Chapter 3

The Case of Function Fields of Positive Characteristics

The main theme of this chapter is the concept of uniform distribution of sequences in $\mathbb{F}_q[x]$, the ring of polynomials over a finite field $\mathbb{F}_q$, and $\mathbb{F}_q((x^{-1}))$, the field of formal Laurent power series over $\mathbb{F}_q$.

In Section 1, we introduce the definitions of these concepts given by L. Carlitz([2]) and J.H. Hodges([6]). In Section 2, we introduce three criteria for the uniform distributivity of sequences in $\mathbb{F}_q[x]$ and $\mathbb{F}_q((x^{-1}))$. These criteria were proved by L. Carlitz([2]) and A. Dijksma([4]).

In Section 3, we give and prove some basic properties of uniform distribution in $\mathbb{F}_q[x]$ and $\mathbb{F}_q((x^{-1}))$. J.H. Hodges([7]) stated and proved a theorem showing a relation between uniform distribution modulo 1 and uniform distribution modulo M. A simple proof of this theorem is given here.

In Section 4, we introduce the sequence $(Z_n)_{n=1}^{\infty}$ in $\mathbb{F}_q[x]$ which plays the same role as the sequence of non-negative integers. The sequence $(Z_n)_{n=1}^{\infty}$ was first constructed by H.G. Meiyer and A. Dijksma [12]. There were several results concerning the sequence $(Z_n)_{n=1}^{\infty}$ in [5] and [19]. In this Section, we extend some of these results.

The last section covers the concept of uniform distribution modulo 1 in $\mathbb{F}_q((x^{-1}))$ in the multidimensional case.

3.1 Introduction

Let $\Phi = \mathbb{F}_q[x]$ denote the ring of polynomials over a finite field $\mathbb{F}_q$ of q elements, where $q = p^r$ for some prime number p and positive integer r.

Let $\mathbb{F}_q(x)$ denote the field of fractions of $\mathbb{F}_q[x]$. Define the real-valued valuation $|.|_\infty$ on $F_q(x)$ by

$$\left|\frac{f(x)}{g(x)}\right|_\infty = \begin{cases} q^{\deg f(x) - \deg g(x)} & \text{if} \quad f(x) \neq 0; \\ 0 & \text{if} \quad f(x) = 0. \end{cases}$$

Let $\Phi' = \mathbb{F}_q((x^{-1}))$ be the completion of $\mathbb{F}_q(x)$ with respect to $|.|_\infty$. Now Φ' consists of all the expressions $\alpha = \sum_{i=-\infty}^{m} c_i x^i \quad (c_i \in \mathbb{F}_q)$. If α has the representation and $c_m \neq 0$, then we define $\deg \alpha = m$; moreover, we define *deg* $0 = -\infty$. Note that for $\alpha \in \Phi'$, we have $|\alpha|_\infty = q^{\deg \alpha}$. The integral part of α is defined as $[\alpha] = \sum_{i=0}^{m} c_i x^i$ and the fractional part as $((\alpha)) = \sum_{i=-\infty}^{-1} c_i x^i$.

Obviously, we have $[\alpha+\beta] = [\alpha] + [\beta]$ and $((\alpha+\beta)) = ((\alpha)) + ((\beta))$ for all $\alpha, \beta \in \Phi'$. If α and β are elements of Φ' we say $\alpha \equiv \beta (\text{mod} 1)$ if $\alpha = \beta + A$ for some $A \in \Phi$. If $\alpha \in \Phi'$ and $\alpha = AB^{-1}$ for some A and $B \neq 0$ in Φ, then α is called *rational*, otherwise α is called *irrational*.

Definition 3.1.1 (L. Carlitz [3]). Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence in Φ'. Then this sequence is said to be *uniformly distributed modulo 1* (abbreviated u.d.mod 1) in Φ' if and only if

$$\lim_{N\to\infty} \frac{1}{N} \cdot |\{n \leq N : \deg(((\alpha_n - \beta))) < -k\}| = \frac{1}{q^k}$$

for all positive integers k and all $\beta \in \Phi'$.

Remark 3.1.2. Let k be a positive integer. For each $\alpha, \beta \in \Phi'$, we define

$$\alpha \sim \beta \quad \text{if and only if} \quad \deg(((\alpha - \beta))) < -k.$$

This is an equivalence relation on Φ', which partitions Φ' into q^k equivalence classes.

Definition 3.1.3 (J.H. Hodges [6]). Let $(A_n)_{n=1}^{\infty}$ be a sequence of elements in Φ. Let $M \in \Phi$ be a polynomial of degree $m \geqslant 1$. Then the sequence $(A_n)_{n=1}^{\infty}$ is said to be *uniformly distributed modulo* M (abbreviated u.d.mod M) in Φ if and only if

$$\lim_{N\to\infty} \frac{1}{N} \cdot |\{n \leq N : A_n \equiv B \ (mod\ M)\}| = \frac{1}{q^m} \qquad \text{for all } B \in \Phi.$$

Furthermore, we say that $(A_n)_{n=1}^{\infty}$ is *uniformly distributed* (abbreviated u.d.) in Φ if $(A_n)_{n=1}^{\infty}$ is u.d.mod M for every $M \in \Phi$ of positive degree.

Remark 3.1.4. Let $M \in \Phi$ be a polynomial of degree $m \geqslant 1$. For each $A, B \in \Phi$,

$$A \sim B \quad \text{if and only if} \quad A \equiv B(mod\ M).$$

This is an equivalence relation on Φ, which partitions Φ into q^m equivalence classes.

3.2 Criteria

In the classical case, Weyl criterion make use of the exponential function of complex numbers. We therefore start this section by defining an analogous function on Φ'. Let $\mu_1, \mu_2, \ldots, \mu_r$ be a fixed basis for the vector space $\mathbb{F}_q$ over $\mathbb{F}_p$. For given $\alpha = \sum_{i=-\infty}^{m} c_i x^i \in \Phi'$ (set $c_{-1} = 0$ if x^{-1} does not appear in the expression for α). Write

$c_{-1} = a_1\mu_1 + a_2\mu_2 + \ldots + a_r\mu_r$ with $a_j \in \mathbb{F}_p$ for $1 \leqslant j \leqslant r$. We define the exponential function $e : \Phi' \to \mathbb{C}$ by $e(\alpha) = e^{\frac{2\pi i a_1}{p}}$. This definition of the function $e(\alpha)$ is taken from L. Carlitz [3]. It is easy to see that for $\alpha, \beta \in \Phi'$, $e(\alpha+\beta) = e(\alpha)\cdot e(\beta)$ and $if\ \alpha \equiv \beta (mod\ 1)$, then $e(\alpha) = e(\beta)$.

The following theorem was first proved by L. Carlitz [2]. Here, we present another proof.

Theorem 3.2.1. *Let $\alpha \in \Phi'$ and $M \in \Phi$ a polynomial of degree $m \geqslant 1$. The sum*

$$\sum_{A\in\Phi,\, \deg(A)<m} e(A\alpha) = \begin{cases} q^m & if \quad \deg(((\alpha))) < -m, \\ 0 & if \quad \deg(((\alpha))) \geq -m. \end{cases}$$

Proof. Case 1. If $\deg(((\alpha))) < -m$, then for every $A \in \Phi$ such that $\deg A < m$, the coefficient of x^{-1} of $A\alpha$ is 0 and hence $e(A\alpha) = 1$; therefore

$$\sum_{A\in\Phi,\ \deg A<m} e(A\alpha) = \sum_{i=1}^{q^m} 1 = q^m.$$

Case 2. $\deg(((\alpha))) \geq -m$. Let $\Psi = \{A \in \Phi : \deg(A) < m\}$. Since for every $A \in \Psi$, $A\alpha = A([\alpha] + ((\alpha))) = A[\alpha] + A((\alpha))$, the coefficient of x^{-1} in $A\alpha$ is equal to the coefficient of x^{-1} of $A((\alpha))$. Let

$$((\alpha)) = \sum_{i=-\infty}^{-k} c_i x^i, \ c_{-k} \neq 0, -k \geq -m.$$

Thus for any $A = a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \ldots + a_1 x + a_0 \in \Psi$, the coefficient of x^{-1} of $A((\alpha))$ is

$$a_{m-1}c_{-m} + a_{m-2}c_{-m+1} + \ldots + a_{k-1}c_{-k}.$$

Note that for fixed $a \in \mathbb{F}_q$,

$$|\{(a_{m-1}, \ldots, a_{k-1}) : a_i \in \mathbb{F}_q, i = k-1, k, \ldots, m-1\ ,\ a = a_{m-1}c_{-m} + \ldots + a_{k-1}c_{-k}\}| = q^{m-k}.$$

Therefore, for fixed $a \in \mathbb{F}_q$,

$|\{A \in \Psi : \text{ the coefficient of } x^{-1} \text{ of } A\alpha \text{ is } a\}|$

$$= |\{(a_0, a_1, \ldots, a_{m-1}) : a_i \in \mathbb{F}_q, i = 0, 1, \ldots, m-1, \quad a = a_{m-1}c_{-m} + \ldots + a_{k-1}c_{-k}\}|$$
$$= q^{m-k} \cdot q^{m-(m-k+1)}$$
$$= q^{m-1}.$$

Moreover, for each $b_1 \in \mathbb{F}_p$,

$$|\{b \in \mathbb{F}_q : b = b_1\mu_1 + b_2\mu_2 + \ldots + b_r\mu_r \text{ for some } b_2, \ldots, b_r \in \mathbb{F}_p\}| = p^{r-1}.$$

Hence,

$$\sum_{A\in\Phi, \deg A<m} e(A\alpha) = q^{m-1}p^{r-1}e^{\frac{2\pi i 0}{p}} + q^{m-1}p^{r-1}e^{\frac{2\pi i}{p}} + \ldots + q^{m-1}p^{r-1}e^{\frac{2\pi i(p-1)}{p}}$$
$$= q^{m-1}p^{r-1} \cdot (\sum_{t=0}^{p-1} e^{\frac{2\pi i t}{p}})$$
$$= q^{m-1}p^{r-1} \cdot (\text{the sum of all roots of the polynomial } x^p - 1)$$
$$= q^{m-1}p^{r-1} \cdot 0$$
$$= 0.$$

□

This last theorem is the main tool in the proof of the following criterion for the uniform distributivity of sequences in Φ'.

Theorem 3.2.2 (L.Carlitz [3]). *The sequence $(\alpha_n)_{n=1}^{\infty}$ of elements of Φ' is u.d.mod 1 in Φ' if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(C\alpha_n) = 0$$

for all $C \in \Phi\backslash\{0\}$.

As to the criteria for the uniform distributivity of sequences in Φ, we have

Theorem 3.2.3 (A.Dijksma [4]). *The sequence* $(A_n)_{n=1}^{\infty}$ *of elements of* Φ *is u.d.mod* M *in* Φ *if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e(CM^{-1}A_n)=0$$

for all $C\in\Phi\backslash\{0\}$ *with* $\deg(C)<\deg(M)$.

Theorem 3.2.4 (A.Dijksma [4]). *The sequence* $(A_n)_{n=1}^{\infty}$ *of elements of* Φ *is u.d. in* Φ *if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e(\alpha A_n)=0$$

for all rational $\alpha\in\Phi'\backslash\{0\}$ *such that deg*$(\alpha)\leq -1$.

Corollary 3.2.5 (A.Dijksma [4]). *If the sequence* $(\alpha_n)_{n=1}^{\infty}$ *of elements of* Φ' *has the property that for some* $M\in\Phi$ *of degree* $m\geqslant 1$ *the sequence* $(M^{-1}\alpha_n)_{n=1}^{\infty}$ *is u.d.mod 1 in* Φ', *then the sequence* $([\alpha_n])_{n=1}^{\infty}$ *is u.d.mod* M *in* Φ.

3.3 Basic Properties

As in the classical case, we give and prove some basic properties of uniform distribution of sequences in Φ and Φ' . The property (3) in the following theorem was stated and proved by J.H. Hodges [6]. However, we give another proof of this property.

Theorem 3.3.1. *Let* $(\alpha_n)_{n=1}^{\infty}$ *be u.d.mod 1 in* Φ' *and* $(A_n)_{n=1}^{\infty}$ *be u.d.mod* M *in* Φ *where* $M\in\Phi$ *with* $\deg(M)>0$.

(1) If $A\in\Phi\backslash\{0\}$ *and* $\alpha\in\Phi'$, *then* $(A\alpha_n+\alpha)_{n=1}^{\infty}$ *is u.d.mod 1 in* Φ'.

(2) If $A,K\in\Phi$ *and* $\gcd(K,M)=1$, *then* $(KA_n+A)_{n=1}^{\infty}$ *is u.d.mod* M *in* Φ.

(3) If $F \in \Phi$ *with* $\deg(F) > 0$ *and* $F|M$, *then* $(A_n)_{n=1}^{\infty}$ *is u.d.mod* F *in* Φ.

(4) If $a \in \mathbb{F}_q \backslash \{0\}$, $A \in \Phi$ *and* $(A_n)_{n=1}^{\infty}$ *is u.d. in* Φ, *then* $(aA_n + A)_{n=1}^{\infty}$ *is u.d. in* Φ.

Proof. (1) Let $A \in \Phi\backslash\{0\}$ and $\alpha \in \Phi'$. For each $C \in \Phi$ with $C \neq 0$, we have, by Theorem 3.2.2,

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} e(C(A\alpha_n + \alpha)) &= \frac{1}{N}\sum_{n=1}^{N} e(CA\alpha_n + C\alpha)) \\ &= \frac{1}{N}\sum_{n=1}^{N} e(CA\alpha_n)\cdot e(C\alpha) \\ &= \frac{e(C\alpha)}{N}\sum_{n=1}^{N} e(CA\alpha_n) \to 0 \text{ as } N \to \infty\end{aligned}$$

since $(\alpha_n)_{n=1}^{\infty}$ is u.d.mod 1 in Φ'. By Theorem 3.2.2, $(A\alpha_n + \alpha)_{n=1}^{\infty}$ is u.d.mod 1 in Φ'.

(2) Let $A, K \in \Phi$ and $(K, M) = 1$. Let $C \in \Phi$ with $C \neq 0$ and $\deg(C) < deg(M)$. Since $(K, M) = 1$ and $\deg(C) < \deg(M)$, $CK \not\equiv 0(mod\ M)$. Write $CK = MD + R$ where R, $D \in \Phi$ with $\deg(R) < \deg(M)$ and $R \neq 0$. Now, by Theorem 3.2.3, we have

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} e(CM^{-1}(KA_n + A)) &= \frac{1}{N}\sum_{n=1}^{N} e(CM^{-1}KA_n)\cdot e(CM^{-1}A) \\ &= \frac{e(CM^{-1}A)}{N}\sum_{n=1}^{N} e(M^{-1}(MD + R)A_n) \\ &= \frac{e(CM^{-1}A)}{N}\sum_{n=1}^{N} e(DA_n)\cdot e(RM^{-1}A_n) \\ &= \frac{e(CM^{-1}A)}{N}\sum_{n=1}^{N} e(RM^{-1}A_n) \to 0 \text{ as } N \to \infty\end{aligned}$$

since $(A_n)_{n=1}^{\infty}$ is u.d.mod M in Φ, by Theorem 3.2.3, $(KA_n + A)_{n=1}^{\infty}$ is u.d.mod M in Φ.

(3)Let $F \in \Phi$, $\deg(F) \geqslant 1$ and $F|M$. Then $M = FD$ for some $D \in \Phi$ with $D \neq 0$. Let $C \in \Phi$ with $C \neq 0$ and $\deg(C) < \deg(F)$. Since $(A_n)_{n=1}^{\infty}$ be u.d.mod M in Φ, by Theorem 3.2.3, we obtain

$$\frac{1}{N}\sum_{n=1}^{N} e(CF^{-1}A_n) = \frac{1}{N}\sum_{n=1}^{N} e(CDM^{-1}A_n)$$
$$= \frac{1}{N}\sum_{n=1}^{N} e((CD)M^{-1}A_n) \to 0 \text{ as } N \to \infty$$

By Theorem 3.2.3, $(A_n)_{n=1}^{\infty}$ is u.d.mod F in Φ

(4) Let $a \in \mathbb{F}_q \backslash \{0\}, a \neq 0$ and $A \in \Phi$. Let $(A_n)_{n=1}^{\infty}$ be u.d. in Φ. To show that $(aA_n + A)_{n=1}^{\infty}$ be u.d. in Φ, let $M \in \Phi$ with $\deg(M) > 0$. Then $(a, M) = 1$. By Theorem 3.3.1(2), $(aA_n+A)_{n=1}^{\infty}$ is u.d.mod M in Φ. Since M is arbitrary, $(aA_n+A)_{n=1}^{\infty}$ is u.d. in Φ. □

Now, we give and prove the following theorem similar to the result in the classical case (see Theorem 2.1.5(ii)).

Theorem 3.3.2. *Let $(\alpha_n)_{n=1}^{\infty}$ be u.d.mod 1 in Φ'. Let $(\beta_n)_{n=1}^{\infty}$ be a sequence in Φ'. If $\lim_{n\to\infty}(((\alpha_n - \beta_n)))$ exists, then $(\beta_n)_{n=1}^{\infty}$ is u.d.mod 1 in Φ'.*

Proof. Let $C \in \Phi$ with $C \neq 0$ and $\deg(C) = k$. Let $\lim_{n\to\infty}(((\alpha_n - \beta_n))) = -\gamma$.
Then $|((\alpha_n - \beta_n)) + \gamma|_\infty \to 0$ as $n \to \infty$. Therefore there is an $m \in \mathbb{Z}^+$ depending on k such that $|((\alpha_n - \beta_n)) + \gamma|_\infty < q^{-k-1}$ for every integer $n \geqslant m$.
Since $|((\alpha_n - \beta_n)) + \gamma|_\infty = q^{\deg(((\alpha_n-\beta_n))+\gamma)}$, $\deg(((\alpha_n - \beta_n)) + \gamma) < -k - 1$ for every integer $n \geqslant m$.
Thus $\deg(C(((\beta_n - \alpha_n)) - \gamma)) = \deg(-C(((\alpha_n - \beta_n)) + \gamma)) = \deg(C((\alpha_n - \beta_n)) + \gamma) < -1$ for every integer $n \geqslant m$.

This implies that $e(C(((\beta_n - \alpha_n)) - \gamma)) = 1$ for every integer $n \geqslant m$. Now, let N be any sufficient large natural number. Then

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N} e(C\beta_n)\\
&= \frac{1}{N}\sum_{n=1}^{m-1} e(C\beta_n) + \frac{1}{N}\sum_{n=m}^{N} e(C\beta_n)\\
&= \frac{1}{N}\sum_{n=1}^{m-1} e(C\beta_n) + \frac{1}{N}\sum_{n=m}^{N} e(C(\alpha_n + (\beta_n - \alpha_n) - \gamma + \gamma))\\
&= \frac{1}{N}\sum_{n=1}^{m-1} e(C\beta_n) + \frac{1}{N}\sum_{n=m}^{N} e(C(\alpha_n + [\beta_n - \alpha_n] + ((\beta_n - \alpha_n)) - \gamma + \gamma))\\
&= \frac{1}{N}\sum_{n=1}^{m-1} e(C\beta_n) + \frac{1}{N}\sum_{n=m}^{N} e(C(\alpha_n)) \cdot e(C([\beta_n - \alpha_n])) \cdot e(C((\beta_n - \alpha_n)) - \gamma) \cdot e(C\gamma))\\
&= \frac{1}{N}\sum_{n=1}^{m-1} e(C\beta_n) + \frac{e(C\gamma)}{N}\sum_{n=m}^{N} e(C\alpha_n).
\end{aligned}$$

Since $(\alpha)_{n=1}^{\infty}$ is u.d.mod 1 in Φ',

$$\frac{1}{N}\sum_{n=1}^{N} e(C\beta_n) = \frac{1}{N}\sum_{n=1}^{m-1} e(C\beta_n) + \frac{e(C\gamma)}{N}\sum_{n=m}^{N} e(C\alpha_n) \to 0 \quad \text{as} \quad N \to \infty.$$

Since C is arbitrary, by Theorem 3.2.2, $(\beta_n)_{n=1}^{\infty}$ be u.d.mod 1 in Φ'. □

J.H. Hodges [7] discovered and proved a relation between u.d.mod 1 and u.d.mod M analogous to the one in the classical case (see Theorem 2.4.8 in Chapter 2). Here, we present an easier proof.

Theorem 3.3.3. *Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence in Φ'. Then the following statements are equivalent:*

(1) $(\alpha_n)_{n=1}^{\infty}$ is u.d.mod 1 in Φ'.

(2) $([M\alpha_n])_{n=1}^{\infty}$ is u.d.mod M for all $M \in \Phi$ of positive degree.

(3) For each $M \in \Phi$ *of positive degree,* $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e(CM^{-1}[M\alpha_n]) = 0$ *for all* $C \in \Phi$ *with* $C \neq 0$ *and* $\deg(C) < \deg(M)$.

(4) For each $A \in \Phi$ *with* $A \neq 0$, *we have* $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e(A\alpha_n) = 0$.

Proof. By Theorem 3.3.1 and Corollary 3.2.5 , we have (1) $\Rightarrow$ (2).

By Theorem 3.2.3, we have (2) $\Rightarrow$ (3).

To show that (3) $\Rightarrow$ (4), let $A \in \Phi$ and $A \neq 0$. Let $\deg(A) = m$ and let $B = x^{m+1} + A$. Then $\deg(A) < \deg(B)$. By (3), we have

$$\begin{aligned} 0 &= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e(AB^{-1}[B\alpha_n]) \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e(AB^{-1}(B\alpha_n - ((B\alpha_n)))) \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e(A\alpha_n)\cdot e(-AB^{-1}((B\alpha_n))) \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e(A\alpha_n) \qquad (\because \mathit{deg}\,(-AB^{-1}((B\alpha_n))) < -1). \end{aligned}$$

By Theorem 3.2.2 , we have (4) $\Rightarrow$ (1). □

3.4 The Sequence $(Z_n)_{n=1}^{\infty}$

Let τ be a one-to-one correspondence between $\mathbb{F}_q$ and the set $\{0, 1, 2, \ldots, q-1\}$ such that $\tau(0) = 0$. We extend the domain and range of τ to Φ and the set of nonnegative integers by defining

$$\tau(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0) = \tau(a_n)q^n + \tau(a_{n-1})q^{n-1} + \ldots + \tau(a_1)q + \tau(a_0).$$

We observe that now τ is a one-to-one correspondence between Φ and $\mathbb{Z}^+ \cup \{0\}$. We can now order $\Phi = \{Z_1, Z_2, \ldots\}$ by setting $Z_n = \tau^{-1}(n-1)$ for all positive integers n. J.H. Hodges [6] showed that the sequence $(Z_n)_{n=1}^{\infty}$ is u.d. in Φ. H.G. Meijer and A. Dijksma [12] proved that the sequence $(Z_n\alpha)_{n=1}^{\infty}$ is u.d.mod 1 in Φ' if and only if α is an irrational element of Φ'. They also showed that the sequence $([Z_n\alpha])_{n=1}^{\infty}$ is u.d. in Φ if and only if either α is irrational or α is a nonzero rational in Φ' with $deg(\alpha) < 0$.

Theorem 2.1.11 is a famous result of Van der Corput. It states that the sequence of real numbers $(x_n)_{n=1}^{\infty}$ is u.d.mod 1 if the sequence $(x_{n+h} - x_n)_{n=1}^{\infty}$ is u.d.mod 1 for all integers $h \geq 1$. This result has been generalized to sequences of elements other than real numbers; see for example Chapter 4 in [8].

Also, A. Dijksma [5] showed that, if the sequence $(g(Z_n + B) - g(Z_n))_{n=1}^{\infty}$ is u.d.mod 1 in Φ' for all $B \in \Phi \setminus \{0\}$, then $(g(Z_n))_{n=1}^{\infty}$ is u.d.mod 1 in Φ'; see Theorem 3.4.2. Moreover, using Theorem 3.4.2, he proved a necessary and sufficient condition for the uniform distributivity modulo 1 in Φ' of the sequence $(f(Z_n))_{n=1}^{\infty}$, where $f(Y)$ is a polynomial over Φ' of degree k with $0 < k < p$; see Theorem 3.4.4.

Additionally, W.A. Webb [19] proved a similar result for uniform distribution modulo M in Φ, see Theorem 3.4.9.

In this section, we give and prove a slight extension of Theorem 3.4.4 and a new theorem, see Theorem 3.4.5 and 3.4.6. Finally, we prove a theorem similar to Theorem 3.4.9 in the case of uniform distribution modulo 1 in Φ'.

Lemma 3.4.1 (A. Dijksma [5]). *Let u be a complex-valued function defined on Φ. Let N and s be positive integers such that $q^s \leqslant N$. If $N = aq^s + b$ where a and b are* *integers such that* $0 \leqslant b \leqslant q^s - 1$, *then*

$$q^s(N + q^s - b)^{-1} \cdot \left|\sum_{n=1}^{N} u(Z_n)\right|^2 \leqslant \sum_{n=1}^{N} |u(Z_n)|^2 + \sum_{h=2}^{q^s}\sum_{k=1}^{N} u(Z_k)\overline{u(Z_k + Z_h)}.$$

Theorem 3.4.2 (A. Dijksma [5]). *Let* $g : \Phi \to \Phi'$ *be a function and put* $g_B(Z_n) = g(Z_n + B) - g(Z_n)$ *where* $B \in \Phi$. *If the sequence* $(g_B(Z_n))_{n=1}^{\infty}$ *is u.d.mod 1 in* Φ' *for all* $B \in \Phi\backslash\{0\}$, *then the sequence* $(g(Z_n))_{n=1}^{\infty}$ *is u.d.mod 1 in* Φ'.

Lemma 3.4.3 (A. Dijksma [5]). *If* $\alpha \in \Phi'$ *is irrational and* $D \in \Phi$ *with* $D \neq 0$, *then the sequence* $(\alpha[Z_n D^{-1}])_{n=1}^{\infty}$ *is u.d.mod 1 in* Φ'.

Theorem 3.4.4 (A. Dijksma [5]). *Let* $f(Y)$ *be a polynomial over* Φ' *of degree* k *with* $0 < k < p$. *Then the sequence* $(f(Z_n))_{n=1}^{\infty}$ *is u.d.mod 1 in* Φ' *if and only if* $f(Y) - f(0)$ *has at least one irrational coefficient.*

Theorem 3.4.5. *Let* $f(Y) = \sum_{l=0}^{k} \alpha_l Y^l$ *be a polynomial over* Φ' *of degree* $k > 0$. *Suppose that* α_l *is rational for all* $l \geqslant p$. *Then the sequence* $(f(Z_n))_{n=1}^{\infty}$ *is u.d. mod 1 if and only if* $f(Y) - f(0)$ *has at least one irrational coefficient.*

Proof. First, assume α_1 is the only irrational coefficient of $f(Y) - \alpha_0$. Then we may write $f(Y) = g(Y) + \alpha_1 Y + \alpha_0$ where $g(Y)$ is a polynomial over Φ' having rational coefficients only. Let D be the least common multiple of the denominators of these coefficients of $g(Y)$. Put $\deg(D) = d$. Let N be a positive integer and let a and b be two integers such that $N = aq^d + b$ and $0 \leq b \leq q^d - 1$. Then, we write

$$\sum_{n=1}^{N} e(f(Z_n)) = \sum_{n=1}^{aq^d} e(f(Z_n)) + \sum_{n=aq^d+1}^{N} e(f(Z_n)) = \Sigma_1 + \Sigma_2$$

where $\Sigma_1 = \sum_{n=1}^{aq^d} e(f(Z_n))$ and $\Sigma_2 = \sum_{n=aq^d+1}^{n} e(f(Z_n))$.

Here $\Sigma_2 = o(N)$, $(N \to \infty)$ and

$$\Sigma_1 = \sum_{n=1}^{aq^d} e(g(Z_n) + \alpha_1 Z_n + \alpha_0) = \sum_{k=0}^{a-1} \sum_{l=kq^d+1}^{(k+1)q^d} e(g(Z_l) + \alpha_1 Z_l + \alpha_0).$$

Note that, for any non-negative integer k and any integer l such that $kq^d + 1 \leq l \leq (k+1)q^d$, we can write $Z_l = AD + C$ where $A, C \in \Phi$ and $\deg(C) < d$, so that $D[Z_l D^{-1}] = D[A + CD^{-1}] = DA = Z_l - C$; this implies that

$$\{Z_l : kq^d+1 \leq l \leq (k+1)q^d\} = \{Z_n+Z_j : 1 \leq j \leq q^d\} = \{D[Z_n D^{-1}]+Z_j : 1 \leq j \leq q^d\}.$$

Thus,

$$\Sigma_1 = \sum_{k=0}^{a-1} q^{-d} \sum_{n=kq^d+1}^{(k+1)q^d} \sum_{j=1}^{q^d} e(g(D[Z_n D^{-1}] + Z_j) + \alpha_1(D[Z_n D^{-1}] + Z_j) + \alpha_0).$$

It follows from the definition of D that $g(DA + B) \equiv g(B) (mod\ 1)$ and hence

$$e(g(DA + B)) = e(g(B)) \qquad \text{for all } A \text{ and } B \text{ in } \Phi.$$

Consequently,

$$\Sigma_1 = q^{-d} \sum_{j=1}^{q^d} e(g(Z_j) + \alpha_1 Z_j + \alpha_0) \sum_{n=1}^{aq^d} e(\alpha_1 D[Z_n D^{-1}]).$$

By Lemma 3.4.3, the sequence $\{\alpha_1 D[Z_n D^{-1}]\}_{n=1}^{\infty}$ is u.d.mod 1 in Φ', thus $\Sigma_1 = o(N)$.

Hence

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e(f(Z_n)) = 0 \qquad \ldots (*)$$

Let A be an arbitrary non-zero element of Φ. Then $(*)$ also holds for $Af(Z_n)$ instead of $f(Z_n)$. Hence, by Theorem 3.2.2, the sequence $\{f(Z_n)\}_{n=1}^{\infty}$ is uniformly distributed modulo 1 in Φ'.

We now proceed by induction on $m \in \{1, 2, \ldots p-1\}$. Let $P(m)$ be the statement "if $h(Y) = \sum_{l=0}^{k} \beta_l Y^l (\beta_l \in \Phi', k > 0,$ and β_l is rational for all $l \geq p)$ and m is the largest integral value of l such that β_l is irrational then the sequence $(h(Z_n))_{n=1}^{\infty}$ is u.d. mod 1 in Φ' ". By the method of the first step, we see that $P(1)$ is true. Now, let $m \in \{1, 2, \ldots, p-2\}$ and assume that $P(m)$ is true. Let $h(Y) = \sum_{l=0}^{k} \beta_l Y^l$ $(\beta_l \in \Phi', k > 0,$ and β_l is rational for all $l \geq p)$ where $m+1$ is the largest integral value of l such that β_l is irrational. Put for an arbitrary $B \in \Phi$, $h_B(Y) = h(Y+B) - h(Y)$. Then $h_B(Y)$ is a polynomial satisfying the condition of the induction hypothesis provided that $B \neq 0$. Hence the sequence $(h_B(Z_n))_{n=1}^{\infty}$ is u.d.mod 1 in Φ' for all $B \neq 0$ in Φ. By Theorem 3.4.2, this implies that $(h(Z_n))_{n=1}^{\infty}$ is u.d.mod 1 in Φ'. Thus, $P(m)$ is true for $m = 1, 2, \ldots, p-1$.

Conversely, suppose $f(Y) - f(0)$ has rational coefficients only. Let D be the least common multiple of the denominators of these coefficients.

Then $e(Df(Z_n)) = e(Df(0))$ for all $Z_n \in \Phi$. Using Theorem 3.2.2, since

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(Df(Z_n)) = e(Df(0)) \neq 0,$$

the sequence $(f(Z_n))_{n=1}^{\infty}$ is not u.d.mod 1 in Φ'. This completes the proof. □

Theorem 3.4.6. *Let* $f(Y) = \sum_{l=0}^{p} \alpha_l Y^l$ *be a polynomial over* Φ' *such that* α_p *is irrational. If there is* $j \in \{2, 3, \ldots, p-1\}$ *such that* α_j *is irrational, then* $(f(Z_n))_{n=1}^{\infty}$ *is u.d.mod 1 in* Φ'.

Proof. Let j^* be the maximum of $j \in \{2, 3, \ldots, p-1\}$ such that α_j is irrational. Let $f_B(Y) = f(Y+B) - f(Y)$ for $B \in \Phi$ and $B \neq 0$. Now, for each $B \in \Phi\backslash\{0\}$, $f_B(Y) = \sum_{l=0}^{p-1} \beta_l Y^l$ with β_{j^*-1} irrational. Thus, by Theorem 3.4.4, the sequence $(f(Z_n + B) - f(Z_n))_{n=1}^{\infty} = (f_B(Z_n))_{n=1}^{\infty}$ is u.d.mod 1 in Φ' for every $B \in \Phi\backslash\{0\}$. Hence, by Theorem 3.4.2, $(f(Z_n))_{n=1}^{\infty}$ is u.d.mod 1 in Φ'. □

Lemma 3.4.7. *Let $u(n)$ be a complex valued function such that $u(n) = 0$ if $n > N$. Then*

$$\frac{\left[\sqrt{N}\right]^2}{N+\left[\sqrt{N}\right]-1}\left|\sum_{n=1}^{N} u(n)\right|^2 \leqslant \left[\sqrt{N}\right]\sum_{n=1}^{N}|u(n)|^2 + 2\Re\left\{\sum_{k=1}^{[\sqrt{N}]-1}(\sqrt{N}-k)\sum_{n=1}^{N-k}u(n)\overline{u(n+k)}\right\},$$

where $\Re z$ denotes the real part of $z \in \mathbb{C}$.

Proof. Substitute $H = \left[\sqrt{N}\right]$ in Lemma 2.1.10. □

Theorem 3.4.8 (W.A. Webb [19]). *Let $(B_n)_{n=1}^{\infty}$ be a sequence in Φ. If $(B_{n+k} - B_n)_{n=1}^{\infty}$ is u.d.mod M (respectively u.d.) in Φ for all integers $k > 0$, then $(B_n)_{n=1}^{\infty}$ is u.d.mod M (respectively u.d.) in Φ.*

Using Lemma 3.4.7, we prove a result similar to Theorem 3.4.8 for the case of u.d.mod 1.

Theorem 3.4.9. *Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence in Φ'. If $(\alpha_{n+k} - \alpha_n)_{n=1}^{\infty}$ is u.d.mod 1 in Φ' for all integers $k > 0$, then $(\alpha_n)_{n=1}^{\infty}$ is u.d.mod 1 in Φ'.*

Proof. Let $C \in \Phi$ and $C \neq 0$. Let N be any sufficient large positive integer. Apply Lemma 3.4.7 with $u(n) = e(C\alpha_n)$ for $1 \leq n \leq N$. Then

$$\sum_{n=1}^{N}|u(n)|^2 = \sum_{n=1}^{N} 1 = N. \tag{3.4.1}$$

Also, for each integer $k > 0$,

$$\begin{aligned}\sum_{n=1}^{N-k} u(n)\overline{u(n+k)} &= \sum_{n=1}^{N-k} e(C\alpha_n)e(-C\alpha_{n+k}) \\ &= \sum_{n=1}^{N-k} e(-C(\alpha_{n+k}-\alpha_n)) \\ &= o(N),\end{aligned} \tag{3.4.2}$$

since $(\alpha_{n+k} - \alpha_n)_{n=1}^{\infty}$ is u.d.mod 1 in Φ'. Now, by Lemma 3.4.7, (3.4.1) and (3.4.2),

$$\frac{\left[\sqrt{N}\right]^2}{N + \left[\sqrt{N}\right] - 1} \left| \frac{1}{N} \sum_{n=1}^{N} u(n) \right|^2$$
$$\leq \frac{\sqrt{N}}{N^2} \sum_{n=1}^{N} |u(n)|^2 + 2Re \left\{ \frac{1}{N} \sum_{k=1}^{[\sqrt{N}]-1} (\sqrt{N} - k) \frac{1}{N} \sum_{n=1}^{N-k} u(n)\overline{u(n+k)} \right\}$$
$$= \frac{1}{\sqrt{N}} + 2Re \left\{ \frac{1}{N} \sum_{k=1}^{[\sqrt{N}]-1} (\sqrt{N} - k) \frac{1}{N} \sum_{n=1}^{N-k} u(n)\overline{u(n+k)} \right\}$$
$$= \frac{1}{\sqrt{N}} + o(1).$$

Since

$$\lim_{N\to\infty} \frac{\left[\sqrt{N}\right]^2}{N + \left[\sqrt{N}\right] - 1} = 1,$$

we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(C\alpha_n) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} u(n) = 0.$$

Hence, $(\alpha_n)_{n=1}^{\infty}$ is u.d.mod 1 in Φ'. □

3.5 The Multidimensional Case

Definition 3.5.1 (L. Carlitz [3]). Let m be a positive integer. Let $(\Omega_n)_{n=1}^{\infty} = ((\omega_1(n), \omega_2(n), \ldots, \omega_m(n)))_{n=1}^{\infty}$ be a sequence in $(\Phi')^m$. The sequence $(\Omega_n)_{n=1}^{\infty}$ is said to be u.d.mod 1 in $(\Phi')^m$ if and only if

$$\lim_{N\to\infty} \frac{1}{N} \cdot |\{n \leq N : \deg(((\omega_j(n) - \beta_j))) < -k_j \text{ for all } j = 1, 2, \ldots, m\}| = q^{-(k_1+k_2+\ldots+k_m)}$$

for all $(\beta_1, \ldots, \beta_m) \in (\Phi')^m$ and all $k_1, \ldots, k_m \in \mathbb{Z}^+$.

Proposition 3.5.2 (L. Carlitz [3]). *A sequence* $(\Omega_n)_{n=1}^{\infty} = ((\omega_1(n), \omega_2(n), \ldots, \omega_m(n)))_{n=1}^{\infty}$ *is u.d.mod 1 in* $(\Phi')^m$ *if and only if for every* $(A_1, \ldots, A_m) \in (\Phi)^m \backslash \{(0, \ldots, 0)\}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(A_1\omega_1(n) + \ldots + A_m\omega_m(n)) = 0.$$

Proposition 3.5.3. *The sequence* $((\omega_1(n), \omega_2(n), \ldots, \omega_m(n)))_{n=1}^{\infty}$ *is u.d.mod 1 in* $(\Phi')^m$ *if and only if the sequence* $(A_1\omega_1(n) + \ldots + A_m\omega_m(n))_{n=1}^{\infty}$ *is u.d.mod 1 in* Φ' *for every* $(A_1, \ldots, A_m) \in (\Phi)^m \backslash \{(0, \ldots, 0)\}$.

Proof. ($\Longrightarrow$) Assume that the sequence $((\omega_1(n), \omega_2(n), \ldots, \omega_m(n)))_{n=1}^{\infty}$ is u.d.mod 1 in $(\Phi')^m$. Let $(A_1, \ldots, A_m) \in (\Phi)^m \backslash \{(0, \ldots, 0)\}$. To show that $(A_1\omega_1(n) + \ldots + A_m\omega_m(n))_{n=1}^{\infty}$ is u.d.mod 1 in Φ' , let $C \in \Phi$ and $\Phi \neq 0$. Then $(CA_1, \ldots, CA_m) \neq (0, \ldots, 0)$. By the assumption and Theorem 3.5.2 ,

$$\sum_{n=1}^{N} e(C(A_1\omega_1(n) + \ldots + A_m\omega_m(n))) = \sum_{n=1}^{N} e(CA_1\omega_1(n) + \ldots + CA_m\omega_m(n)) = o(N).$$

Hence, by Theorem 3.2.2 , the sequence $(A_1\omega_1(n) + \ldots + A_m\omega_m(n))_{n=1}^{\infty}$ is u.d.mod 1 in Φ'.

($\Longleftarrow$) Assume that $(A_1\omega_1(n) + \ldots + A_m\omega_m(n))_{n=1}^{\infty}$ is u.d.mod 1 in Φ' for every $(A_1, \ldots, A_m) \in (\Phi)^m \backslash \{(0, \ldots, 0)\}$. Let $(A_1, \ldots, A_m) \in (\Phi)^m \backslash \{(0, \ldots, 0)\}$. Then $(A_1\omega_1(n) + \ldots + A_m\omega_m(n))_{n=1}^{\infty}$ is u.d.mod 1 in Φ'. By Theorem 3.2.2, we have

$$\sum_{n=1}^{N} e(C(A_1\omega_1(n) + \ldots + A_m\omega_m(n))) = o(N) \quad \text{for all } C \in \Phi \text{ with } C \neq 0. \qquad (*)$$

Choose $C = 1$ in (*), we have $((\omega_1(n), \omega_2(n), \ldots, \omega_m(n)))_{n=1}^{\infty}$ is u.d.mod 1 in $(\Phi')^m$ by Theorem 3.5.2. □

Proposition 3.5.4. *If the sequence* $((\omega_1(n), \omega_2(n), \ldots, \omega_m(n)))_{n=1}^{\infty}$ *is u.d.mod 1 in* $(\Phi')^m$*, then* $(\omega_j(n))_{n=1}^{\infty}$ *is u.d.mod 1 in* Φ' *for all* $j = 1, 2, \ldots, m$*, whereas the converse need not hold.*

Proposition 3.5.5. *If* $\xi_1, \ldots, \xi_m \in \Phi'$ *and* $1, \xi_1, \ldots, \xi_m$ *are linearly independent over* Φ*, then the sequence* $(Z_n\xi_1, Z_n\xi_2, \ldots, Z_n\xi_m)_{n=1}^{\infty}$ *is u.d.mod 1 in* $(\Phi')^m$ *where* $(Z_n)_{n=1}^{\infty}$ *is the sequence defined in section 4.*

Proof. We prove this theorm by using Theorem 3.5.3. Let $(A_1, \ldots, A_m) \in (\Phi')^m \backslash \{(0, \ldots, 0)\}$. We want to show that $(A_1 Z_n \xi_1 + \ldots + A_m Z_n \xi_m)_{n=1}^{\infty}$ is u.d.mod 1 in Φ'. Consider $A_1 Z_n \xi_1 + \ldots + A_m Z_n \xi_m = Z_n(A_1\xi_1 + \ldots + A_m\xi_m)$. Since $(A_1, \ldots, A_m) \neq 0$ and $1, \xi_1, \ldots, \xi_m$ are linearly independent, $A_1\xi_1 + \ldots + A_m\xi_m$ is irrational. Thus, by Theorem 3.4.4, $(A_1 Z_n \xi_1 + \ldots + A_m Z_n \xi_m)_{n=1}^{\infty} = (Z_n(A_1\xi_1 + \ldots + A_m\xi_m))_{n=1}^{\infty}$ is u.d.mod 1 in Φ'. □