (5.6) which will be proved as follows.
The first series follows

$$\begin{aligned}
\sum_{i=2}^{\infty}\sum_{j=1}^{i-1}\frac{1}{i^{\frac{2}{p}}}EX_jEX_iI(X_i > i^{\frac{1}{p}}) &\leq \sum_{i=1}^{\infty}\left(\frac{1}{i^{\frac{1}{p}}}\sum_{j=1}^{i}EX_j\right)\frac{1}{i^{\frac{1}{p}}}EX_iI(X_i > i^{\frac{1}{p}}) \\
&\leq \left(\sup_{n\in\mathbb{N}}\frac{1}{n^{\frac{1}{p}}}\sum_{j=1}^{n}EX_j\right)\sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{p}}}EX_iI(X_i > i^{\frac{1}{p}}) \\
&\leq \left(\sup_{n\in\mathbb{N}}\frac{1}{n^{\frac{1}{p}}}\sum_{j=1}^{n}EX_j\right)\sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{p}}}EX_iI(X_i > i^{\alpha}) \\
&< \infty. \qquad \text{(by the assumptions } (i) \text{ and } (ii))
\end{aligned} \tag{5.7}$$

For the second series, we apply Lemma 5.3 as follows.

$$\begin{aligned}\sum_{i=2}^{\infty}\sum_{j=1}^{i-1}\frac{1}{i^{\frac{2}{p}}}EX_iEX_jI(X_j>j^{\frac{1}{p}}) &= \sum_{j=1}^{\infty}EX_jI(X_j>j^{\frac{1}{p}})\sum_{i=j+1}^{\infty}\frac{1}{i^{\frac{2}{p}}}EX_i\\ &\le \sum_{j=1}^{\infty}EX_jI(X_j>j^{\frac{1}{p}})\frac{2}{j^{\frac{1}{p}}}\sup_{n\in\mathbb{N}}\frac{1}{n^{\frac{1}{p}}}\sum_{i=1}^{n}EX_i\\ &\le 2(\sup_{n\in\mathbb{N}}\frac{1}{n^{\frac{1}{p}}}\sum_{i=1}^{n}EX_i)\sum_{j=1}^{\infty}\frac{1}{j^{\frac{1}{p}}}EX_jI(X_j>j^{\alpha})\\ &<\infty. \qquad \text{(by the assumptions } (i) \text{ and } (ii))\end{aligned} \tag{5.8}$$

By (5.6) – (5.8), we have

$$\sum_{i=2}^{\infty}\sum_{j=1}^{i-1}\frac{\rho_{ij}}{i^{\frac{2}{p}}}<\infty. \tag{5.9}$$

Combining (5.4), (5.5) and (5.9), $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\rho_{ij}}{(\max(i^{\frac{1}{p}},j^{\frac{1}{p}}))^2}<\infty.$

By Theorem 5.2,

$$\frac{1}{n^{\frac{1}{p}}}(T_n-ET_n)\longrightarrow 0 \ a.s. \tag{5.10}$$

It follows from (5.1) – (5.3) and (5.10) that $\frac{S_n-ES_n}{n^{\frac{1}{p}}}\longrightarrow 0 \ a.s.$ □

From Theorem 5.4, we can replace the condition of random variables from non-negative random variables with $EX_iX_j \le EX_iEX_j$ to pairwise NQD random variables. Then we obtain the following corollary.

Corollary 5.5. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of pairwise NQD random variables. For $0<p<2$, if*

(i) $\sup_{n\in\mathbb{N}}\frac{1}{n^{\frac{1}{p}}}\sum_{i=1}^{n}E|X_i|<\infty,$

(ii) $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{p}}}E|X_n|I(|X_n|>n^{\alpha})<\infty$ *for some* $\alpha\in\left(0,\min\left(\frac{1}{2},\frac{1}{p}-\frac{1}{2}\right)\right)$,

then $\frac{S_n-ES_n}{n^{\frac{1}{p}}}\longrightarrow 0 \ a.s.$

Proof. By Remark 3.7, $(X_n^+)_{n\in\mathbb{N}}$ and $(X_n^-)_{n\in\mathbb{N}}$ are sequences of pairwise NQD. By Proposition 3.6(i), $EX_i^+X_j^+ \leq EX_i^+EX_j^+$ and $EX_i^-X_j^- \leq EX_i^-EX_j^-$. By conditions ($i$) and ($ii$) and the fact that $E|X_i| = EX_i^+ + EX_i^-$, we can see that

$$\sup_{n\in\mathbb{N}} \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} EX_i^{\pm} < \infty \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{p}}} EX_n^{\pm} I(X_n^{\pm} > n^{\alpha}) < \infty$$

Therefore the sequences $(X_n^+)_{n\in\mathbb{N}}$ and $(X_n^-)_{n\in\mathbb{N}}$ satisfy the conditions of Theorem 5.4. Hence the almost surely convergence holds for S_n^+ and S_n^- and therefore for S_n. The theorem is then proved. □

Remark 5.6. If we replace the assumption (ii) of Theorem 5.4 by the weaker assumption

(ii') $\displaystyle \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} EX_i I(X_i > i^{\alpha}) \longrightarrow 0$ as $n \to \infty$ for some $\alpha \in \left(0, \min\left(\frac{1}{2}, \frac{1}{p} - \frac{1}{2}\right)\right)$,

then the asymtotic result reduces to convergence in probability, i.e. $\displaystyle \frac{S_n - ES_n}{n^{\frac{1}{p}}} \longrightarrow 0$ in probability which is called Marcinkiewicz-Zygmund WLLN. The proof follows Chandra and Goswami[4] directly.

5.2 Generalization of Order

In this section, we consider the rate of convergence for the SLLN when $(X_n)_{n\in\mathbb{N}}$ is a sequence of pairwise NQD random variables.

We first introduce these following lemmas which are used in the proof of our main result of this section.

Lemma 5.7. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that for any $r \geq 0$, $\displaystyle \inf_{i\in\{1,2,\ldots,n\}} a_i \geq -n^r$ and $\displaystyle \sum_{n=1}^{\infty} \frac{(a_1 + a_2 + \cdots + a_n)^2}{n^{\delta}} < \infty$ for some $\delta \in (0, \infty)$. Then*

$$\sum_{i=1}^{n} \frac{a_i}{n} = o(n^{\frac{4r+\delta-3}{4}}).$$

Proof. To prove this lemma, we can follow the proof of Lemma 11 in Sancetta[14]. Since $\displaystyle \sum_{n=1}^{\infty} \frac{(a_1 + a_2 + \cdots + a_n)^2}{n^{\delta}} < \infty$ and the Kronecker Lemma(Lemma 5.1), we have that $\frac{1}{n^\delta}\sum_{i=1}^{n}(a_1+a_2+\cdots+a_i)^2 \longrightarrow 0$ as $n \to \infty$. By Hölder's inequality(Remark 2.14), for each $x_i \in \mathbb{R}$, $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$. Then

$$\begin{aligned}
\frac{1}{n^{\frac{\delta+1}{2}}}\sum_{i=1}^{n}(a_1+a_2+\cdots+a_i) &= \frac{1}{n^{\frac{\delta}{2}}}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(a_1+a_2+\cdots+a_i)\right)\\
&\leq \frac{1}{n^{\frac{\delta}{2}}}\left(\sum_{i=1}^{n}(a_1+a_2+\cdots+a_i)^2\right)^{\frac{1}{2}}\\
&= \left(\frac{1}{n^{\delta}}\sum_{i=1}^{n}(a_1+a_2+\cdots+a_i)^2\right)^{\frac{1}{2}}\\
&\longrightarrow 0 \quad \text{as } n \to \infty. \qquad (5.11)
\end{aligned}$$

Define $t_n = \sum_{i=1}^{n} a_i$ and $w_n = \sum_{i=1}^{n} t_i$.
For $k \in \mathbb{N}$,

$$\begin{aligned}
\sum_{i=n+1}^{n+k}(t_i - t_n) &= \sum_{i=n+1}^{n+k}(a_{n+1}+\cdots+a_i)\\
&\geq (-(n+k)^r) + (-2(n+k)^r) + (-3(n+k)^r) + \cdots + (-k(n+k)^r)\\
&= \frac{-(n+k)^r k(k+1)}{2}\\
&\geq -k^2(n+k)^r
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=n-k+1}^{n}(t_n - t_i) &= \sum_{i=n-k+1}^{n-1}(a_{i+1}+\cdots+a_n)\\
&\geq (-(n+k)^r) + (-2(n+k)^r) + \cdots + (-(k-1)(n+k)^r)\\
&= \frac{-(n+k)^r k(k-1)}{2}\\
&\geq -k^2(n+k)^r.
\end{aligned}$$

Therefore

$$w_{n+k} - w_n = \sum_{i=n+1}^{n+k} t_i - kt_n + kt_n = \sum_{i=n+1}^{n+k}(t_i - t_n) + kt_n \geq -k^2(n+k)^r + kt_n$$

and

$$w_n - w_{n-k} = \sum_{i=n-k+1}^{n} t_i - kt_n + kt_n = \sum_{i=n-k+1}^{n} (t_i - t_n) + kt_n \leq k^2(n+k)^r + kt_n.$$

Then

$$\frac{w_n - w_{n-k}}{nk} - \frac{k}{n}(n+k)^r \leq \frac{t_n}{n} \leq \frac{w_{n+k} - w_n}{nk} + \frac{k}{n}(n+k)^r.$$

Define

$$\sigma_n := \max_{1 \leq i \leq 2n} |w_i| \text{ and } k_n := k = 1 + \lfloor \sqrt{\sigma_n} \rfloor$$

where $\lfloor \sigma_n \rfloor$ is the integer part of σ_n.

From (5.11), we have that for $i = 1, \ldots, 2n$, $w_i = \sum_{k=1}^{i} (a_1 + \cdots + a_k) = o(n^{\frac{\delta+1}{2}})$.

Then $\sigma_n = o(n^{\frac{\delta+1}{2}})$, $k_n = o(n^{\frac{\delta+1}{4}})$ and $\frac{\sigma_n}{k_n^2} = O(1)$.

Thus

$$\begin{aligned} \frac{|t_n|}{n} &\leq \frac{2\sigma_n}{nk_n} + \frac{k_n(n+k_n)^r}{n} \\ &= \left(\frac{2\sigma_n}{k_n^2(n+k)^r} + 1 \right) \frac{k_n(n+k)^r}{n} \\ &= O\left(\frac{k_n}{n^{1-r}} \right) \\ &= o(n^{\frac{4r+\delta-3}{4}}). \end{aligned}$$

Hence $\sum_{i=1}^{n} \frac{a_i}{n} = o(n^{\frac{4r+\delta-3}{4}})$. □

Lemma 5.8. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables. Assume that $\sum_{n=1}^{\infty} \frac{Var(X_1 + \cdots + X_n)}{n^\delta} < \infty$ for some $\delta \in (0, \infty)$ and $\inf_{i \in \mathbb{N}} (X_i - EX_i) > -\infty$. Then for any $r \geq 0$,*

$$\frac{S_n - ES_n}{n} = o(n^{\frac{4r+\delta-3}{4}}) \text{ a.s.}$$

Proof. We prove by following Lemma 9 in Sancetta[14] and use the result from Lemma 5.7. □

Theorem 5.9. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of pairwise NQD random variables. If*

$$\sup_{i \in \mathbb{N}} E|X_i| < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{Var(X_1 + \cdots + X_n)}{n^{\delta}} < \infty \text{ for some } \delta \in (0, \infty).$$

Then for any $r \geq 0$,

$$\frac{S_n - ES_n}{n} = o(n^{\frac{4r+\delta-3}{4}}) \ a.s.$$

Proof. Note that $X_j = X_j^+ - X_j^-$ and $|X_j| = X_j^+ + X_j^-$.
Also, we see that

$$Var|S_n| = Var(S_n^+) + Var(S_n^-) + 2Cov(S_n^+, S_n^-) \tag{5.12}$$

$$Var(S_n) = Var(S_n^+) + Var(S_n^-) - 2Cov(S_n^+, S_n^-). \tag{5.13}$$

By (5.12), (5.13) and the fact that $Var|S_n| \leq Var(S_n)$, we have

$$Var|S_n| - Var(S_n) = 4Cov(S_n^+, S_n^-) \leq 0.$$

This implies that $Cov(S_n^+, S_n^-) \leq 0$ and $Var(S_n^+) + Var(S_n^-) \leq Var(S_n)$.
Therefore

$$\sum_{n=1}^{\infty} \frac{Var(X_1^{\pm} + \cdots + X_n^{\pm})}{n^{\delta}} < \infty.$$

Following the proof of Theorem 2 in Sancetta[14] and applying Lemma 5.8, we get

$$\begin{aligned} \frac{S_n - ES_n}{n} &= \frac{1}{n}\left(\sum_{i=1}^{n}(X_i^+ - EX_i^+) - \sum_{i=1}^{n}(X_i^- - EX_i^-)\right) \\ &= o(n^{\frac{4r+\delta-3}{4}}) \ a.s. \end{aligned}$$

□

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VITA

Name	Miss Thanaporn Thanodomdech
Date of Birth	22 October 1990
Place of Birth	Bangkok, Thailand
Education	B.Sc. (Mathematics) (Second Class Honours), Chulalongkorn University, 2013
Scholarship	The His Royal Highness Crown Prince Maha Vajiralongkorn Scholarship from the Graduate School, Chulalongkorn University to commemorate the 72$^{\text{nd}}$ anniversary of his Majesty King Bhumibala Aduladeja
Publication	Thanodomdech, T., Suntornchost, J., Chaidee, N.: The Strong Law of Large Numbers for Non-identically Distributed random Variables, *Proceedings of the* 21$^{\text{th}}$ *Annual Meeting in Mathematics*, 33–42 (2016).
Conference	**Oral Presentation** • *The Strong Law of Large Numbers for Non-identically Distributed random Variables* at the 21$^{\text{th}}$ Annual Meeting in Mathematics, 23-25 May 2016 at Chulalongkorn University, Thailand