CHAPTER III

LINEAR AND ALGEBRAIC INDEPENDENCE

From here, our main concern is the function field $\mathbb{F}_q((x^{-1}))$ denoted by $\mathbf{F}$. The infinite valuation $|\cdot|$ over $\mathbb{F}_q(x)$, the field of rational functions over $\mathbb{F}_q$, is defined as follows: for $f(x)/g(x) \in \mathbb{F}_q(x)\backslash\{0\}$, set

$$|0| = 0, \qquad \left|\frac{f(x)}{g(x)}\right| = q^{\deg f(x) - \deg g(x)}.$$

Then $\mathbf{F}$ is the completion of $\mathbb{F}_q(x)$, with respect to this valuation. The extension of the valuation to $\mathbf{F}$ is also denoted by $|\cdot|$. The continued fractions considered here are **RCF**.

3.1 Linear independence

First we start with a definition.

Definition 3.1. Let E be an extension field of K and $\alpha_1, \alpha_2, \ldots, \alpha_n \in E$.
Then $\alpha_1, \alpha_2, \ldots, \alpha_n$ are **linearly independent** over K if for all $a_1, a_2, \ldots, a_n \in K$,

$$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0 \Rightarrow a_1 = a_2 = \ldots = a_n = 0.$$

Recently, Hančl, [11], has given an interesting criterion for linear independence of real continued fractions as follows:

Theorem 3.2. *Let $\epsilon > 1$ be a real number, $N \in \mathbb{N}$ and $\{a_{n,j}\}_{n=0}^{\infty}$ $(j = 1, 2, \ldots, N)$ be N sequences of positive integers such that*

$$a_{n,j+1} > a_{n,j}\left(1 + \frac{\epsilon}{n log\ n}\right) \tag{3.1}$$

$$a_{n+1,1} > a_{n,N}^{N-1}\left(1 + \frac{1}{n}\right) \tag{3.2}$$

hold for every sufficiently large positive integer n and $j \in \{1, 2, \ldots, N-1\}$. Then the continued fractions, $\alpha_j := [a_{0,j}; a_{1,j}, \ldots]$ $(j = 1, 2, \ldots, N)$ and the number 1 are linearly independent over $\mathbb{Q}$.

We have known that rationality can be characterized by considering (Ruban) continued fractions, that is, finite continued fractions represent rationals and conversely. Moreover, it is well-known that (infinite) periodic (Ruban) continued fractions represent quadratic irrationals and conversely. There are quadratic irrationals linearly dependent as well as quadratic irrationals which are linearly independent over $\mathbb{F}_q(x)$ as seen the following examples.

Examples 1) Let $f = [\overline{x}], g = [2x, \overline{x}] \in \mathbb{F}_3((x^{-1}))$. Then $f = 2x + 2\sqrt{x^2+1}$ and $g = 2\sqrt{x^2+1}$ and so $x + f = g$ i,e, $f, g, 1$ are linearly dependent over $\mathbb{F}_3(x)$.

2) Let $u = [x, \overline{2x}], v = [x^2, \overline{2x^2}] \in \mathbb{F}_3((x^{-1}))$. Then $u = \sqrt{x^2+1}$ and $v = \sqrt{x^4+1}$. If $u, v, 1$ are linearly dependent, then there are $A, B, C \in \mathbb{F}_3[x]$ not all zero such that $A + B\sqrt{x^2+1} = C\sqrt{x^4+1}$. Thus $A^2 + 2AB\sqrt{x^2+1} + B^2(x^2+1) = C^2(x^4+1)$ and so $AB = 0$. If $B = 0$, then $A = C\sqrt{x^4+1}$ is irrational which is a contradiction. Hence $B \neq 0$ and so $A = 0$ yielding

$$\frac{x^2+1}{x^4+1} = \left(\frac{C}{B}\right)^2$$

which is a contradiction since x^2+1 and x^4+1 are relatively prime and both are not perfect squares. Hence $u, v, 1$ are linearly independent over $\mathbb{F}_3(x)$.

The criterion for linear independence using partial quotients of **RCF** is not easy to find for such elements. Here, we propose such a criterion for linear independence. We observe that, this criterion cannot be used for the above examples.

Extending the result of Hančl [11], in this section we establish a sufficient condition for linear independence of continued fractions in **F**. This criterion is based on a suitable growth condition of the partial quotients involved.

Theorem 3.3. *Let $N \in \mathbb{N}$ and $\{a_{n,j}\}_{n=0}^{\infty}$ $(j = 1, 2, \ldots, N)$ be N sequences of non-constant polynomials over $\mathbb{F}_q$. Assume that there exists an increasing sequence of positive integers $n_0 = 0 < n_1 < n_2 < \ldots$ with the following properties:*

$$|a_{n_k,j+1}| \geq |a_{n_k,j}|\varepsilon_{n_k}, \tag{3.3}$$

$$|a_{n,j+1}| \geq |a_{n,j}|c_n \qquad (n_k < n < n_{k+1}\ ;\ k \in \mathbb{N}_0), \tag{3.4}$$

$$|a_{n_k+1,1}| \geq |a_{n_k,N}|^{N-1}\delta_{n_k}, \tag{3.5}$$

$$|a_{n+1,1}| \geq |a_{n,N}|^{N-1}d_n \quad (n_k < n < n_{k+1}\ ; k \in \mathbb{N}_0), \tag{3.6}$$

where $\varepsilon_{n_k}, \delta_{n_k}, c_n, d_n$ *are positive real numbers subject to the conditions that*

$$c_n \geq c > 0 \qquad (n_k < n < n_{k+1}\ ; k \in \mathbb{N}_0),$$

$$\prod_{i=0}^{\infty}(c_{n_i+1}\cdots c_{n_{i+1}-1}\varepsilon_{n_{i+1}}) = \infty = \prod_{i=0}^{\infty}(d_{n_i+1}\cdots d_{n_{i+1}-1}\delta_{n_{i+1}}).$$

Then $\alpha_j := [a_{0,j}, a_{1,j}, \ldots]$ $(j = 1, 2, \ldots, N)$ *and* 1 *are linearly independent over* $\mathbb{F}_q(x)$.

Proof. We start with the case $N = 1$. Here $\alpha_1 := [a_{0,1}, a_{1,1}, \ldots]$ is an infinite continued fraction and so α_1 is irrational, i.e., α_1 and 1 are linearly independent over $\mathbb{F}_q(x)$. Henceforth, take $N \geq 2$. Assume that $1, \alpha_1, \alpha_2, \ldots, \alpha_N$ are linearly dependent over $\mathbb{F}_q[x]$. Then there exist $A_1, A_2, \ldots, A_N, A_{N+1} \in \mathbb{F}_q[x]$ not all zero such that

$$A_{N+1} = \sum_{j=1}^{N} A_j\alpha_j. \tag{3.7}$$

Write each continued fraction α_j $(j = 1, 2, \ldots, N)$ as

$$\alpha_j = \frac{C_{n,j}}{D_{n,j}} + R_{n,j}, \tag{3.8}$$

where $C_{n,j}/D_{n,j} = [a_{0,j}, a_{1,j}, \ldots, a_{n,j}]$ is the n^{th} convergence of α_j and $R_{n,j}$ is its remainder. Note that

$$|R_{n,j}| = \left|\alpha_j - \frac{C_{n,j}}{D_{n,j}}\right| = \frac{1}{\left|a_{n+1,j}D_{n,j}^2\right|} \neq 0. \tag{3.9}$$

Substituting (3.8) into (3.7), we obtain

$$A_{N+1} = \sum_{j=1}^{N} A_j\left(\frac{C_{n,j}}{D_{n,j}} + R_{n,j}\right).$$

Multiplying both sides of the last equation by $\prod_{j=1}^{N} D_{n,j}$, we obtain

$$M_n := \left(A_{N+1} - \sum_{j=1}^{N} A_j\frac{C_{n,j}}{D_{n,j}}\right)\prod_{j=1}^{N} D_{n,j} = \prod_{j=1}^{N} D_{n,j}\sum_{j=1}^{N} A_jR_{n,j} \tag{3.10}$$

in $\mathbb{F}_q[x]$, which we next show to be nonzero. By (3.3) and (3.4), for $j \in \{1, 2, \ldots, N-1\}$, $k \in \mathbb{N}_0$, we have

$$\begin{aligned}
|D_{n_k,j+1}| &= |a_{n_k,j+1}a_{n_k-1,j+1}\cdots a_{1,j+1}| \\
&= \prod_{i=1}^{k}|a_{n_i,j+1}|\,\left|a_{n_k-1,j+1}\cdots a_{n_{k-1}+1,j+1}\right|\cdots\left|a_{n_1-1,j+1}\cdots a_{n_0+1,j+1}\right| \\
&\geq \prod_{i=0}^{k}\varepsilon_{n_i}\,|a_{n_i,j}|\prod_{i=0}^{k-1}(c_{n_i+1}c_{n_i+2}\cdots c_{n_{i+1}-1})\left|a_{n_i+1,j}a_{n_i+2,j}\cdots a_{n_{i+1}-1,j}\right|.
\end{aligned}$$

We conclude that, for $j \in \{1, 2, \ldots, N-1\}$, $k \in \mathbb{N}_0$,

$$|D_{n_k,j+1}| \geq \prod_{i=0}^{k-1} (c_{n_i+1}c_{n_i+2}\cdots c_{n_{i+1}-1}\varepsilon_{n_{i+1}})\,|D_{n_k,j}|. \tag{3.11}$$

Since $\prod_{i=0}^{\infty}(c_{n_i+1}c_{n_i+2}\cdots c_{n_{i+1}-1}\varepsilon_{n_{i+1}}) = \infty$, there exists $N_0 \in \mathbb{N}$ such that for all $j \in \{1, 2, \ldots, N-1\}$, and all $k \geq N_0$, we have

$$|D_{n_k,j+1}| > |D_{n_k,j}|. \tag{3.12}$$

Let l be the least positive integer such that $A_l \neq 0$. For $j \in \{l+1, l+2, \ldots, N\}$, $k \in \mathbb{N}_0$, by (3.4) and (3.11),

$$\begin{aligned}
\left|\frac{R_{n_k,l}}{R_{n_k,j}}\right| &= \left|\frac{a_{n_k+1,j}D^2_{n_k,j}}{a_{n_k+1,l}D^2_{n_k,l}}\right| \geq c_{n_k+1}^{(j-l)}\left|\frac{D_{n_k,j}}{D_{n_k,j-1}}\frac{D_{n_k,j-1}}{D_{n_k,j-2}}\cdots\frac{D_{n_k,l+1}}{D_{n_k,l}}\right|^2 \\
&\geq c_{n_k+1}^{j-l}\left[\prod_{i=0}^{k-1}(c_{n_i+1}c_{n_i+2}\cdots c_{n_{i+1}-1}\varepsilon_{n_{i+1}})\right]^{2(j-l)} \\
&\geq c^{j-l}\left[\prod_{i=0}^{k-1}(c_{n_i+1}c_{n_i+2}\cdots c_{n_{i+1}-1}\varepsilon_{n_{i+1}})\right]^{2(j-l)}.
\end{aligned}$$

Since $\prod_{i=0}^{\infty}(c_{n_i+1}c_{n_i+2}\cdots c_{n_{i+1}-1}\varepsilon_{n_{i+1}}) = \infty$, there exists $N_1 \geq N_0$ such that for all $j \in \{l+1, l+2, \ldots, N\}$, and all $k \geq N_1$, we have

$$\left|\frac{R_{n_k,l}}{R_{n_k,j}}\right| > \left|\frac{A_j}{A_l}\right|$$

i.e.

$$|R_{n_k,l}A_l| > |R_{n_k,j}A_j| \quad (j \in \{l+1, l+2, \ldots, N\}). \tag{3.13}$$

Then from (3.10) and (3.13), for all $k \geq N_1$,

$$|M_{n_k}| = \left|\sum_{i=l}^{N}\left(\prod_{j=1}^{N} D_{n_k,j}\right) A_i R_{n_k,i}\right| = \max_{l\leq i\leq N}\left\{\prod_{j=1}^{N}|D_{n_k,j}||A_iR_{n_k,i}|\right\} \tag{3.14}$$

$$= \prod_{j=1}^{N}|D_{n_k,j}||A_lR_{n_k,l}| \neq 0. \tag{3.15}$$

Now we prove that $|M_{n_k}| < 1$ for k sufficiently large. From (3.14), we obtain

$$
\begin{aligned}
|M_{n_k}| &= \prod_{j=1}^{N} |D_{n_k,j}||A_l R_{n_k,l}| = \frac{|A_l|\prod_{j=1}^{N}|D_{n_k,j}|}{|a_{n_k+1,l}D^2_{n_k,l}|} && \text{(using (3.9))}\\
&\le \frac{|A_l|\prod_{j=2}^{N}|D_{n_k,j}|}{c^{l-1}_{n_k+1}|a_{n_k+1,1}||D_{n_k,1}|} \le \frac{|A_l|\prod_{j=2}^{N}|D_{n_k,N}|}{c^{l-1}_{n_k+1}|a_{n_k+1,1}||D_{n_k,1}|} && \text{(by (3.4) and (3.12))}\\
&\le \frac{|A_l|}{c^{l-1}_{n_k+1}|a_{1,1}|}\left|\frac{(a_{n_k,N}a_{n_k-1,N}\cdots a_{1,N})^{N-1}}{a_{n_k+1,1}a_{n_k,1}\cdots a_{2,1}}\right|\\
&= \frac{|A_l|}{c^{l-1}_{n_k+1}|a_{1,1}|}\left|\frac{(a_{n_k,N}a_{n_k-1,N}\cdots a_{n_1,N})^{N-1}}{a_{n_k+1,1}a_{n_k,1}\cdots a_{n_1+1,1}}\right|\left|\frac{(a_{n_k-1,N}a_{n_k-2,N}\cdots a_{n_{k-1}+1,N})^{N-1}}{a_{n_k,1}a_{n_k-1,1}\cdots a_{n_{k-1}+2,1}}\right|\cdots\\
&\qquad\cdots\left|\frac{(a_{n_1-1,N}a_{n_1-2,N}\cdots a_{n_0+1,N})^{N-1}}{a_{n_1,1}a_{n_1-1,1}\cdots a_{n_0+2,1}}\right|\\
&\le \frac{|A_l|}{c^{l-1}_{n_k+1}|a_{1,1}|}\prod_{i=0}^{k-1}(d_{n_i+1}d_{n_i+2}\cdots d_{n_{i+1}-1}\delta_{n_{i+1}})^{-1} && \text{(by (3.5) and (3.6))}\\
&\le \frac{|A_l|}{c^{l-1}|a_{1,1}|}\prod_{i=0}^{k-1}(d_{n_i+1}d_{n_i+2}\cdots d_{n_{i+1}-1}\delta_{n_{i+1}})^{-1}.
\end{aligned}
$$

Since $\prod_{i=0}^{\infty}(d_{n_i+1}d_{n_i+2}\cdots d_{n_{i+1}-1}\delta_{n_{i+1}}) = \infty$, there exists $N_2 \ge N_1$ such that, for all $k \ge N_2$, $|M_{n_k}| < 1$. From this and (3.14), we obtain $0 < |M_{n_k}| < 1$ for $k \ge N_2$, which is not tenable because $M_{n_k} \in \mathbb{F}_q[x]\setminus\{0\}$. □

The criterion of Hančl follows, in the case of $\mathbf{F}$, by choosing $c_n = \varepsilon_n = 1+\varepsilon/(n\log n)$, where ε is a positive real number > 1 and $\delta_n = 1 + 1/n$.

Corollary 3.4. *Let $\varepsilon > 1$ be a real number, $N \in \mathbb{N}, \{a_{n,j}\}_{n=0}^{\infty}$ $(j = 1, 2, \ldots, N)$ be N sequences of non-constant polynomials over $\mathbb{F}_q$ such that*

$$|a_{n,j+1}| \ge |a_{n,j}|\left(1 + \frac{\varepsilon}{n\log n}\right) \tag{3.16}$$

$$|a_{n+1,1}| \ge |a_{n,N}|^{N-1}\left(1 + \frac{1}{n}\right) \tag{3.17}$$

hold for every sufficiently large positive integer n and $j = 1, 2, \ldots, N-1$. Then $\alpha_j := [a_{0,j}, a_{1,j}, \ldots]$ $(j = 1, 2, \ldots, N)$ and 1 are linearly independent over $\mathbb{F}_q(x)$.

An immediate consequence of our main result is the following particularly pleasing result which holds for both the real number and the formal series cases.

Corollary 3.5. *Let* $\alpha_1 = [a_0, a_1, a_2, \ldots]$, $\alpha_2 = [b_0, b_1, b_2, \ldots]$ *be two continued fractions whose partial quotients are subject to the conditions*

$$q|a_n| \le |b_n| \le q^{-1}|a_{n+1}|.$$

Then α_1, α_2 *and* 1 *are linearly independent.*

3.2 Algebraic independence

In this section we start with a definition.

Definition 3.6. Let E be an extension field of K and $\alpha_1, \alpha_2, \ldots, \alpha_n \in E$. Then $\alpha_1, \ldots, \alpha_n$ are **algebraically independent** over K if for all $h(y_1, y_2, \ldots, y_n) \in K[y_1, y_2, \ldots, y_n]$,

$$h(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0 \Rightarrow h \equiv 0.$$

In the real case, Laohakosol [14] gave a criterion for algebraic independence for two continued fractions.

Theorem 3.7. *Let* $A = [a_0,\ a_1,\ a_2, \ldots]$ *and* $B = [b_0,\ b_1,\ b_2, \ldots]$ *be continued fractions with positive integral partial quotients. Let* $r > 1$, (n_j) *be an increasing sequence of positive integers and let* $f(n)$ *be an integer-valued function on the non negative integers* n *with* $f(n_j) \to \infty$ *as* $j \to \infty$. *If*

$$r^{-1}a_n \ge b_n \ge a_{n-1}^{f(n-1)} \quad (n = 1, 2, 3, \ldots),$$

then A *and* B *are algebraically independent over* $\mathbb{Q}$.

Later Adams [1] extended this result as follows:

Theorem 3.8. *Let* $\alpha_1, \ldots, \alpha_N$ *be* N *real numbers. Assume that we are given integers* $p_{n,j}, q_{n,j}$ $(n = 1, 2, \ldots; 1 \le j \le N)$ *with* $q_{n,j} \to \infty$ $(n \to \infty)$, *and that , for all* $j = 2, \ldots, N$,

$$\lim_{n\to\infty} \left|\alpha_{j-1} - \frac{p_{n,j-1}}{q_{n,j-1}}\right| \Big/ \left|\alpha_j - \frac{p_{n,j}}{q_{n,j}}\right| = 0. \tag{3.18}$$

Further assume that for each $j = 1, 2, \ldots, N$ *and all positive integers* D *there is an* $N_0 = N_0(D)$ *such that, for all* $n > N_0$,

$$0 < |\alpha_j - p_{n,j}/q_{n,j}| < 1/(q_{n,1}q_{n,2}\cdots q_{n,j})^D. \tag{3.19}$$

Then $\alpha_1, \ldots, \alpha_N$ *are algebraically independent.*

A p-adic analogue of an independence criterion of Adams was established by Laohakosol [16] as follows:

Theorem 3.9. *Let* $\alpha_1, \ldots, \alpha_N$ *be* N *numbers in* $p\mathbb{Z}_p \setminus \{0\}$. *Let* $A_{n,j}, B_{n,j}$ $(n = 1, 2, \ldots; j = 1, 2, \ldots, N)$ *be rational integers with*

$$M_{n,j} = \max\{|A_{n,j}|, |B_{n,j}|\} \to \infty \quad (n \to \infty).$$

For $j = 2, \ldots, N$, *assume that*

$$\lim_{n\to\infty} \frac{|\alpha_{j-1} - A_{n,j-1}/B_{n,j-1}|_p}{|\alpha_j - A_{n,j}/B_{n,j}|_p} = 0. \tag{3.20}$$

Further assume that for each $j = 1, 2, \ldots, N$ *and all positive integers* D, *there is an* $N_0 = N_0(D)$ *such that, for all* $n > N_0$,

$$0 < |\alpha_j - A_{n,j}/B_{n,j}|_p < (M_{n,1} \cdots M_{n,j})^{-D}. \tag{3.21}$$

Then $\alpha_1, \ldots, \alpha_N$ *are algebraically independent.*

Theorem 3.10. *Let* $\alpha_1, \ldots, \alpha_N$ *be* N *numbers in* $p\mathbb{Z}_p \setminus \{0\}$ *with* ***RCFs***

$$\alpha_j = [a_{0,j}, a_{1,j}, a_{2,j}, \ldots] \quad (j = 1, \ldots, N).$$

Suppose there are constants $\tau, r > 1$ *and a function* $g(i)$ *for* $i = 1, 2, \ldots$ *with* $g(i) \to \infty$ $(i \to \infty)$, *and an increasing sequence of positive integers* $n_1 < n_2 < \ldots$ *such that for all* $n = 0, 1, \ldots$ *and* $j = 2, 3, \ldots, N$ *we have*

$$|a_{n,1}|_p \geq (\sqrt{2}|a_{n-k,1}|_p)^{\tau^k} \qquad (k = 1, 2, \ldots, N), \tag{3.22}$$

$$|a_{n,j-1}|_p \geq r|a_{n,j}|_p, \tag{3.23}$$

$$|a_{n_i,j}|_p \geq |a_{n_i-1,1}|_p^{g(i)}. \tag{3.24}$$

Then $\alpha_1, \ldots, \alpha_N$ *are algebraically independent.*

Theorem 3.11. *Let* $\alpha_1, \ldots, \alpha_N$ *be* N *numbers in* $p\mathbb{Z}_p \setminus \{0\}$ *with* ***SCFs***

$$\alpha_j = [0; c_{0,j}, d_{0,j}; c_{1,j}, d_{1,j}; c_{2,j}, d_{2,j}; \ldots] \quad (j = 1, \ldots, N).$$

Suppose there are constants $\tau, r > 1$ *and a function* $g(i)$ *for* $i = 1, 2, \ldots$ *with* $g(i) \to \infty$ $(i \to \infty)$ *and an increasing sequence of positive integers* $n_1 < n_2 < \ldots$ *such that, for* *all* $n = 0, 1, \dots,$ *and* $j = 2, 3, \dots, N$ *we have*

$$c_{n,1} \geq c_{n-k,1}^{r^k} \quad (k = 1, \dots, n), \tag{3.25}$$

$$c_{n,j-1} \geq r c_{n,j}, \tag{3.26}$$

$$c_{n_i,j} \geq c_{n_i-1,1}^{g(i)}. \tag{3.27}$$

Then $\alpha_1, \dots, \alpha_N$ *are algebraically independent.*

The following lemma can be easily proved by using Lemma 2.8 (iii).

Lemma 3.12. *Let* $\zeta = [\alpha_0, \alpha_1, \dots, \alpha_n, \dots]$, *and* $\xi = [\beta_0, \beta_1, \dots, \beta_n, \dots]$ *be continued fractions in* $\boldsymbol{F}$. *If* $|\alpha_n| \geq r_n |\beta_n|$ *for all* $n \in \mathbb{N}$, *then* $|D_n(\zeta)| \geq r_1 r_2 \cdots r_n |D_n(\xi)|$ *for all* $n \in \mathbb{N}$.

Our main result is:

Theorem 3.13. *Let* $\zeta = [\alpha_0, \alpha_1, \dots, \alpha_n, \dots]$, *and* $\xi = [\beta_0, \beta_1, \dots, \beta_n, \dots]$ *be continued fractions in* $\boldsymbol{F}$. *Assume*

(i) $|\alpha_n| \geq r_n \, |\beta_n| \quad (n \in \mathbb{N})$ *where* $r_n \in \mathbb{R}$ *with*

$$\lim_{n\to\infty} \frac{1}{(r_1 \cdots r_n)^2 r_{n+1}} = 0,$$

(ii) for any nonnegative integers A, B, *there is an increasing sequence of positive integers* (n_j) *such that*

$$\lim_{j\to\infty} \frac{|\beta_{n_j+1}|}{|\alpha_1 \cdots \alpha_{n_j}|^A \, |\beta_1 \cdots \beta_{n_j}|^B} = \infty.$$

Then ζ *and* ξ *are algebraically independent over* $\mathbb{F}_q(x)$.

Proof. Assume the result false, then there would exist a nonzero polynomial

$$P(S,T) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} w_{ij} S^i T^j \in \mathbb{F}_q[x][S,T]$$

such that $P(\zeta, \xi) = 0$. We may assume that $P(S,T)$ is one with minimum total degree $m_1 + m_2$ among such polynomials.

Consider, for fixed n,

$$P_n := P\left(\frac{C_n(\zeta)}{D_n(\zeta)}, \frac{C_n(\xi)}{D_n(\xi)}\right) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} w_{ij} \left(\frac{C_n(\zeta)}{D_n(\zeta)}\right)^i \left(\frac{C_n(\xi)}{D_n(\xi)}\right)^j.$$

Now putting $\delta_1 := \delta_1(n,\zeta) = \zeta - C_n(\zeta)/D_n(\zeta)$ and $\delta_2 := \delta_2(n,\xi) = \zeta - C_n(\xi)/D_n(\xi)$. Then we get

$$P_n = P(\zeta - \delta_1, \xi - \delta_2) = \sum_{i,j} w_{ij} (\zeta - \delta_1)^i (\xi - \delta_2)^j = w_1\delta_1 + w_2\delta_2 + O(|\delta|^2), \quad (3.28)$$

where $|\delta| = \max(|\delta_1|, |\delta_2|)$,

$$w_1 = -\sum_{i,j} i w_{ij} \zeta^{i-1}\xi^j \quad \text{and} \quad w_2 = -\sum_{i,j} j w_{ij} \zeta^i \xi^{j-1}.$$

If $w_1 = 0$ or $w_2 = 0$, then ζ and ξ would satisfy

$$\sum_{i,j} i w_{ij} S^{i-1} T^j = 0 \quad \text{or} \quad \sum_{i,j} j w_{ij} S^i T^{j-1} = 0,$$

whose total degree is lower than $m_1 + m_2$. Then w_1, w_2 are not zero.

By Lemma 3.12, we have

$$\left|\frac{\delta_1}{\delta_2}\right| = \left|\frac{D_n(\xi)D_{n+1}(\xi)}{D_n(\zeta)D_{n+1}(\zeta)}\right| \leq \frac{1}{(r_1\cdots r_n)^2 r_{n+1}} \to 0 \ \ (n \to \infty).$$

Consequently,

$$\left|\frac{O(|\delta|^2)}{\delta_2}\right| \to 0$$

as $n \to \infty$. Then there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$,

$$\left|w_1\frac{\delta_1}{\delta_2}\right|, \left|\frac{O(|\delta|^2)}{\delta_2}\right| < |w_2|$$

By (3.28), for all $n \geq N_0$, we get

$$\left|\frac{P_n}{\delta_2}\right| = |w_2| \neq 0.$$

By this and Lemma 2.8 (iii), for all $n \geq N_0$, we have

$$\frac{1}{|\alpha_1\cdots\alpha_n|^{m_1}|\beta_1\cdots\beta_n|^{m_2}} = \frac{1}{|D_n(\zeta)|^{m_1}|D_n(\xi)|^{m_2}} \leq |P_n| = \frac{|w_2|}{|\beta_1\cdots\beta_n|^2|\beta_{n+1}|},$$

i.e.

$$\frac{|\beta_{n+1}|}{|\alpha_1\cdots\alpha_n|^{m_1}|\beta_1\cdots\beta_n|^{m_2-2}} \leq |w_2|$$

which is a contradiction. □

Corollary 3.14. *Let $\zeta = [\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots]$ and $\xi = [\beta_0, \beta_1, \ldots, \beta_n, \ldots]$ be continued fractions in $\boldsymbol{F}$. Let r be a real number such that $r > 1$, (n_j) be an increasing sequence of* *positive integers and let* $f(n)$ *be an integer-valued function with* $f(n_j) \to \infty \quad (j \to \infty)$. *If*

$$r^{-1}|\alpha_n| \geq |\beta_n| \geq |\alpha_{n-1}|^{f(n-1)} \quad (n = 1, 2, 3, \ldots),$$

then ζ *and* ξ *are algebraically independent over* $\mathbb{F}_q(x)$.

A general Liouville type algebraic criterion:

Theorem 3.15. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_N \in \boldsymbol{F}$. *Assume that we are given polynomials* $C_{n,j}, D_{n,j}$ $(n = 1, 2, 3, \ldots; 1 \leq j \leq N)$ *with* $|D_{n,j}| \to \infty$ $(n \to \infty)$. *Assume that, for all* $j = 2, \ldots, N$,

$$\lim_{n\to\infty} \left|\alpha_{j-1} - \frac{C_{n,j-1}}{D_{n,j-1}}\right| \Big/ \left|\alpha_j - \frac{C_{n,j}}{D_{n,j}}\right| = 0. \tag{3.29}$$

Further assume that for each $j = 1, 2, \ldots, N$ *and all positive integers* M *there is an* $N_0 = N_0(M)$ *such that, for all* $n \geq N_0$,

$$0 < \left|\alpha_j - \frac{C_{n,j}}{D_{n,j}}\right| < \frac{1}{|D_{n,1}D_{n,2}\cdots D_{n,j}|^M}. \tag{3.30}$$

Then $\alpha_1, \ldots, \alpha_N$ *are algebraically independent over* $\mathbb{F}_q(x)$.

Proof. We proceed by induction on N.

For $N = 1$, suppose that α_1 is algebraic over $\mathbb{F}_q(x)$. If α_1 is algebraic of degree 1, then $\alpha_1 \in \mathbb{F}_q(x)$, say A/B. By (3.30), there is an $N_0 = N_0(2)$ such that, for all $n \geq N_0$,

$$0 < \left|\frac{A}{B} - \frac{C_{n,1}}{D_{n,1}}\right| < \frac{1}{|D_{n,1}|^2}$$

so that, for all $n \geq N_0$,

$$\frac{1}{|BD_{n,1}|} \leq \left|\frac{AD_{n,1} - BC_{n,1}}{BD_{n,1}}\right| < \frac{1}{|D_{n,1}|^2}.$$

Now we have $|D_{n,1}| < |B|$ which is a contradiction since $|D_n| \to \infty \quad (n \to \infty)$. If α_1 is algebraic of degree $m \geq 1$, then, by Mahler [21], there is a constant $K > 0$ such that for all $n \in \mathbb{N}$,

$$\left|\alpha_1 - \frac{C_{n,1}}{D_{n,1}}\right| \geq \frac{K}{|D_{n,1}|^m}.$$

By (3.30), there is an $N_0 = N_0(m+1)$ such that, for all $n \geq N_0$,

$$\frac{K}{|D_{n,1}|^m} \leq \left|\alpha_1 - \frac{C_{n,1}}{D_{n,1}}\right| < \frac{1}{|D_{n,1}|^{m+1}} \quad \to 0 \ \ (n \to \infty.)$$

which is a contradiction. We conclude that α_1 is thus transcendental, and we are done in this case.

Now consider $N > 1$. Assume that it is true for $N-1$. Suppose that the result were false for N, then there would exist a nonzero polynomial $f(T_1, T_2, \ldots, T_N)$ with integral coefficients of minimal total degree such that $f(\alpha_1, \alpha_2, \ldots, \alpha_N) = 0$. Expanding the polynomial f about $\alpha_1, \ldots, \alpha_N$, we get

$$f(T_1, T_2, \ldots, T_N) = \sum h_{(\nu)}(T_1 - \alpha_1)^{\nu_1} \cdots (T_N - \alpha_N)^{\nu_N},$$

where $(\nu) = (\nu_1, \nu_2, \ldots, \nu_N)$, and

$$h_{(\nu)} = \frac{1}{(\nu_1 + \nu_+ \ldots + \nu_N)!}\left(\frac{\partial^{\nu_1+\nu_2+\ldots+\nu_N}}{\partial T_1^{\nu_1} \partial T_2^{\nu_2} \cdots \partial T_N^{\nu_N}} f(\alpha_1, \alpha_2, \ldots, \alpha_N)\right).$$

Clearly, $h_{(0,\ldots,0)} = f(\alpha_1, \alpha_2, \ldots, \alpha_N) = 0$. For $i = 1, 2, \ldots, N$, set $H_i = h_{(0,\ldots,0,\underbrace{1}_{ith},0,\ldots,0)}$. Let $\mathcal{H}_N(T_1, \ldots, T_N) := \frac{\partial}{\partial T_N} f(T_1, \ldots, T_N)$. We observe that T_N occurs in f. Then $\mathcal{H}_N \neq 0$ and $H_N = \mathcal{H}_N(\alpha_1, \alpha_2, \ldots, \alpha_N)$. We claim that $H_N \neq 0$. Suppose not. If T_N occurs in $\mathcal{H}_N(T_1, \ldots, T_N)$, then $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a root of a nonzero polynomial of smaller degree than f, which is a contradiction. Thus T_N does not occur in $\mathcal{H}_N(T_1, \ldots, T_N)$. It means $\alpha_1, \alpha_2, \ldots, \alpha_{N-1}$ are algebraically dependent, contradicting the induction hypothesis, and the claim is verified.

Now putting

$$\delta_j(n) = \frac{C_{n,j}}{D_{n,j}} - \alpha_j \quad (j = 1, 2, \ldots, N).$$

By (3.30) and n sufficiently large, we get $|\delta_n(n)| \neq 0$. Consider

$$\begin{aligned}
&f\left(\frac{C_{n,1}}{D_{n,1}}, \frac{C_{n,2}}{D_{n,2}}, \ldots, \frac{C_{n,N}}{D_{n,N}}\right) \\
&= \sum h_{(\nu)}(\delta_1(n))^{\nu_1} \cdots (\delta_N(n))^{\nu_N} \\
&= \sum_{i=1}^{N} H_i(\delta_i(n)) + \sum_{h_{(\nu)} \neq H_i, (\nu) \neq 0} h_{(\nu)}(\delta_1(n))^{\nu_1} \cdots (\delta_N(n))^{\nu_N} \\
&= \delta_N(n)\left\{\left(H_1 \frac{\delta_1(n)}{\delta_N(n)} + \ldots + H_{n-1}\frac{\delta_{N-1}(n)}{\delta_N(n)} + H_N\right) + O(|\delta_N(n)|)\right\},
\end{aligned}$$

for n sufficiently large. Since

$$\begin{aligned}
&\left|H_1 \frac{\delta_1(n)}{\delta_N(n)} + \ldots + H_{n-1}\frac{\delta_{N-1}(n)}{\delta_N(n)} + O(|\delta_N(n)|)\right| \\
&\leq \max\left\{\left|H_1\frac{\delta_1(n)}{\delta_N(n)}\right|, \ldots, \left|H_{n-1}\frac{\delta_{N-1}(n)}{\delta_N(n)}\right|, O(|\delta_N(n)|)\right\} \\
&\to 0 \quad (n \to \infty),
\end{aligned}$$

we have

$$\max\left\{\left|H_1\frac{\delta_1(n)}{\delta_N(n)}\right|,\ldots,\left|H_{n-1}\frac{\delta_{N-1}(n)}{\delta_N(n)}\right|,O(|\delta_N(n)|)\right\}<|H_N|$$

and so

$$\left|f\left(\frac{C_{n,1}}{D_{n,1}},\frac{C_{n,2}}{D_{n,2}},\ldots,\frac{C_{n,N}}{D_{n,N}}\right)\right|=|\delta_N(n)H_N|\neq 0$$

for n sufficiently large. Let $m_1,m_2,\ldots,m_N$ denote the degrees of f in $T_1,T_2,\ldots,T_N$, respectively. Then, for n sufficiently large,

$$\frac{1}{\left|D_{n,1}^{m_1}\cdots D_{n,N}^{m_N}\right|}\leq\left|f\left(\frac{C_{n,1}}{D_{n,1}},\frac{C_{n,2}}{D_{n,2}},\ldots,\frac{C_{n,N}}{D_{n,N}}\right)\right|=|\delta_N(n)H_N|=|H_N|\left|\alpha_N-\frac{C_{n,N}}{D_{n,N}}\right|.$$

Choose $M=\max\{m_1,m_2,\ldots,m_N\}+1$. By (3.30), there exists $N_1=N_1(M)$ such that for all $n\geq N_1$,

$$\frac{1}{\left|D_{n,1}^{m_1}\cdots D_{n,N}^{m_N}\right|}\leq|H_N|\left|\alpha_N-\frac{C_{n,N}}{D_{n,N}}\right|<\frac{|H_N|}{|D_{n,1}\cdots D_{n,N}|^M}$$

i.e.

$$|H_N|>\left|D_{n,1}^{M-m_1}\cdots D_{n,N}^{M-m_N}\right|\to\infty\quad(n\to\infty),$$

which is a contradiction. Hence $\alpha_1,\ldots,\alpha_N$ are algebraically independent. □

We apply Theorem 3.15 to construct a class of algebraically independent of Liouville type.

Theorem 3.16. *For $j=1,2,\ldots,N$, let $\alpha_j=[a_{0,j},a_{1,j},\ldots]$ be continued fractions in* $\boldsymbol{F}$, $C_{n,j}/D_{n,j}=[a_{0,j},a_{1,j},\ldots,a_{n,j}]$. *Let $f_1(i)$, $f_2(i)$ be the integer-valued function with $f_1(i),f_2(i)\to\infty\quad(i\to\infty)$. Assume that there is an increasing sequence of positive integers (n_i) such that for all $i=1,2,\ldots$,*

$$|a_{n_i+1,j}|\geq|D_{n_i,1}|^{f_1(i)}\qquad(j=1,2,\ldots,N),\tag{3.31}$$

$$|D_{n,j-1}|\geq f_2(i)\,|D_{n,j}|\qquad(j=2,3,\ldots,N;n=n_i,n_i+1).\tag{3.32}$$

Then $\alpha_1,\ldots,\alpha_N$ are algebraically independent over $\mathbb{F}_q(x)$.

Proof. For each $j=2,3,\ldots,N$ we have

$$\left|\frac{\alpha_{j-1}-C_{n_i,j-1}/D_{n_i,j-1}}{\alpha_j-C_{n_i,j}/D_{n_i,j}}\right|=\left|\frac{D_{n_i+1,j}D_{n_i,j}}{D_{n_i+1,j-1}D_{n_i,j-1}}\right|\leq\frac{1}{f_2(i)^2}\to 0\ \ (i\to\infty).$$

Now let $j \in \{2,3,\dots,N\}$ and $M \in \mathbb{N}$. To verify (3.30) of Theorem 3.15 we use

$$\left|\alpha_j - \frac{C_{n_i,j}}{D_{n_i,j}}\right| = \frac{1}{\left|D^2_{n_i,j} a_{n_i+1,j}\right|}.$$

Since $f_1(i) \to \infty \quad (i \to \infty)$, $\exists N_0 = N_0(M) \in \mathbb{N}$ such that, for all $i \geq N_0$, $f_1(i) \geq jM \geq 1$. From this and (3.31), (3.32), we have

$$|D_{n_i,1}D_{n_i,2}\cdots D_{n_i,j}|^M \leq |D_{n_i,1}|^{jM} \leq |D_{n_i,1}|^{f_1(i)} \leq |a_{n_i+1,j}|\,.$$

Hence for $i \geq N_0$, we get

$$\left|\alpha_j - \frac{C_{n_i,j}}{D_{n_i,j}}\right| \leq \frac{1}{|D_{n_i,j}|^2\,|D_{n_i,1}D_{n_i,2}\cdots D_{n_i,j}|^M} < \frac{1}{|D_{n_i,1}D_{n_i,2}\cdots D_{n_i,j}|^M}.$$

By Theorem 3.15 we have $\alpha_1,\dots,\alpha_N$ are algebraically independent over $\mathbb{F}_q(x)$. □

Moreover we obtain the following theorem.

Corollary 3.17. *Let $\alpha_j = [a_{0,j}, a_{1,j},\dots,a_{n,j},\dots] \in \boldsymbol{F}$ and $C_{n,j}/D_{n,j}$ $(n = 1,2,3,\dots; 1 \leq j \leq N)$ be the continued fractions and their convergent, respectively. Suppose there are constants $\tau, r > 1$ and a function $g(i), i \in \mathbb{N}$, with $g(i) \to \infty \quad (i \to \infty)$ and an increasing sequence of positive integers $n_1 < n_2 < \dots$ such that, for all $j = 1,\dots,N$, we have*

$$|a_{n_i+1,1}| \geq |a_{n_i-k,1}|^{\tau^{k-1}} \quad (k = 1,2,\dots,n_i-1), \tag{3.33}$$

$$|a_{n,j-1}| \geq r\,|a_{n,j}| \quad (n \in \mathbb{N}), \tag{3.34}$$

$$|a_{n_i+1,j}| \geq |a_{n_i,1}|^{g(i)}. \tag{3.35}$$

Then $\alpha_1, \alpha_2,\dots,\alpha_N$ are algebraically independent over $\mathbb{F}_q(x)$.

Proof. By Lemma 2.8 (iv) and (3.34) we have

$$|D_{n,j-1}| = |a_{n,j-1}a_{n-1,j-1}\cdots a_{1,j-1}| \geq r^n\,|a_{n,j}a_{n-1,j}\cdots a_{1,j}| = r^n\,|D_{n,j}|\,.$$

Choose $f_2(i) = r^{n_i}$. Then we have (3.31) of Theorem 3.16. By (3.33),we have

$$|D_{n_i,1}| = \left|\prod_{k=1}^{n_i} a_{n_i-(k-1),1}\right| \leq \left|\prod_{k=1}^{n_i} a_{n_i,1}^{1/\tau^{k-1}}\right| = |a_{n_i,1}|^{\sum_{k=1}^{n_i}\frac{1}{\tau^{k-1}}} \leq |a_{n_i,1}|^{\tau/(\tau-1)}.$$

From this and (3.35), we get

$$|D_{n_i,1}|^{g(i)\tau/(\tau-1)} \leq |a_{n_i,1}|^{g(i)} \leq |a_{n_i+1,j}|.$$

Choose $f_1(i) = g(i)\tau/(\tau-1)$. Then we have (3.32) of Theorem 3.16. □

As an application, we shall prove

Corollary 3.18. *Let $(k_\nu)_{\nu=1}^{\infty}$ be a strictly increasing sequence of positive integers such that $\limsup_{\nu\to\infty} k_{\nu+1}/k_\nu = \infty$. Let $g_1, g_2, \ldots, g_N \in \boldsymbol{F}$ be such that $|g_N| < |g_{N-1}| < \ldots < |g_1|$. Set $\alpha_j = \sum_{\nu=1}^{\infty} g_j^{-k_\nu}$ $(j = 1, 2, \ldots, N)$. Then $\alpha_1, \ldots, \alpha_N$ are algebraically independent over $\mathbb{F}_q(x)$.*

Proof. Let $1 \le j \le N$ and $M > 0$. Set $D_{n,j} = g_j^{k_n}$. Then

$$\left|\frac{1}{g_j^{k_{n+1}}}\right| = \left|\alpha_j - \frac{C_{n,j}}{D_{n,j}}\right|. \tag{3.36}$$

Since $\limsup_{\nu\to\infty} k_{\nu+1}/k_\nu = \infty$, there is $J_0 = J_0(M)$ such that, for all $l \ge J_0$,

$$|g_1 \cdots g_n|^M \le |g_j^{(k_{n_l+1}/k_{n_l})}| \quad (j = 1, 2, \ldots, N).$$

Then, for all $l \ge J_0$, we obtain

$$\frac{1}{g_j^{k_{n_l+1}}} \le \frac{1}{|g_1 \cdots g_n|^{k_{n_l} M}}.$$

By (3.36) we conclude that, for all $l \ge J_0$,

$$0 < \left|\alpha_j - \frac{C_{n_l,j}}{D_{n_l,j}}\right| = \left|\frac{1}{g_j^{k_{n_l+1}}}\right| \le \frac{1}{|g_1 \cdots g_N|^{k_{n_l} M}} = \frac{1}{|D_{n_l,1} \cdots D_{n_l,j}|^M}.$$

By (3.36) we have, for all $j = 2, \ldots, N$,

$$\left|\frac{\alpha_{j-1} - C_{n_l,j-1}/D_{n_l,j-1}}{\alpha_j - C_{n_l,j}/D_{n_l,j}}\right| = \left|\frac{g_j^{K_{n_l+1}}}{g_{j-1}^{K_{n_l+1}}}\right| \to 0 \quad (l \to \infty).$$

By Theorem 3.15, $\alpha_1, \ldots, \alpha_N$ are algebraically independent over $\mathbb{F}_q(x)$. □