# CHAPTER II

# AN ELEMENT OF THEORY OF FRÉCHET DIFFERENTIATION

In this chapter, we give a definitions of Fréchet derivative and present some basic theorems on Fréchet differentiable functions which are essential for our work. The proof of the theorems whose proof are omitted can be found in Flett [3] .

Throughout the chapter, $X$, $Y$, and $Z$ will denote Banach spaces over either the real or the complex field, unless otherwise stated.

*Let* $f : A \to Y$ *where* $A \subseteq X$ *and* $a_o$ *an interior point of* $A$. *Then* $f$ *is said to be* ***Fréchet differentiable at*** $a_o$ *if there is a continuous linear function* $L : X \to Y$ *such that for every positive real number* $\varepsilon$ *there is a positive real number* $\delta(\varepsilon)$ *such that for every vector* $x \in X$ *satisfying* $||x - a_o|| \leq \delta(\varepsilon)$ *we have* $x \in A$ *and*

$$||f(x) - f(a_o) - L(x-a_o)|| \leq \varepsilon ||x - a_o|| . \qquad \text{............... (1)}$$

**Remark :** *The condition* (1) *in the above definition can be expressed by the equation*

$$\lim_{||x - a_o|| \to 0} \frac{||f(x) - f(a_o) - L(x-a_o)||}{||x - a_o||} = 0 \qquad \text{............... (2)}$$

If such a linear function $L$ in (1) or (2) exists, then it is unique. The linear function $L$ is called the ***Fréchet differential of $f$ at $a_o$*** or the ***Fréchet derivative of $f$ at $a_o$*** and will be denoted by $df(a_o)$.

**Example.** Consider the space $l_2$, of all sequences $(x_n)$ of real numbers such that

$$\sum_{i=1}^{\infty} x_i^2 < \infty .$$

For $x = (x_n)$ and $y = (y_n)$ in $l_2$, the inner product of x and y is defined by $\langle x, y\rangle = \sum x_i y_i$. It is known that $l_2$ with the inner product defined above is a Hilbert space and it is a Banach space with the norm $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$.

Define $f : l_2 \to \mathbf{R}$ by $f(x) = \| x \|^2$ for each $x \in l_2$.

Let $a_0$ be any element in $l_2$. We note that

$$\begin{aligned}
| f(x) - f(a_0) - 2\langle a_0, x - a_0\rangle | &= | \| x \|^2 - \| a_0 \|^2 - 2\langle a_0, x - a_0\rangle| \\
&= | \| x \|^2 - \| a_0 \|^2 - 2\langle a_0, x\rangle + 2\langle a_0, a_0\rangle | \\
&= | \| x \|^2 + \| a_0 \|^2 - 2\langle a_0, x\rangle| \\
&= \| x - a_0\|^2,
\end{aligned}$$

and then

$$\begin{aligned}
\lim_{\|x-a_0\|\to 0} \frac{|f(x) - f(a_0) - 2\langle a_0, x - a_0\rangle|}{\|x - a_0\|} &= \lim_{\|x-a_0\|\to 0} \frac{\|x - a_0\|^2}{\|x - a_0\|} \\
&= \lim_{\|x-a_0\|\to 0} \|x - a_0\| \\
&= 0.
\end{aligned}$$

Therefore $f$ is Fréchet differentiable at $a_0$ and $df(a_0)$ is the continuous linear function $df(a_0) : u \mapsto 2\langle a_0, u\rangle$, for each $u \in l_2$.

For normed spaces $X$ and $Y$, we denote by $\mathcal{L}(X,Y)$ the space of all continuous linear functions from X to Y. It is known that $\mathcal{L}(X,Y)$ is a normed space under the operations of addition of functions and multiplication of functions by scalars, with the norm

$$\| T \| = \sup \{ \| Tx \| \mid x \in X , \| x \| \leq 1 \}.$$

It is well known that, if $X$ and $Y$ are Banach spaces, then $\mathcal{L}(X,Y)$ is also a Banach space. This follows from the following proposition.

**Proposition. ([5])** *If $X$ is a normed space and $Y$ is a Banach space then $\mathcal{L}(X,Y)$ is a Banach space.*

Let $E$ be the set of interior points $x$ of $A$ such that $df(x)$ exists. The function from $E$ to $\mathcal{L}(X,Y)$ which assigns for each $x$ in $E$ the linear function $df(x)$ is called the ***Fréchet differential of f*** and denoted by $df$.

If $A$ is an open set in $X$, a function which is Fréchet differentiable at each point of $A$ is said to be ***Fréchet differentiable on A***.

Note that if function $f : A \subseteq \mathbf{R}^n \to \mathbf{R}^m$, then the Fr?échet derivative of $f$ at $a_0$ is the derivative of $f$ at $a_0$.

The next theorem is immediatly obtained from the definition of Fréchet differentiability of a point $a_0$ in $A^0$.

**Theorem 2.1.** *Let* $A \subseteq X$ , *and let* $f: A \rightarrow Y$ *be Fréchet differentiable at a point* $a_o$ *in* $A^0$. *Then for each* $\varepsilon > 0$ *there exists a neighbourhood* $U$ *of* $a_o$ *in* $X$, *contained in* $A$, *such that for all* $x \in U$,

$$\| f(x) - f(a_o) \| \leq ( \| df(a_o) \| + \varepsilon ) \| x - a_o \|.$$

As it is known that a linear function from a normed space $X$ to a normed space $Y$ is continuous if and only if it is bounded. Hence if $df(a_0)$ is continuous then there is a constant real number K such that $\| df(a_o) \| \leq$ K.

We note that the inequality in Theorem 2.1 yields that the Fréchet differentiability of a function $f$ at a point implies the continuity of $f$ at that point. So we have the following theorem.

**Theorem 2.2.** *If* $f$ *is Fréchet differentiable at* $a_o$ , *then* $f$ *is continuous at* $a_o$ .

If a function $f: X \rightarrow Y$ is a linear function then the function $f$ itself satisfies the condition for the Fréchet differential of $f$ at any point in $X$. Since for each $a_0 \in X$

$$f(x) - f(a_0) - f(x - a_0) = 0,$$

for any $x$ in a neighbourhood of $a_0$, hence the left hand side of (1) is zero and thus (1) holds.

**Theorem 2.3.** *(i) If* $f : X \to Y$ *is a linear function then* $f$ *is Fréchet differentiable on* $X$ *and* $df(x) = f$ *for any* $x$ *in* $X$.

*(ii) If* $f : X \to Y$ *is a constant function then* $f$ *is Fréchet differentiable on* $X$ *and for any* $x$ *in* $X$, $df(x)$ *is the zero function.*

The next result gives some equivalent formulations of the differentiability.

**Theorem 2.4.** *Let* $f$ *be a function from a set* $A \subseteq X$ *into* $Y$, *let* $a_o$ *be an interior point of* $A$, *let* $T$ *be a linear transformation from the set* $X$ *into* $Y$, *and let*

$$R(h) = f(a_o + h) - f(a_o) - T(h) \qquad (a_o + h \in A)$$

*Then the following statements are equivalent :*

*(i)* $f$ *is Fréchet differentible at* $a_o$, *with* $df(a_o) = T$,

*(ii) for each bounded set* $E \subseteq X$, $R(th)/t \to 0$ *as* $t \to 0$ *in* $\mathbf{R}$, *uniformly for* $h$ *in* $E$ ;

*(iii) for each sequence* $(t_n)$ *in* $\mathbf{R} \setminus \{0\}$ *converging to* 0 *and for each bounded sequence* $(h_n)$ *in* $X$, $R(t_n h_n)/t_n \to 0$ *as* $n \to \infty$.

Further, *if $X$ is finite - dimensional, then each of (i) - (iii) is equivalent to (iv) for each compact set $E \subseteq X$, $\mathbf{R}(th)/t \to 0$ as $t \to 0$ in $\mathbf{R}$, uniformly for $h$ in $E$.*

**Theorem 2.5.** *(i) If $A \subseteq X$, and $f : A \to Y$ is Fréchet differentiable at $a_o$, then for each scalar $\alpha$ the function $\alpha f$ is Fréchet differentiable at $a_o$ and its differential at $a_o$ is $\alpha df(a_o)$.*

*(ii) If $A, B \subseteq X$, and $f : A \to Y$ and $g : B \to Y$ are Fréchet differentiable at $a_o$ then $f + g$ is Fréchet differentiable at $a_o$ and*

$$d(f+g)(a_o) = df(a_o) + dg(a_o).$$

These two results in theorem 2.5 assert that if $\mathcal{D}$ is the set of all functions which are Fréchet differentiable at a point $a_o$ of $A$, then $\mathcal{D}$ is a vector space under the operations of addition of functions and multiplication of functions by scalar. Moreover, the function $f \mapsto df(a_o)$ is a linear function from $\mathcal{D}$ into $\mathcal{L}(X, Y)$.

The next result is the chain rule, which asserts that the derivative of the composition of two Fréchet differentiable functions is the composition of their derivatives.

**Theorem 2.6.** ( The chain rule ).

*Let $A \subseteq X$, $B \subseteq Y$, let $f : A \to Y$ be Fréchet differentiable at $a_o$, and let $g : B \to Z$ be Fréchet differentiable at the point $b_0 = f(a_o)$. Then the function $g \circ f$ is Fréchet differentiable at $a_o$, and*

$$d(g \circ f)(a_o) = dg(b_o) \circ df(a_o).$$

**Theorem 2.7** *Let $A \subseteq X$, $f : A \to \mathbf{R}$, $g : A \to R$. If $f$ and $g$ are Fréchet differentiable at $c$ then the product function $f \cdot g$ which assigns for each $x$ in $A$ the product $f(x)g(x)$ is Fréchet differentiable at $c$ and for each $u \in X$,*

$$d(f \cdot g)(c) : u \mapsto g(c)df(c)(u) + f(c)dg(c)(u). \qquad \text{.........................(3)}$$

**Proof.** Let $d$ and $g$ be Fréchet differentiable at $c$. Then $df(c)$ and $dg(c)$ are continuous linear functions and hence $d(f \cdot g)(c)$ defined in (3) is a continuous linear function from $X$ into $\mathbf{R}$. Since

$$(f \cdot g)(x) - (f \cdot g)(c) - [g(c)df(c)(x - c) - f(c)dg(c)(x - c)]$$

$$= g(x)[f(x) - f(c) - df(c)(x-c)] + f(c)[g(x) - g(c) - dg(c)(x - c)] + (g(x) - g(c))[df(c)(x - c)],$$

then

$$\frac{|(f \cdot g)(x) - (f \cdot g)(c) - [g(c)df(c)(x-c) - f(c)dg(c)(x-c)]|}{||x-c||}$$

$$\leq \left| g(x)\left[\frac{f(x) - f(c) - df(c)(x-c)}{||x - c||}\right]\right| + \left| f(x)\left[\frac{g(x) - g(c) - dg(c)(x-c)}{||x - c||}\right]\right|$$

$$+ \; |g(x) - g(c)| \left\| df(c)\frac{(x-c)}{||x-c||} \right\|$$

$$\leq |g(x)| \frac{|f(x)-f(c)-df(c)(x-c)|}{\|x-c\|} + |f(x)| \frac{|g(x)-g(c)-dg(c)(x-c)|}{\|x-c\|}$$

$$+ |g(x) - g(c)| \, \| df(c) \| . \quad \ldots\ldots\ldots\ldots (4)$$

By the definition of $df(c)$, we have $\| df(c) \|$ is finite. Hence it is clear from (4) that $\lim\limits_{x\to c} \frac{|(f\cdot g)(x)-(f\cdot g)(c)-[g(c)df(c)(x-c)-f(c)dg(c)(x-c)]|}{\|x-c\|} = 0.$

That is, $f\cdot g$ is Fréchet differentiable at $c$ and for each $u \in X$,

$$d(f\cdot g)(c)(u) = g(c)df(c)(u) + f(c)dg(c)(u).$$

■

**Theorem 2.8.** *Let $A \subseteq X$, $f: A \to \mathbf{R}$ be such that $f(x) \neq 0$ for any $x \in A$. If $f$ is Fréchet differentiable at $c$ then the function $\frac{1}{f} : x \mapsto \frac{1}{f(x)}$ for each $x \in A$ is Fréchet differentiable at $c$ and*

$$d(\frac{1}{f})(c)(u) = -\frac{1}{f^2(x)} df(c)(u), \quad \text{for all } u \in X. \quad \ldots\ldots\ldots (5)$$

**Proof.** Let $f$ be Fréchet differentiable at $c$. Then the function $d(\frac{1}{f})(c)$ is a continuous linear function from $X$ into $\mathbf{R}$. Since

$$\frac{1}{f(x)} - \frac{1}{f(c)} + \frac{1}{f^2(c)} df(c)(x-c)$$

$$= -\left(\frac{f(x)-f(c)-df(c)(x-c)}{f(x)-f(c)}\right) - \frac{df(c)(x-c)}{f(x)f(c)} + \frac{1}{f^2(c)} df(c)(x-c),$$

then $\frac{1}{\|x-c\|}\left[\frac{1}{f(x)} - \frac{1}{f(c)} + \frac{1}{f^2(c)} df(c)(x-c)\right]$

$$= -\frac{1}{f(x)f(c)}\left[\frac{f(x)-f(c)-df(c)(x-c)}{\|x-c\|}\right] - df(c)\frac{(x-c)}{\|x-c\|}\left[\frac{1}{f(x)f(c)} - \frac{1}{f^2(c)}\right].$$

We note that $\left| df(c)\frac{(x-c)}{||x-c||} \right| \leq ||\, df(c)\,||$ implies

$$\frac{1}{||x-c||}\left[\frac{1}{f(x)}-\frac{1}{f(c)}+\frac{1}{f^2(c)}df(c)(x-c)\right]$$

$$\leq \frac{1}{|f(x)-f(c)|}\left|\frac{f(x)-f(c)-df(c)(x-c)}{||x-c||}\right| + ||\, df(c)\,|| \left|\frac{1}{f(x)f(c)}-\frac{1}{f^2(c)}\right| .$$

Therefore

$$\lim_{x\to c}\frac{1}{||x-c||}\left[\frac{1}{f(x)}-\frac{1}{f(c)}+\frac{1}{f^2(c)}df(c)(x-c)\right]$$

$$\leq \lim_{x\to c}\frac{1}{|f(x)f(c)|}\lim_{x\to c}\frac{|f(x)-f(c)-df(c)(x-c)|}{||x-c||} + ||\, df(c)\,|| \lim_{x\to c}\left|\frac{1}{f(x)f(c)}-\frac{1}{f^2(c)}\right|$$

$$= \frac{1}{|f^2(c)|}\cdot 0 + ||\, df(c)\,|| \cdot 0$$

$$= 0.$$

Hence $\frac{1}{f}$ is Frchet differenteable at $c$ and for each $u \in X$,

$$d(\frac{1}{f})(c)(u) \;=\; -\frac{1}{f^2(c)}df(c)(u).$$

■

**Corollary 2.9.** *Let $A \subseteq X$, $f: A \to R$, $g: A \to R$ be such that $g(x) \neq 0$ for each $x \in A$. If $f$ and $g$ are Fréchet differentiable at $c$ then the quotient $\frac{f}{g}$ is Fréchet differentiable at $c$ and for each $u \in X$,*

$$d(\frac{f}{g})(c)(u) \;=\; \frac{g(c)df(c)(u)-f(c)dg(c)(u)}{g^2(c)}. \qquad \text{.........................(6)}$$

**Proof.** Let $f$ and $g$ be Fréchet differentiable at $c$. Then the function $d(\frac{f}{g})(c)$ defined in (6) is a continuous linear function from $X$ into $\mathbf{R}$. By applying theorem 2.7 and 2.8, we have

$$\begin{aligned} d(\frac{f}{g})(c)(u) &= d(f \cdot \frac{1}{g})(c)(u) \\ &= \frac{1}{g(c)} df(c)(u) + f(c)(-\frac{1}{g^2(c)} dg(c)(u)) \\ &= \frac{g(c)df(c)(u) - f(c)dg(c)(u)}{g^2(c)} \end{aligned}$$

Hence $\frac{f}{g}$ is Fréchet differentiable at $c$ with $d(\frac{f}{g})(c)$ is as defined in (6). ■

The next theorem is a mean value inequality for functions $f$ from a subset of a Banach space into a Banach space.

**Theorem 2.10.** *Let $a$, $b$ be distinct points of $X$, let $S$ be the closed line segment in $X$ with endpoints $a$ and $b$, and let $f$ be a function from a subset of $X$ containing $S$ into $Y$ which is continuous on $S$ and Fréchet differentiable on $S$. Then there exist a point $c \in S$ such that*

$$\| f(b) - f(a) \| \leq \| df(c)(b-a) \| .$$

Recall that if $f$ is a function from a set $A \subseteq X$ into $Y$ and differentiable at every point in a subset $E$ of $A$, then the differential, $df$, of $f$ is a function from the set $E$ into the Banach space $\mathcal{L}(X, Y)$. That is $df : E \to \mathcal{L}(X, Y)$. So we can define the Fréchet differential of $df$ at a point $a_0$ in $E^0$. This is the way the higher Fréchet differential defined.

*Let* $f : A \to Y$ *where* $A \subseteq X$ *and* $a_o$ *an interior point of* $A$. *Then* $f$ *is said to be* ***twice Fréchet differentiable at*** $a_o$ *if* $df$ *is defined for* $x$ *in a neighbourhood of* $a_o$ *in* $X$ *and there exists a continuous linear function* $T : X \to \mathcal{L}(X, Y)$ *such that*

$$\lim_{x \to a_o} \frac{\| df(x) - df(a_o) - T(x-a_o) \|}{\| x - a_o \|} = 0.$$

*We call the continuous linear function* $T$, *the* ***second Fréchet differential of*** $f$ ***at*** $a_o$ *and denote by* $d^2 f(a_o)$.

*Generally, the* ***k-th Fréchet differential*** $d^k f$ ***of*** $f$ *is defined inductively by the formulae*

$$d^1 f = df, \quad d^k f = d(d^{k-1} f) \qquad \text{where } n = 2, 3, 4, \ldots$$

*If* $d^k f$ *is defined at the point* $a_o \in X$, *we say that* $f$ *is* ***k-times Fréchet differentiable at*** $a_o$, *and we call the value* $d^k f(a_o)$ *of* $d^k f$ *there, the* ***k th Fréchet differential of*** $f$ ***at*** $a_o$.

Note that $\quad df : X \to \mathcal{L}(X,Y)$

$$d^2 f = d(df) : X \to (X, \mathcal{L}(X,Y))$$

$$d^3 f = d(d^2 f) : X \to (X, \mathcal{L}(X, \mathcal{L}(X,Y))$$

$$\vdots$$

$$d^{k+1} f = d(d^k f) : X \to (X, \mathcal{L}(X, \ldots, \mathcal{L}(X,Y))$$

We can simplify this notation by using the isometric between

$$\mathcal{L}(X, \mathcal{L}(X, Y)) \quad \text{and} \quad \mathcal{L}(X \times X, Y).$$

$$\mathcal{L}(X, \mathcal{L}(X, \mathcal{L}(X, Y)) \quad \text{and} \quad \mathcal{L}(X \times X \times X, Y)$$

$$\vdots$$

$$\mathcal{L}(X, \mathcal{L}(X, \dots, \mathcal{L}(X, Y)) \quad \text{and} \quad \mathcal{L}(X \times X \times \dots \times X, Y).$$

The next theorem come naturally from the above definition as the consequence of the theorem 2.5.

**Theorem 2.11.**

*(i) If $f: A \to Y$ is n-times Fréchet differentiable at $a_o$, where $A \subseteq X$, and $\alpha$ is a scalar, then $\alpha f$ is n-times Fréchet differentiable at $a_o$.*

*(ii) If $f: A \to Y$ and $g: A \to Y$ are n-times Fréchet differentiable at $a_o$, where $A \subseteq X$, then $f + g$ is n-times Fréchet differentiable at $a_o$.*

**Theorem 2.12.** *Let $A \subseteq X$, $B \subseteq Y$, let $f: A \to Y$ be n-times Fréchet differentiable at the point $a_o$, and let $g: B \to Z$ be n-times Fréchet differentiable at the point $b_o = f(a_o)$. Then $g \circ f$ is n-times Fréchet differentiable at $a_o$.*