CHAPTER I

INTRODUCTION

1.1 Prologue

A *group divisible design* $\mathsf{GDD}(v = v_1 + v_2 + v_3 + \cdots + v_g, g, k; \lambda_1, \lambda_2)$ is originally an ordered pair $(V, \mathcal{B})$ where V is a set of v elements partitioned into g sets of sizes $v_1, v_2, v_3, \ldots, v_g$ (each set is called a *group*) while $\mathcal{B}$ is a collection of k-subsets of V (each k-subset is called a *block*) such that each pair of elements in the same group occurs together in exactly λ_1 blocks and each pair of elements from different groups occurs together in exactly λ_2 blocks. The parameters λ_1 and λ_2 are called the *first* and the *second index*, respectively. Pairs occurring together in the same group are called *first associates* and pairs occurring in different groups are called *second associates.*

The existence problem of group divisible designs has been of interest over the years. In 1952, Bose and Shimamoto [1] began the classification of such designs. In 2000, Fu, Rodger and Sarvate [2, 3] completely solved the existence of GDDs with g groups of the same size, having all blocks of size three, or $\mathsf{GDD}(v = n+n+n+\cdots+n, g, 3; \lambda_1, \lambda_2)$. When $\lambda_1 = \lambda_2$, a $\mathsf{GDD}(v = n+n+n+\cdots+n, g, 3; \lambda_1, \lambda_1)$ is simply a well-known design, namely a *triple system* or $\mathsf{TS}(v; \lambda_1)$. The existence problem of a $\mathsf{TS}(v; \lambda)$ is reviewed in Chapter II. Recently, GDDs with blocks of size three and two groups of different sizes, or $\mathsf{GDD}(v = m + n, 2, 3; \lambda_1, \lambda_2)$, have been extensively studied (see [6, 8, 9]). Later on, in 2012, Pabhapote [5] provided the existence of such GDDs for all $m \neq 2$ and $n \neq 2$, with the larger first index $\lambda_1 \geq \lambda_2$. In this work, we generalize the study of GDDs in [5]. Those GDDs have unique first index. We introduce GDDs with two first indices λ_1 and λ_1'. In details, we define our group divisible designs with three associate classes, GDD$(m, n; \lambda_1, \lambda_1', \lambda_2)$, which is a 3-tuple $(M, N, \mathcal{B})$ where M and N are disjoint sets (called *groups*) such that $|M| = m$ and $|N| = n$ while $\mathcal{B}$ is a collection of 3-subsets (each 3-subset is called *block* or *triple)* of $M \cup N$ satisfying:

(i) each pair of distinct elements in M occurs together in exactly λ_1 blocks,

(ii) each pair of distinct elements in N occurs together in exactly λ_1' blocks, and

(iii) each $x \in M$ and $y \in N$ occur together in exactly λ_2 blocks.

Analogously to [5], we focus on GDD$(m, n; \lambda_1, \lambda_1', \lambda_2)$ with $m \neq 2$, $n \neq 2$, $\lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$.

1.2 Graphical Illustration

Throughout this thesis, we use the following notations to describe a GDD$(m, n; \lambda_1, \lambda_1', \lambda_2)$ graphically.

A *graph* G is an ordered pair $(V(G), E(G))$ where $V(G)$ is a nonempty finite set of vertices and $E(G)$ is a finite set of edges. A graph H is said to be a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Note that our graphs may have multiple edges. When we want to emphasize that a graph G has multiple edges, we will call G a *multigraph*. Let λK_v denote the multigraph with v vertices in which every pair of vertices is joined by λ edges. When specified, we use $\lambda K_v(V)$ to denote the complete multigraph having the vertex set V where $|V| = v$. Let G_1 and G_2 be graphs. The graph $G_1 \vee_\lambda G_2$ is obtained from the union of G_1 and G_2 by joining every vertex in G_1 to every vertex in G_2 with λ edges. A *decomposition* of a graph G into a graph H is a partition of the edges of G such that each element of the partition induces a copy of H. Then, the existence of a GDD$(m, n; \lambda_1, \lambda_1', \lambda_2)$

is equivalent to the existence of a decomposition of the graph $\lambda_1 K_m \vee_{\lambda_2} \lambda_1' K_n$ into K_3's, simply called a *K_3-decomposition*. Hence, triples in the designs are triangles or K_3's in their graph representations.

Example 1.1. Let $M_2 = \{x_1, x_2\}$ and $N_3 = \{y_1, y_2, y_3\}$ be two disjoint sets and let $\mathcal{B} = \{\{x_1, x_2, y_1\}, \{x_1, x_2, y_2\}, \{x_1, x_2, y_3\}, \{y_1, y_2, y_3\}, \{y_1, y_2, y_3\}\}$. Then, $(M_2, N_3, \mathcal{B})$ is a $\mathsf{GDD}(2, 3; 3, 2, 1)$ represented as a graph $3K_2 \vee_1 2K_3$ in Figure 1.1.

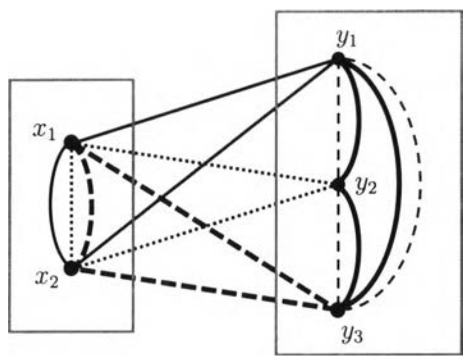

Figure 1.1: The graph $3K_2 \vee_1 2K_3$

Throughout this thesis, we let $M_m = \{x_1, x_2, x_3, \ldots, x_m\}$ and $N_n = \{y_1, y_2, y_3, \ldots, y_n\}$ be two disjoint sets of elements. Therefore, $\lambda_1 K_m(M_m)$ and $\lambda_1' K_n(N_n)$ are complete multigraphs lying on the sets M_m and N_n, respectively.

1.3 Necessity

In order to show the existence of a $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$, we find necessary conditions for the existence of such GDDs and then prove that such conditions are sufficient by constructing the GDDs satisfying such conditions. The necessity part is easily obtained by considering a GDD graphically, as shown in the following theorem.

Theorem 1.2. (Necessary Conditions) *Let m and n be positive integers such that $m \neq 2$ and $n \neq 2$. Let λ_1, λ_1' and λ_2 be nonnegative integers such that $\lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$. If there exists a $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$, then*

(i) $2|(\lambda_1(m-1) + \lambda_2 n)$,

(ii) $2|(\lambda_1'(n-1) + \lambda_2 m)$,

(iii) $6|(\lambda_1 m(m-1) + \lambda_1' n(n-1) + 2\lambda_2 mn)$ *and*

(iv) *if $\lambda_2 = 0$, then there exists a $\mathsf{TS}(m; \lambda_1)$ and a $\mathsf{TS}(n; \lambda_1')$.*

Proof. Assume that a $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$ exists. Then, there is a decomposition of the graph $\lambda_1 K_m \vee_{\lambda_2} \lambda_1' K_n$ into triangles. This implies that the number of edges in the graph must be divisible by three, that is $3|(\lambda_1\binom{m}{2} + \lambda_1'\binom{n}{2} + \lambda_2 mn)$. Thus, (iii) holds. Since each triangle contributes two to the value of the vertex degree, every vertex in the graph has even degree. Therefore, (i) and (ii) hold.

Moreover, if $\lambda_2 = 0$, then the graph has 2 components, which are $\lambda_1 K_m$ and $\lambda_1' K_n$. Since the design exists, both of them must have a decomposition into triangles. Hence, a $\mathsf{TS}(m; \lambda_1)$ and a $\mathsf{TS}(n; \lambda_1')$ exist. □

The remainder of this thesis is devoted to the construction of all GDDs satisfying all conditions in Theorem 1.2. First, we note that conditions (i) and (ii) in Theorem 1.2 imply the following four facts:

(1) if m and n are odd, then λ_2 is even,

(2) if m and n are even, then λ_1 and λ_1' are even,

(3) if m is odd and n is even, then $\lambda_1' + \lambda_2$ is even and

(4) if m is even and n is odd, then $\lambda_1 + \lambda_2$ is even.

Together with condition (iii) in Theorem 1.2 and regarding m and n as integers modulo 6, we rewrite the conditions of $m, n, \lambda_1, \lambda_1'$ and λ_2 for the existence of a $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$ when $m \neq 2$, $n \neq 2$, $\lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$, as shown in Table 1.1. Note that a $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$ is equivalent to a $\mathsf{GDD}(n, m; \lambda_1', \lambda_1, \lambda_2)$, thus, we will display only one of them in the table.

$m \setminus n$	0	1	2	3	4	5
0	$2\|\lambda_1$, $2\|\lambda_1'$	$2\|(\lambda_1+\lambda_2)$	$2\|\lambda_1$, $6\|\lambda_1'$	$2\|(\lambda_1+\lambda_2)$	$2\|\lambda_1$, $2\|\lambda_1'$	$3\|\lambda_1'$ $2\|(\lambda_1+\lambda_2)$
1		$6\|\lambda_2$	$2\|(\lambda_1'+\lambda_2)$ $3\|(\lambda_1'+2\lambda_2)$	$2\|\lambda_2$	$2\|(\lambda_1'+\lambda_2)$ $3\|\lambda_2$	$3\|(\lambda_1'+2\lambda_2)$ $2\|\lambda_2$
2			$2\|\lambda_1$, $2\|\lambda_1'$ $3\|(\lambda_1+\lambda_1'+\lambda_2)$	$3\|\lambda_1$ $2\|(\lambda_1+\lambda_2)$	$2\|\lambda_1$, $2\|\lambda_1'$ $3\|(\lambda_1+2\lambda_2)$	$2\|(\lambda_1+\lambda_2)$ $3\|(\lambda_1+\lambda_1'+\lambda_2)$
3				$2\|\lambda_2$	$2\|(\lambda_1'+\lambda_2)$	$3\|\lambda_1'$ $2\|\lambda_2$
4					$2\|\lambda_1$, $2\|\lambda_1'$ $3\|\lambda_2$	$3\|(\lambda_1'+2\lambda_2)$ $2\|(\lambda_1+\lambda_2)$
5						$2\|\lambda_2$ $3\|(\lambda_1+\lambda_1'+\lambda_2)$

Table 1.1: Necessity

For the sufficiency part of our problem, we provide the construction in Chapter III and Chapter IV. For Chapter II, some background which will be useful for our construction is provided here. Chapter III is to construct GDDs with $m \equiv 0$ or 1 (mod 3) while the GDDs with both m and $n \equiv 2$ (mod 3) are constructed in Chapter IV. Lastly, we summarize our result and give an open problem in Chapter V.