CHAPTER I

INTRODUCTION AND PRELIMINARIES

This work is concerned with Rolle's theorem and its consequences developed for functions from a subset of $\mathbf{R}^n$ into a Banach space.

In 1995, M.Furi and M.Martelli [4] proved a multi-dimensional version of Rolle's theorem. The basic idea of their result is to assume a certain behavior of the function f on the boundary of $\mathcal{D}$ in $\mathbf{R}^n$ and then obtain information on the derivative of f at an interior point of $\mathcal{D}$. The theorem assume that a function f is defined on a subset $\mathcal{D}$ of $\mathbf{R}^n$ with values in $\mathbf{R}^q$. If f is continuous on $\mathcal{D}$, differentiable on the interior of $\mathcal{D}$ and there is a point v in $\mathbf{R}^n$ such that v is orthogonal to $f(x)$ for every x in the boundary of $\mathcal{D}$ then there is a point c in the interior of $\mathcal{D}$ such that v is orthogonal to $df(c)(u)$ for any u in $\mathbf{R}^n$.

We prove theorems analogous to that of Furi and Martelli for a function from a subset of $\mathbf{R}^n$ into a Banach space. And as its consequence we develop a theorem which is a generalization of the Mean Value theorem of Sanderson [6].

Now we give the definitions of differentiation in $\mathbf{R}^n$ and present some basic theorems of differentiable functions in $\mathbf{R}^n$ which are needed in our work.

The inner product and the norm on $\mathbf{R}^n$ are the Euclidean inner product and the Euclidean norm in $\mathbf{R}^n$, respectively, that is

$$x \cdot y = \sum_{i=1}^{n} x_i y_i \quad \text{where } x = (x_1, x_2, \ldots, x_n), \quad y = (y_1, y_2, \ldots, y_n) \text{ in } \mathbf{R}^n,$$

and $\|x\|$ = the norm of $x = \sqrt{x \cdot x}$ for x in $\mathbf{R}^n$.

Definition 1.1. *Let f be a function with domain A in $\mathbf{R}^n$ and range in $\mathbf{R}^m$, and let a_0 be an interior point of A. We say that* ***f is differentiable at a_0*** *if there exists a linear function $L : \mathbf{R}^n \to \mathbf{R}^m$ such that for every $\varepsilon > 0$ there exists a positive real number $\delta(\varepsilon)$ such that if $x \in \mathbf{R}^n$ is any vector satisfying $\|x - a_0\| \le \delta(\varepsilon)$, then $x \in A$ and*

$$\|f(x) - f(a_0) - L(x - a_0)\| \le \varepsilon \|x - a_0\|.$$

If such a linear function L exists then it is unique. It is called the ***derivative* of f at a_0** and denoted by **$df(a_0)$**.

Lemma 1.2. *If $f : A \to \mathbf{R}^m$ is differentiable at $a_0 \in A$, then there exist strictly positive real numbers δ, K such that if $\|x - a_0\| \le \delta$, then*

$$\|f(x) - f(a_0)\| \le K \|x - a_0\|.$$

It follows that if f is differentiable at a_0 then f is continuous at a_0.

Theorem 1.3 *Let $A \subseteq \mathbf{R}^n$ and let a_0 be an interior point of A.*

(a) If f and g are defined on A with values in $\mathbf{R}^m$ and are differentiable at a_0, and if $\alpha, \beta \in \mathbf{R}$, then the function $h = \alpha f + \beta g$ is differentiable at a_0 and

$$dh(a_0) = \alpha df(a_0) + \beta dg(a_0).$$

(b) If $\varphi : A \to \mathbf{R}$ and $f : A \to \mathbf{R}^m$ are differentiable at a_0, then the product function $k = \varphi f : A \to \mathbf{R}^m$ is differentiable at a_0 and

$$dk(a_0)(u) = [d\varphi(a_0)(u)]f(a_0) + \varphi(a_0)[df(a_0)(u)] \qquad \text{for } u \in \mathbf{R}^n.$$

The next result is the chain rule which asserts that the derivative of the composition of two differentiable functions is the composition of their derivatives.

Theorem 1.4. *Let f have domain $A \subseteq \mathbf{R}^n$ and range in $\mathbf{R}^m$, and let g have domain $B \subseteq \mathbf{R}^m$ and range in $\mathbf{R}^r$. Suppose that f is differentiable at a_0 and that g is differentiable at $b = f(a_0)$. Then the composition $h = g \circ f$ is differentiable at a_0 and*

$$dh(a_0) = dg(b) \circ df(a_0),$$

that is,

$$d(g \circ f)(a_0) = dg(f(a_0)) \circ df(a_0).$$

The next two theorems are the Mean Value Theorems.

Theorem 1.5. *Let f be a real - valued function defined on an open subset* Ω *of* $\mathbf{R}^n$. *Suppose that the set* Ω *contains the points a, b and the line segment S joining them, and that f is differentiable at every point of this line segment. Then there exists a point c on S such that*

$$f(b) - f(a) = df(c)(b - a).$$

Theorem 1.6. *Let* $\Omega \subseteq \mathbf{R}^n$ *be an open set and let* $f : \Omega \to \mathbf{R}^m$. *Suppose that* Ω *contains the points a, b and the line segment S joining these points, and that f is differentiable at every point of S. Then there exists a point c on S such that*

$$\| f(b) - f(a) \| \leq \| df(c)(b - a)\|.$$

Throughout this research, we denote by $D(x_0 , r)$, $B(x_0 , r)$ and $S(x_0 , r)$, the closed ball, the open ball and the sphere, centered at x_0 with radius r in $\mathbf{R}^n$, respectively.

That is $D(x_o, r) = \{ x \in \mathbf{R}^n \mid \| x - x_o \| \leq r \}$,

$B(x_o, r) = \{ x \in \mathbf{R}^n \mid \| x - x_o \| < r \}$,

$S(x_o, r) = \{ x \in \mathbf{R}^n \mid \| x - x_o \| = r \} = \partial D(x_o, r)$.

The following three theorems are the multi-dimensional version of Rolle's theorem proved by Furi and Martelli [4] .

Theorem 1.7. *Let $x_0 \in \mathbf{R}^n$, $f : D(x_o, r) \to \mathbf{R}^p$ be continuous on $D(x_o, r)$ and differentiable on $B(x_o, r)$. Assume that there exists a vector $v \in \mathbf{R}^p$ such that v is orthogonal to $f(x)$ for every $x \in S(x_o, r)$. Then there exists a vector $c \in B(x_o, r)$ such that $v \cdot df(c)(u) = 0$ for every $u \in \mathbf{R}^n$.*

Theorem 1.8. *Let $x_0 \in \mathbf{R}^n$, $f : D(x_o, r) \to \mathbf{R}^p$ be continuous on $D(x_o, r)$ and differentiable on $B(x_o, r)$. Assume that there exists a vector $v \in \mathbf{R}^p$ such that $v \cdot f(x)$ is constant for every $x \in S(x_o, r)$. Then there exists a vector $c \in B(x_o, r)$ such that $v \cdot df(c)(u) = 0$ for every $u \in \mathbf{R}^n$.*

Theorem 1.9. *Let $x_0 \in \mathbf{R}^n$, $f : D(x_o, r) \to \mathbf{R}^p$ be continuous on $D(x_o, r)$ and differentiable on $B(x_o, r)$. Let $v \in \mathbf{R}^p$ and $z_o \in B(x_o, r)$ be such that $v \cdot (f(x) - f(z_o))$ does not change sign on $S(x_o, r)$. Then there exists a vector $c \in B(x_o, r)$ such that $v \cdot df(c)(u) = 0$ for every $u \in \mathbf{R}^n$.*

It is noticed that a straightforward reformulation of Rolle's theorem in $\mathbf{R}^n$, for $n \geq 2$, fails.

For example, let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$f(x,y) = (x(x^2 + y^2 - 1), y(x^2 + y^2 - 1))$$

for any (x,y) in $\mathbf{R}^2$. Then f is continuous on $D(0,1)$, differentiable on $B(0,1)$ and $f(x,y) = (0,0) = 0$ for every (x,y) in $S(0,1)$. However the derivative of f

at $\bar{x} = (x, y)$ is the linear transformation $df(\bar{x})$, from $\mathbf{R}^2$ into $\mathbf{R}^2$, represented by the matrix

$$\begin{bmatrix} 3x^2 + y^2 - 1 & 2xy \\ 2xy & 3y^2 + x^2 - 1 \end{bmatrix}$$

for every $\bar{x}$ in B(0,1) and it is obvious that $df(\bar{x}) \neq 0$ for any $\bar{x} \in$ D(0,1).

In proving our main theorems, we need the following two basic theorems for real-valued function on a subset of $\mathbf{R}^n$.

Theorem 1.10 *Let $K \subseteq \mathbf{R}^n$ and let $f : K \to \mathbf{R}$ be continuous on K. If K is compact, then there are points x_1, x_2 in K such that $f(x_1) = sup\{ f(x) : x \in K \}$, $f(x_2) = inf\{f(x) : x \in K \}$.*

Theorem 1.11 *Let $A \subseteq \mathbf{R}^n$, and $f : A \to \mathbf{R}$. If an interior point c of A is a point of extremum of f, and if the derivative $df(c)$ exists, then $df(c) = 0$, the zero function.*