CHAPTER I
INTRODUCTION

The problem of characterizing elements in various fields is not only significant from the theoretical point of view but it is also of utmost importance in diverse applications such as statistics and digital computations. In the case of real numbers, one of the earliest known classification is that of rational and irrational numbers. Regarding the representation of real numbers, it is well-known that each real number is representable as a simple continued fraction in exactly two different shapes. Should the last partial quotient be taken greater than or equal to two, the representation becomes unique. Combining independence with representation, there arises a natural problem of characterizing elements via their unique representations.

The work in this thesis centers around these two concepts, namely, continued fraction representation and their independence, in the field, $\mathbb{F}_q((x^{-1}))$, of Laurent series over a finite field, referred to here as a function field, which is the completion of the field of rational functions, over a finite base field, with respect to the infinite valuation. There are two best-known kinds of continued fractions, termed Ruban and Schneider continued fractions, abbreviated by **RCF** and **SCF**, respectively. Rational elements, i.e. elements in $\mathbb{F}_q(x)$, are precisely those having finite **RCF** and **SCF** expansions. That periodic **RCF**'s correspond exactly to quadratic irrationals is well-known. The same question for **SCF**'s is difficult and still open. The first objective of this thesis is to derive more information, such as their pre-periods and palindromic properties of periodic SCF's which could be useful for the resolution of this question.

A number of papers related to transcendence, irrationality and independence of continued fraction expansions in different settings have appeared, see e.g. [1], [2],[5],[6],[7],[9], [12],[14],[15],[16],[24],[31],[32],[33] and [34]. We are here interested in the result of Hančl [12], where a linear independence criterion for classical continued fractions is obtained, and results of [1], [14],[15],[16] and [33] for algebraic independence criteria. The next main objectives of the thesis are to establish two general independence criteria, one for linear and the other for algebraic independence and to the extensive computation of intriguing examples.

In Chapter II, we begin by collecting those definitions and results about valuation, the function field $\mathbb{F}_q((x^{-1}))$, and continued fractions, mainly without proofs, to be used throughout the entire thesis. We describe the construction of continued fractions in the classical case, in the field of p-adic numbers and in the field of Laurant series $\mathbb{F}_q((x^{-1}))$, where $\mathbb{F}_q$ is a finite field of q elements. There are two well-known continued fractions for p-adic numbers, namely the one due to Ruban [28] and the other due to Schneider [30]. As seen from their algorithms, both kinds of continued fractions can be constructed in $\mathbb{F}_q((x^{-1}))$. In the classical case, real numbers are rational if and only if their continued fractions are finite. In the p-adic case, the situation, though already settled, is more complicated for there are rational numbers whose p-adic continued fractions are infinite periodic, see e.g. Bundschuh [6], Laohakosol [13], Lianxiang [17], de Weger [10], and Browkin [4]. In this chapter, we describe the construction of the **RCF** in $\mathbb{F}_q((x^{-1}))$. The construction of this continued fraction mimics that of the classical simple continued fraction in the real case. Next, we derive basic properties and show that the continued fraction terminates if and only if it represents a rational element. As to the characterization of quadratic irrationals, such continued fraction is (infinite) periodic if and only if it represents a quadratic irrational. We also describe the construction of the **SCF** in $\mathbb{F}_q((x^{-1}))$. A rationality characterization is considered with similar results to those of **RCF**. As to the characterization of quadratic irrationals, that periodic **SCF** represents a quadratic irrational is trivial, while the converse is still not known. In an attempt towards a possible resolution of this problem, using a matrix approach of van der Poorten [26], we obtain information about pre-period, palindromic and approximation properties of periodic **SCF**.

In Chapter III, we concentrate on the investigation of independence criteria. We derive a criterion for linear independence similar to the one in [11] but with slightly weaker restrictions, and derive a criterion for algebraic independence along the line of [1], [14], and [16]. The linear independence criterion states roughly that if the partial quotients grow at a moderately fast rate, their continued fractions are linearly independent. This linear independence criterion when applied to the real case encompasses that obtained by Hančl in 2002. As to algebraic independence, a general Liouville-type sufficient condition through rational approximations is proved. Historically, this algebraic independence criterion is a culmination of earlier works in [1], [8], [9],[14], [15], [16], [17], [18], [19], [24], [33] and [34]. When applied to continued fractions, there are a great deal of possibilities depending on the parameters to choose in order to manufacture a particular criterion. To be specific, we strategically deduce a number of criteria which show roughly that exponentially growing partial quotients imply algebraic independence. Works in this direction for p-adic numbers can be found in [5], [8], [10], and [12]. It seems plausible that the algebraic independence criteria so obtained could be subject to improvements as well to various extensions so as to accommodate representations other than those of continued fractions

In the final chapter, Chapter IV, two interesting types of explicit continued fractions are worked out as examples illustrating the strength of the criteria so obtained. These elements have explicit shapes both for their series and continued fraction expansions, and are of genuine number theoretical interests within themselves. The first main kind of these examples, which involves explicit continued fractions that have series expansions containing exponents with certain predictable base representations, has quite a long history starting possibly in 1932 with the work of Böhmer [3] and continued in [7]. The second main kind of examples, which involves explicit continued fractions that have series expansions of certain lacunary type, was first discovered by Shallit [31], [32], later extended in [25] and [27]. It is to be noted that these examples are also of great significance as they illustrate clearly the very important difference between the function field $\mathbb{F}_q((x^{-1}))$, which has characteristic p and the real and the p-adic number fields which have characteristic 0. This difference was originally made explicit by the work of Mahler [21] in 1949. It states briefly that the Liouville diophantine approximation property cannot be improved in the case of fields with characteristic p.