## CHAPTER III

## NUMERICAL SCHEME

### Boundary Discretization

The problem has been formulated in the form of integral equations in the previous chapter as expressed in Eqs. 16, 17 and 19. To solve these integral equations numerically, each section of the boundary, such as section I, is divided, as shown in Fig. 3, into $N^I$ intervals of equal arc length $2\omega^I\xi^I$. The center of each interval, being at $(\xi^I, \alpha_k^I)$ ;k = 1,2,3,...,$N^I$ ;I = 1,2,3,...,L, is to be considered as a nodal point. Approximating each unknown function by a set of discrete constant values over each interval, and identifying the integral equations, Eqs. 16, 17 and 19, successively at each nodal point on the boundary and each point of the column location leads to a set of $(2\sum_{J=1}^{L} N^J + N_c)$ algebraic equations in the form:

$$\frac{1}{2} w(\rho^J, \theta_1^J) = \int_A q(\xi,\alpha)\, w^*(\rho^J, \theta_1^J; \xi, \alpha)\, dA(\xi,\alpha)$$

$$+ \sum_{I=1}^{L} \sum_{k=1}^{N^I} \Big[ V_\rho(\xi^I, \alpha_k^I) \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} w^*(\rho^J, \theta_1^J; \xi^I, \alpha_k^I)\xi^I d\lambda$$

$$- M_\rho(\xi^I, \alpha_k^I) \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} \frac{\partial w^*(\rho^J, \theta_1^J; \xi^I, \alpha_k^I)}{\partial \xi}\xi^I d\lambda$$

$$- w(\xi^I,\alpha_k^I) \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} V_\xi^*(\rho^J,\theta_l^J;\xi^I,\alpha_k^I)\xi^I d\lambda$$

$$+ \frac{\partial w(\xi^I,\alpha_k^I)}{\partial\rho} \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} M_\xi^*(\rho^J,\theta_l^J;\xi^I,\alpha_k^I)\xi^I d\lambda \ ]$$

$$+ \sum_{n=1}^{N_c} R_c(\xi_n,\alpha_n)\ w^*(\rho^J,\theta_l^J;\xi_n,\alpha_n) \quad ;\ l=1,2,3,\ldots,N^J \quad ;\ J=1,2,3,\ldots,L, \qquad (20)$$

$$\frac{1}{2}\cdot\frac{\partial w(\rho^J,\theta_l^J)}{\partial\rho} = \int_A q(\xi,\alpha)\ \frac{\partial w^*}{\partial\rho}(\rho^J,\theta_l^J;\xi,\alpha)\ dA(\xi,\alpha)$$

$$+ \sum_{I=1}^{L}\sum_{k=1}^{N^I} [\ V_\rho(\xi^I,\alpha_k^I) \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} \frac{\partial w^*}{\partial\rho}(\rho^J,\theta_l^J;\xi^I,\alpha_k^I)\xi^I d\lambda$$

$$- M_\rho(\xi^I,\alpha_k^I) \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} \frac{\partial^2 w^*}{\partial\rho\partial\xi}(\rho^J,\theta_l^J;\xi^I,\alpha_k^I)\xi^I d\lambda$$

$$- w(\xi^I,\alpha_k^I) \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} \frac{\partial V_\xi^*}{\partial\rho}(\rho^J,\theta_l^J;\xi^I,\alpha_k^I)\xi^I d\lambda$$

$$+ \frac{\partial w(\xi^I,\alpha_k^I)}{\partial\rho} \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} \frac{\partial M_\xi^*}{\partial\rho}(\rho^J,\theta_l^J;\xi^I,\alpha_k^I)\xi^I d\lambda \ ]$$

$$+ \sum_{n=1}^{N_c} R_c(\xi_n,\alpha_n)\ \frac{\partial w^*}{\partial\rho}(\rho^J,\theta_l^J;\xi_n,\alpha_n) \quad ;\ l=1,2,3,\ldots,N^J \quad ;\ J=1,2,3,\ldots,L, \qquad (21)$$

$$
\begin{aligned}
0 = & \int_A q(\xi,\alpha)\, w^*(\rho_m,\theta_m;\xi,\alpha)\, dA(\xi,\alpha) \\
& + \sum_{I=1}^{L} \sum_{k=1}^{N^I} \Big[ V_\rho(\xi^I,\alpha_k^I) \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} w^*(\rho_m,\theta_m;\xi^I,\alpha_k^I)\xi^I d\lambda \\
& - M_\rho(\xi^I,\alpha_k^I) \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} \frac{\partial w^*}{\partial \xi}(\rho_m,\theta_m;\xi^I,\alpha_k^I)\xi^I d\lambda \\
& - w(\xi^I,\alpha_k^I) \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} V_\xi^*(\rho_m,\theta_m;\xi^I,\alpha_k^I)\xi^I d\lambda \\
& + \frac{\partial w}{\partial \rho}(\xi^I,\alpha_k^I) \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} M_\xi^*(\rho_m,\theta_m;\xi^I,\alpha_k^I)\xi^I d\lambda \Big] \\
& + \sum_{n=1}^{N_c} R_c(\xi_n,\alpha_n)\, w^*(\rho_m,\theta_m;\xi_n,\alpha_n) \qquad ;\ m=1,2,3,\ldots,N_c, \qquad (22)
\end{aligned}
$$

which involves and can be solved for the same number of unknown discrete values, i.e., either two out of the four sets of unknowns, $V_\rho(\xi^I,\alpha_k^I)$, $M_\rho(\xi^I,\alpha_k^I)$, $w(\xi^I,\alpha_k^I)$ and $\partial w(\xi^I,\alpha_k^I)/\partial\rho$ in each boundary sections and the reaction, $R_c$ , of each column.

## Treatment of Improper Integral

It should be noted that, all the integrals of the influence functions in Eqs. 20 to 22 may be evaluated in a straightforward manner as given in appendix B, except that of the function $\partial V_\xi^*/\partial\rho$

in Eq. 21 which, for $(\rho^J,\theta_1^J)$ and $(\xi^I,\alpha_k^I)$ being coincident, is improper. This improper integral can be evaluated by physical interpretation as follow:

Figure 4 depicts that the term $\partial V_\xi^*/\partial\rho$ in Eq. 21, is in physical sense, the Kirchhoff's shear due to a half of a unit couple applied at a nodal point of the edge of the plate. Now, if a free body is considered cut along a circumference as shown in the figure, then the total transverse shear and corner forces acting on the free body must be in equilibrium, i.e.,

$$\left[ \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} \frac{\partial V_\xi^*(\rho^J,\theta_1^J;\xi^I,\alpha_k^I)}{\partial\rho}\xi^I d\lambda \right]_{k=1}^{I=J} = - \int_{(\pi+\omega^I)/2}^{(3\pi-\omega^I)/2} \frac{\partial V_r^*(\bar{r},\bar{\theta})}{\partial\rho}\,\bar{r}d\bar{\theta}$$

$$- \left[ \frac{\partial M_{\xi\alpha}^*(\rho^J,\theta;\xi^I,\alpha)}{\partial\rho}\Bigg|_{\alpha-\theta=\omega^I}^{I=J} - \frac{\partial M_{r\theta}^*}{\partial\rho}(\bar{r},\frac{\pi+\omega^I}{2}) \right]$$

$$+ \left[ \frac{\partial M_{\xi\alpha}^*(\rho^J,\theta;\xi^I,\alpha)}{\partial\rho}\Bigg|_{\alpha-\theta=-\omega^I}^{I=J} - \frac{\partial M_{r\theta}^*}{\partial\rho}(\bar{r},\frac{3\pi-\omega^I}{2}) \right]. \qquad (23)$$

In which $\partial V_r^*(\bar{r},\bar{\theta})/\partial\rho = -(3-\nu)\cos\bar{\theta} \,/\, 4\pi\bar{r}^2$ is the Kirchhoff's shear around the indented circumference of the free body and

$$\left[ \frac{\partial M_{\xi\alpha}^*(\rho^J,\theta;\xi^I,\alpha)}{\partial\rho}\Bigg|_{\alpha-\theta=\omega^I}^{I=J} - \frac{\partial M_{r\theta}^*}{\partial\rho}(\bar{r},\frac{\pi+\omega^I}{2}) \right]$$

$$= \frac{(1-\nu)}{8\pi\xi^{I}} \cdot \frac{\sin\omega^{I}\cos\omega^{I}}{1-\cos\omega^{I}} - \frac{(1-\nu)}{4\pi\bar{r}}\frac{\sin(\pi+\omega^{I})}{2}$$

and

$$\left[ \frac{\partial M_{\xi\alpha}{}^{*}(\rho^{J},\theta;\xi^{I},\alpha)}{\partial\rho}\Bigg|^{I=J}_{\alpha-\theta=-\omega^{I}} - \frac{\partial M_{r\theta}{}^{*}}{\partial\rho}\left(\bar{r},\frac{3\pi-\omega^{I}}{2}\right) \right]$$

$$= \frac{(1-\nu)}{8\pi\xi^{I}} \cdot \frac{\sin(-\omega^{I})\cos(-\omega^{I})}{1-\cos(-\omega^{I})} - \frac{(1-\nu)}{4\pi\bar{r}}\sin\frac{(3\pi-\omega^{I})}{2}$$

represent the two corner forces.

Substitution of the above expressions into the last equation yields

$$\left[ \int_{\alpha_k{}^{I}-\omega^{I}}^{\alpha_k{}^{I}+\omega^{I}} \frac{\partial V_{\xi}{}^{*}(\rho^{J},\theta_1{}^{J};\xi^{I},\alpha_k{}^{I})}{\partial\rho}\xi^{I}d\lambda \right]^{I=J}_{k=1}$$

$$= \frac{-1}{2\pi\xi^{I}}\left[ \cot\frac{\omega^{I}}{2} - \frac{(1-\nu)}{2} \cdot \frac{\sin\omega^{I}\cos\omega^{I}}{1-\cos\omega^{I}} \right] \quad (24)$$

## Evaluation of the Domain Integrals

The domain integrals which appear in Eqs. 20 to 22 may be separately considered in two types of applied loads as follow:

In the case of uniformly distributed load of intensity $q_o$, if the unit load in the virtual system acts at a point $(\rho,\theta)$, the domain integrals may be computed as:

$$\int_A q(\xi,\alpha)\, w^*(\rho,\theta;\xi,\alpha)\, dA(\xi,\alpha) = q_o \int_0^{2\pi}\int_b^a w^*(\rho,\theta;\xi,\alpha)\, \xi d\xi d\alpha$$

$$= \frac{q_o a^4}{64D}\left[ \frac{\rho^4}{a^4} + 4\frac{\rho^2}{a^2}(2\ln a + 1) + 4\ln a - 1 \right]$$

$$- \frac{q_o b^4}{16D}\left[ 2\frac{\rho^2}{b^2}\ln \rho + \ln \rho + 1 \right] \quad ;b\leq\rho\leq a \qquad (25a)$$

$$\int_A q(\xi,\alpha)\, \frac{\partial w^*}{\partial \rho}(\rho,\theta;\xi,\alpha)\, dA(\xi,\alpha) = q_o \int_0^{2\pi}\int_b^a \frac{\partial w^*}{\partial \rho}(\rho,\theta;\xi,\alpha)\, \xi d\xi d\alpha$$

$$= \frac{q_o a^3}{16D}\left[ \frac{\rho^3}{a^3} + 2\frac{\rho}{a}(2\ln a + 1) \right]$$

$$- \frac{q_o b^3}{16D}\left[ 4\frac{\rho}{b}\ln \rho + 2\frac{\rho}{b} + \frac{b}{\rho} \right] \quad ;b\leq\rho\leq a \qquad (25b)$$

If a singular load, $P$, acts at a point $(\xi_o,\alpha_o)$ while the unit load in the virtual system acts at a point $(\rho,\theta)$, we replace $q(\xi,\alpha)$ by a Dirac delta function, $\delta(\xi,\alpha;\xi_o,\alpha_o)$, for which

$$\int_A \delta(\xi,\alpha;\xi_o,\alpha_o)\, w^*(\rho,\theta;\xi,\alpha)\, dA(\xi,\alpha) = w^*(\rho,\theta;\xi_o,\alpha_o) \qquad (26a)$$

$$\int_A \delta(\xi,\alpha;\xi_o,\alpha_o) \frac{\partial w^*}{\partial \rho}(\rho,\theta;\xi,\alpha)\, dA(\xi,\alpha) = \frac{\partial w^*}{\partial \rho}(\rho,\theta;\xi_o,\alpha_o) \tag{26b}$$

<u>Domain Solutions</u>

Accordingly, the deflection of the plate at any point, $(\rho,\theta)$, as written in Eq. 13, is to be approximated by:

$$\begin{aligned} w(\rho,\theta) = & \int_A q(\xi,\alpha)\, w^*(\rho,\theta;\xi,\alpha)\, dA(\xi,\alpha) \\ & + \sum_{I=1}^{L} \sum_{k=1}^{N^I} \Big[ V_\rho(\xi^I,\alpha_k^I) \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} w^*(\rho,\theta;\xi^I,\alpha_k^I)\xi^I d\lambda \\ & - M_\rho(\xi^I,\alpha_k^I) \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} \frac{\partial w^*}{\partial \xi}(\rho,\theta;\xi^I,\alpha_k^I)\xi^I d\lambda \\ & - w(\xi^I,\alpha_k^I) \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} V_\xi^*(\rho,\theta;\xi^I,\alpha_k^I)\xi^I d\lambda \\ & + \frac{\partial w}{\partial \rho}(\xi^I,\alpha_k^I) \int_{\alpha_k^I-\omega^I}^{\alpha_k^I+\omega^I} M_\xi^*(\rho,\theta;\xi^I,\alpha_k^I)\xi^I d\lambda \Big] \\ & + \sum_{n=1}^{N_c} R_c(\xi_n,\alpha_n)\, w^*(\rho,\theta;\xi_n,\alpha_n). \end{aligned} \tag{27}$$

Eventually, desired stress resultants may be obtained by appropriate differentiation following Eqs. 1 to 7.