APPENDIX A

(18)

The Trotter Product Formula

THEOREM: (Trotter product formula) Let A and B be linear operators on a Banach space X such that A, B and $A+B$ are the infinitesimal generators of the contraction semigroups P^t, Q^t, and R^t respectively. Then for all $\psi \in X$

$$R^t\psi = \lim_{n\to\infty} \left(P^{t/n} Q^{t/n} \right)^n \psi \qquad (A.1)$$

In this appendix we define some of the terms used above, indicate how a proof of the theorem goes, and examine some of its consequences.

A semigroup is a set closed under a binary , associative operation. Were inverses required to be in the set it would form a group. A semigroup may or maynot posses an identity.

<u>Definition</u>: A contraction semigroup on Banach space X is a family of bounded everywhere defined linear operators P^t, $0 \leq t \leq \infty$ mapping $X \to X$ such that

$$P^0 = 1 ; \quad P^t P^s = P^{t+s} , \quad 0 \leq t \leq \infty \qquad (A.2)$$

$$\| P^t \| \leq 1 \qquad 0 \leq t \leq \infty \qquad (A.3)$$

$$\lim_{t\to 0} P^t\psi = \psi \qquad \psi \in X \qquad (A.4)$$

The norm used above is defined as follows:

$$\| Q \| = \inf_{\rho\in B} \rho \quad \text{where } B = \left\{ \rho / \| Qx \| \leq \rho \|x\| \text{ for all } x \in X \right\} . \qquad (A.5)$$

and $\|x\|$ is the norm in X. The term "contraction" comes from the fact that $\|P^t\| \leq 1$, since vectors do not grow as they evolve under P^t. The infinitesimal A of P^t is defined by

$$A\psi = \lim_{t \to 0} \frac{1}{t}(P^t\psi - \psi) \tag{A.6}$$

on the domain $D(A)$ of all $\psi \in X$ for which the limit exists.

Remarks on the Proof of the Theorem. Let h be a positive real number and let P, Q, R be as defined in the statement of the theorem. By the definition of genenrators we have

$$(P^hQ^h - 1)\psi = (P^h - 1)\psi + P^h(Q^h - 1)\psi = h(A+B)\psi + o(h) \tag{A.7}$$

where $o(h)$ denotes vectors x such that $\lim_{h \to 0} \|x\|/h = 0$. Then since

$$(R^h - 1)\psi = h(A+B)\psi + o(h) \tag{A.8}$$

it follows that

$$(P^hQ^h - R^h)\psi = o(h) \tag{A.9}$$

Now we must establish the uniformity of the bound $o(h)$. By using properties necessarily possesed by infinitesimal generators we show that for ψ in some compact subset of $D(A+B)$, $h^{-1}\|(P^hQ^h - R^h)\psi\|$ is uniformly bounded. For some $\psi \in D(A+B)$, $\{R^s\psi\}$, $0 \leq s < t$, is compact and in $D(A+B)$, hence $\|(P^hQ^h - R^h)R^s\psi\| = o(h)$ unifromly in s.

Let $h = t/n$. Then we wish to show that

$$\| \{ (P^{h} Q^{h})^{n} - R^{hn} \} \psi \| \rightarrow \quad \text{for } n \rightarrow \infty \qquad (A.10)$$

To this end we examine

$$(P^{h}Q^{h})^{n} - R^{hn} = (P^{h}Q^{h} - R^{h}) R^{h(n-1)} + P^{h}(P^{h}Q^{h} - R^{h}) R^{h(n-2)} + \dots$$
$$+ (P^{h}Q^{h})^{n-1} (P^{h}Q^{h} - R^{h}) \qquad (A.11)$$

Next apply this to ψ, and use the fact that $\| P^{h}Q^{h} \| \leq 1$. This implies

$$\| \{ (P^{h}Q^{h})^{n} - R^{hn} \} \psi \| \leq \| (P^{h}Q^{h} - R^{h}) R^{h(n-1)} \psi \| + \dots = n o\left(\frac{t}{n}\right) \rightarrow 0 \qquad (A.12)$$

where the limit is uniform.

For the physical application of this formula we let

$$A = i\Delta \quad , \quad B = -iV$$

$$-(A+B) = i(-\Delta + V) = iH \qquad (A.13)$$

Thus it is necessary to know whether A , B, and $A+B$ generate contractive semigroups. Basically what is evolved is to examining $\| e^{Ct} \|$ where C is the proposed infinitesimal generator. Recall that

$$\| Q \| = \sup_{\psi \in X} \frac{\| Q \psi \|}{\| \psi \|} \qquad (A.14)$$

For V there is no problem; V is a multiplication operator and

$$\| e^{-iV}\psi\|^2 = \int dx\, e^{iV(x)}\psi^*(x)\, e^{-iV(x)}\psi(x) = \int dx\,|\psi(x)|^2 = \|\psi\|^2 \tag{A.15}$$

implies $\| e^{-iVt}\| = 1$ for all t.

To show that the Laplacian Δ generates a contractive semi-group we review some elementary facts and intuitions about the norm. Generally speaking the norm looks for the largest eigenvalue. If X is finite (M) dimensional we have $Q\psi_i = q_i\psi_i$, $i = 1,\dots,M$. Then the worst case is the biggest $|q_i|$, call it $|\hat{q}|$ and its eigenvector $\tilde{\psi}$, for then $\|Q\| = \max_\psi \|Q\psi\|/\|\psi\| = \|Q\hat{\psi}\|/\|\tilde{\psi}\| = |\hat{q}|$. Say $Q = e^{C}$. Its eigenvalues are e^{c_i} where c_i are the eigenvalues of C. Then $\|e^{C}\| = \max_i |e^{c_i}| = \max_i e^{\mathrm{Re}\, c_i}$. Thus

$$\| e^{tC}\| = \max_i e^{t\,\mathrm{Re}\, c_i} \tag{A.16}$$

so that the condition for C to be the generator of a contractive semi-group is that $\mathrm{Re}\, c_i < 0$ for all i.

On Hilbert space, the thing that would be the eigenvector is not always in the space and the definition of the norm as the limit ("sup") must be invoked. For example, let M be the multiplication operator by the function $\exp(-x^2)$. Thus

$$M\psi(x) = e^{-x^2}\psi(x) . \tag{A.17}$$

Clearly $\| M\psi(x)\| \leqslant \|\psi\|$ so that $\|M\| \leqslant 1$. Let

$$\psi = \frac{N}{2}\,\theta\!\left(\frac{1}{N} - |x|\right) \tag{A.18}$$

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then

$$\|\psi\|^2 = \frac{N^2}{4}\frac{2}{N} = \frac{N}{2} < \infty \qquad (A.19)$$

For all finite N

$$\frac{\|e^{-x^2}\psi\|^2}{\|\psi\|^2} = \frac{N}{2}\int_{-1/N}^{1/N} e^{-x^2}\,dx > \frac{N}{2}\int_{-1/N}^{1/N}(1-x^2)\,dx = 1-\frac{1}{3N^2} \qquad (A.20)$$

But for large enough N this gets arbitrarily close to 1, hence $\|M\| = 1$. However, there is no ψ in the Hilbert space such that $M\psi = \psi$. Thus 1 is not an eigenvalue of M , but it does have some special properties with respect to M: it is the spectrum. The spectrum is defined as the complement of the resolvent set where the resolvent set of an operator A is the set of λ for which $(\lambda - A)^{-1}$ exists.

Above we have a condition on the eigenvalues of a finite dimensional matrix C so that it generated a contractive semigroup . In a Hilbert space it is most convenient to state the condition in terms of the spectrum. The condition on an operator C is that Re$\lambda < 0$ for λ in the spectrum. If C = iK, in terms of the eigenvalues (or spectrum) k_i of K this means Im $k_i > 0$ for all i . If e^{-tC} is also to be a contractive semigroup-evolution in both directions-we must have

$$\mathrm{Im}\, k = 0 \qquad (A.21)$$

That is , K has only real spectrum, a condition guaranteed by the usual requirement that the Hamiltonian be self-adjoint. Thus $\|e^{-it\Delta}\| = 1$ which is the statement that the free particle propagation is norm preserving. To determine whether $A + B$ generates a contractive semigroup we examine whether $-\Delta+V$ is self-adjoint (so that its spectrum would be real). This question is what mathematicians call perturbation theory (and physicists never bother to ask). If it is self-adjoint, then the conditions for the Trotter formula are satisfied.

APPENDIX B

Gaussian Integration

Functional gaussian integrations will be understood to be the product of many regular gaussian integrals. The simplest is

$$G(a) = \int_{-\infty}^{\infty} dx \exp\left(-ax^2\right) \qquad (B.1)$$

which, using Poisson's trick of taking the square and of expressing the integrand in polar coordinates, is seen to be

$$G(a) = \sqrt{\frac{\pi}{a}} \qquad (B.2)$$

We can generalize it to N degrees of freedom. Let

$$G(a) = \int_{-\infty}^{\infty} dx_1 dx_2 \ldots dx_N \exp\left(-x_i a_{ij} x_j\right) \qquad (B.3)$$

where A is the real symmetric N x N matrix with elements a_{ij}. We write

$$x_i a_{ij} x_j = x^T A x \qquad \text{with} \qquad A^T = A \qquad (B.4).$$

A can be diagonalized by means of a rotation

$$A = R^T D R \qquad , \quad R^T R = R R^T = 1 \qquad . \qquad (B.5)$$

with D a diagonal matrix with entries $d_1, d_2,$ $, d_N$. Then

$$G(a) = \int dx_1 dx_2 \ldots dx_N \exp\left(-x^T R^T D R x\right) \qquad (B.6)$$

$$G'(A) = \int dx_1 dx_2 \ldots dx_N \exp\left(-y^T D y\right) \qquad (B.7)$$

with $y = Rx$. The jacobian is one. In the y variables, $G'(A)$ is separable in the N -fold product of

$$G'_N(A) = G(d_1)\, G(d_2) \ldots G(d_N) \qquad (B.8)$$

$$= \pi^{N/2} \left(d_1 d_2 \ldots d_N \right)^{-1/2} \qquad (B.9)$$

$$= \pi^{N/2} \left(\det A \right)^{-1/2} \qquad (B.10)$$

provided that all the eigenvalues of A are positive. In a similar way we can prove that if z are N complex variables

$$\int \prod_i^N dz_i\, dz_i^* \, e^{-z^T C z} = (2\pi)^N \left(\det C \right)^{-1} \qquad (B.11)$$

where C is an hermitian N x N matrix with positive eigenvalues. Formally, one then defines gaussian path integrals by taking the limit $N \to \infty$.

These formulae are valid when the determinant does not vanish. If it does, it means that some d_i is equal to zero, leading to an infinity from integrating over an infinite interval. Ideally we would like to devide out the culprit infinite integral. Suppose the symmetric N x N matric A has n zero eigenvalues. In the y variables define the restricted gaussian integral

$$G'_{rest}(A) = \int dy_1 \ldots dy_N \exp\left(- x^T(y)\, A\, x(y) \right) \qquad (B.12)$$

where we integrate only over the variables corresponding to a non zero eigenvalue of A . This representation of $G_{rest}(A)$ is awkward since it depends on the right system of coordinates " y " . To make up of this, invent new variables y_{N-n+1} , ... , y_N and rewrite Eq. (B.12) as

$$G'_{rest}(A) = \int dy_1 dy_2 \dots dy_{N-n} dy_{N-n+1} \dots dy_N \, \delta(y_{N-n+1}) \dots \delta(y_N) \, e^{-x^T(y) A x(y)} \tag{B.13}$$

Now change variables from y to x , using the Jacobi formula

$$dy_1 dy_2 \dots dy_N = dx_1 dx_2 \dots dx_N \left| \frac{dy}{dx} \right| \tag{B.14}$$

to obtain the final expression

$$G_{rest}(A) = \int \prod_i^N dx_i \, \det\left[\frac{dy}{dx}\right] \prod_{a=N-n+1}^{N} \delta(y_a) \, e^{-x^T A x} \tag{B.15}$$

This integral is perfectly well defined. The y_a are some arbitary functions of x , and the extra factors $\det \left| \frac{dy}{dx} \right| \prod \delta(y)$ in the measure effectively restrict the integration from an N-dimensional space to an N - n dimensional one. As the construction has shown, $G_{rest}(A)$ does not depend on the specific form of $y_a(x) \geq N-n$. It goes without saying that the $y_a(x)$ should be cleverly chosen so as to do the job, i.e., restrict the integration region ; if they do not, the Jacobian $\det\left[\frac{dy}{dx}\right]$ is seen to be singular.

Finally we prove one more expression . Consider now

$$F(A,w) = \int \prod_i^N dx_i \, e^{-x^T A x + \omega^T x} \tag{B.16}$$

We rewrite the exponent by completing the squares

$$x^T A x - \omega^T x = \left(x - \tfrac{1}{2}A^{-1}\omega\right)^T A \left(x - \tfrac{1}{2}A^{-1}\omega\right) - \tfrac{1}{4}\omega^T A^{-1}\omega \qquad (B.17)$$

provided that A^{-1} exists. Letting

$$x' = A - \tfrac{1}{2}A^{-1}\omega \qquad (B.18)$$

so that $dx_i' = dx_i$, we find

$$F(A,\omega) = e^{-\frac{1}{4}\omega^T A^{-1}\omega} \int \prod_i^N dx_i \, e^{-x'^T A x'}$$

$$= \pi^{N/2} e^{-\frac{1}{2}\omega^T A^{-1}\omega} \left[\det A\right]^{-1/2} \qquad (B.19)$$

Again the path integral result is formally obtained in the limit $N \to \infty$.

APPENDIX C

General Quadratic Lagrangian [5]

C.1. Derivation of the propagator by polygonal paths approach.

In this appendix we apply the polygonal paths approach to the path integral one-dimensional quadratic action characterized by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left\{a(t)\,\dot{x}^2 - b(t)\,x^2\right\} + c(t)\,x \qquad (C.1)$$

where $a(t) > 0$, $b(t)$ and $c(t)$ are well-behaved functions of time.

According to the coventional polygonal approach the propagator is to be evaluated as:

$$K(b,a) = \lim_{N\to\infty} K_N(b,a) \qquad (C.2)$$

where K_N is defined as

$$K_N = A_N \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{\infty} \prod_{k=1}^{N-1} dx_k \exp\left\{\frac{i}{\hbar} S_N\right\} \qquad (C.3)$$

Here A_N, the normalization factor required to define the path-differential measure, is given by

$$\prod_{k=1}^{N} \left(a_k / 2\pi i \hbar\epsilon\right)^{1/2} \quad , \quad a_k = a(t_k) \qquad (C.4)$$

and S_N , the discretized form of the action defined over the partition of the time interval $[t_1, t_2]$ into N subintervals each of lengh ϵ. With the notation, $x_k = x(t_k)$, $x_0 = x_a$, $x_N = x_b$, and similarly for the coefficients a , b , c , we have

$$S_N = [1/2\varepsilon]\left\{\sum_{k=1}^{N}\left[a_k(x_k - x_{k-1})^2 + 2\varepsilon^2 c_k x_k\right]\right\} \qquad (C.5)$$

The expressions for A_N and S_N are then inserted in Eq. (C.3) and the integrations over x_k ($k = 1, 2, \ldots, N-1$) are performed to obtain K_N. In order to carry out this task it is convenient to introduce the column vectors $\mathbb{X}$ and $\mathbb{Y}$ having $(N-1)$ components:

$$\mathbb{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix}, \quad \mathbb{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \end{bmatrix} \qquad (C.6)$$

where

$$\begin{aligned} y_1 &= -c_1\varepsilon^2 + a_1 x_0 \\ y_k &= -c_k\varepsilon^2 \qquad (2 \leq k \leq N-2) \\ y_{N-1} &= -c_{N-1}\varepsilon^2 + a_N x_N \end{aligned} \qquad (C.7)$$

and an (N - 1)-dimensional symmetric matrix $\mathbb{P}$ with the following structure

$$\begin{aligned} P_{k,k} &= a_k + a_{k+1} - b_k\varepsilon^2 \\ P_{k+1,k} &= -a_{k+1}, \quad P_{j,k} = 0 \quad (j \neq k, k \pm 1). \end{aligned} \qquad (C.8)$$

We may then write

$$K_N(b,a) = \left(\prod_{k=1}^{N} a_k\right)^{1/2} (\alpha/i\pi)^{N/2} \exp\left\{i\alpha\left[a_1 x_0^2 + (a_N - b_N\varepsilon^2)x_N^2\right.\right.$$

$$\cdots + 2\varepsilon^2 c_N x_N]\Big\} \int_{-\infty}^{\infty} d\mathbb{X} \exp\Big\{ i\alpha [\mathbb{X}^T \mathbb{P} \mathbb{X} - 2\mathbb{X}^T \mathbb{Y}]\Big\}. \quad (C.9)$$

with the symbol $d\mathbb{X} = \prod_{k=1}^{N-1} dx_k$, $\mathbb{X}^T$ denoting the transpose of $\mathbb{X}$ and $\alpha = (2\hbar\varepsilon)^{-1}$. The gaussian integral in Eq. (C.9) may be readily evaluated as

$$\int_{-\infty}^{\infty} d\mathbb{X} \exp\Big\{ i\alpha [\mathbb{X}^T \mathbb{P} \mathbb{X} - 2\mathbb{X}^T \mathbb{Y}]\Big\} = (i\pi/\alpha)^{(N-1)/2} (\det \mathbb{P})^{-1/2} \exp(-i\alpha \mathbb{Y}^T \mathbb{P}^{-1} \mathbb{Y})$$

(C.10)

We obtain thereby

$$K_N = (D_N / i\pi)^{1/2} \exp(i\chi_N) \quad (C.11)$$

where

$$D_N = \Big\{ \Big(\prod_{k=1}^{N} a_k\Big) (\alpha/\det \mathbb{P}) \Big\}^{1/2} \quad (C.12)$$

$$\chi_N = \alpha [a_1 x_0^2 + (a_N - b_N \varepsilon^2) x_N^2 + 2\varepsilon^2 c_N x_N - (\mathbb{Y}^T \mathbb{P} \mathbb{Y})]. \quad (C.13)$$

Thus to obtain the propagator K we need to evaluate the exponent χ_N and the normalization factor D_N in the limit $\varepsilon \to 0$, $N \to \infty$ with $N\varepsilon = t_b - t_a$. We denote these limits by χ and D rexpectively.

C.2. Exponent

We introduce a new vector $\mathbb{U}$ such that

$$\mathbb{P}\mathbb{U} = \mathbb{Y} \quad (C.14)$$

Written in the component form Eq. (C.14) reads as

$$-a_k u_{k-1} + (a_k + a_{k+1} - b_k \varepsilon^2) u_k - a_{k+1} u_{k+1} = -c_k \varepsilon^2 , k=1,\ldots,N-1. \qquad (C.15)$$

with the end-point values defined as

$$u_0 = x_0 \ , \ u_N = x_N \qquad (C.16)$$

Rearranging the terms in Eq.(C.15) and dividing throughout by ε^2 we take limit $\varepsilon \to 0$ to obtain the differential equation

$$\frac{d}{dt}\left\{ a(t)\dot{u}(t) \right\} + b(t)u(t) = c(t) \qquad (C.17)$$

along with the end-point conditions

$$u(t_a) = x_0 = x_a \ , \ u(t_b) = x_N = x_b \qquad (C.18)$$

Note that Eq. (C.17) is just the classical equation of motion obtained from the lagrangian in Eq. (C.18). Now consider the exponent χ of Eq. (C.13) which may be re-written as

$$\chi_N = (1/2\hbar\varepsilon)\left\{ a_1 x_0^2 + (a_N - b_N \varepsilon^2) x_N^2 + 2\varepsilon^2 c_N x_N - \sum_{k=1}^{N-1} \varepsilon c_k u_k + \varepsilon c_N x_N \right\}$$

Using the definitions of y_k in Eq.(C.7) and noting that $u_0 = x_0$ and $u_N = x_N$, we may write

$$\chi_N = (1/2\hbar)\left\{ a_N x_N (u_N - u_{N-1})/\varepsilon - a_1 x_1 (u_1 - u_0)/\varepsilon + \sum_{k=1}^{N} \varepsilon c_k u_k + \varepsilon c_N x_N \right\}$$

It is thus clear that on taking $\varepsilon \to 0$ we have

$$\chi = \lim_{\varepsilon \to 0} \chi_N = (1/2\hbar)\left\{ a_b x_b u_b - a_a x_a u_a + \int_{t_a}^{t_b} c(t)u(t)\,dt \right\} \qquad (C.19)$$

It is interesting to note here that the exponent χ is determined entirely by the solution $u(t)$ of the classical equation of motion in Eq. (C.17). Further it is easy to verify that

$$\chi = S_{cl}/\hbar \qquad (C.20)$$

where S_{cl} is the action $\int_{t_a}^{t_b} \mathcal{L}dt$ evaluated along the classical path determined from Eq.(C.17).

The expression in Eq. (C.19) for χ may be cast ed in the form that brings out the correct dependence of χ on the end points x_a and x_b. For this purpose we need a formal solution of the classical equation Eq. (C.17) which depends on the end points x_a and x_b according to Eq. (C.18) By means of a substitution $v = u\sqrt{a}$, Eq. (C.17) may be cast in the simpler form

$$\ddot{v} + \Omega^2(t)\, v = c/\sqrt{a} \qquad (C.21a)$$

with the end-point conditions Eq. (C.18) as

$$v(t_a) = \sqrt{a_a}\, x_a \,, \quad v(t_b) = \sqrt{a_b}\, x_b \,. \qquad (C.21b)$$

and

$$\Omega^2(t) = \frac{1}{2}\left(\dot{a}^2/2a^2 - \ddot{a}/a \right) + b/a. \qquad (C.21c)$$

Consequently, if v_1 and v_2 be two independent solutions of the homogeneous equation

$$\ddot{v} + \Omega^2(t)\, v = 0 \qquad (C.22a)$$

satisfying the initial conditions

$$\vartheta_1(t_a) = \vartheta_{1a} = 0 \quad , \quad \vartheta_2(t_b) = \vartheta_{2b} = 0 \qquad (C.22b)$$

we may introduce the Green's function of Eq. (C.21) $G(t,s)$ such that $G(t_a,s) = G(t_b,s) = 0$,

$$G(t,s) = \begin{cases} [\vartheta_1(t)\,\vartheta_2(s)]'/Q & , t<s \\ [\vartheta_1(s)\,\vartheta_2(t)]/Q & , t>s \end{cases} \qquad (C.23)$$

$$Q = \dot{\vartheta}_1(s)\,\vartheta_2(s) - \vartheta_1(s)\,\dot{\vartheta}_2(s) \qquad (C.24)$$

Now Eq. (C.21) implies that Q is independent of s and hence we may set either $s = t_a$ or $s = t_b$ obtaining thereby

$$Q = \dot{\vartheta}_{1a}\,\vartheta_{2a} = \vartheta_{1a}\,\dot{\vartheta}_{2b} \qquad (C.25)$$

Hence the formal solution of Eq. (C.21a) satisfying the conditions Eq. (C.21b) is

$$\vartheta(t) = \frac{\sqrt{a_a}\,x_a\,\vartheta_2}{\vartheta_{2a}} + \frac{\sqrt{a_b}\,x_b\,\vartheta_1}{\vartheta_{1b}} - \int_{t_a}^{t_b} \frac{G(t,s)\,c(s)}{\sqrt{a(s)}}\,ds \qquad (C.26)$$

This solution and the relation $u = \vartheta/\sqrt{a}$ is inserted in Eq. (C.19) to arrive at the following form for the exponent χ :

$$\chi = -\frac{1}{4T}\left\{\frac{\dot{a}_b \mathcal{X}_b^2}{a_b} + \frac{\dot{a}_A \mathcal{X}_A^2}{a_A}\right\} + \frac{1}{2h}\left\{\frac{\dot{\vartheta}_{1b}\mathcal{X}_b^2}{\vartheta_{1b}} + \frac{\dot{\vartheta}_2 \mathcal{X}_A^2}{\vartheta_{2a}} - \frac{2\mathcal{X}_A \mathcal{X}_b \dot{\vartheta}_{1a}}{\vartheta_{1A}}\right.$$
$$\left. + \frac{2\mathcal{X}_A}{\vartheta_{2a}}\int_{t_a}^{t_b}\frac{c(s)\,\vartheta_2(s)}{\sqrt{a(s)}}\,ds + \frac{2\mathcal{X}_b}{\vartheta_{1b}}\int_{t_A}^{t_b}\frac{c(s)\,\vartheta_1(s)}{\sqrt{a(s)}} - \int_{t_A}^{t_b}dt\int_{t_A}^{t_b}ds\,\frac{c(t)\,G(t,s)\,c(s)}{\sqrt{[a(s)\,a(t)]}}\right. \qquad (C.27)$$

where $\mathcal{X} = \sqrt{a}\,x$ and we have used the relation $\vartheta_{2a}\dot{\vartheta}_{1a} = -\vartheta_{1b}\dot{\vartheta}_{2b}$.

C.3. Normalization factor $\mathcal{D}$.

In order to determine the normalization factor $\mathcal{D}$, it is necessary to obtain an expression for det $\mathbb{P}$. From the definition Eq.(C.8) of the matrix $\mathbb{P}$, it is easy to see that Δ_k, the k_{th} minor of det $\mathbb{P}$ satisfies the recursion relation

$$\Delta_k = (a_k + a_{k+1} - b_k \varepsilon^2)\Delta_{k-1} - a_k^2 \Delta_{k-2} \quad , \quad k > 1 \qquad (C.28)$$

with

$$\Delta_0 = 1 \quad , \quad \Delta_{-1} = 0 \ .$$

It is conveniant to set

$$\Delta_k = (1/\varepsilon)\left(\prod_{j=1}^{k+1} a_j\right)\psi_{k+1} \qquad (C.29)$$

in Eq. (C.28) so that it becomes

$$a_{k+1}\psi_{k+1} - (a_k + a_{k-1})\psi_k + a_k\psi_{k-1} + b_k\varepsilon^2\psi_k = 0 \qquad (C.30a)$$

with the auxiliary conditions

$$\psi_0 = 0 \quad , \quad \psi_1 = \varepsilon/a_1 . \qquad (C.30b)$$

Rearranging the terms in Eq.(C.30a) and dividing throughout by ε^2 we arrive the limit $\varepsilon \to 0$, the differential equation

$$\frac{d}{dt}(a\dot{\psi}) + b\psi = 0 \qquad (C.31a)$$

On the other hand, Eq.(C.30b) implies in the limit $\varepsilon \to 0$, the end-point conditions to be satisfied by $\psi(t)$ .viz.,

$$\psi(t_a) = \psi_a = 0 \quad , \quad \dot{\psi}_a = 1/a_a . \qquad (C.31b)$$

Eq. (C.31a) is just the homogeneous differential equation corresponding to Eq. (C.17). Con-sequently, its solution satisfying the conditions Eq. (C.31b) may be written immediately in terms of ϑ_1 as

$$\psi(t) = \frac{\vartheta_1(t)}{\sqrt{[a(t)\,a(t_a)]}}\;\frac{1}{\dot{\vartheta}_1(t_a)}, \tag{C.31c}$$

Finally, since det $\mathbb{P} = \Delta_{N-1}$, we have

$$\begin{aligned} D &= \lim_{\varepsilon\to 0} D_N = \lim_{\varepsilon\to 0} \left\{ \left(\prod_{k=1}^{N} a_k \right) \frac{\alpha}{\det \mathbb{P}} \right\}^{1/2} \\ &= \left\{ \frac{1}{2\hbar\psi_b} \right\}^{1/2} = \left\{ \frac{(\dot{\vartheta}_{1a}^{2}(a_a a_b))^{1/4}}{\sqrt{(2\hbar\vartheta_{1b})}} \right\} \end{aligned} \tag{C.32}$$

It is easy to check from Eqs. (C.27) and (C.32) that

$$D = \left\{ \frac{1}{2} \left[\frac{\partial^2 \chi}{\partial x_a \partial x_b} \right] \right\}^{1/2} = \left\{ \frac{1}{2\hbar} \left[\frac{\partial^2 S_{cl}}{\partial x_a \partial x_b} \right] \right\}^{1/2} \tag{C.33}$$

Thus the propagator for the quadratic action Eq. (C.1) is given by

$$K(b,a) = (D/\pi i)^{1/2} \exp(i\chi) \tag{C.34}$$

where χ and D are as in Eqs. (C.27) and (C.32) respectively. It is clear from Eqs. (C.20) and (C.33) that the propagator in Eq. (C.34) is precisely given by the Van Vleck-Pauli formula [8] [9].

$$K(b,a) = \left\{ \frac{1}{2\pi i\hbar} \left[\frac{\partial^2 S_{cl}}{\partial x_a \partial x_b} \right] \right\}^{1/2} \exp\left\{ \frac{i}{\hbar} S_{cl} \right\} \tag{C.35}$$

This result is generally true for a quadratic action and is useful for obtaining propagators for two- and three-dimensional quadratic actions without explicit path integrations.[20]

APPENDIX D

Solution of Eq.(III.17 b)

From the homogeneous differential equation

$$\ddot{\gamma}^c_{c1}(\tau) + \Omega J \dot{\gamma}^c_{c1}(\tau) + \nu^2 \gamma^c_{c1}(\tau) = 0 \qquad (D.1)$$

or

$$\left(D^2 + \Omega J D + \nu^2 I\right) \gamma^c_{c1}(\tau) = 0 \qquad (D.2)$$

when $D = \frac{\partial}{\partial \tau}$. The solution for D is

$$D = -\frac{\Omega}{2} J \pm i\omega \qquad (D.3)$$

where $\omega^2 = \frac{\Omega^2}{4} + \nu^2$. We will have

$$\frac{\partial}{\partial \tau} \gamma^c_{c1}(\tau) = \left(-\frac{\Omega}{2} J \pm i\omega\right) \gamma^c_{c1}(\tau)$$

or

$$\frac{\partial \gamma^c_{c1}(\tau)}{\gamma^c_{c1}(\tau)} = \left(-\frac{\Omega}{2} J \pm i\omega\right) \partial \tau \qquad (D.4)$$

The solution is

$$\gamma^c_{c1}(\tau) = e^{-\frac{\Omega}{2} J \tau + i\omega\tau} A + e^{-\frac{\Omega}{2} J \tau - i\omega\tau} B \qquad (D.5)$$

with the boundaries

$$\gamma_{c1}(0) = \gamma^c_1(0) = \gamma_a \quad \text{and} \quad \gamma_{c1}(t) = \gamma^c_{c1}(t) = \gamma_b .$$

After using the boundary conditions, we have

$$A = \gamma_a + \frac{1}{2i \sin(\omega t)} \left\{ e^{\frac{\Omega}{2} J t} \gamma_b - e^{i\omega t} \gamma_a \right\} \qquad (D.6)$$

and

$$B = - \frac{1}{2i \sin(\omega t)} \left\{ e^{\frac{\Omega}{2} J t} r_b - e^{i\omega t} r_a \right\} \qquad (D.7)$$

Then Eq. (D.5) becomes

$$r^{d}_{el}(\tau) = \frac{e^{-\frac{\Omega}{2} J \tau}}{\sin(\omega t)} \left\{ \sin(\omega [t-\tau]) r_a + \sin(\omega \tau) e^{\frac{\Omega}{2} J t} r_b \right\}$$

(D.8).

APPENDIX E

Evaluation of $G(\tau, \tau')$

The Green's function $G(\tau,\tau')$ of our problem can be found from the differential equation

$$\left(D^2 + \Omega J D + \nu^2 I \right) G(\tau,\tau') = \delta(\tau-\tau') . \quad (E.1)$$

We can write $G(\tau,\tau')$ as

$$G(\tau,\tau') = G_1(\tau,\tau') + G_2(\tau,\tau') \quad (E.2)$$

where

$$\left(D^2 + \Omega J D + \nu^2 I \right) G_1(\tau,\tau') = 0 , \quad \tau' > \tau , \quad (E.3)$$

and

$$\left(D^2 + \Omega J D + \nu^2 I \right) G_2(\tau,\tau') = 0 , \quad \tau > \tau' \quad (E.4).$$

with the boundary conditions

$$\left.\begin{aligned} G_1(0,\tau') &= 0 , \\ G_2(t,\tau') &= 0 , \\ G_1(\tau',\tau') &= G_2(\tau',\tau') \\ \text{and} \quad \left\{ \frac{\partial}{\partial\tau} G_2(\tau,\tau') \right\} &- \left\{ \frac{\partial}{\partial\tau} G_1(\tau,\tau') \right\}_{\tau=\tau'} = -1 \end{aligned}\right\} \quad (E.5)$$

The solution of Eq. (E.2) is

$$G_1(\tau,\tau') = e^{-\frac{\Omega}{2}J\tau + i\omega\tau} C + e^{-\frac{\Omega}{2}J\tau - i\omega\tau} D . \quad (E.6)$$

and the solution of Eq. (E.3) is

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$$G_2(z,z') = e^{-\frac{\Omega}{2}Jz + i\omega z} E + e^{-\frac{\Omega}{2}Jz - i\omega z} F . \quad (E.7)$$

Using the boundary conditions in Eq. (E.5), we have

$$C = -\frac{\sin(\omega(t-z'))}{2i\omega\sin(\omega t)} e^{\frac{\Omega}{2}Jz'} \quad (E.8)$$

$$D = \frac{\sin(\omega(t-z'))}{2i\omega\sin(\omega t)} e^{\frac{\Omega}{2}JS} \quad (E.9)$$

$$E = \frac{\sin(\omega z')}{2i\omega\sin(\omega t)} e^{\frac{\Omega}{2}Jz' - i\omega t} \quad (E.10)$$

and

$$F = -\frac{\sin(\omega z')}{2i\omega\sin(\omega t)} e^{\frac{\Omega}{2}Jz' + i\omega t} \quad (E.11)$$

From Eq. (E.6) we get

$$G_1(z,z') = -\frac{e^{-\frac{\Omega}{2}J(z-z')}}{\omega\sin(\omega t)} \sin(\omega(t-z'))\sin(\omega S) \quad (E.12)$$

From Eq. (E.7) we get

$$G_2(z,z') = -\frac{e^{-\frac{\Omega}{2}J(z-z')}}{\omega\sin(\omega t)} \sin(\omega(t-z))\sin(\omega z') \quad (E.13)$$

Then we get the Green's function

$$G(z,z') = -\frac{e^{-\frac{\Omega}{2}J(z-z')}}{\omega\sin(\omega t)} \left\{ \sin(\omega(t-z))\sin(\omega z')H(z-z') + \sin(\omega(t-z'))\sin(\omega z)H(z'-z) \right\} \quad (E.14).$$

where $H(\cdots)$ is the Heaviside step function.

APPENDIX F

Evaluation of $F_{eff}(t,0)$ in Eq. (III.23).

From Van Vleck-Pauli's result, we have

$$F_{eff}(t,0) = \left\{ \det \left[\frac{i}{2\pi\hbar} \left\{ \frac{\partial^2 S_{el}^{eff}(a,b)}{\partial \gamma_a \partial \gamma_b} \right\} \right] \right\}^{1/2} \tag{F.1}$$

We can see from Eqs. (III.22a) and (III.22b) that

$$\frac{\partial^2 S_{el}^{eff}(a,b)}{\partial \gamma_a \partial \gamma_b} = \frac{\partial^2 S_{el}^{0}(a,b)}{\partial \gamma_a \partial \gamma_b}$$

$$= - \frac{m\omega}{\sin(\omega t)} e^{\frac{\Omega}{2} J t} \tag{F.2}$$

Then Eq. (F.1) becomes

$$F_{eff}(t,0) = \left\{ \det \left[\frac{-m\omega}{\sin(\omega t)} e^{\frac{\Omega}{2} J t} \right] \right\}^{1/2}$$

$$= \frac{-m\omega}{\sin(\omega t)} \tag{F.3}$$

when

$$\det \left\{ e^{\frac{\Omega}{2} J t} \right\} = 1 \tag{F.4}$$

APPENDIX G

Evaluation of the Exact Propagator

From Eq.(III.25), we heve the effective propagator

$$K_{eff}(a,b) = \left\{ \frac{m\omega}{2\pi i\hbar \sin(\omega t)} \right\} \exp\left(\frac{i}{\hbar} S^{o}_{el}(a,b) \right)$$

$$\otimes \exp\left\{ \frac{i}{\hbar} \int_0^t \gamma^{cT}_{cl}(\tau) f(\tau) d\tau + \frac{i}{2\hbar m} \int_0^t \int_0^t f^T(\tau) G(\tau,\tau') f(\tau') d\tau d\tau' \right\}$$

$$\otimes \exp\left\{ \frac{iF^T}{2\hbar m} \int_0^t \int_0^t G'(\tau,\tau') d\tau d\tau' + \frac{i}{\hbar} \left[\int_0^t \gamma^{c}_{cl}(\tau) d\tau + \frac{1}{m} \int_0^t \int_0^t f^T(\tau) G(\tau,\tau') d\tau d\tau' \right] F \right\}. \qquad (G.1)$$

From the notation in Eq.(III.6) ;

$$K(a,b) = \langle K_{eff}(a,b) \rangle_F \qquad (G.2)$$

we have

$$K(a,b) = \left(\frac{m\omega}{2\pi i\hbar \sin(\omega t)} \right) e^{\frac{i}{\hbar} S^{o}_{el}(a,b)}$$

$$\otimes \exp\left\{ \frac{i}{\hbar} \int_0^t \gamma^{cT}_{el}(\tau) f(\tau) d\tau + \frac{i}{2\hbar m} \int_0^t \int_0^t f^T(\tau) G(\tau,\tau') f(\tau') d\tau d\tau' \right\}$$

$$\otimes \frac{\int_{-\infty}^{\infty} dF \exp\left\{ \frac{-it F^T}{2\hbar m \nu^2} \left[I - \frac{\nu^2}{t} \int_0^t \int_0^t G(\tau,\tau') d\tau d\tau' \right] F \right\}}{\int_{-\infty}^{\infty} dF \exp\left\{ \frac{-it}{2\hbar m \nu^2} F^T F \right\}}$$

$$\otimes \exp\left\{ \frac{i}{\hbar} \left[\int_0^t \gamma^{cT}_{cl}(\tau) d\tau + \frac{1}{m} \int_0^t \int_0^t f^T(\tau) G(\tau,\tau') d\tau d\tau' \right] F \right\} \qquad (G.3)$$

Using the property of gaussian integration , we get

$$K(a,b) = \left\{\frac{it}{2\hbar m\nu^2}\right\}\left\{\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right\}$$

$$\times\left[\det\left\{\frac{it}{2\hbar m\nu^2}\left[I - \frac{\nu^2}{t}\int_0^t\int_0^t G(\tau,\tau')d\tau d\tau'\right]\right\}\right]^{-1/2}$$

$$\times\exp\left\{\frac{i}{\hbar}S^0_{el}(a,b) + \frac{i}{\hbar}\int_0^t \gamma^{cT}_{el}(\tau)f(\tau)d\tau + \frac{i}{2\hbar m}\int_0^t\int_0^t f^T(\tau)G(\tau,\tau')f(\tau')d\tau d\tau'\right.$$

$$- \frac{1}{\hbar^2}\left(\int_0^t \gamma^{cT}_{el}(\tau)d\tau + \frac{1}{m}\int_0^t\int_0^t f^T(\tau)G(\tau,\tau')d\tau d\tau'\right)$$

$$\times\left[\frac{it}{2\hbar m\nu^2}\left\{I - \frac{\nu^2}{t}\int_0^t\int_0^t G(\tau,\tau')d\tau d\tau'\right\}\right]^{-1}$$

$$\left.\times\left(\int_0^t \gamma^{c}_{el}(\tau)d\tau + \frac{1}{m}\int_0^t\int_0^t G(\tau,\tau')f(\tau')d\tau d\tau'\right)\right\}$$

(G.4)

We can write Eq. (G.4) in standard representation as

$$K(a,b) = F(t,0)\exp\left(\frac{i}{\hbar}S_{el}(a,b)\right) \qquad (G.5)$$

where

$$S_{el}(a,b) = S^0_{el}(a,b) + \int_0^t R^{cT}(\tau)f(\tau)d\tau + \frac{1}{2m}\int_0^t\int_0^t f^T(\tau)\mathcal{G}(\tau,\tau')f(\tau')d\tau d\tau'$$

(G.6)

when

$$S^0_{el}(a,b) = S^{0*}_{el}(a,b) + \int_0^t\int_0^t \gamma^{cT}_{el}(\tau)\gamma^{c}_{el}(\tau')d\tau' d\tau, \qquad (G.7)$$

$$R^{cT}(\tau) = \gamma^{cT}_{el}(\tau) + \frac{\nu^4\sin(\omega t)}{2\omega\left(\cos(\frac{\Omega}{2}t)-\cos(\omega t)\right)}\int_0^t \gamma^{cT}_{el}(\tau')d\tau'\int_0^t G(\tau'',\tau)d\tau'', \qquad (G.8)$$

$$\mathcal{G}(\tau,\tau') = G(\tau,\tau') + \frac{\nu^4\sin(\omega t)}{2\omega\sin(\omega t)\left(\cos(\frac{\Omega}{2}t)-\cos(\omega t)\right)}\int_0^t\int_0^t G(\tau,\tau'')G(\tau'',\tau')d\tau'' d\tau''$$

(G.9)

and

$$F(t,0) = \frac{m\nu^2 t}{4\pi i\hbar\left(\cos(\frac{\Omega}{2}t)-\cos(\omega t)\right)} \qquad (G.10)$$

From appendix D, we have

$$\int_0^t r^c_{e1}(\tau)d\tau = \frac{1}{\sin(\omega t)}\int_0^t e^{-\frac{\Omega}{2}J\tau}\left\{\sin(\omega(t-\tau))r_a + \sin(\omega\tau)e^{\frac{\Omega}{2}Jt} r_b\right\}ds$$

$$= \frac{\omega}{\nu^2\sin(\omega t)}\Big\{\left[\cos(\tfrac{\Omega}{2}t)-\cos(\omega t)\right](r_b+r_a)$$

$$+\frac{1}{\omega}\left[\omega\sin(\tfrac{\Omega}{2}t)-\tfrac{\Omega}{2}\sin(\omega t)\right]J(r_b-r_a)\Big\}$$

(G.11)

From appendix E, we have

$$\int_0^t G(\tau',\tau)d\tau' = -\frac{e^{\frac{\Omega}{2}J\tau}}{\omega\sin(\omega t)}\Big\{\sin(\omega\tau)\int_\tau^t e^{-\frac{\Omega}{2}J\tau'}\sin(\omega(t-\tau'))d\tau'$$

$$+\sin(\omega(t-\tau))\int_0^\tau e^{-\frac{\Omega}{2}J\tau'}\sin(\omega\tau')d\tau'\Big\}$$

$$= \frac{1}{\nu^2\sin(\omega t)}\Big\{\sin(\omega t) - e^{\frac{\Omega}{2}J\tau}\Big[\sin(\omega(t-\tau))$$

$$+ e^{-\frac{\Omega}{2}Jt}\sin(\omega\tau)\Big]\Big\}. \quad \text{(G.12)}$$

Using Eq. (G.11) in Eq. (G.10), we get

$$S^0_{e1}(a,b) = S^{0*}_{e1}(a,b) + \frac{m\omega}{4\sin(\omega t)}\Big\{\left(\cos(\tfrac{\Omega}{2}t)-\cos(\omega t)\right)(r_b+r_a)^2$$

$$-\frac{1}{4\omega}\left(\omega\sin(\tfrac{\Omega}{2}t)-\tfrac{\Omega}{2}\sin(\omega t)\right)r_b^TJr_a + \frac{1}{\omega^2}\frac{\left(\omega\sin(\frac{\Omega}{2}t)-\frac{\Omega}{2}\sin(\omega t)\right)^2}{\left(\cos(\frac{\Omega}{2}t)-\cos(\omega t)\right)}(r_b-r_a)^2\Big\}$$

(G.13)

Using Eqs. (G.11) and (G.12) in Eq. (G.7), we get

$$R^{cT}(\tau) = r^{cT}_{e1}(\tau) + \frac{1}{2\sin(\omega t)}\Big\{(r_b+r_a)^T - \frac{1}{\omega}\frac{\left(\omega\sin(\frac{\Omega}{2}t)-\frac{\Omega}{2}\sin(\omega t)\right)}{\left(\cos(\frac{\Omega}{2}t)-\cos(\omega t)\right)}(r_b-r_a)^TJ\Big\}$$

$$\otimes\Big\{\sin(\omega t) - e^{\frac{\Omega}{2}J\tau}\Big[\sin(\omega(t-\tau)) + e^{-\frac{\Omega}{2}Jt}\sin(\omega\tau)\Big]\Big\}.$$

(G.14)

Using Eq. (G.12) in Eq. (G.8), we get

$$\mathcal{G}(z,z') = G(z,z') + \frac{1}{2\omega \sin(\omega t)\left(\cos(\frac{\Omega}{2}t) - \cos(\omega t)\right)}$$

$$\otimes \left\{ \sin(\omega t) - e^{-\frac{\Omega}{2}Jz}\left[\sin(\omega(t-z)) + e^{\frac{\Omega}{2}Jt}\sin(\omega z)\right]\right\}$$

$$\otimes \left\{ \sin(\omega t) - e^{\frac{\Omega}{2}Jz'}\left[\sin(\omega(t-z')) + e^{-\frac{\Omega}{2}Jt}\sin(\omega z')\right]\right\}.$$

(G.15)

APPENDIX H

H.1; Limiting procedure of Eq. (IV.12).

From Eq. (G.13), we have

$$S^{o}_{el}(a,b) = S^{o*}_{el}(a,b) + \frac{m\omega}{4\sin(\omega t)}\Big\{ \left(\cos(\tfrac{\Omega}{2}t)-\cos(\omega t)\right)(r_b+r_a)^2$$
$$-\frac{1}{4\omega}\left(\omega \sin(\tfrac{\Omega}{2}t)-\tfrac{\Omega}{2}\sin(\omega t)\right) r_b^T J r_a$$
$$+\frac{1}{\omega^2}\frac{\left(\omega\sin(\tfrac{\Omega}{2}t)-\tfrac{\Omega}{2}\sin(\omega t)\right)^2}{\left(\cos(\tfrac{\Omega}{2}t)-\cos(\omega t)\right)}(r_b-r_a)^2\Big\}$$

where (H.1)

$$S^{o*}_{el}(a,b) = \frac{m\omega}{2\sin(\omega t)}\Big\{\cos(\omega t)\,(r_b^2+r_a^2) - 2r_a^T e^{\frac{\Omega}{2}Jt} r_b\Big\}. \qquad (H.2)$$

Let the magnetic field going to zero, we get

$$\lim_{\Omega\to 0} S^{o}_{el}(a,b) = \frac{m\nu}{2\sin(\nu t)}\Big\{\cos(\nu t)(r_b^2+r_a^2) - 2r_a^T r_b\Big\}$$
$$+\frac{m\nu}{4\sin(\nu t)}\Big\{\left(1-\cos(\nu t)\right)(r_b+r_a)^2\Big\}$$
$$= \frac{m\nu}{4}\cot(\tfrac{\nu}{2}t)\Big\{r_b-r_a\Big\}^2 \qquad (H.3)$$

H.2; Limiting procedure of Eq. (IV.13)

From Eq. (G.14), we have

$$R^{cT}(\tau) = r^{cT}_{el}(\tau) + \frac{1}{2\sin(\omega t)}\Big\{(r_b+r_a)^T - \frac{1}{\omega}\frac{\left(\omega\sin(\tfrac{\Omega}{2}t)-\tfrac{\Omega}{2}\sin(\omega t)\right)}{\left(\cos(\tfrac{\Omega}{2}t)-\cos(\omega t)\right)}(r_b-r_a)^T J\Big\}$$
$$\times\Big\{\sin(\omega t) - e^{\frac{\Omega}{2}J\tau}\left[\sin(\omega(t-\tau)) + e^{-\frac{\Omega}{2}Jt}\sin(\omega\tau)\right]\Big\}$$

(H.4)

where $\gamma_{el}^{cT}(\tau) = \frac{1}{\sin(\omega t)} e^{-\frac{\Omega}{2}J\tau} \left\{ \sin(\omega(t-\tau))\gamma_a^T + \sin(\omega\tau)\gamma_b^T e^{\frac{\Omega}{2}Jt} \right\}$. (H.5)

When the magnetic field going to zero, we get

$$\lim_{\Omega \to 0} \gamma^{cT}(\tau) = \frac{1}{\sin(\nu t)}\left\{ \sin(\nu(t-\tau))\gamma_a^T + \sin(\nu\tau)\gamma_b^T \right\}$$
$$+ \frac{1}{2\sin(\nu t)}(\gamma_b+\gamma_a)^T \left\{ \sin(\nu t) - \sin(\nu(t-\tau)) - \sin(\nu\tau) \right\}$$
$$= \frac{1}{\sin(\nu t)}\left\{ \gamma_a^T\left[\sin(\nu(t-\tau)) - 2\sin(\tfrac{\nu}{2}t)\sin(\tfrac{\nu}{2}(t-\tau))\sin(\tfrac{\nu\tau}{2}) \right] + \gamma_b^T\left[\sin(\nu\tau) - 2\sin(\tfrac{\nu}{2}t)\sin(\tfrac{\nu}{2}(t-\tau))\sin(\tfrac{\nu}{2}\tau) \right] \right\}$$
(H.6)

H.3; Limiting procedure of Eq. (IV.14).

From Eq. (G.15) we have

$$\mathcal{G}(\tau,\tau') = G(\tau,\tau') + \frac{1}{2\omega\sin(\omega t)(\cos(\frac{\Omega}{2}t) - \cos(\omega t))}$$
$$\times \left\{ \sin(\omega t) - e^{-\frac{\Omega}{2}J\tau}\left[\sin(\omega(t-\tau)) + e^{\frac{\Omega}{2}Jt}\sin(\omega\tau) \right] \right\}$$
$$\times \left\{ \sin(\omega t) - e^{\frac{\Omega}{2}J\tau'}\left[\sin(\omega(t-\tau')) + e^{-\frac{\Omega}{2}Jt}\sin(\omega\tau') \right] \right\}$$
(H.7)

when

$$G(\tau,\tau') = -\frac{e^{-\frac{\Omega}{2}J(\tau-\tau')}}{\omega\sin(\omega t)}\left\{ \sin(\omega(t-\tau))\sin(\omega\tau')H(\tau-\tau') + \sin(\omega(t-\tau'))\sin(\omega\tau)H(\tau'-\tau) \right\}$$
(H.8)

When the magnetic field going to zero, we get

$$\lim_{\Omega \to 0} G(z, z') = -\frac{1}{\nu \sin(\nu t)}\Big[\sin(\nu(t-z))\sin(\nu z')H(z-z')$$

$$+\sin(\nu(t-z'))\sin(\nu z)H(z'-z)\Big]$$

$$+\frac{1}{2\nu\sin(\nu t)(1-\cos(\nu t))}$$

$$\otimes\Big\{\sin(\nu t)-\sin(\nu(t-z))-\sin(\nu z)\Big\}$$

$$\otimes\Big\{\sin(\nu t)-\sin(\nu(t-z'))-\sin(\nu z')\Big\}$$

$$=-\frac{1}{\nu\sin(\nu t)}\Big\{\sin(\nu(t-z))\sin(\nu z')H(z-z')$$

$$+\sin(\nu(t-z'))\sin(\nu z)H(z'-z)$$

$$-4\sin\left(\frac{\nu}{2}(t-z)\right)\sin\left(\frac{\nu}{2}z\right)\sin\left(\frac{\nu}{2}(t-z')\right)\sin\left(\frac{\nu}{2}z'\right)\Big\}. \qquad (H.9)$$

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