CHAPTER III
PROBLEM SOLVING METHOD

3.1 Partial Difference Equation (PDE) Theory

Linear partial difference equation are of the second order are frequently referred to as being of the elliptic parabolic, and hyperbolic. Such a classification is possible is the equation has been rearranged, to the form

$$\sum_{i=1}^{n} A_i \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i} + Cu + D = 0 \qquad (3.1)$$

in which the coefficient A_i, evaluated at the point (x_1, x_2, x_3 ....x_n).u is dependent variable and x_i are independent variable. Since the coefficient A_i, B_i ,Cand D are function of variable(x_1, x_2, x_3 ....x_n),the classification of Partial Difference Equation may vary according to the particular point being consider in the (x_1, x_2, x_3 ....x_n) space. Very frequently, one of the independent variables will be time t and the remainder will be distance coordinate x y and z. When using a finite-difference technique to solve a Partial Difference Equation a network of grid point is the first established thought out the region of interest occupied by the independent. (Wilkes, 1969)

3.2 The Finite Difference Approximation

Suppose for simplicity that u=u(x,y).Assuming that u propose a sufficient number of partial derivatives, the value of u at the two points(x, y) and (x+h , y+k) are related by Tarlor's expansion:

$$u(x+h, y+k) = u(x, y) + (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})u(x, y) + \frac{1}{2!}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^2 u(x, y) + ...$$

where the remaining term is given by

$$R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n u(x + \xi h, y + \xi k) \tag{3.2}$$

Expanding in Talor's series for $u_{i-1,j}$and about the central value , we obtain

$$u_{i-1,j} = u_{i,j} - \Delta x u_x + \frac{(\Delta x)^2}{2!}u_{xx} - \frac{(\Delta x)^3}{3!}u_{xxx} + \ldots.. \tag{3.3}$$

$$u_{i+1,j} = u_{i,j} + \Delta x u_x + \frac{(\Delta x)^2}{2!}u_{xx} + \frac{(\Delta x)^3}{3!}u_{xxx} + \ldots.. \tag{3.4}$$

Here $u_x = \partial u / \partial x$, $u_{xx} = \partial^2 u / \partial x^2$,etc, all derivatives are evaluated at grid-point (i,j).By taking these equations single, by adding or subtracting one form the other, we obtain the following finite-difference formulas are obtained for the first and second order derilative at (i,j):

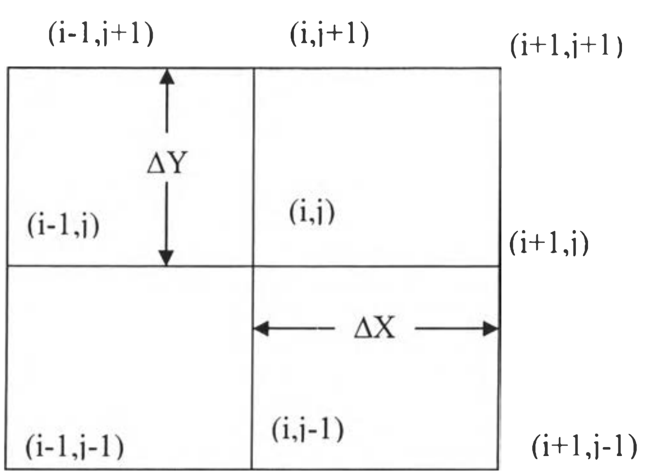

Figure 3.1 The coordinate in x and y direction.

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x) \tag{3.5}$$

$$\frac{\partial u}{\partial x} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} + O(\Delta x) \quad (3.6)$$

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x)^2 \quad (3.7)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + O(\Delta x)^2 \quad (3.8)$$

Formulas are known as forward backward and central difference form respectively. Similar forms exist for $\partial u / \partial y$,and $\partial^2 u / \partial y^2$ It also be shown that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}}{4\Delta x \Delta y} + O(\Delta x)^2 \quad (3.9)$$

By taking more and neighboring points, an unlimited number of other approximation can be obtained, but above forms are the most compact.

For convenience, the central difference operator defined by δ_x will be used occasionally. It is defined by

$$\delta_x u_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{\Delta x} \quad (3.10)$$

where

$$\delta_x^2 u_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \quad (3.11)$$

(Wilkes, 1969)

3.3 The Implicit Form of the Difference Equation

The explicit method previously described ,$v_{i,n}$ depends only on $v_{i-1,n-1}$,$v_{i,n-1}$ and $v_{i+1,n-1}$ area A can have any influence on the value of $v_{i,n}$ whereas it is known that the solution u(x,y) of the partial difference equation depend on the value of u both in A and in B in the time earlier than t_n.

Furthermore, the convergence criterion $0<\Delta t/(\Delta x)^2<=1/2$,places n undesirable restriction on the time increment which can be used. For problems extending over large values of time, this could result in excessive amount of computation.

The implicit method, overcomes both these difficulties at the expense of some what more complicate calculation procedure from evaluating at the advance point of time t_{n+1} instead of at t_n as in the implicit method. The difference equation with finite difference approximation becomes

$$\frac{v_{i,n+1} - v_{i,n}}{\Delta t} = \frac{v_{i-1,n+1} - 2v_{i,n+1} + v_{i+1,n+1}}{(\Delta x)^2}$$

That is ,the following relation existed between the value of v at four points shown in the space time grid

$$-\lambda v_{i-1,n+1} + (1+2\lambda)v_{i,n+1} - \lambda v_{i+1,n+1} = v_{i,n} \qquad (3.14)$$

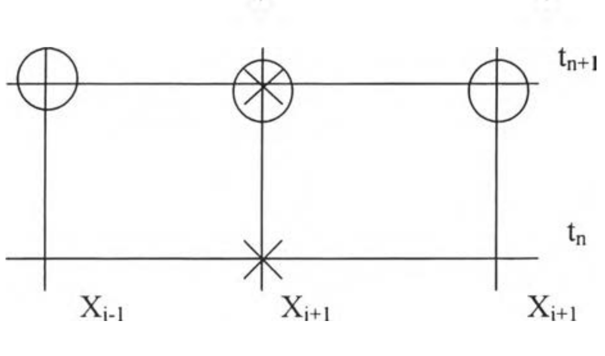

Figure 3.2 The implicit form.

(Wilkes, 1969)

3.4 The Implicit Alternating-Direction Method

The implicit alternating-direction method (IAD) provides a means for solving the parabolic equation in two dimensional by using tridiagonal matrices. Each time step consists of two half-time steps. For the first half-time step, the equation (3.15) is approximated by

$$\frac{\Phi^*_{i,j} - \Phi_{i,j,n}}{\Delta t/2} = \delta_x^2 \Phi^*_{i,j} + \delta_y^2 \Phi_{i,j,n} \tag{3.15}$$

Followed by

$$\frac{\Phi_{i,j,n+1} - \Phi^*_{i,j}}{\Delta t/2} = \delta_x^2 \Phi^*_{i,j} + \delta_y^2 \Phi_{i,j,n+1} \tag{3.16}$$

Rearrange and simplify, these equations to get equation 3.17 and 3.18

$$-\Phi^*_{i-1,j} + 2\left(\frac{1}{\lambda_{x_{i,j}}} + 1\right)\Phi^*_{i,j} - \Phi^*_{i+1,j} = \Phi_{i,j-1,n} + 2\left(\frac{1}{\lambda_{y_{i,j}}} - 1\right)\Phi_{i,j,n} + \Phi_{i,j+1,n} \tag{3.17}$$

and

$$\Phi_{i,j-1,n+1} + 2\left(\frac{1}{\lambda_{y_{i,j}}} + 1\right)\Phi_{i,j,n+1} + \Phi_{i,j+1,n+1} = -\Phi^*_{i-1,j} + 2\left(\frac{1}{\lambda_{x_{i,j}}} - 1\right)\Phi^*_{i,j} - \Phi^*_{i+1,j} \tag{3.18}$$

3.5 Equation Resulting from the Implicit Method

For the solution of equation resulting from the implicit alternating-direction method, the equation is approximated by

$$(1+2\lambda)\Phi^{*}_{1,n+1} - 2\lambda\Phi^{*}_{2,n+1} = \Phi_{1,n} + \lambda g_0(t_{n+1})$$

$$-\lambda_{x_{i,j}}\Phi^{*}_{i-1,n+1} + (1+2\lambda)\Phi^{*}_{i,n+1} - \lambda\Phi^{*}_{i+1,n+1} = \Phi_{i,n} \text{ for } 2 \le i \le M-2 \tag{3.19}$$

$$\lambda_{x_{i,j}}\Phi^{*}_{m-2,n+1} + (1+2\lambda)\Phi^{*}_{m-1,n+1} - \lambda\Phi^{*}_{m,n+1} = \Phi_{m-1,n} + \lambda g_1(t_{n+1})$$

Express more clearly, system of equations are special form system

$$(1+2\lambda_{x_{i,j}})\Phi^{*}_{0,j} - 2\lambda_{x_{i,j}}\Phi^{*}_{1,j} = d_0$$

$$-\lambda_{x_{i,j}}\Phi^{*}_{0,j} + (1+2\lambda_{x_{i,j}})\Phi^{*}_{1,j} - \lambda_{x_{i,j}}\Phi^{*}_{2,j} = d_1$$

....

$$-\lambda_{x_{i,j}}\Phi^{*}_{i-1,j} + (1+2\lambda_{x_{i,j}})\Phi^{*}_{i,j} - \lambda_{x_{i,j}}\Phi^{*}_{i+1,j} = d_i \tag{3.20}$$

....

$$-\lambda_{x_{i,j}}\Phi^{*}_{m-3,j} + (1+2\lambda_{x_{i,j}})\Phi^{*}_{m-2,j} - \lambda_{x_{i,j}}\Phi^{*}_{m,j} = d_{m-2}$$

$$-2\lambda_{x_{i,j}}\Phi^{*}_{m-2,j} + (1+2\lambda_{x_{i,j}})\Phi^{*}_{m-1,j} = d_{m-1}$$