INTRODUCTION

An element a of a semigroup S is called an idempotent of S if $a^2 = a$. For a semigroup S, E(S) will denote the set of all idempotents of S, that is;

$$E(S) = \{a \in S \mid a^2 = a\}.$$

A semigroup S is a semilattice if for all a, b $\in$ S, $a^2 = a$ and ab = ba.

A semigroup S is called a left [right] zero semigroup if ab = a[ab = b] for all a, b $\in$ S. A semigroup S with zero 0 is called a zero semigroup if ab = 0 for all a, b $\in$ S.

Let S be a semigroup, and let 1 be a symbol not representing any element of S. The notation $S \cup 1$ denotes the semigroup obtained by extending the binary operation on S to one by defining 11 = 1 and 1a = a1 = a for all a $\in$ S. For a semigroup S, the notation S^1 denotes the following semigroup :

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity,} \\ S \cup 1 & \text{if } S \text{ has no identity.} \end{cases}$$

Let S be a semigroup. A subgroup of S is a subsemigroup of S which is also a group under the same operation.

Let S be a semigroup with identity 1. An element a of S is called a unit of S if there exists a' $\in$ S such that aa' = a'a = 1. Let G be the set of all units of S. Then

$$G = \{a \in S \mid aa' = a'a = 1 \text{ for some } a' \in S\}$$

and G is the greatest subgroup of S which has 1 as its identity, and it is called the <u>group of units</u> of S.

An element a of a semigroup S is <u>regular</u> if a = axa for some $x \in S$. A semigroup S is <u>regular</u> if every element of S is regular.

In any semigroup S, if a, $x \in S$ such that a = axa, then ax and xa are idempotents of S. Hence, if S is a regular semigroup, then $E(S) \neq \phi$.

Let a be an element of a semigroup S. An element x of S is called an <u>inverse</u> of a if a = axa and x = xax. If a is a regular element of a semigroup S, then a = axa for some $x \in S$, and hence xax is an inverse of a. Therefore, a semigroup S is regular if and only if every element of S has an inverse. A semigroup S is called an <u>inverse semigroup</u> if every element of S has a unique inverse, and the unique inverse of the element a of S is denoted by a^{-1}. A semigroup S is an inverse semigroup if and only if S is regular and any two idempotents of S commute with each other [1, Theorem 1.17]. Hence, if S is an inverse semigroup, then E(S) is a semilattice. For any elements a, b of an inverse semigroup S and $e \in E(S)$, the following hold :

$$e^{-1} = e, \ (a^{-1})^{-1} = a \text{ and } (ab)^{-1} = b^{-1}a^{-1}$$

[1, Lemma 1.18].

If a is an element of an inverse semigroup S, then aa^{-1} and $a^{-1}a$ are idempotents of S.

Every group is an inverse semigroup and the identity of the group is its only idempotent.

The relation $\leq$ defined on an inverse semigroup S by

$$a \leq b \text{ if and only if } aa^{-1} = ab^{-1}$$

is a partial order on S [2, Lemma 7.2], and this partial order is called the <u>natural partial order</u> on the inverse semigroup S. We note that the restriction of the natural partial order $\leq$ on an inverse semigroup S to E(S) is as follows : For e, f $\in$ E(S),

$$e \leq f \text{ if and only if } e = ef \ (= fe).$$

Then, if S is a semilattice, $a \leq b$ in S if and only if $a = ab$ $(=ba)$.

Let X be a set. Let $A \subseteq X$, $B \subseteq X$ and $\alpha : A \rightarrow B$ be an onto map. Then α is a <u>partial transformation</u> of X, and we denote A and B by $\Delta\alpha$ and $\nabla\alpha$; respectively. If $\Delta\alpha = \nabla\alpha = \phi$, then α is called the <u>empty transformation</u> of X and is denoted by 0. Let T_X be the set of all partial transformations of X (including 0). For α, $\beta \in T_X$, define the product $\alpha\beta$ as follows : If $\nabla\alpha \cap \Delta\beta = \phi$, we define $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \phi$, let $\alpha\beta : (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \rightarrow (\nabla\alpha \cap \Delta\beta)\beta$ be the composition map. Obviously, $\nabla(\alpha\beta) = (\nabla\alpha \cap \Delta\beta)\beta$. Then T_X is a semigroup with zero 0 and it is called the partial <u>transformation semigroup</u> on the set X.

For any set X, T_X is a regular semigroup with zero and identity,

$$E(T_X) = \{\alpha \in T_X \mid \nabla\alpha \subseteq \Delta\alpha \text{ and } \alpha \text{ is the identity map on } \nabla\alpha\}.$$

An element $\alpha \in T_X$ is a <u>one-to-one partial transformation</u> of X if α is a one-to-one map from $\Delta\alpha$ onto $\nabla\alpha$. Let I_X be the set of all one-to-one partial transformation of X. Then under the composition of maps, I_X is an inverse subsemigroup of T_X, which is called the symmetric inverse semigroup on the set X;

$E(I_X) = \{\alpha \in I_X \mid \alpha$ is the identity map on $\Delta\alpha\}$ [1, page 29].

An element $\alpha \in T_X$ is a full transformation of X if $\Delta\alpha = X$. Let $\mathcal{T}_X$ be the set of all full transformations of X. Then under the composition of maps, $\mathcal{T}_X$ is a subsemigroup of T_X and it is called the full transformation semigroup on X. For any set X, $\mathcal{T}_X$ is also a regular semigroup.

Let X be a set. The notation G_X denotes the permutation group on X. Then G_X is the group of units of T_X, also of I_X and of $\mathcal{T}_X$.

Let S and T be semigroups and $\psi : S \to T$ be a map. The map ψ is a homomorphism from S into T if

$$(ab)\psi = (a\psi)(b\psi)$$

for all a, b $\in$ S, and ψ is called an isomorphism if ψ is a homomorphism and one-to-one. The semigroups S and T are isomorphic if there is an isomorphism from S onto T and we write $S \cong T$.

A semigroup T is a homomorphic image of a semigroup S if there exists a homomorphism from S onto T.

A homomorphic image of a regular semigroup is clearly a regular semigroup.

Let a semigroup T be a homomorphic image of a semigroup S by a homomorphism ψ. If S is an inverse semigroup, then $T = S\psi$ is an inverse semigroup, for any $a \in S$, $(a\psi)^{-1} = a^{-1}\psi$ [2, Theorem 7.36], and moreover, for each $f \in E(T)$, there is $e \in E(S)$ such that $e\psi = f$

[2, Lemma 7.34], and hence

$$E(T) = \{e\psi \mid e \in E(S)\}.$$

Let S be a semigroup. A relation ρ on S is called <u>left compatible</u> if for a, b, c $\in$ S, $a\rho b$ implies $ca\rho cb$. <u>Right compatible</u> is defined dually. An equivalence relation ρ on S is called a <u>congruence</u> on S if it is both left compatible and right compatible.

Arbitrary intersection of congruences on a semigroup S is a congruence on S.

Let S be a semigroup. If $i = \{(a, a) \mid a \in S\}$, then i is a congruence on S and we call it the <u>identity congruence</u> on S. If $\omega = S \times S$, then ω is a congruence on S we call it the <u>universal congruence</u> on S.

If ρ is a congruence on a semigroup S, then the set

$$S/\rho = \{a\rho \mid a \in S\}$$

with operation defined by

$$(a\rho)(b\rho) = (ab)\rho \qquad (a, b \in S)$$

is a semigroup, and is called the <u>quotient semigroup relative to</u> the congruence ρ.

Let ρ be a congruence on a semigroup S. Then the mapping $\psi : S \to S/\rho$ defined by

$$a\psi = a\rho \qquad (a \in S)$$

is an onto homomorphism.

Conversely, if $\psi : S \to T$ is a homomorphism from a semigroup S into a semigroup T, then the relation ρ on S defined by

$$a\rho b \text{ if and only if } a\psi = b\psi \qquad (a, b \in S)$$

is a congruence on S and $S/\rho \cong S\psi$.

Let ρ be a congruence on an inverse semigroup S. Then S/ρ is an inverse semigroup and hence for $a \in S$, $(a\rho)^{-1} = a^{-1}\rho$ and

$$E(S/\rho) = \{e\rho \mid e \in E(S)\}.$$

A nonempty subset A of a semigroup S is called a <u>left ideal</u> of S if $SA \subseteq A$. A <u>right ideal</u> of a semigroup S is defined dually. An <u>ideal</u> of a semigroup S is both a left ideal and a right ideal of S.

Let S be a semigroup. The relation $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ on S are defined as follows :

$$a\mathcal{L}b \iff S^1a = S^1b.$$

$$a\mathcal{R}b \iff aS^1 = bS^1.$$

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}$$

The relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are called <u>Green's relations</u> on S, and they are equivalence relations on S. Moreover, $\mathcal{L}$ is right compatible and $\mathcal{R}$ is left compatible. For each $a \in S$, let

$$L_a = \{x \in S \mid x\mathcal{L}a\} ;$$

and R_a, H_a are defined similarly.

Every $\mathcal{L}$-class and every $\mathcal{R}$-class of an inverse semigroup S contain exactly one idempotent [1, Theorem 1.17].

In any semigroup S, the $\mathcal{H}$-class of S containing an idempotent e of S is a subgroup of S [1, Theorem 2.16], and

$$H_e = \{a \in S \mid ae = ea = a \text{ and } aa' = e = a'a \text{ for some } a' \in S\}$$

which is the maximum subgroup of S having e as its identity. If a semigroup S has an identity 1, then H_1, the $\mathcal{H}$-class of S containing 1, is the group of units of S.

Let C be a class of semigroups and ρ be a congruence on a semigroup S. Then ρ is called a C congruence if $S/\rho \in C$. Then, a congruence ρ on a semigroup S is a semilattice congruence on S if S/ρ is a semilattice.

Every semigroup S has a minimum semilattice congruence which is the intersection of all semilattice congruences of S.

Let Y be a semilattice and a semigroup $S = \bigcup_{\alpha \in Y} S_\alpha$ be a disjoint union of subsemigroups S_α of S. S is called a semilattice Y of semigroups S_α if $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$; or equivalently, for all $\alpha, \beta \in Y$, $a \in S_\alpha$, $b \in S_\beta$ imply $ab \in S_{\alpha\beta}$.

If $S = \bigcup_{\alpha \in Y} S_\alpha$ is a semilattice Y of semigroups S_α, then the relation ρ defined by

$a\rho b$ if and only if $a, b \in S_\alpha$ for some $\alpha \in Y$ $(a, b \in S)$ is a semilattice congruence on S, for each $\alpha \in Y$, S_α is a ρ-class, and $S/\rho \cong Y$.

Let ρ be a semilattice congruence on a semigroup S. Then S is a semilattice Y of semigroups S_α where $Y = S/\rho$ and for each $\alpha \in Y$, S_α is a ρ-class.

A semilattice of inverse semigroups is an inverse semigroup [2, Theorem 7.52]. Then a semilattice Y of groups is an inverse semigroup.

A subsemigroup T of a semigroup S is called a filter of S if for any a, b $\in$ S, ab $\in$ T implies a, b $\in$ T.

A semigroup S is said to be factorizable if there exist a subgroup G of S and a set E of idempotents of S such that S = GE.

We give general properties of factorizable semigroups in the first chapter. It is shown that every factorizable semigroup is a regular semigroup and has a left identity. It is also proved that if a semigroup S is factorizable as GE, then G is a maximal subgroup of S; and G becomes the group of units of S if S also has an identity. An ideal of a factorizable semigroup is not necessarily factorizable. Necessary and sufficient conditions of an ideal of a factorizable semigroup to be factorizable are given in this chapter.

Minimum semilattice congruences on factorizable semigroups are studied in the second chapter. It is shown that in any factorizable semigroup S with identity 1, the group of units of S is the class of minimum semilattice congruence of S containing 1. The group of units of a regular semigroup with identity need not be a class of its minimum semilattice congruence. A counter example is given.

In the third chapter, we study semilattice congruences on factorizable inverse semigroups. A congruence on the set of all idempotents of a factorizable inverse semigroup S is not necessary able to be extended to a semilattice congruence on S. The following are proved : Let S be a factorizable inverse semigroup. Then every congruence on E(S) can be extended to a semilattice congruence on S if and only if S is a semilattice of groups. Moreover, if any such extension of a given congruence on E(S) exists, it is unique.

The significant result of this thesis is given in Chapter IV. It is proved that for any set X, the partial transformation semigroup on the set X is factorizable if and only if X is a finite set; and also, the full transformation semigroup on the set X is factorizable if and only if X is a finite set.