CHAPTER I

PRELIMINARIES

In this chapter we shall give some notations, definitions and theorems used in this thesis. Our notations are

$\mathbb{Z}$ is the set of all integers,

$\mathbb{Z}^+$ is the set of all positive integers,

$\mathbb{Q}$ is the set of all rational numbers,

$\mathbb{Q}^+$ is the set of all positive rational numbers,

$\mathbb{R}^+$ is the set of all positive real numbers,

$\mathbb{Z}_n$, $n \in \mathbb{Z}^+$, is the set of congruence classes modulo n in $\mathbb{Z}$,

$\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$,

$\mathbb{Q}_0^+ = \mathbb{Q}^+ \cup \{0\}$.

Definition 1.1. A triple $(S,+,\cdot)$ is said to be a right seminear-ring iff

(a) $(S,+)$ and $(S,\cdot)$ are semigroups and

(b) $(x+y)z = xz + yz$ for all $x,y,z \in S$. (right distributive law)

A left seminear-ring is defined similarly. If S is a right and left seminear-ring then we call S a distributive seminear-ring.

Throughout this thesis the word "seminear-ring" will mean a right seminear-ring. Each statements for right seminear-rings has a dual statement for left seminear-rings.

Example 1.2. $\mathbb{Z},\mathbb{Z}^+,\mathbb{Z}_0^+,\mathbb{Q}^+$ and $\mathbb{R}^+$ with the usual addition and multiplication are seminear-rings.

Example 1.3. Let S be a nonempty set. Define + and · on S by $x + y = x$ and $x \cdot y = y$ for all $x,y \in S$. Then $(S,+,\cdot)$ is a seminear-ring.

Example 1.4. Let $(S,+)$ be a semigroup. Let $M(S) = \{f : S \longrightarrow S \mid f \text{ is a map}\}$. Define + and · on M(S) by $(f+g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(g(x))$ for all $x \in S$. Then $(M(S),+,\cdot)$ is a seminear-ring.

Definition 1.5. A seminear-ring $(D,+,\cdot)$ is said to be a ratio seminear-ring iff $(D,\cdot)$ is a group. (In J. Hattakosol's thesis [1], ratio seminear-rings are called division seminear-rings)

Example 1.6. $\mathbb{Q}^+$ and $\mathbb{R}^+$ with the usual addition and multiplication are ratio seminear-rings. Also, if we define $x + y = \min\{x,y\}$ for all x,y $\mathbb{Q}^+$ or $\mathbb{R}^+$ and use the usual multiplication we still obtain a ratio seminear-ring.

Example 1.7. Let $(D,\cdot)$ be a group. Define + on D by $x + y = y$ for all $x,y \in D$ or $x + y = x$ for all $x,y \in D$. Then $(D,+,\cdot)$ is a ratio seminear-ring.

Example 1.8. Let $D = \{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \mid x,z \in \mathbb{Q}^+, y \in \mathbb{Q}\}$ with the usual addition and multiplication. Then $(D,+,\cdot)$ is a ratio seminear-ring.

Example 1.9. Let D_1 and D_2 be ratio seminear-rings. Then $D_1 \times D_2$ with the usual product structure is a ratio seminear-ring.

Definition 1.10. Let $(S,+,\cdot)$ be a seminear-ring and T a nonempty subset of S. T is said to be a subseminear-ring of S iff $(T,+,\cdot)$

is a seminear-ring. A subseminear-ring is said to be a ratio subseminear-ring iff it is a ratio seminear-ring.

Definition 1.11. Let S be a semigroup. S is said to be a band iff $x^2 = x$ for all $x \in S$. S is said to be a rectangular band iff $xyx = x$ for all $x,y \in S$.

Lemma 1.12. If $(D,+,\cdot)$ is a finite ratio seminear-ring then $(D,+)$ is a band.

See [1], page 17.

Definition 1.13. Let G be a group and G_1, G_2 subgroups of G. Then G is said to be a Zappa-Szep product of G_1 and G_2, denoted by $G = G_1 * G_2$, iff $G = G_1G_2$ and $G_1 \cap G_2 = \{e\}$ where e is the identity of G.

Example 1.14. Let S_3 be the symmetric group on three letters. Thus $S_3 = \{(1),(12),(13),(23),(123),(132)\}$. Let $A = \{(1),(12)\}$ and $A_3 = \{(1),(123),(132)\}$. Then A and A_3 are subgroups of S_3. Since $(13) = (132)(12) \in A_3A$ and $(23) = (123)(12) \in A_3A_2$, $S_3 = A_3A$. Hence $S_3 = A_3 * A$.

Theorem 1.15. Let D be a finite ratio seminear-ring. Then there exist unique ratio subseminear-rings D_1, $D_2 \subseteq D$ such that

(1) $x + y = x$ for all $x,y \in D_1$,

(2) $x + y = y$ for all $x,y \in D_2$,

(3) $(D,\cdot) = (D_1,\cdot) * (D_2,\cdot)$ $(= (D_2,\cdot) * (D_1,\cdot))$.

Furthermore,

(4) $(D,+) \cong (D_1,+) \times (D_2,+)$,

(5) $D_2 + D_1 = \{e\}$ (where e is the identity of $(D,\cdot)$),

(6) $(D,+)$ is a rectangular band, $D_1 = \{x \in D \mid x + e = x\}$

and $D_2 = \{x \in D | x + e = e\}$.

See [1], page 17 - 19.

Definition 1.16. Let S be a seminear-ring and $x \in S$. x is said to be left additively cancellative (L.A.C.) iff for all $y,z \in S$ ($x+y = x+z$ implies $y = z$). A right additively cancellative (R.A.C.) element is defined similarly. x is said to be additively cancellative (A.C.) iff it is both left and right additively cancellative. S is said to be left additively cancellative (L.A.C.) iff y is L.A.C. for all $y \in S$. S is said to be right additively cancellative (R.A.C.) iff y is R.A.C. for all $y \in S$. S is said to be additively cancellative (A.C.) iff S is both left and right additively cancellative.

x is said to be a left additive zero of S iff $x + y = x$ for all $y \in S$. A right additive zero of S is defined similarly. x is said to be an additive zero of S iff it is both a left and a right additive zero.

x is said to be a left additive identity of S iff $x + y = y$ for all $y \in S$. A right additive identity of S is defined similarly. x is said to be an additive identity of S iff it is both a left and a right additive identity of S.

x is said to be a left multiplicative zero of S if $xy = x$ for all $y \in S$. A right multiplicative zero of S is defined similarly. x is said to be a mutiplicative zero of S iff it is both a left and a right multiplicative zero of S.

Let $d \in S$. Then x is said to be a left additive identity of d iff $x + d = d$. A right additive identity of d is defined similarly. x is said to be an additive identity of d iff it is both a left and a right additive identity of d.

The set of all left (right) additive identities of d is denoted by $LI_S(d)$ ($RI_S(d)$). The set of all additive identities of d is denoted by $I_S(d)$.

Example 1.17. $\mathbb{Z}^+$, $\mathbb{Q}^+$ and $\mathbb{R}^+$ with the usual addition and multiplication are A.C.

Proposition 1.18. Let D be a ratio seminear-ring.

(1) If one element of D is a left (right) additive zero then every element of D is a left (right) additive zero so (D,+) is a left (right) zero semigroup.

(2) If one element of D is a left (right) additive identity then every element of D is a left (right) additive identity so (D,+) is a right (left) zero semigroup.

Proof. To prove (1), let d be a left additive zero of D. Then $d + x = d$ for all $x \in D$. Let $y \in D$. We must show that y is a left additive zero of D. $y + z(d^{-1}y) = d(d^{-1}y) + z(d^{-1}y) = (d+z)(d^{-1}y) = d(d^{-1}y) = y$ for all $z \in D$. Let $C = \{zd^{-1}y \mid z \in D\}$. Since $(D,\cdot)$ is a group, $C = D$. Thus $y + w = y$ for all $w \in D$. Hence y is a left additive zero of D. Since y is an arbitary element in D, we obtain (1).

To prove (2), let d be a left additive identity of D. Then $d + x = x$ for all $x \in D$. Let $y \in D$. We must show that y is a left additive identity of D. $y + zd^{-1}y = d(d^{-1}y) + z(d^{-1}y) = (d+z)d^{-1}y = zd^{-1}y$ for all $z \in D$. Let $C = \{zd^{-1}y \mid z \in D\}$. Since $(D,\cdot)$ is a group, $C = D$. Thus $y + w = w$ for all $w \in D$. Hence y is a left additive identity of S. Since y is an arbitary element in D, we obtain (2).

Proposition 1.19. If D is a ratio seminear-ring of order greater than 1 then D contains no additive zero.

See [1], page 22.

Corollary 1.20. Let D be a ratio seminear-ring and e the identity of $(D,\cdot)$. If e is an additive zero of D then $D = \{e\}$.

Proof. It immediately follows from Proposition 1.19.
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Proposition 1.21. If D is a ratio seminear-ring of order greater than 1 then D contains no additive identity.

See [1], page 22.

Definition 1.22. An ideal I of a semigroup S is called completely prime iff for any $a,b \in S$, $ab \in I$ implies that $a \in I$ or $b \in I$.

Definition 1.23. Let S be a semigroup and F a nonempty subset of S. F is said to be a filter in S iff for all $x,y \in S$ ($x,y \in F$ iff $xy \in F$).

It is well-known that F is a filter in S implies that $S \setminus F = \emptyset$ or $S \setminus F$ is a completely prime ideal in S.

Example 1.24. $\mathbb{Z}^+$ with the usual multiplication is a semigroup. Let $F = \{n \in \mathbb{Z}^+ \mid n \text{ is odd}\}$. Then F is a filter in $\mathbb{Z}^+$, so $\mathbb{Z}^+ \setminus F$ is a completely prime ideal in $\mathbb{Z}^+$.

Proprosition 1.25. Let S be a seminear-ring and $d \in S$. Then the following statements hold :

(1) $LI_S(d) = \emptyset$ or $LI_S(d)$ is an additive subsemigroup of S.

(2) $RI_S(d) = \emptyset$ or $RI_S(d)$ is an additive subsemigroup of S.

(Therefore $I_S(d) = \emptyset$ or $I_S(d)$ is an additive subsemigroup of S.)

(3) If S is a seminear-ring with identity e then the following statements hold :

(3.1) $LI_S(e) = \emptyset$ or $LI_S(e)$ is a subseminear-ring of S.

(3.2) $RI_S(e) = \emptyset$ or $RI_S(e)$ is a subseminear-ring of S.

(Therefore $I_S(e) = \emptyset$ or $I_S(e)$ is a subseminear-ring of S.)

(3.3) $LI_S(e) \cdot d \subseteq LI_S(d)$, $RI_S(e) \cdot d \subseteq RI_S(d)$ and $I_S(e) \cdot d \subseteq I_S(d)$.

(4) If S is a ratio seminear-ring then the following statements hold :

(4.1) $LI_S(d) = LI_S(e) \cdot d$ and $LI_S(e) \cdot LI_S(d) \subseteq LI_S(d)$.

(4.2) $RI_S(d) = RI_S(e) \cdot d$ and $RI_S(e) \cdot RI_S(d) \subseteq RI_S(d)$.

(4.3) $I_S(d) = I_S(e) \cdot d$ and $I_S(e) \cdot I_S(d) \subseteq I_S(d)$.

(4.4) If $RI_S(d) = S$ and $F \subseteq LI_S(d)$ is a filter in $(S,+)$ then $F = S = \{e\}$.

(4.5) If $LI_S(d) = S$ and $F \subseteq RI_S(d)$ is a filter in $(S,+)$ then $F = S = \{e\}$.

<u>Proof</u>. (1) Suppose that $LI_S(d) \neq \emptyset$. Let $x,y \in LI_S(d)$. Then $x + d = y + d = d$, so $(x+y) + d = x + (y+d) = x + d = d$. Hence $x + y \in LI_S(d)$.

(2) The proof of (2) is similar to the proof of (1).

(3) Assume that S is a seminear-ring with identity e.

(3.1) Suppose that $LI_S(e) \neq \emptyset$. By (1), $LI_S(e)$ is an additive subsemigroup of S so we need only show that $LI_S(e)$ is a multiplicative subsemigroup of S. Let $x,y \in LI_S(e)$. Then $x + e = y + e = e$, so $xy + e = xy + (y+e) = (xy+y) + e = (x+e)y + e = ey + e = y + e = e$. Hence $xy \in LI_S(e)$.

(3.2) The proof of (3.2) is similar to the proof of (3.1).

(3.3) Let $x \in LI_S(e) \cdot d$. Then $x = yd$ for some

$y \in LI_S(e)$. So $y + e = e$. Thus $x + d = yd + d = (y+e)d = ed = d$. Hence $x \in LI_S(d)$. Therefore $LI_S(e) \cdot d \subseteq LI_S(d)$.

Similarly, we can show that $RI_S(e) \cdot d \subseteq RI_S(d)$.

Since $I_S(e) \cdot d \subseteq LI_S(e) \cdot d \subseteq LI_S(d)$ and $I_S(e) \cdot d \subseteq RI_S(e) \cdot d \subseteq RI_S(d)$, $I_S(e) \cdot d \subseteq LI_S(d) \cap RI_S(d) = I_S(d)$.

(4) Assume that S is a ratio seminear-ring

(4.1) By (3.3), $LI_S(e) \cdot d \subseteq LI_S(d)$ so we need only show that $LI_S(d) \subseteq LI_S(e) \cdot d$. Let $x \in LI_S(d)$. Then $x + d = d$, so $e = dd^{-1} = (x+d)d^{-1} = xd^{-1} + e$. Thus $xd^{-1} \in LI_S(e)$. Hence $x = (xd^{-1})d \in LI_S(e) \cdot d$. Therefore $LI_S(d) \subseteq LI_S(e) \cdot d$.

To show that $LI_S(e) \cdot LI_S(d) \subseteq LI_S(d)$, note that $LI_S(e)$ is a multiplicative subsemigroup of S so $LI_S(e) \cdot LI_S(d) \subseteq LI_S(e)$. Thus $LI_S(e) \cdot LI_S(d) = LI_S(e) \cdot (LI_S(e) \cdot d) = [LI_S(e) \cdot LI_S(e)] \cdot d \subseteq LI_S(e) \cdot d = LI_S(d)$.

The proofs of (4.2) and (4.3) are similar to the proof of (4.1).

(4.4) Assume that $RI_S(d) = S$ and $F \subseteq LI_S(d)$ is a filter in $(S,+)$. Then $d + x = d$ for all $x \in S$. Thus d is a left additive zero of S and S is a ratio seminear-ring. By Proposition 1.18, $(S,+)$ is a left zero semigroup. Thus $LI_S(d) = \{d\}$. Since $F \neq \emptyset$, $F = \{d\}$. Hence $\{d\}$ is a filter in $(S,+)$. Claim that $S \setminus \{d\} = \emptyset$. Suppose that $S \setminus \{d\} \neq \emptyset$. Let $y \in S \setminus \{d\}$. Since $RI_S(d) = S$, $d + y = d$. Thus $d + y \in \{d\}$. Since $\{d\}$ is a filter in $(S,+)$, $y = d$, a contradiction. Hence $S \setminus \{d\} = \emptyset$. Therefore $S = \{d\}$. Since $e \in S$, $d = e$. Consequently, $S = \{e\} = F$.

The proof of (4.5) is similar to the proof of (4.4). #

<u>Definition 1.26</u>. A seminear-ring $(K,+,\cdot)$ is said to be a <u>seminear-field</u> iff there exists an element a in K such that $a^2 = a$ and $(K\setminus\{a\},\cdot)$ is a group. Such an element a is called a <u>special element</u> of K.

<u>Example 1.27</u>. (1) $\mathbb{Q}_0^+$ and $\mathbb{R}_0^+$ with usual the addition and multiplication are seminear-fields.

(2) Let $(G,\cdot)$ be a group with zero element ∞. Define + on G by

(i) $x + y = \infty$ for all $x,y \in G$ or

(ii) $x + y = \begin{cases} x \text{ if } x = y \\ \infty \text{ if } x \neq y \end{cases}$ for all $x,y \in G$.

Then $(G,+,\cdot)$ is a seminear-field with ∞ as a special element.

(3) $K = \{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \mid x,z \in \mathbb{Q}^+,\ y \in \mathbb{Q}\} \cup \{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$

with the usual addition and multiplication is a seminear-field with $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ as a special element.

(4) Let $(G,\cdot)$ be a group and let a be a symbol not representing any element of G. Let $K = G \cup \{a\}$. Define + on K and extend $\cdot$ to K by

(i) $ax = xa = a$ and $x + y = x$ for all $x,y \in K$,

(ii) $ax = xa = a$ and $x + y = y$ for all $x,y \in K$,

(iii) $ax = xa = x$ and $x + y = x$ for all $x,y \in K$ or

(iv) $ax = xa = x$ and $x + y = y$ for all $x,y \in K$.

Then $(K,+,\cdot)$ is a seminear-field with a as a special element.

<u>Theorem 1.28</u>. If K is a seminear-field of order greater than 2 then there exists a unique special element of K.

See [1], page 26.

<u>Theorem 1.29</u>. Let K be a seminear-field with a as a special element. Then ($a \cdot x = a$ for all $x \in K$ or $a \cdot x = x$ for all $x \in K$) and ($x \cdot a = a$ for all $x \in K$ or $x \cdot a = x$ for all $x \in K$).

See [1], page 28.

From Theorem 1.29 we get that there are 4 types of special elements a in a seminear-field. They have exactly one of the following properties.

(1) $ax = xa = a$ for all $x \in K$,

(2) $ax = xa = x$ for all $x \in K$,

(3) $ax = a$ and $xa = x$ for all $x \in K$ and

(4) $ax = x$ and $xa = a$ for all $x \in K$.

We call a special element a satisfying (1),(2),(3) or (4) a <u>category I,II,III or IV special element of K</u> respectively (In [1] a seminear-field with a category I,II,III or IV special element is called a category I,II,III or IV seminear-field, respectively.). Clearly a category I special element is unique and so is a category II special element.

Note that Example 1.27(1), Example 1.27(2), Example 1.27(3) and Example 1.27(4(i)), (4(ii)) are seminear-fields with a category I special element. Example 1.27(4(iii)), (4(iv)) are seminear-fields with a category II special element.

A seminear-field with a category III or IV special element is of order 2 (see [1], page 29). For a complete classification up to isomorphism see [1], page 29 - 34.

<u>Theorem 1.30</u>. Let K be a seminear-field with a as a category I special element. Then (either $a + x = a$ for all $x \in K$ or $a + x = x$ for all $x \in K$) and (either $x + a = a$ for all $x \in K$ or $x + a = x$

for all $x \in K$).

See [1], page 35.

From Theorem 1.30 we see that a category I special element a has exactly one of the following properties.

(1) $a + x = x + a = x$ for all $x \in K$. In this case we say that a is a zero special element (a 0-special element).

(2) $a + x = x + a = a$ for all $x \in K$. In this case we say that a is an infinity special element (an ∞-special element).

(3) $a + x = a$ and $x + a = x$ for all $x \in K$. Then for all $x,y \in K$, $x + y = (x+a) + y = x + (a+y) = x + a = x$. Thus $(K,+)$ is a left zero semigroup.

(4) $a + x = x$ and $x + a = a$ for all $x \in K$. Then for all $x,y \in K$, $x + y = x + (a+y) = (x+a) + y = a + y = y$. Thus $(K,+)$ is a right zero semigroup.

Definition 1.31. Let K be a seminear-field with a as a category I special element. If K contains a 0-special element then K is called a 0-seminear-field. If K contains on ∞-special element then K is called a ∞-seminear-field. If K contains a category I special element a satisfying (3) then K is called an additive left zero seminear-field with a category I special element. If K contains a category I special element a satisfying (4) then K is called an additive right zero seminear-field with a category I special element.

Theorem 1.32. Let K be a seminear-field with a as a category II special element. Then $(K\setminus\{a\},+,\cdot)$ is a ratio seminear-ring.

See [1], page 53 - 54.

Theorem 1.32. Let K be a seminear-field with a as a category II special element. Then $(K \setminus \{a\}, +, \cdot)$ is a ratio seminear-ring.

See [1], page 53 - 54.

Theorem 1.33. Let K be a seminear-field with a as a category II special element and let e denote the identity of $(K \setminus \{a\}, \cdot)$. Then the following statements hold :

(1) If $a + a = a$ then $(K, +)$ is a band.

(2) If $a + a \neq a$ then $a + a = e + e$.

(3) $e + a = a$ or $e + a = e + e$.

(4) $a + e = a$ or $a + e = e + e$.

(5) For all $x, y \in K \setminus \{a\}$, $x + x = y + y$ iff $x = y$.

(6) For all $x \neq a$, $x + a = a$ or $x + a = x + e$.

(7) For all $x \neq a$, $a + x = a$ or $a + x = e + x$.

See [1], page 54.