Chapter 4

Derivation of the Clocked SE Using the Normal SE

4.1 Straightforward Substitution

In this section, we present the direct derivation of the clocked SE from the normal SE by straight forward substitution. Let us consider the traversal wave function of Sokolovski which gives the probability amplitude for the particle in x to spending in $\Omega \equiv [a, b]$ prior to time t, a net time duration τ,

$$\psi(x, t\,|\tau) = \int dx_0 \int D[x(t)]\delta(\tau - t^{cl}_{ab}[x])e^{iS[x(t)]/\hbar}\psi_0(x_0) \tag{4.1}$$

where δ is the *Dirac-delta* function, S is an action of the particle and $\psi_0(x_0)$ is the initial wave function at $t = 0$. Summation of all values of τ, the wave function $\psi(x, t)$ must be restored,

$$\psi(x, t) = \int_0^t d\tau\psi(x, t\,|\tau)\,. \tag{4.2}$$

Exactly the restored wave function $\psi(x, t)$ must satisfy the normal SE

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x, t) + V(x)\psi(x, t) - i\hbar\frac{\partial}{\partial t}\psi(x, t) = 0. \tag{4.3}$$

We substitute Eq.(4.2) into Eq.(4.3) to obtain

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\int_0^t d\tau\psi(x, t|\tau) - i\hbar\frac{\partial}{\partial t}\int_0^t d\tau\psi(x, t|\tau) = 0. \tag{4.4}$$

Now a theorem of differential calculus (See Appendix B) is used. Let $\Phi(\lambda) = \int_{v(\lambda)}^{u(\lambda)} dx f(x, \lambda)$ where $u(\lambda)$ and $v(\lambda)$ are differentiable functions in a closed interval

(λ_0, λ_1); $f(x, \lambda)$ and $f'(x, \lambda)$ are continuous in the region $\lambda_0 \leq \lambda \leq \lambda_1$, $u \leq x \leq v$, then

$$\frac{\partial}{\partial\lambda}\Phi(\lambda) = \frac{\partial u}{\partial\lambda}f(x, u) - \frac{\partial v}{\partial\lambda}f(x, v) + \int_{v(\lambda)}^{u(\lambda)} dx \frac{\partial}{\partial\lambda}f(x, \lambda). \tag{4.5}$$

By the properties of the traversal wave function, $\psi(x, t|\tau)$ and $\frac{\partial}{\partial t}\psi(x, t|\tau)$ are continuous at time t (see Appendix C). Following the theorem of differential calculus, the last term of Eq.(4.4) becomes

$$\begin{aligned} -i\hbar\frac{\partial}{\partial t}\int_0^t d\tau\psi(x, t|\tau) &= -i\hbar(\frac{\partial t}{\partial t})\psi(x, t|\tau = t) + i\hbar(\frac{\partial 0}{\partial t})\psi(x, t|\tau = 0) \\ &\quad - \int_0^t dx i\hbar(\frac{\partial}{\partial t})_\tau\psi(x, t|\tau) \\ &= -i\hbar\psi(x, t|\tau = t) - \int_0^t dx i\hbar(\frac{\partial}{\partial t})_\tau\psi(x, t|\tau) \end{aligned}$$

where $(\frac{\partial}{\partial t})_\tau$ is the partial differential with respect to t at constant τ.

So Eq.(4.4) can be written as

$$\int_0^t d\tau\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x, t|\tau) + V(x)\psi(x, t|\tau) - i\hbar(\frac{\partial}{\partial t})_\tau\psi(x, t|\tau)\right] = i\hbar\psi(x, t|\tau = t). \tag{4.6}$$

Now we consider the exact integral

$$\int_0^t d\tau\frac{\partial}{\partial\tau}\psi(x, t|\tau) = \psi(x, t|\tau = t) - \psi(x, t|\tau = 0). \tag{4.7}$$

Then, multiplying Eq.(4.7) by $-i\hbar\Theta_{ab}[x]$, we have

$$-\int_0^t d\tau\Theta_{ab}[x]i\hbar\frac{\partial}{\partial\tau}\psi(x, t|\tau) = -i\hbar\Theta_{ab}[x]\psi(x, t|\tau = t) + i\hbar\Theta_{ab}[x]\psi(x, t|\tau = 0). \tag{4.8}$$

Adding Eq.(4.8) into Eq.(4.6), we obtain

$$\int_0^t d\tau\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x, t|\tau) + \int_0^t d\tau\left[-i\hbar(\frac{\partial}{\partial t})_\tau - i\hbar\Theta_{ab}[x]\frac{\partial}{\partial\tau}\right]\psi(x, t|\tau)$$

$$= i\hbar[1-\Theta_{ab}[x]]\psi(x,t|\tau=t)+i\hbar\Theta_{ab}[x]\psi(x,t|\tau=0). \qquad (4.9)$$

Using the meaning of the traversal wave function Eq.(4.32). we obtain the boundary conditions at $\tau = 0$ and $\tau = t$.

(*i*) $\psi(x,t|\tau = t)$ is the probability amplitude for the particle at x to have spent in the given region $\Omega \equiv [a,b]$ prior to time t, a net time duration $\tau = t$. Then it is only possible that the particle is in Ω for all t. If the particle is outside Ω, it is impossible for the particle to have a net time duration $\tau = t$ in Ω. For the traversal wave function $\psi(x,t|\tau = t)$, it is impossible to find the particle outside Ω.

(*ii*) $\psi(x,t|\tau = 0)$ is the probability amplitude for the particle at x not to have been in Ω prior to time t. If the particle is inside Ω it is impossible for the particle to have a net time duration $\tau = 0$. For the traversal wave function the $\psi(x,t|\tau = 0)$, it is impossible to find the particle inside Ω.

So we obtain the boundary condition for $\psi(x,t|\tau = t)$ and $\psi(x,t|\tau = 0)$ as

$$\begin{aligned}\psi(x,t|\tau &= t) = 0 \quad \text{for } x < a \text{ or } x > b \qquad (4.10)\\ \psi(x,t|\tau &= 0) = 0 \quad \text{for} \quad a \le x \le b.\end{aligned}$$

By using the boundary condition Eq.(4.10) and the definition of $\Theta_{ab}[x]$, the right hand side of Eq.(4.9) is zero over the complete region of x.

$$\begin{aligned}[1-\Theta_{ab}[x]]\psi(x,t|\tau &= t) = 0, \qquad (4.11)\\ \Theta_{ab}[x]\psi(x,t|\tau &= 0) = 0.\end{aligned}$$

Then we obtain

$$\int_0^t d\tau \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - V(x) + i\hbar\Theta_{ab}[x](\frac{\partial}{\partial\tau})_t + i\hbar(\frac{\partial}{\partial t})_\tau\right)\psi(x,t|\tau) = 0. \qquad (4.12)$$

This equation is true for all possible value of t, therefore it implies

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) - i\hbar\Theta_{ab}[x](\frac{\partial}{\partial\tau})_t - i\hbar(\frac{\partial}{\partial t})_\tau\right)\psi(x,t|\tau) = 0. \qquad (4.13)$$

Eq.(4.13) is the clocked SE derived by starting with the SE and using the boundary conditions in Eq.(4.10). Therefore the SE appears to be capable of analyzing such physical quantities and aspects of quantum motion that can be discussed by a theory based exclusively on Feynman's quantum mechanics. This is in contrast to what has been claimed by Sokolovski [19].

4.2 The Reduction of the Composite System

We consider the composite system for the measurement problem. If we want to know *how long a particle spends in a given region of space* then it is necessary to use an apparatus which has some interaction with the observed particle. Therefore, we need to extend consideration of a single system to a combined system. This is divided into three parts: *i*) the observed system, *ii*) the apparatus system or the pointer and *iii*) the actual observer. Therefore, we have chosen the measurement system and the coupling interaction in the form of a weak measurement for deriving the clocked SE. The weak measurement interaction coupling operator is written as

$$\mathbf{V}_{int}(t) = g(t)\mathbf{P}\mathbf{A} \tag{4.14}$$

where $g(t) = g$, for $t \geq 0$ and 0 otherwise. $\mathbf{P}$ is the generator of translation (momentum operator) for the apparatus and $\mathbf{A}$ is an operator that we wish to measure, acting on an observed state. This form of weak measurement was proposed by von Neumann [25] and developed by Aharonov, Albert and Vaidman [24]. For measuring the traversal time, it is necessary to turn on the interaction for a time interval t. We use the operator $\mathbf{A} = \Theta_{ab}[\mathbf{x}] - \Lambda(t)$ where $\Lambda(t)$ is an arbitrary function of time t. The coupling interaction then takes the form

$$\mathbf{V}_{int}(t) = g(t)\mathbf{P}(\Theta_{\mathbf{ab}}[\mathbf{x}] - \Lambda(t)). \tag{4.15}$$

The Hamiltonian of the actual observer has coupling in the form of a weak measurement with the apparatus system, where $\Lambda(t)$ is the variable of the actual observer, incorporated into coupling interaction. It implies that we assume the actual observer to have a very large mass so that the kinetic energy term of the observer can be neglected. The quantum variable $\mathbf{\Lambda}$ is then changed to the classical variable $\Lambda(t)$. If the state of the actual observer can be represented by a wave packet narrow enough in Λ space, the variable Λ will move in an essentially classical way, very nearly following the definite function $\Lambda(t)$. This method was discussed by Aharonov and Bohm [26] in the process of time measurement by considering the time-measuring variable. The classical variable $\Lambda(t)$ will be used to determine the traversal time after we reduce the SE of the whole system. In this way, we can derive the clocked SE and the traversal wave function from the SE. Let us first derive the clocked SE. Suppose the initial state before $t = 0$ is known as the (tensor) product:

$$|\Psi_0(t < 0)\rangle = |\phi_0\rangle \otimes |\psi_0\rangle \tag{4.16}$$

where $|\psi_0\rangle$ and $|\phi_0\rangle$ are the initial states of the observed and of the apparatus system, respectively. The total Hamiltonian for the whole system is

$$\mathbf{H}_{total} = \mathbf{H}_0 + \mathbf{H}_A + \mathbf{V}_{int}(t) \tag{4.17}$$

where $\mathbf{H}_0$ represents the Hamiltonian of the observed particle. $\mathbf{H}_A$ represents that of the apparatus particle. By following von Neumann, we assume that $\mathbf{H}_A$ and $\mathbf{P}$ commute, $[\mathbf{H}_A, \mathbf{P}] = 0$, so that $\mathbf{H}_A$ may be written as

$$\mathbf{H}_A = \frac{\mathbf{P}^2}{2M} \tag{4.18}$$

where M is mass of the apparatus particle. The total Hamiltonian is time dependent and the $\mathbf{H}_{total}$'s at different time commute, so that the time-evolution operator can be written as

$$\mathbf{U}(t,0) = \exp\{-\frac{i}{\hbar}\int_0^t(\mathbf{H}_0+\mathbf{H}_A+\mathbf{V}_{int}(t'))dt'\} \tag{4.19}$$
$$= e^{-\frac{i}{\hbar}\int_0^t(\mathbf{H}_0+\mathbf{H}_A+g\mathbf{P}(\Theta_{ab}[\mathbf{x}]-\Lambda(t')))dt'}.$$

The total state evolves with time as

$$|\Psi_{total}(t)\rangle = \mathbf{U}(t,0)\,|\phi_0\rangle\otimes|\psi_0\rangle \tag{4.20}$$
$$= e^{-\frac{i}{\hbar}\int_0^t(\mathbf{H}_0+\mathbf{H}_A+g\mathbf{P}(\Theta_{ab}[\mathbf{x}]-\Lambda(t')))dt'}\,|\phi_0\rangle\otimes|\psi_0\rangle\,.$$

Let p' be the eigenvalue of the momentum operator $\mathbf{P}$ of the apparatus system with the eigen kets $|p'\rangle$

$$\mathbf{P}\,|p'\rangle = p'\,|p'\rangle\,. \tag{4.21}$$

We can express the total state at t. Eq.(4.20) by using the completeness relation $I=\int dp'\,|p'\rangle\,\langle p'|$. We obtain

$$|\Psi_{total}(t)\rangle = \int dp'\,|p'\rangle\,\langle p'|\,\mathbf{U}(t,0)\,|\phi_0\rangle\otimes|\psi_0\rangle \tag{4.22}$$
$$= \int dp'\,|p'\rangle\,\langle p'|\,e^{-\frac{i}{\hbar}\int_0^t(\mathbf{H}_0+\mathbf{H}_A+g\mathbf{P}(\Theta_{ab}[\mathbf{x}]-\Lambda(t')))dt'}\,|\phi_0\rangle\otimes|\psi_0\rangle\,.$$

By using the Baker-Hausdorf lemma [43]. $e^{(\mathbf{A}+\mathbf{B})}=e^{\mathbf{A}}e^{\mathbf{B}}e^{-\frac{1}{2}[\mathbf{A},\mathbf{B}]}$ for $[[\mathbf{A},\mathbf{B}],\mathbf{A}] = [[\mathbf{A},\mathbf{B}],\mathbf{B}] = 0$ and $[\mathbf{H}_A,\mathbf{H}_0+g\mathbf{P}(\Theta_{ab}[\mathbf{x}]-\Lambda(t'))]=0$ and the completeness relation $I=\int dp'\,|p'\rangle\,\langle p'|$. Eq.(4.22) can be written as

$$|\Psi_{total}(t)\rangle = \int dp'\,|p'\rangle\,\langle p'|\,e^{-\frac{i}{\hbar}\int_0^t\mathbf{H}_A dt'}e^{-\frac{i}{\hbar}\int_0^t(\mathbf{H}_0+g\mathbf{P}(\Theta_{ab}[\mathbf{x}]-\Lambda(t')))dt'}\,|\phi_0\rangle\otimes|\psi_0\rangle$$
$$= \int dp'\,|\phi(p'),t\rangle\otimes|\psi_{p'},\tau,t\rangle\,. \tag{4.23}$$

where $|\psi_{p'},\tau,t\rangle$ is given by

$$|\psi_{p'},\tau,t\rangle = e^{\frac{i}{\hbar}gp'\tau}e^{-(\frac{i}{\hbar})\int_0^t(\mathbf{H}_0+gp'\Theta_{ab}[\mathbf{x}])dt'}\,|\psi_0\rangle \tag{4.24}$$

and the phase factor is defined as

$$\tau = \int_0^t \Lambda(t')dt'. \qquad (4.25)$$

We multiply $e^{\frac{i}{\hbar}gp'\tau}$ with the time evolution of the observed system, so the state of the apparatus in time is of the form

$$|\phi(p'),t\rangle = e^{-\frac{i}{\hbar}\frac{p'^2}{2M}t}\,|p'\rangle\,\langle p'|\,\phi_0\rangle. \qquad (4.26)$$

Eq. (4.26), represents an entangled state of the whole system. The variable $\Lambda(t)$ defines the phase factor for the apparatus state but we can choose to associate this phase factor with the observed state. Then $|\psi_{p'},\tau,t\rangle$ is dependent on the arbitrary function τ. Here $|\psi_{p'},\tau,t\rangle$ is regarded merely as the coefficient of the expansion of the total state into a series of pointer states $|\phi_0(p'),t\rangle$. Usually, the total state in the form Eq.(4.23) satisfies the SE for $t > 0$,

$$\mathbf{H}_{total}\,|\Psi_{total}(t)\rangle = i\hbar\frac{\partial}{\partial t}\,|\Psi_{total}(t)\rangle\,. \qquad (4.27)$$

We define the SE for the observed particle, which contains the effects of the measurement, by eliminating the apparatus system using the identity

$$\frac{g}{2\pi\hbar}\int dp\frac{\langle\phi(p),t|\,\mathbf{H}_{total} - i\hbar\frac{\partial}{\partial t}\,|\Psi_{total}(t)\rangle}{\langle\phi(p),t|\,\phi(p),t\rangle} = 0. \qquad (4.28)$$

By using Eq.(4.28), we can reduce the SE for the whole system into an effective SE for the observed particle. Straight forward substitution of the total state in Eq.(4.23) into Eq.(4.28) and differentiation by t leads to (see in Appendix D)

$$\frac{g}{2\pi\hbar}\int dp\left\{\mathbf{H}_0 + gp\Theta_{ab}[\mathrm{x}] - i\hbar(\frac{\partial}{\partial t})_\tau\right\}|\psi_p,\tau,t\rangle = 0. \qquad (4.29)$$

Without loss of generality we integrate over all $p-space$ and obtain

$$\left\{\mathbf{H}_0 - i\hbar\frac{\partial}{\partial\tau}\Theta_{ab}[\mathrm{x}] - i\hbar(\frac{\partial}{\partial t})_\tau\right\}|\psi,t|\,\tau\rangle = 0 \qquad (4.30)$$

where $|\psi, t|\,\tau\rangle = \frac{g}{2\pi\hbar}\int dp\; e^{\frac{i}{\hbar}gp\tau} e^{-\left(\frac{i}{\hbar}\right)\int\limits_0^t (\mathbf{H}_0 + gp\Theta_{ab}[\mathbf{x}])dt'} |\psi_0\rangle$ and the effective wave function can be written as

$$\begin{aligned}
\psi(x,t\,|\tau) &= \frac{g}{2\pi\hbar}\int dp \langle x\,|\psi_p, \tau, t\rangle \qquad (4.31)\\
&= \frac{g}{2\pi\hbar}\int dp\,\langle x|\, e^{\frac{i}{\hbar}gp\tau} e^{-\left(\frac{i}{\hbar}\right)\int\limits_0^t (\mathbf{H}_0 + gp\Theta_{ab}[\mathbf{x}])dt'} |\psi_0\rangle\,.
\end{aligned}$$

This is equivalent to Sokolovski's definition of the traversal wave function. We can show by inserting the completeness relation of the coordinate of the observed system, $I = \int dx_k\, |x_k\rangle \langle x_k|$, in each time segment, say between t_k and t_{k+1}, leads to

$$\psi(x,t\,|\tau) = \int dx_0 \int D[x(t)]\delta(\tau - t^{cl}_{ab}[x])e^{iS[x(t)]/\hbar}\psi_0(x_0), \qquad (4.32)$$

with $S = \int\limits_0^t \mathbf{H}_0 dt'$ being the action of the observed system and $\psi_0(x_0)$ being the initial wave function of the observed system. Eq.(4.30) is the clocked Schroedinger equation and Eq.(4.32) is the traversal wave function, so that we have shown that it is possible to derive the clocked SE and the traversal wave function by directly using the SE for the composite system. After the SE for the whole system is reduced to the effective SE for the observed particle, then the phase factor or the variable $\Lambda(t)$ appears as a constraint for the evolution of the observed state. Those paths of the observed particle which give the value of $\Theta_{ab}[x(t)]$ in the time period t, synchronizing with $\Lambda(t)$, are the same as the constrained paths of the observed system which have spent a duration τ in the region $a \leq x \leq b$. In this way the measurement system is used to constrain the evolution of the observed system, just as the Dirac-delta function in the derivation of Sokolovski. This also implies that we can obtain the propagator with constraint by reducing the total propagator for the whole system into the effective propagator. Let us consider the propagator of the whole system which is written as

$$K_{total}(x, x_0; y, y_0; t) = \langle y| \otimes \langle x|\, \mathbf{U}(t, 0)\, |y_0\rangle \otimes |x_0\rangle\,. \qquad (4.33)$$

We can reduce this propagator into the effective propagator by using the relation

$$K_{eff}(x,x_0;t)=\frac{\int dy\int dy_0 K_{total}K_a^*}{\int dy\int dy_0 K_a K_a^*} \tag{4.34}$$

where $K_a(y,y_0;t)=\langle y|\exp\{-\frac{i}{\hbar}\int\limits_0^t \mathbf{H}_A dt'\}|y_0\rangle$ is the propagator of the pointer particle, in the form of a free particle propagator. The meaning of Eq.(4.34) is analogous to the probability theorem

$$P(M)=\sum_i P(M\,|A_i)\,P(A_i) \tag{4.35}$$

when the total propagator, Eq.(4.33), is analogous to the conditional probability $P(M\,|A_i)$ of an event M assuming A_i. Straightforward substitution of the total propagator of Eq.(4.33) into Eq.(4.34), leads to

$$\begin{aligned} K_{eff}(x,x_0;t) &= \int dy\int dy_0[\ \langle y|\otimes\langle x|\, e^{[-\frac{i}{\hbar}\int\limits_0^t(\mathbf{H}_0+\mathbf{H}_A+g\mathbf{P}(\Theta_{ab}[\mathbf{x}]-\Lambda(t')))dt']}|y_0\rangle \\ &\quad\times\langle y_0|\, e^{\frac{i}{\hbar}\int\limits_0^t \mathbf{H}_A dt'}|y\rangle\otimes|x_0\rangle\ ] \end{aligned}$$

$$\times\left[\int dy\int dy_0\,\langle y|\exp\{-\frac{i}{\hbar}\int\limits_0^t \mathbf{H}_A dt'\}|y_0\rangle\,\langle y_0|\exp\{\frac{i}{\hbar}\int\limits_0^t \mathbf{H}_A dt'\}|y\rangle\right]^{-1}. \tag{4.36}$$

By using the completeness relation $I=\int dy_0\,|y_0\rangle\,\langle y_0|$, Eq.(4.36) becomes

$$K_{eff}(x,x_0;t)=\int dy\,\langle y|\otimes\langle x|\, e^{[-\frac{i}{\hbar}\int\limits_0^t(\mathbf{H}_0+g\mathbf{P}(\Theta_{ab}[\mathbf{x}]-\Lambda(t')))dt']}|y\rangle\otimes|x_0\rangle\times\left[\int dy\,\langle y|\,y\rangle\right]^{-1}. \tag{4.37}$$

and by inserting the completeness relation $I=\int dp\,|p\rangle\,\langle p|$ between $\langle y|$ and $|y\rangle$, leads to (see Appendix E)

$$K_{eff}(x,x_0;t)=\int D[x]\delta(\tau-t_{ab}^{cl}[x])e^{iS[x(t)]/\hbar}/\delta(0) \tag{4.38}$$

The effective propagator in the form of Eq.(4.38) corresponds to the propagator of Sokolovski[19]. By definition of the initial traversal wave function

$$\psi(x,t=0\,|\tau)=\psi_0(x)\delta(0) \tag{4.39}$$

and the effective propagator Eq.(4.38) we obtain

$$\psi(x,t\,|\tau) = \int dx_0 \int D[x]\delta(\tau - t_{ab}^{cl}[x])e^{iS[x(t)]/\hbar}\psi_0(x_0). \tag{4.40}$$

Thus we can obtain the clocked SE and the traversal time wave function by starting from the normal SE in two ways. We have derived the clocked SE in two ways by starting with the normal SE. These are (i) straight forward substitution and (ii) reduction of the composite system. By straight forward substitution, we use the boundary condition and the wave function in Eq.(4.2) for providing the clocked SE. For the composite system which has the coupling interaction in the form of a weak measurement, we find an effective SE. Thus we obtain the clocked SE and the effective wave function, Eq.(4.31) which is equivalent to defining the traversal wave function of Sokolovski. By the use of the conditional probability theorem, we have obtained the effective propagator, Eq.(4.38) which is analogous to the constraint propagator of Sokolovski. We derive the clocked SE and the traversal wave function from the SE by breaking the composite system into three parts: the observed system, the apparatus and the actual observer (following the measuring process of von Neumann [25]). By assuming the observer to have a very large mass and the wave packet being narrow enough, in Λ space, the quantum variable $\mathbf{\Lambda}$ can be replaced by the classical variable $\Lambda(t)$. This leads to a constraint on the observed system when we reduce the complete system. The result of the reduction is that the observed particle has the value of $\Theta_{ab}[x]$ in the time period t, synchronizing with the $\Lambda(t)$ to be selected.