CHAPTER III

PROPERTIES OF QUOTIENT O-SEMIFIELDS

In this chapter we shall study the relationship between properties on M.C. (O-M.C.) semirings and properties of their quotient ratio semiring (quotient O-semifields). In this chapter the words "O-M.C. semirings" means a semiring with zero element O, which is O-M.C. and of order greater than one.

We saw in Chapter II that $\mathbb{Q}^+[x]$ and $\mathbb{Q}_0^+[x]$ are M.C. semirings whose quotient ratio semirings, $\mathbb{Q}^+(x)$ and $\mathbb{Q}_0^+(x)$, are not total, i.e. $D(\mathbb{Q}^+(x))$ and $D(\mathbb{Q}_0^+(x))$ are not fields. We also saw in Chapter II that $\mathbb{Z}^+$ is an M.C. semiring whose quotient ratio semiring is isomorphic to $\mathbb{Q}^+$ which is total i.e. $D(\mathbb{Q}^+)$ is a field. So we see that some M.C. semiring S have the property that QR(S) is total whereas in other M.C. semirings S, QR(S) is not total. We shall now find a necessary and sufficient condition on an M.C. semiring which guarantees that its quotient ratio semiring is total.

<u>Definition 3.1.</u> Let S be a semiring then S is called <u>derivable</u> if and only if for all $a_1,a_2,b_1,b_2 \in S$ such that $a_1b_2 \neq a_2b_1$ there exist $x_1,x_2,y_1,y_2 \in S$ such that

$$a_2b_2x_2y_2+a_1b_2x_2y_1+a_2b_1x_1y_2 = a_1b_2x_1y_2+a_2b_1x_2y_1.$$

<u>Proposition 3.2.</u> Let S be an M.C. semiring. Then QR(S) is total if and only if S is derivable.

Proof. Assume S is derivable. Let $\alpha, \beta \in QR(S)$ be such that $\alpha \neq \beta$. Choose $(a_1,a_2) \in \alpha$, $(b_1,b_2) \in \beta$. Then $a_1b_2 \neq a_2b_1$. Since S is derivable, there exist $x_1,x_2,y_1,y_2 \in S$ such that $a_2b_2x_2y_2+a_1b_2x_2y_1+a_2b_1x_1y_2 = a_1b_2x_1y_2+a_2b_1x_2y_1$. Hence in QR(S), $1+\frac{a_1y_1}{a_2y_2}+\frac{b_1x_1}{b_2x_2} = \frac{a_1x_1}{a_2x_2}+\frac{b_1y_1}{b_2y_2}$ which implies that

$$1+\alpha[(y_1,y_2)]+\beta[(x_1,x_2)] = \alpha[(x_1,x_2)]+\beta[(y_1,y_2)] .$$

Hence QR(S) is total.

Conversely, suppose that QR(S) is total. To show that S is derivable. Let $a_1,a_2,b_1,b_2 \in S$ be such that $a_1b_2 \neq a_2b_1$. Then $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ in QR(S). Since QR(S) is total, there exist $x_1,x_2,y_1,y_2 \in S$ such that $1+\frac{a_1y_1}{a_2y_2}+\frac{b_1x_1}{b_2x_2} = \frac{a_1x_1}{a_2x_2}+\frac{b_1y_1}{b_2y_2}$. Hence $a_2b_2x_2y_2+a_1b_2x_2y_1+a_2b_1x_1y_2 = a_1b_2x_1y_2+a_2b_1x_2y_1$ so S is derivable. #

Using the same proof as in the cases $\mathbb{Q}_0^+[x]$ and $\mathbb{Z}^+$. We see that $\mathbb{Q}_0^+[x]\cup\{0\}$ is 0-M.C. semiring whose quotient 0-semifield $\mathbb{Q}_0^+(x)\cup\{0\}$ is not total whereas $\mathbb{Z}_0^+$ is a 0-M.C. semiring whose quotient 0-semifield is isomorphic to $\mathbb{Q}_0^+$ which is total. We shall now give a necessary and sufficient condition on a 0-M.C. semiring S which guarantees that its quotient 0-semifield is total.

<u>Definition 3.3.</u> Let S be a semiring with 0. Then S is called <u>0-derivable</u> if and only if for all $a_1,a_2,b_1,b_2 \in S$ such that $a_1b_2 \neq a_2b_1$ and $a_2,b_2 \neq 0$ there exist $x_1,x_2,y_1,y_2 \in S$ such that $x_2,y_2 \neq 0$ and $a_2b_2x_2y_2+a_1b_2x_2y_1+a_2b_1x_1y_2 = a_1b_2x_1y_2+a_2b_1x_2y_1$.

<u>Proposition 3.4.</u> Let S be a 0-M.C. semiring. Then Q(S) is total if and only if S is 0-derivable.

Proof. Similar to the proof of Proposition 3.2. #

Proposition 3.5. Let S be an M.C. semiring of order greater than one. Then QR(S) has no zero element.

See [1], page 12.

We see from the preceding Proposition that if S is an M.C. semiring of order greater than one then QR(S) cannot be a field since a field must contain a zero element. We see from the remark after Definition 1.28 that O-M.C. semirings are the only semirings which can possible have the property that their quotient O-semifield are fields. Every O-M.C. ring R (e.g. $\{2n \mid n \in \mathbb{Z}\}$ or any integral domain) has the property that Q(R) is a field. There exist O-M.C. semirings S which are not rings such that Q(S) is a field. Before we give an example we shall need a lemma.

Lemma 3.6. Let K be a O-semifield. If there exists an $x \in K-\{0\}$ such that x has an additive inverse, then K is a field.

See [1], page 22.

Example 3.7. Let $S = \{a_0+a_1x+\ldots+a_nx^n \mid n \in \mathbb{Z}_0^+$ and $a_1,\ldots,a_n \in \mathbb{Z}$, $a_0 \in \mathbb{Z}_0^+\}$ with the usual addition and multiplication. Then S is an A.C. and O-M.C. semiring but S is not a ring (since $1 \in S$ but its additive inverse does not exist in S).

Consider the O-semifield Q(S), $[(1,1)]$, $[(x,-x)] \in Q(S)-\{0\}$ and $[(1,1)]+[(x,-x)] = [(-x+x,-x)] = [(0,-x)] = 0$ in Q(S). By Lemma 3.6, Q(S) is a field. Claim that $D(S) \cong \mathbb{Z}[x]$. Define $f:D(S)\longrightarrow\mathbb{Z}[x]$ as follows: Let $\alpha \in D(S)$. Choose $(p(x),q(x)) \in \alpha$. Then $p(x),q(x) \in S \subseteq \mathbb{Z}[x]$ define $f(\alpha) = p(x)-q(x)$.

1) if $(r(x),s(x)) \in \alpha$ also, then $p(x)+s(x) = q(x)+r(x)$ so $p(x)-q(x) = r(x)-s(x)$. Therefore f is well-defined.

2) Let $p(x) \in \mathbb{Z}[x]$. Then $p(x) = a_0+a_1x+\dots+a_nx^n$ for some $a_0 \in \mathbb{Z}_0^+$, $a_1,\dots,a_n \in \mathbb{Z}$, $n \in \mathbb{Z}_0^+$.

Case 1. $a_0 \geq 0$. Then $f([(p(x),0)]) = p(x)-0 = p(x)$.

Case 2. $a_0 < 0$. Then $-a_0 > 0$ so $-p(x), -2p(x) \in S$.

Thus $f([(-p(x),-2p(x))]) = -p(x)-(-2p(x)) = p(x)$.

Hence f is onto.

3) Let $\alpha, \beta \in D(S)$ be such that $f(\alpha) = f(\beta)$.

Choose $(p(x),q(x)) \in \alpha$, $(r(x),s(x)) \in \beta$. Then $p(x)-q(x) = r(x)-s(x)$ so $p(x)+s(x) = q(x)+r(x)$ so $\alpha = \beta$. Hence f is 1-1.

4) Let $\alpha, \beta \in D(S)$. Choose $(p(x),q(x)) \in \alpha$ and $(r(x),s(x)) \in \beta$.

Then $f(\alpha\beta) = f([(p(x)r(x)+q(x)s(x), p(x)s(x)+q(x)r(x))]) = p(x)r(x)+q(x)s(x)-p(x)s(x)-q(x)r(x) = (p(x)-q(x))(r(x)-s(x)) = f(\alpha)f(\beta)$. And

$f(\alpha+\beta) = f([(p(x)+r(x), q(x)+s(x))]) = p(x)+r(x)-q(x)-s(x) = f(\alpha)+f(\beta)$. Hence f is a homomorphism.

Therefore we have the claim, so we see that D(S) is not a field.

$\mathbb{Z}_0^+$ is a 0-M.C. semiring such that $Q(\mathbb{Z}_0^+)$ ($= \mathbb{Q}_0^+$) is not a field. This shows that the quotient 0-semifield of 0-M.C. semiring may or may not be a field. We shall now give a necessary and sufficient condition on a 0-M.C. semiring which guarantees that Q(S) is a field.

Definition 3.8. Let S be a semiring with 0. Then S is called extensive if and only if for all $x \in S$ there exist $a, b \in S$ with $b \neq 0$ such that $bx+a = 0$.

Theorem 3.9. Let S be a 0-M.C. semiring. Then Q(S) is a field if and only if S is extensive.

Proof. Assume Q(S) is a field. Let $x \in S \subseteq Q(S)$. Then there exist $a, b \in S$ with $b \neq 0$ such that $x+\frac{a}{b} = 0$. Hence $xb+a = 0$ so S is extensive.

Conversely, suppose that S is extensive and let $\alpha \in Q(S)$. Choose $(x,y) \in \alpha$. Then $y \neq 0$. Since S is extensive, there exist $a, b \in S$ with $b \neq 0$ such that $bx+a = 0$ ___(*). Since $y, b \in S-\{0\} \subseteq Q(S)-\{0\}$, the multiplicative inverses of y and b exist in Q(S). Multiplying (*) by $\frac{1}{yb}$ we get that $\frac{x}{y} + \frac{a}{b} = 0$ in Q(S). Hence $-\alpha$ exists in Q(S). Thus Q(S) is a field. #

We now give an example of an A.C., 0-M.C. semiring S which is not a 0-semifield and D(S) is a field.

Example 3.10. Let $S = [1, \infty) \cup \{0\}$ with the usual addition and multiplication. Then S is an A.C. and 0-M.C. semiring. S is not a 0-semifield since $2 \in S-\{0\}$ and the multiplicative inverse of 2 does not exist. Using the same proof as in Example 2.1 we get that $D(S) \cong \mathbb{R}$ which is a field.

Note that $Q(S) \cong \mathbb{R}_0^+$ (Using the same proof as Example 2.32) so Q(S) is not a field.

We already gave an example of an A.C., 0-M.C. semiring S which is not a ring such that Q(S) is a field in Example 3.7 and if S is the semiring in Example 3.7 then $D(S) \cong \mathbb{Z}[x]$ so D(S) is not a field.

Clearly, if S is a field then D(S) and Q(S) are fields. It is natural to ask is the converse true i.e. suppose that S is an A.C., O-M.C. semiring such that D(S) and Q(S) are field, must S be a field. The answer is yes as the next Theorem shows.

<u>Theorem 3.11</u>. Let S be an A.C. and O-M.C. semiring. Then D(S) and Q(S) are field if and only if S is a field.

<u>Proof</u>. Assume that S is a field. Clearly D(S) and Q(S) are fields.

Conversely, assume that D(S) and Q(S) are fields. To show that S is a field, it suffices to show that S is a ring. Let $u \in S-\{0\}$ then $[(u,u)] \in Q(S)$, the additive inverse of $[(u,u)]$ exists. Let $x,y \in S$ be such that $[(u,u)]+[(x,y)] = [(0,u)]$.
Then $0 = u(uy+ux) = u^2(y+x)$. Hence $x+y = 0$ ______(1).
Since D(S) is a field, S is exact. Let $a,b \in S$ be such that
for all $p,q \in S$, $ap+bq+q = aq+bp+p$ ______(2).
and for all distinct $p,q \in S$ there exist $u,v \in S$ such that $b+qu+pv = a+qv+pu$. Since $a+x \neq a$, there exist $u,v \in S$ such that $b+(a+x)u+av = a+(a+x)v+au$. Hence $b+au+xu+av = a+av+xv+au$
which implies that $b+xu = a+xv$ ______(3).
By (1), $0 = xv+yv$ so $a = a+xv+yv = b+xu+yv$. Since $a \neq b$, we get that $w = xu+yv$ $S-\{0\}$. Hence $a = b+w$.
By(3), $b+xu = b+w+xv$, so $xu = w+xv$ ______(4).
By(2), for all $p,q \in S$, $(b+w)p+bq+q = (b+w)q+bp+p$ so
$bp+wp+bq+q = bq+wq+bp+p$, so $wq+q = wq+p$ ______(5).
Let $z \in S$ be arbitrary. By(1), $0 = zu(x+y) = z(ux+uy) = z(w+xv+uy) = zw+z(xv+uy) = z+wz(xv+uy)$ (by (5)). Hence for all $z \in S$, z has an additive inverse so S is a ring, S is a field. #

Let S be an A.C. and S.M.C. semiring. Then D(S) is a O-M.C. so Q(D(S)) exists. Since S is S.M.C., S is either M.C. or O-M.C..

Case 1. S is M.C.. Then QR(S) exists. Since S is A.C. and S.M.C., QR(S) is A.C. and precise so D(QR(S)) is O-M.C.. Hence Q(D(QR(S))) exists.

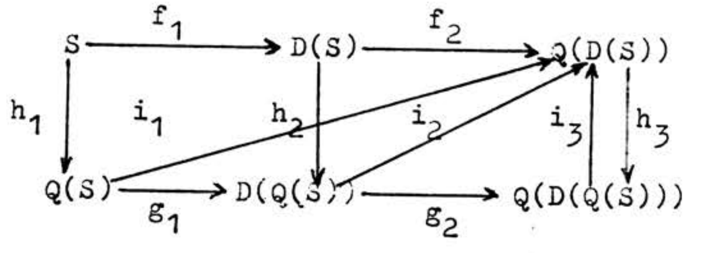
Case 2. S is O-M.C.. Then Q(S) exists. Since S is A.C. and S.M.C., Q(S) is A.C. and precise so D(Q(S)) is O-M.C.. Hence Q(D(Q(S))) exists.

We see from the above that if S is an A.C., S.M.C. semiring with O then Q(D(S)) and Q(D(Q(S))) are fields. It is natural to ask, are these two fields isomorphic? Answer yes!

Note that if S has no O, then Q(D(QR(S))) and Q(D(S)) are fields which are isomorphic as we shall show.

Theorem 3.12. Let S be an A.C. and S.M.C. semiring with O. Then the fields Q(D(Q(S))),Q(D(S)) are naturally isomorphic.

Proof.



Let $h_1:S\longrightarrow Q(S)$, $g_1:Q(S)\longrightarrow D(Q(S))$, $g_2:D(Q(S))\longrightarrow Q(D(Q(S)))$ $f_1:S\longrightarrow D(S)$, $f_2:D(S)\longrightarrow Q(D(S))$ be the embedding given by the constructions. Since $g_1h_1:S\longrightarrow D(Q(S))$, there exists a monomorphism $h_2:D(S)\longrightarrow D(Q(S))$ such that $h_2f_1 = g_1h_1$ and for all $[(x,y)] \in D(S)$, $h_2([(x,y)]) = g_1h_1(x)-g_1h_1(y)$.

Since $g_2h_2:D(S)\longrightarrow Q(D(Q(S)))$ where $Q(D(Q(S)))$ is a field containing $D(S)$, there exists a monomorphism $h_3:Q(D(S))\longrightarrow Q(D(Q(S)))$ such that $h_3f_2 = g_2h_2$ where h_3 is given as follows: For $[(\beta_1, \beta_2)] \in Q(D(S))$, $h_3([(\beta_1,\beta_2)]) = \dfrac{g_2h_2(\beta_1)}{g_2h_2(\beta_2)}$.

Since $f_2f_1:S\longrightarrow Q(D(S))$ where $Q(D(S))$ is a field containing S, $Q(S)$ is a quotient 0-semifield containing S. Then there exists a monomorphism $i_1:Q(S)\longrightarrow Q(D(S))$ such that $i_1h_1 = f_2f_1$ where

$$i_1([(x,y)]) = \frac{f_2f_1(x)}{f_2f_1(y)} \quad \text{for all } [(x,y)] \in Q(S).$$

Since $Q(D(S))$ is a field, it is also a ring so there exists a monomorphism $i_2:D(Q(S))\longrightarrow Q(D(S))$ define as follows

For $[(\beta_1, \beta_2)] \in D(Q(S))$, $i_2([(\beta_1, \beta_2)]) = i_1(\beta_1)-i_1(\beta_2)$.

Hence $i_2g_1 = i_1$. Since $Q(D(Q(S)))$ is the smallest field containing $D(Q(S))$, there exists a monomorphism $i_3:Q(D(Q(S)))\longrightarrow Q(D(S))$ defined as follows:

For $[(\beta_1,\beta_2)] \in Q(D(Q(S)))$, $i_3([(\beta_1,\beta_2)]) = \dfrac{i_2(\beta_1)}{i_2(\beta_2)}$.

Hence $i_3g_2 = i_2$. Claim that $i_2h_2 = f_2$. To prove this, let $[(x,y)] \in D(S)$. Then $i_2h_2([(x,y)]) = i_2(g_1h_1(x)-g_1h_1(y)) = i_2g_1h_1(x)-i_2g_1h_1(y) = i_1h_1(x)-i_1h_1(y) = f_2f_1(x)-f_2f_1(y) = f_2(f_1(x)-f_1(y)) = f_2([(x,y)])$. Hence $i_2h_2 = f_2$.

Next we shall show that $i_3h_3:Q(D(S))\longrightarrow Q(D(S))$ is the identity map. Let $[(\beta_1, \beta_2)] \in Q(D(S))$. Then

$$i_3h_3([(\beta_1, \beta_2)]) = \frac{i_3g_2h_2(\beta_1)}{i_3g_2h_2(\beta_2)} = \frac{i_2h_2(\beta_1)}{i_2h_2(\beta_2)} = \frac{f_2(\beta_1)}{f_2(\beta_2)} = [(\beta_1,\beta_2)].$$

In Theorem 3.12 if S has no 0 we shall prove that Q(D(QR(S))) is isomorphic to Q(D(S)). Before we can prove this we shall need a lemma.

Lemma 3.13. Let S be an A.C. and M.C. semiring and QR(S) the quotient ratio semiring of S and $f:S \longrightarrow QR(S)$ the embedding given by the construction. Let K be any 0-semifield and $i:S \longrightarrow K$ a homomorphism. Then there exists a monomorphism $g:QR(S) \longrightarrow K$ such that $g \circ f = i$.

Proof. Claim that $i(x) \neq 0$ for all $x \in S$. Suppose not, let $x \in S$ be such that $i(x) = 0$. Let $y \in S-\{x\}$. Then $i(x) = i(x)i(x) = 00 = 0i(y) = i(x)i(y) = i(xy)$, so $xx = xy$ Thus $x = y$, a contradiction. Hence we have the claim.
Let $\alpha \in QR(S)$. Choose $(x,y) \in \alpha$, Define $g(\alpha) = \frac{i(x)}{i(y)}$.
Suppose $(u,v) \in \alpha$. Then $uy = vx$, $\frac{i(u)}{i(v)} = \frac{i(x)}{i(y)}$ so g is well-defined.
Let $\alpha, \beta \in QR(S)$. Choose $(x,y) \in \alpha$, $(z,w) \in \beta$. Then

$$g(\alpha + \beta) = g([(xw+yz,yw)]) = \frac{i(xw+yz)}{i(yw)} = \frac{i(x)}{i(y)} + \frac{i(z)}{i(w)} = g(\alpha)+g(\beta).$$

$$g(\alpha\beta) = g([(xz,yw)]) = \frac{i(xz)}{i(yw)} = \frac{i(x)}{i(y)}\frac{i(z)}{i(w)} = g(\alpha)g(\beta).$$

Hence g is a homomorphism.
If $g(\alpha) = g(\beta)$ then $\frac{i(x)}{i(y)} = \frac{i(z)}{i(w)}$. Hence $xw = yz$. Thus $\alpha = \beta$. so g is one-one map.
Let $x \in S$. Fix $a \in S$, $g(f(x)) = g([(xa,a)]) = \frac{i(x)i(a)}{i(a)} = i(x)$, so $g \circ f = i$. #

<u>Theorem 3.14.</u> Let S be an A.C. and S.M.C. semiring such that S has no 0. Then the fields Q(D(QR(S))), Q(D(S)) are naturally isomorphic.

<u>Proof.</u> By Lemma 3.13 then there exist a monomorphism $i_1 : QR(S) \longrightarrow Q(D(S))$ such that $i_1 h_1 = f_2 f_1$. Using the same proof in Theorem 3.12, we have Theorem 3.14. #

Now we shall study relationships between the properties A.C., M.C., S.M.C., precise, total, unitive, exact, derivable, 0-derivable and extensive in a semiring.

<u>Proposition 3.15.</u> Let S be a semiring with 1. Then the following hold:

i) if S is S.M.C. then S is precise.

ii) if S is total then S is unitive, exact, derivable (or 0-derivable if S has a 0)

<u>Proof.</u> i) Let $u, v \in S$ be such that $1+uv = u+v$. Then $11+uv = 1v+1u$ so $u = 1$ or $v = 1$.

ii) Assume S is total. Then, we proved in Chapter II that S is unitive and exact. To show that S is derivable, let $u_1, u_2, v_1, v_2 \in S$ be such that $u_1v_2 \neq u_2v_1$. Then there exist $x, y \in S$ such that $1+xu_1v_2+yu_2v_1 = xu_2v_1+yu_1v_2$ so $u_2v_2+u_2v_2xu_1v_2+u_2v_2yu_2v_1 = u_2v_2xu_2v_1+u_2v_2yu_1v_2$. Therefore $u_2v_2+u_1v_2xu_2v_2+u_2v_1yu_2v_2 = u_1v_2yu_2v_2+u_2v_1xu_2v_2$. Choose $y_1 = xu_2v_2$, $x_1 = yu_2v_2$, $y_2 = x_2 = 1$ then $u_2v_2x_2y_2+u_1v_2x_2y_1+u_2v_1x_1y_2 = u_1v_2x_1y_2+u_2v_1x_2y_1$. Hence S is derivable. A similar proof shows that if S has a 0 then S is total which implies that S is 0-derivable. #

Proposition 3.16. Let S be a ratio semiring. Then the following hold:

i) if S is precise then S is S.M.C..

ii) if S is derivable then S is total and so is exact

Proof. i) Assume S is precise. Let $x_1, x_2, y_1, y_2 \in S$ be such that $x_1y_1+x_2y_2 = x_1y_2+x_2y_1$. Then $1+\frac{x_2y_2}{x_1y_1} = \frac{x_1y_2}{x_1y_1}+\frac{x_2y_1}{x_1y_1} = \frac{y_2}{y_1}+\frac{x_2}{x_1}$. Since S is precise, $\frac{x_2}{x_1} = 1$ or $\frac{y_2}{y_1} = 1$. Hence $x_1 = x_2$ or $y_1 = y_2$. Therefore S is S.M.C..

ii) Assume S is derivable. We must show that S is total. Let $x, y \in S$ be such that $x \neq y$. Then $1x \neq 1y$. Since S is derivable, there exist u_1, u_2, v_1, v_2 S such that $yu_2v_2+xu_2v_1+yu_1v_2 = xu_1v_2+yu_2v_1$, $1+\frac{xu_2v_1}{yu_2v_2}+\frac{yu_1v_2}{yu_2v_2} = \frac{xu_1v_2}{yu_2v_2}+\frac{yu_2v_1}{yu_2v_2}$. Therefore $1+x\frac{v_1}{yv_2}+y\frac{u_1}{yu_2} = x\frac{u_1}{yu_2}+y\frac{v_1}{yv_2}$. Hence S is total. #

Remark 3.17. In a O-semifield K, if K is O-derivable then K is total so exact.

Proposition 3.18. Let S_1, S_2 be semiring with multiplicative identities. Then $S_1 \times S_2$ is not precises.

Proof. Let $x \in S_1-\{1\}$ and $y \in S_2-\{1\}$ then $(1,1)+(x,1)(1,y) = (1,1)+(x,y) = (1+x,1+y) = (x,1)+(1,y)$. But $(x,1) \neq (1,1)$ and $(1,y) \neq (1,1)$. Hence $S_1 \times S_2$ is not precise. #

<u>Proposition 3.19.</u> Let S be a semiring with 1. If $S = S_1 \times S_2$ where S_1, S_2 have order > 1 and $1 \in S_1 \cap S_2$. Then S is not precise.

<u>Proof.</u> Let $x \in S_1 - \{1\}$ and $y \in S_2 - \{1\}$.
Then $(1,1)+(x,1)(1,y) = (1,1)+(x,y) = (1+x,1+y) = (x,1)+(1,y)$ but $(x,1) \neq (1,1)$ and $(1,y) \neq (1,1)$. Hence S is not precise. #

The converse of this proposition is not always true as the next example shows.

<u>Example 3.20.</u> Let $S = \{1,2,3,6,12\}$. Define $a+b = \text{g.c.d}\{a,b\}$ (greatest common divisor of a,b), $a \cdot b = \text{l.c.m}\{a,b\}$ (least common multiple of a,b). Then S is a semiring with 1. S is not precise since $1+2\cdot 3 = 1 = 2+3$ but $2 \neq 1$ and $3 \neq 1$.
Clearly $S \neq S_1 \times S_2$ where S_1, S_2 are semirings of order > 1.

<u>Example 3.21.</u> Let $S = [1,\infty)$, define $x+y = \min\{x,y\}$ and $x \cdot y = \max\{x,y\}$ then $(S,+,\cdot)$ is a semiring and $1 \in S$ is a multiplicative identity and an additive zero.

1) S is precise. Let $u,v \in S$ be such that $1+uv = u+v$ then $1 = u$ or v.

2) S is unitive. Let $a = b = 1$ then for all $x,y \in S$, $1x+1y+y = 1y+1x+x = x+y$.

3) S is exact. Since for all $x,y \in S$, $1+xx+yy = 1 = 1+xy+xy$.

4) S is derivable. Let $a_1,a_2,b_1,b_2 \in S$ be such that $a_1b_2 \neq a_2b_1$. Let $x_1 = x_2 = y_1 = y_2 = \max\{a_1,a_2,b_1,b_2\}$. Then $a_2b_2x_2y_2+a_1b_2x_2y_1+a_2b_1x_1y_2 = x_1 = a_1b_2x_1y_2+a_2b_1x_2y_1$.

5) S is not A.C. and not M.C. since $1+2 = 1+3$ but $2 \neq 3$ and $1\cdot 3 = 2\cdot 3$ but $1 \neq 2$.

6) S is not S.M.C.. Since $1\cdot3+2\cdot4 = 3 = 1\cdot4+2\cdot3$ but $1 \neq 2$ and $3 \neq 4$.

7) S is not total. Since $2,3 \in S$ and there do not exist $x,y \in S$ such that $1+2y+3x = 2x+3y$ (since $1+2y+3x = 1$ but $2x+3y \geq 2$).

<u>Example 3.22.</u> Let $S_\infty = [1,\infty)\cup\{\infty\}$ where ∞ is a symbol not representing any element of $[1,\infty)$.

Define $x+y = \min\{x,y\}$ if $x,y \in [1,\infty)$,

$xy = \max\{x,y\}$ if $x,y \in [1,\infty)$, $\infty+\infty = \infty\infty = \infty$,

$x+\infty = \infty+x = x$ and $\infty x = x\infty = \infty$ for all $x \in [1,\infty)$

Then $(S_\infty,+,\cdot)$ is a semiring where $1 \in S_\infty$ is a multiplicative identity and an additive zero and ∞ is a zero element.

Using the same proof as in Example 3.21, we can show that S_∞ is precise, unitive, exact and derivable.

We must show that S_∞ is O-derivable, let $a_1,a_2,b_1,b_2 \in S$ be such that $a_2,b_2 \neq \infty$ and $a_1b_2 = a_2b_1$. if $a_1 = \infty$ then $b_1 \neq \infty$

Choose $x_1 = x_2 = y_1 = y_2 = \max\{a_1,b_1,b_2\}$. We get that

$a_2b_2x_2y_2+a_1b_2x_2y_1+a_2b_1x_1y_2 = x_1 = a_1b_2x_1y_2+a_2b_1x_2y_1$.

If $b_1 = \infty$. Choose $x_1 = x_2 = y_1 = y_2 = \max\{a_1,a_2,b_2\}$.

Then $a_2b_2x_2y_2+a_1b_2x_2y_1+a_2b_1x_1y_2 = a_1b_2x_1y_2+a_2b_1x_2y_1$.

If $a_1 \neq \infty$ and $b_1 \neq \infty$. Choose $x_1 = x_2 = y_1 = y_2 = \max\{a_1,a_2,b_1,b_2\}$

Then $a_2b_2x_2y_2+a_1b_2x_2y_1+a_2b_1x_1y_2 = a_1b_2x_1y_2+a_2b_1x_2y_1$.

S_∞ is not A.C., O-M.C., S.M.C., total, extensive (since $2 \in S$ but there do not exist $b,a \in S_\infty, b \neq \infty$ such that $2b+a = \infty$)

<u>Example 3.23.</u> In Example 2.13, K is a O-semifield. (we called K is the Boolean semifield). Claim that K is S.M.C.,

Let $x_1,x_2,y_1,y_2 \in K$ be such that $x_1y_1+x_2y_2 = x_1y_2+x_2y_1$.

We must show that $x_1 = x_2$ or $y_1 = y_2$. If $x_1 \neq x_2$, then assume that $x_1 = 0$ and $x_2 = 1$. Hence $y_1 = y_2$. Thus K is S.M.C.. Since K is total (see the proof in Example 2.15), then K is unitive, exact, derivable, 0-derivable. But K is not extensive (since $1 \in K$ but there do not exist $b, a \in K$, $b \neq 0$ such that $b1+a = 0$). Clearly K is not A.C..

<u>Now we get the following implications:</u>

S.M.C. implies precise, M.C., 0-M.C.

- does not imply total, derivable ($\mathbb{Q}^+(x)$ with the usual addition and multiplication)
- does not imply unitive, exact ($2\mathbb{Z}$ with the usual addition and multiplication)
- does not imply 0-derivable ($\mathbb{Q}^+(x) \cup \{0\}$ with the usual addition and multiplication)
- does not imply extensive, A.C. (K the Boolean semifield).

precise implies unitive

- does not imply A.C., M.C., S.M.C., total (Example 3.21)
- does not imply 0-M.C., extensive (Example 3.22)
- does not imply derivable, 0-derivable ($\mathbb{Q}^+(x)$ and $\mathbb{Q}^+(x) \cup \{0\}$ with the usual addition and multiplication).
- does not imply exact ($\mathbb{Z}$ with the usual addition and multiplication)

total implies unitive, exact, derivable, 0-derivable.

- does not imply A.C. $((\mathbb{Q}^+(x), \min, \cdot))$
- does not imply precise, S.M.C. $(\mathbb{Q}^+ \times \mathbb{Q}^+ \times \mathbb{Q}^+ \times \mathbb{Q}^+, \min, \cdot)$
- does not imply extensive (K the Boolean semifield)

unitive does not imply A.C., M.C., S.M.C., total (Example 3.21)

does not imply O-M.C., extensive (Example 3.22)

does not imply precise ($Q^+ \times Q^+$ with the usual addition and multiplication)

does not imply derivable, O-derivable ($Q^+(x)$, $Q^+(x) \cup \{0\}$ with the usual addition and multiplication).

does not imply exact ($\mathbb{Z}$ with the usual addition and multiplication)

exact implies unitive

does not imply A.C., M.C., S.M.C., total (Example 3.21)

does not imply O-M.C., extensive (Example 3.22)

does not imply precise ($Q^+ \times Q^+$, min, $\cdot$)

The following is an unsolved problem: Suppose that S is exact. Is S derivable or O-derivable ?

derivable does not imply A.C., M.C., S.M.C., total (Example 3.21)

does not imply O-M.C., extensive (Example 3.22)

does not imply precise (($Q^+ \times Q^+$, min, $\cdot$))

does not imply unitive, exact ($2\mathbb{Z}^+$ with the usual addition and multiplication)

O-derivable does not imply A.C., O-M.C., S.M.C., total, extensive (Example 3.22)

does not imply precise (($Q^+ \times Q^+ \cup \{0\}$, min, $\cdot$))

does not imply unitive, exact ($2\mathbb{Z}$ with the usual addition and multiplication)

Extensive implies derivable

does not imply A.C. $((\{A \subseteq \mathbb{Z} \mid A \text{ is finite}\}, \cup, \cap))$

does not imply O-M.C. ($\mathbb{Z}_4$ = the set of congruence classes modulo 4 with the usual addition and multiplication)

does not imply S.M.C., precise ($\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ with the usual addition and multiplication)

does not imply total, exact ($\mathbb{Z}$ with the usual addition and multiplication)

does not imply unitive ($2\mathbb{Z}$ with the usual addition and multiplication)

The following is an unsolved problem.

Suppose that S is extensive. Is S O-derivable?