CHAPTER II

SEMINEAR-FIELDS WITH A CATEGORY II SPECIAL ELEMENT

In this chapter we shall study seminear-fields with a category II special element.

We recall the definition of a seminear-field with a category II special element.

Let K be a seminear-field and let a be a special element of K. K is said to be a seminear-field with a as a category II special element iff a has the property that $ax = xa = x$ for all $x \in K$.

Example 2.1. Let D be a ratio seminear-ring and let a be a symbol not representing any element of D. Extend + and $\cdot$ from D to $D \cup \{a\}$ by

(i) $ax = xa = x$ for all $x \in D \cup \{a\}$,

(ii) $a + x = e + x$ and $x + a = x + e$ for all $x \in D$ and

(iii) $a + a = e + e$.

Then $(D \cup \{a\},+,\cdot)$ is a seminear-field with a as a category II special element.

Proposition 2.2. Let K be a seminear-field with a as a category II special element. If $a + x = x + a = a$ for all $x \in K\setminus\{a\}$ or $a + x = x + a = x$ for all $x \in K\setminus\{a\}$ then $|K| = 2$.

Proof. By Theorem 1.32, $(K\setminus\{a\},+,\cdot)$ is a ratio seminear ring. Let e denote the identity of $(K\setminus\{a\},\cdot)$. Assume that $a + x = x + a = a$ for all $x \in K\setminus\{a\}$. Multiply this equation on the right by e, we get that $x + e = e + x = e$ for all $x \in K\setminus\{a\}$. By Corollary 1.20, $|K\setminus\{a\}| = 1$. Hence $|K| = 2$.

Assume that $a + x = x + a = x$ for all $x \in K\setminus\{a\}$. Claim that $e + x = x + e = e$ for all $x \in K\setminus\{a\}$. Let $x \in K\setminus\{a\}$. Then $a + x^{-1} = x^{-1} + a = x^{-1}$, so $x + e = e + x = e$. By Corollary 1.20, $|K\setminus\{a\}| = 1$. Hence $|K| = 2$.

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<u>Proposition 2.3</u>. If K is a seminear-field with a category II special element of order greater than 2 then K contains no additive zero.

<u>Proof</u>. Let a be a category II special element of K. By Theorem 1.32, $(K\setminus\{a\},+,\cdot)$ is a ratio seminear-ring of order greater than 1. By Proposition 1.19, $K\setminus\{a\}$ contains no additive zero.

If a is an additive zero of K then, by Proposition 2.2, $|K| = 2$, a contradiction.

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<u>Proposition 2.4</u>. If K is a seminear-field with a category II special element of order greater than 2 then K contains no additive identity.

<u>Proof</u>. This proof is similar to the proof of the Proposition 2.2, except that we use Proposition 1.21 instead of Proposition 1.19.

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<u>Theorem 2.5</u>. Let K be a seminear-field with a as a category II special element and let e be the identity of $(K\setminus\{a\},\cdot)$. Then

(1) $|K| = 2$ if and only if e is a multiplicative zero of K.

(2) $|K| > 2$ if and only if K contains no multiplicative zero.

<u>Proof</u>. To prove (1), assume that $|K| = 2$. Then $K = \{e,a\}$. Since a is a category II special element, a is not a multiplicative zero.

Now $e \cdot e = e$ and $e \cdot a = a \cdot e = e$, so e is a multiplicative zero.

Conversely, assume that e is a multiplicative zero. Then $e \cdot x = x \cdot e = e$ for all $x \in K$. To show that $|K| = 2$. Let $y \in K\setminus\{a\}$. Then $y = e \cdot y = e$. Therefore $|K| = 2$.

To prove (2), it is clear that if K contains no multiplicative zero then $|K| > 2$. Conversely, suppose that there exists a multiplicative zero y_o in K. Then $xy_o = y_o x = y_o$ for all $x \in K$. Thus $y_o^2 = y_o$ and so $y_o = e$ or $y_o = a$.

If $y_o = a$ then $e = e \cdot a = e \cdot y_o = y_o = a$, a contradiction. Hence $y_o = e$ so by (1), $|K| = 2$. Therefore if $|K| > 2$ so K contains no multiplicative zero.

<u>Theorem 2.6</u>. Let K be a seminear-field and let a be a category II special element of K. Then the following statements hold :

(1) If K is L.A.C. then $x + y = y$ for all $x,y \in K\setminus\{a\}$.

(2) If K is R.A.C. then $x + y = x$ for all $x,y \in K\setminus\{a\}$.

(3) If $a + a = a$ then the following statements hold :

(3.1) K is L.A.C. if and only if $x + y = y$ for all $x,y \in K$.

(3.2) K is R.A.C. if and only if $x + y = x$ for all $x,y \in K$.

(3.3) K connot be A.C.

<u>Proof</u>. Let e be the identity of $(K\setminus\{a\},\cdot)$.

(1) Assume that K is L.A.C. Claim that $z + a = a$ for all $z \in K\setminus\{a\}$. Let $z \in K\setminus\{a\}$. If $z + a \neq a$ then by Theorem 1.33 $z + a = z + e$ and so $a = e$, a contradiction. Hence $z + a = a$ for all $z \in K\setminus\{a\}$. Let $x,y \in K\setminus\{a\}$. Then $xy^{-1} + a = a$, so $y = ay = (xy^{-1} + a)y = x + ay = x + y$. Hence $x + y = y$ for all

$x,y \in K\setminus\{a\}$.

The proof of (2) is similar to the proof of (1).

(3) Assume that $a + a = a$.

(3.1) Assume that K is L.A.C. Let $x,y \in K$.

Case 1. $x = y = a$. Then $x + y = a + a = a = y$.

Case 2. $x \neq a$, $y = a$. In the proof of (1), we showed that $z + a = a$ for all $z \in K\setminus\{a\}$. Thus $x + y = x + a = a = y$.

Case 3. $x = a$, $y \neq a$. By (1), we get that $e + y = y$. If $a + y = a$ then $a + y = a + a$. Thus $y = a$, a contradiction. Hence $a + y = e + y = y$. Therefore $x + y = a + y = e + y = y$.

Case 4. $x \neq a$, $y \neq a$. By (1), $x + y = y$.

Hence $x + y = y$ for all $x,y \in K$.

The converse is obvious.

The proof of (3.2) is similar to the proof of (3.1).

(3.3) Suppose that K is A.C. In the proof of (1), we showed that $z + a = a$ for all $z \in K\setminus\{a\}$. Now $e + a = a$. $a + a = a = e + a$. Since K is R.A.C., $a = e$, a contradiction. Hence K is not A.C.

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Theorem 2.7. Let K be a seminear-field with a as a category II special element and let e be the identity of $(K\setminus\{a\},\cdot)$. Then K is A.C. if and only if $K = \{a,e\}$ with structure

$\cdot$	e	a
e	e	e
a	e	a

and

+	e	a
e	e	a
a	a	e

Proof. Assume that K is A.C. Claim that $a + x = x + a = a$ for all $x \in K\setminus\{a\}$. Let $x \in K\setminus\{a\}$. If $x + a = x + e$ then $a = e$, a contradiction. Hence $x + a = a$. If $a + x = e + x$ then $a = e$, a contradiction. Hence $a + x = a$. Therefore $a + x = x + a = a$ for all $x \in K\setminus\{a\}$. By Proposition 2.2, $|K| = 2$. By Theorem 2.6 (3.3), we get that $a + a = e + e = e$ since $(K\setminus\{a\},+,\cdot)$ is a finite ratio seminear-ring. Therefore we have the above structure.

Conversely, one can easily check that the seminear-field given above is A.C.

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Definition 2.8. Let K be a seminear-field with a as a special element. Let $D = K\setminus\{a\}$. Then $\{x \in D | x + a = a\}$ ($\{x \in D | a + x = a\}$) is called the left (right) fundamental set of a in K. The set $\{x \in K | x + a = a + x = a\}$ is called the fundamental set of a in K. If a is a category II special element of K then we shall denote the left (right) fundamental set of a in K by S_L (S_R) and denote the fundamental set of a in K by S.

Remark 2.9. Let K be a seminear-field with a as a category II special element and let $D = K\setminus\{a\}$.

(1) If $y \in D\setminus S_L$ then y is not L.A.C.

(2) If $y \in D\setminus S_R$ then y is not R.A.C.

(Therefore if $y \in D\setminus S$ then y is not A.C.)

Proof. (1) If $y \in D\setminus S_L$ then $y + a = y + e$ where e is the identity of $(D,\cdot)$. Since $a \neq e$, y is not L.A.C.

The proof of (2) is similar to the proof of (1).

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Proposition 2.10. Let K be a seminear-field with a as a category II special element, $D = K\setminus\{a\}$ and e the identity of $(D,\cdot)$. Then the following statements hold :

(1) $S_L \subseteq LI_D(e)$ and $S_R \subseteq RI_D(e)$. (Therefore $S \subseteq I_D(e)$)

(2) $S_L = \emptyset$ or S_L is a filter in $(D,+)$. (Hence $D\setminus S_L = \emptyset$

or $D \setminus S_L$ is a completely prime ideal of $(D,+)$.)

(3) $S_R = \emptyset$ or S_R is a filter in $(D,+)$. (Hence $D \setminus S_R = \emptyset$ or $D \setminus S_R$ is a completely prime ideal of $(D,+)$.)

(4) $S = \emptyset$ or S is a filter in $(D,+)$. (Hence $D \setminus S = \emptyset$ or $D \setminus S$ is a completely prime ideal of $(D,+)$.)

(5) If $e \in S_L$ then $S_L = LI_D(e)$.

(6) If $e \in S_R$ then $S_R = RI_D(e)$.

(Therefore if $e \in S$ then $S = I_D(e)$.)

<u>Proof</u>. (1) To show that $S_L \subseteq LI_D(e)$, let $x \in S_L$. Then $x + a = a$, so $e = ae = (x+a)e = x + ae = x + e$. Thus $x \in LI_D(e)$. Hence $S_L \subseteq LI_D(e)$. Similarly, we can show that $S_R \subseteq RI_D(e)$.

(2) Suppose that $S_L \neq \emptyset$. Let $x,y \in D$. Assume that $x,y \in S_L$. Then $x + a = a = y + a$, so $(x+y) + a = x + (y+a) = x + a = a$. Thus $x + y \in S_L$.

Conversely, assume that $x + y \in S_L$. Then $(x+y) + a = a$. If $y + a \neq a$ then $x + (y+a) \in D$ (since D is closed w.r.t. addition) so we get a contradiction. Thus $y + a = a$ and so $x + a = a$. Hence $x,y \in S_L$.

The proofs of (3) and (4) are similar to the proof of (2).

(5) Assume that $e \in S_L$. Then $e + a = a$. By (1), it suffices to show that $LI_D(e) \subseteq S_L$. Let $x \in LI_D(e)$. Then $x + e = e$, so $x + a = x + (e+a) = (x+e) + a = e + a = a$. Hence $x \in S_L$. Therefore $LI_D(e) \subseteq S_L$.

The proof of (6) is similar to the proof of (5).

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We shall give an example of a filter F in $LI_D(e)$ where D is a ratio seminear-ring.

<u>Example 2.11</u>. $\mathbb{R}_o^+[X] = \{ \sum_{i=0}^{n} a_i X^i \mid n \in \mathbb{Z}_o^+, a_i \in \mathbb{R}^+ \cup \{0\},$ $i = 0,1,2,\ldots,n\}$.

Let $\mathbb{R}^+[X]$ denote the set $\mathbb{R}_o^+[X] \setminus \{0\}$. Let $\cdot$ be the usual multiplication in $\mathbb{R}^+[X]$. Define $\oplus$ on $\mathbb{R}^+[X]$ by

$$f \oplus g = \begin{cases} f & \text{if } \deg f < \deg g \\ f + g \text{ (usual addition)} & \text{if } \deg f = \deg g \\ g & \text{if } \deg f > \deg g. \end{cases}$$

To show that $\oplus$ is associative, let $f,g,h \in \mathbb{R}^+[X]$.

<u>Case 1</u>. $\deg f = \deg g = \deg h$. Then $\deg (g+h) = \deg f = \deg h = \deg (f+g)$. $f \oplus (g \oplus h) = f \oplus (g+h) = f + (g+h) = (f+g) + h = (f+g) \oplus h = (f \oplus g) \oplus h$.

<u>Case 2</u>. $\deg f = \deg g$, $\deg g < \deg h$. Then $\deg (f+g) = \deg g < \deg h$. $f \oplus (g \oplus h) = f \oplus g = f + g$, $(f \oplus g) \oplus h = (f+g) \oplus h = f + g$.

<u>Case 3</u>. $\deg f = \deg g$, $\deg g > \deg h$. Then $\deg (f+g) = \deg g > \deg h$. $f \oplus (g \oplus h) = f \oplus h = h$, $(f \oplus g) \oplus h = (f+g) \oplus h = h$.

<u>Case 4</u>. $\deg f = \deg h$, $\deg h < \deg g$. Then $\deg f < \deg g$. $f \oplus (g \oplus h) = f \oplus h = f + h$, $(f \oplus g) \oplus h = f \oplus h = f + h$.

<u>Case 5</u>. $\deg f = \deg h$, $\deg h > \deg g$. Then $\deg f > \deg g$. $f \oplus (g \oplus h) = f \oplus g = g$, $(f \oplus g) \oplus h = g \oplus h = g$.

<u>Case 6</u>. $\deg f$, $\deg g$, $\deg h$ are all distincts.

Let $k = \min \{\deg f, \deg g, \deg h\}$.

$$f \oplus (g \oplus h) = (f \oplus g) \oplus h = \begin{cases} f \text{ if } k = \deg f, \\ g \text{ if } k = \deg g, \\ h \text{ if } k = \deg h. \end{cases}$$

To show that $\oplus$ is commutative, let $f,g,h \in \mathbb{R}^+[X]$.

Case 1. deg f = deg g.

$f \oplus g = f + g = g + f = g \oplus f$.

Case 2. deg f < deg g. $f \oplus g = f = g \oplus f$.

Case 3. deg f > deg g. $f \oplus g = g = g \oplus f$.

Hence $\oplus$ is commutative.

To show that $\mathbb{R}^+[X]$ is distributive, let $f,g,h \in \mathbb{R}^+[X]$.

Case 1. deg f = deg g. Then deg (fh) = deg (gh).

$(f \oplus g)h = (f+g)h = fh + gh = fh \oplus gh$.

Case 2. deg f < deg g. Then deg (fh) < deg (gh).

$(f \oplus g)h = fh = fh \oplus gh$.

Case 3. deg f > deg g. Then deg (fh) > deg (gh).

$(f \oplus g)h = gh = fh \oplus gh$.

Hence $\mathbb{R}^+[X]$ is distributive.

It is clear that fg = fh implies that g = h.

Define $\sim$ on $\mathbb{R}^+[X] \times \mathbb{R}^+[X]$ by $(f,g) \sim (f',g')$ iff $fg' = gf'$. Then $\sim$ is clearly an equivalence relation.

Let $\mathbb{R}^+(X) = \dfrac{\mathbb{R}^+[X] \times \mathbb{R}^+[X]}{\sim}$

Define $\oplus'$ and $\odot$ on $\mathbb{R}^+(X)$ as follows : given $[(f,g)]$, $[(f',g')] \in \mathbb{R}^+(X)$, define $[(f,g)] \odot [(f',g')] = [(ff',gg')]$, and

$$[(f,g)] \oplus' [(f',g')] = [(fg' \oplus gf',gg')].$$

Using the same proof as in Theorem 2.11 in [3] page 12 - 14, we obtain that $\mathbb{R}^+(X)$ is a ratio seminear-ring.

Now $[(1,1)]$ is the identity of $(\mathbb{R}^+(X), \odot)$.

We shall compute $I_{\mathbb{R}^+(X)}([(1,1)])$.

$$I_{\mathbb{R}^+(X)}([(1,1)]) = \{[(f,g)] \in \mathbb{R}^+(X) \mid [(f,g)] \oplus' [(1,1)] = [(1,1)]\}$$

$$= \{[(f,g)] \in \mathbb{R}^+(X) \mid [(f \oplus g, g)] = [(1,1)]\}$$
$$= \{[(f,g)] \in \mathbb{R}^+(X) \mid (f \oplus g, g) \sim (1,1)\}$$
$$= \{[(f,g)] \in \mathbb{R}^+(X) \mid f \oplus g = g\}$$
$$= \{[(f,g)] \in \mathbb{R}^+(X) \mid \deg f > \deg g\}.$$

Fix $n \in \mathbb{Z}_o^+$, let $F_n = \{[(f,g)] \in \mathbb{R}^+(X) \mid \deg f > \deg g + n\}$.

Let $[(f,g)] \in F_n$. Then $\deg f > \deg g + n$. To show that deg is well-defined, let $[(f',g')] \in \mathbb{R}^+(X)$ be such that $[(f,g)] = [(f',g')]$. Then $fg' = gf'$, so $\deg g + \deg f' = \deg (gf')$ $= \deg (fg') = \deg f + \deg g' > (\deg g + n) + \deg g'$. Thus $\deg f' > \deg g' + n$. Hence deg is well-defined. Clearly $F_n \subseteq I_{\mathbb{R}^+(X)}([(1,1)])$. To show that F_n is a filter in $\mathbb{R}^+(X)$, let $[(f,g)],[(f',g')] \in \mathbb{R}^+(X)$. Suppose that $[(f,g)], [(f',g')] \in F$. Then $\deg f > \deg g + n$ and $\deg f' > \deg g' + n$. Now $[(f,g)] \oplus' [(f',g')] = [(fg' \oplus gf', gg')]$.

Case 1. $\deg (fg') > \deg (gf')$. Then $fg' \oplus gf' = gf'$.
$\deg (fg' \oplus gf') = \deg (gf') = \deg g + \deg f' > \deg g + (\deg g' + n)$ $= \deg (gg') + n$. Hence $[(f,g)] \oplus' [(f',g')] \in F_n$.

Case 2. $\deg (fg') = \deg (gf')$. Then $fg' \oplus gf' = fg' + gf'$.
$\deg (fg' \oplus gf') = \deg (fg' + gf') = \deg (fg') = \deg f + \deg g' >$ $(\deg g + n) + \deg g' = \deg (gg') + n$. Hence $[(f,g)] \oplus' [(f',g')] \in F_n$.

Case 3. $\deg (fg') < \deg (gf')$. Then $fg' \oplus gf' = fg'$.
$\deg (fg' \oplus gf') = \deg (fg') = \deg f + \deg g' > (\deg g + n) + \deg g'$ $= \deg (gg') + n$. Hence $[(f,g)] \oplus' [(f',g')] \in F_n$.

Conversely, suppose that $[(f,g)] \oplus' [(f',g')] \in F_n$. Then $[(fg' \oplus gf', gg')] \in F_n$, so $\deg (fg' \oplus gf') > \deg (gg') + n$.

Case 1. $\deg (fg') > \deg (gf')$. Then $fg' \oplus gf' = gf'$, so $\deg (gf') > \deg (gg') + n$. Thus $\deg g + \deg f' > (\deg g + \deg g') + n$.

Hence deg $f' > \deg g' + n$. Therefore $[(f',g')] \in F_n$. Since $\deg(fg') > \deg(gf') > \deg(gg') + n$, $\deg f > \deg g + n$. Hence $[(f,g)] \in F_n$.

Case 2. $\deg(fg') = \deg(gf')$. Then $fg' \oplus gf' = fg' + gf'$, so $\deg(fg' + gf') > \deg(gg') + n$. Since $\deg(fg') = \deg(fg' + gf') > \deg(gg') + n$, $\deg f > \deg g + n$ and since $\deg(gf') = \deg(fg' + gf') > \deg(gg') + n$, $\deg f' > \deg g' + n$. Hence $[(f,g)], [(f',g')] \in F_n$.

Case 3. $\deg(fg') < \deg(gf')$.

This proof is similar to Case 1.

Hence F_n is a filter in $\mathbb{R}^+(X)$.

Proposition 2.12. Let K be a seminear-field with a as a category II special element, $D = K\setminus\{a\}$ and let e denote the identity of $(D,\cdot)$.

(1) If $S_L = \emptyset$ and $S_R \neq \emptyset$ then $e \in S_R$ iff $a + a = a$.

(2) If $S_R = \emptyset$ and $S_L \neq \emptyset$ then $e \in S_L$ iff $a + a = a$.

(3) If $\emptyset \neq S_R \subset D$ and $S_L \subset S_R$ then $e \in S_R$ iff $a + a = a$.

(4) If $\emptyset \neq S_L \subset D$ and $S_R \subset S_L$ then $e \in S_L$ iff $a + a = a$.

(5) If $S_L \nsubseteq S_R$ and $S_R \nsubseteq S_L$ then $a + a = e + e$.

Proof. (1) Assume that $S_L = \emptyset$ and $S_R \neq \emptyset$. Suppose that $e \in S_R$. Since $S_R \subseteq RI_D(e)$, $e = e + e$. Since $S_L = \emptyset$, $e + a = e + e = e$. Thus $a + a = (a+e) + a = a + (e+a) = a + e = a$.

Conversely, assume that $a + a = a$. If $S_R = D$ then $e \in S_R$. Suppose that $S_R \subset D$. To show that $e \in S_R$, suppose that $e \in D\setminus S_R$. Then $a + e = e + e$. Let $x \in S_R$. Then $a + x = a$ and $x + a = x + e$. Thus $a = a + a = (a+x) + a = a + (x+a) = a + (x+e) = (a+x) + e = a + e = e + e$, a contradiction. Hence $e \in S_R$.

The proof of (2) is similar to the proof of (1).

(3) Assume that $\emptyset \neq S_R \subset D$ and $S_L \subset S_R$. Suppose that $e \in S_R$. Then $a + e = a$. Let $x \in S_R \setminus S_L$. Then $a + x = a$ and $x + a = x + e$. Thus $a + a = (a+x) + a = a + (x+a) = a + (x+e) = (a+x) + e = a + e = a$

Assume that $e \in D \setminus S_R$. Then $a + e = e + e$. To show that $a + a \neq a$, let $y \in S_R \setminus S_L$. Then $a + y = a$ and $y + a = y + e$. Thus $a + a = (a+y) + a = a + (y+a) = a + (y+e) = (a+y) + e = a + e = e + e \neq a$.

The proof of (4) is similar to the proof of (3).

(5) Assume that $S_L \not\subseteq S_R$ and $S_R \not\subseteq S_L$. Claim that $a + e = e + e$. Since $S_L \not\subseteq S_R$, there is an element x in $S_L \setminus S_R$. Thus $x + a = a$ and $a + x = e + x$. Since $S_L \subseteq LI_D(e)$, $x + e = e$. Thus $a + e = a + (x+e) = (a+x) + e = (e+x) + e = e + (x+e) = e + e$. Since $S_R \not\subseteq S_L$, there is an element y in $S_R \setminus S_L$. Then $a + y = a$ and $y + a = y + e$. Since $S_R \subseteq RI_D(e)$, $e + y = e$. So $a + a = (a+y) + a = a + (y+a) = a + (y+e) = (a+y) + e = a + e = e + e$.

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<u>Theorem 2.13</u>. Let D be a ratio seminear-ring and let a be a symbol not representing any element of D. Let $F_L \subseteq LI_D(e)$ be either $\emptyset$ or a filter in $(D,+)$ and let $F_R \subseteq RI_D(e)$ be either $\emptyset$ or a filter in $(D,+)$. Then the binary operations on D can be extended to $K = D \cup \{a\}$ in such a way that the following properties hold :

(1) K is a seminear-field and a is a category II special element of K.

(2) F_L is the left fundamental set of a in K and F_R is the right fundamental set of a in K.

(3) If $(D,+)$ is not a band then $a + a = e + e$.

(4) If $(D,+)$ is a band then

$$a + a = \begin{cases} a \text{ or } e & \text{if } F_L = F_R = \emptyset, \\ a & \text{if } F_L = \emptyset,\ F_R = D \text{ (in this case } (D,+) \text{ is a right zero semigroup.)}, \\ a & \text{if } F_L = \emptyset,\ \emptyset \neq F_R \subset D,\ e \in F_R, \\ e & \text{if } F_L = \emptyset,\ \emptyset \neq F_R \subset D,\ e \in D \setminus F_R, \\ a \text{ or } e & \text{if } F_L = F_R = D \text{ (in this case } D = \{e\}), \\ a & \text{if } F_L = D,\ F_R = \emptyset \text{ (in this case } (D,+) \text{ is a left zero semigroup.)}, \\ a & \text{if } \emptyset \neq F_L \subset D,\ e \in F_L,\ F_R = \emptyset, \\ e & \text{if } \emptyset \neq F_L \subset D,\ e \in D \setminus F_L,\ F_R = \emptyset, \\ a \text{ or } e & \text{if } \emptyset \neq F_L \subset D,\ F_L = F_R, \\ a & \text{if } \emptyset \neq F_L \subset D,\ \emptyset \neq F_R \subset D, \text{ (either } F_L \subset F_R,\ e \in F_R\ F_R \subset F_L,\ e \in F_L), \\ e & \text{if } \emptyset \neq F_L \subset D,\ \emptyset \neq F_R \subset D, \text{ (either } F_L \subset F_R,\ e \in D \setminus F_R \text{ or } F_R \subset F_L,\ e \in D \setminus F_L), \\ e & \text{if } F_L \not\subseteq F_R,\ F_R \not\subseteq F_L. \end{cases}$$

Furthermore, any extension of addition on D to K such that (1) and (2) hold must be as given above.

<u>Proof</u>. Suppose that $F_L = F_R = \emptyset$. Extend $+$ and $\cdot$ from D to K by

(1) $ax = xa = x$ for all $x \in K$,

(2) $x + a = x + e$, $a + x = e + x$ for all $x \in D$ and

(3) $a + a = \begin{cases} a \text{ or } e & \text{if } (D,+) \text{ is a band}, \\ e + e & \text{if } (D,+) \text{ is not a band}. \end{cases}$

Clearly $(K,\cdot)$ is a semigroup. To show that K is a seminear-field, we shall show that (b_1) $x + (y+z) = (x+y) + z$ for all $x,y,z \in K$ and (c_1) $(x+y)z = xz + yz$ for all $x,y,z \in K$.

To prove (b_1), let $x,y,z \in K$. Consider the follwing cases :

Case 1. $x = y = z = a$.

Subcase 1.1. $a + a = a$.

$x + (y+z) = a + (a+a) = a + a = (a+a) + a = (x+y) + z$.

Subcase 1.2. $a + a = e$.

$x + (y+z) = a + (a+a) = a + e = e + e = e + a = (a+a) + a = (x+y) + z$.

Subcase 1.3. $a + a = e + e$.

$x + (y+z) = a + (a+a) = a + (e+e) = e + (e+e) = (e+e) + e = (e+e) + a$
$= (a+a) + a = (x+y) + z$.

Case 2. $x = y = a,\ z \neq a$.

Subcase 2.1. $a + a = a$. Then $(D,+)$ is a band.

$x + (y+z) = a + (a+z) = a + (e+z) = e + (e+z) = (e+e) + z = e + z$
$= a + z = (a+a) + z = (x+y) + z$.

Subcase 2.2. $a + a = e$. Then $(D,+)$ is a band.

$x + (y+z) = a + (a+z) = a + (e+z) = e + (e+z) = (e+e) + z = e + z$
$= (a+a) + z = (x+y) + z$.

Subcase 2.3. $a + a = e + e$.

$x + (y+z) = a + (a+z) = a + (e+z) = e + (e+z) = (e+e) + z = (a+a) + z$
$= (x+y) + z$.

Case 3. $x = z = a,\ y \neq a$.

$x + (y+z) = a + (y+a) = a + (y+e) = e + (y+e) = (e+y) + e = (e+y) + a$
$= (a+y) + a = (x+y) + z$.

Case 4. $x \neq a,\ y = z = a$.

This proof is similar to Case 2.

Case 5. $x \neq a,\ y \neq a,\ z = a$.

$x + (y+z) = x + (y+a) = x + (y+e) = (x+y) + e = (x+y) + a = (x+y) + z$.

Case 6. $x \neq a,\ y = a,\ z \neq a$.

$x + (y+z) = x + (a+z) = x + (e+z) = (x+e) + z = (x+a) + z = (x+y) + z$.

<u>Case 7</u>. $x = a,\ y \neq a,\ z \neq a$.

This proof is similar to Case 5.

<u>Case 8</u>. $x \neq a,\ y \neq a,\ z \neq a$.

$x + (y+z) = (x+y) + z$.

To prove (c_1), let $x,y,z \in K$.

<u>Case 1</u>. $z = a$.

$(x+y)z = (x+y)a = x + y = xa + ya = xz + yz$.

<u>Case 2</u>. $z \neq a$.

<u>Subcase 2.1</u>. $x = y = a$.

<u>Subcase 2.1.1</u>. $a + a = a$. Then $(D,+)$ is a band.

$(x+y)z = (a+a)z = az = z = z + z = az + az = xz + yz$.

<u>Subcase 2.1.2</u>. $a + a = e$. Then $(D,+)$ is a band.

$(x+y)z = (a+a)z = ez = z + z = az + az = xz + yz$.

<u>Subcase 2.1.3</u>. $a + a = e + e$.

$(x+y)z = (a+a)z = (e+e)z = z + z = az + az = xz + yz$.

<u>Subcase 2.2</u>. $x = a,\ y \neq a$.

$(x+y)z = (a+y)z = (e+y)z = z + yz = az + yz = xz + yz$.

<u>Subcase 2.3</u>. $x \neq a,\ y = a$.

This proof is similar to Subcase 2.2.

<u>Subcase 2.4</u>. $x \neq a,\ y \neq a$.

$(x+y)z = xz + yz$.

Hence K is a seminear-field and we obtain (1) - (4).

Suppose that $F_L = \emptyset$ and $F_R = D$. Since $e \in F_R = RI_D(e)$, $e + e = e$. Thus $(D,+)$ is a band. Extend $+$ and $\cdot$ from D to K by

(1) $ax = xa = x$ for all $x \in K$,

(2) $x + a = x + e$ and $a + x = a$ for all $x \in D$ and

(3) $a + a = a$.

To show that K is a seminear-field, we shall show that (b_2) $x + (y+z) = (x+y) + z$ for all $x,y,z \in K$ and (c_3) $(x+y)z = xz + yz$ for all $x,y,z \in K$.

To prove (b_2), let $x,y,z \in K$. Note that $a + t = a$ for all $t \in K$.

<u>Case 1</u>. $x = a$.

$x + (y+z) = a + (y+z) = a = a + z = (a+y) + z = (x+y) + z$.

<u>Case 2</u>. $x \neq a$.

<u>Subcase 2.1</u>. $y = z = a$.

$x + (y+z) = x + (a+a) = x + a = x + e = (x+e) + e = (x+e) + a = (x+a) + a = (x+y) + z$.

<u>Subcase 2.2</u>. $y = a$, $z \neq a$. Since $z \in D = RI_D(e)$, $e = e + z$.

$x + (y+z) = x + (a+z) = x + a = x + e = x + (e+z) = (x+e) + z = (x+a) + z = (x+y) + z$.

<u>Subcase 2.3</u>. $y \neq a$, $z = a$.

$x + (y+z) = x + (y+a) = x + (y+e) = (x+y) + e = (x+y) + a = (x+y) + z$.

<u>Subcase 2.4</u>. $y \neq a$, $z \neq a$.

$x + (y+z) = (x+y) + z$.

To prove (c_3), let $x,y,z \in K$.

<u>Case 1</u>. $z = a$.

$(x+y)z = (x+y)a = x + y = xa + ya = xz + yz$.

<u>Case 2</u>. $z \neq a$.

<u>Subcase 2.1</u>. $x = y = a$

$(x+y)z = (a+a)z = az = z = z + z = az + az = xz + yz$.

<u>Subcase 2.2</u>. $x = a$, $y \neq a$.

$(x+y)z = (a+y)z = az = z = ez = (e+y)z = z + yz = az + yz = xz + yz$.

<u>Subcase 2.3</u>. $x \neq a$, $y = a$.

$(x+y)z = (x+a)z = (x+e)z = xz + z = xz + az = xz + yz$.

<u>Subcase 2.4</u>. $x \neq a$, $y \neq a$.

$(x+y)z = xz + yz$.

Hence K is a seminear-field and we obtain (1),(2) and (4).

Suppose that $F_L = \emptyset$ and F_R is a proper filter in $(D,+)$. Then $D \setminus F_R$ is an ideal of $(D,+)$. Extend + and · from D to K by

(1) $ax = xa = x$ for all $x \in K$,

(2) $x + a = x + e$ for all $x \in D$,

$a + x = a$ for all $x \in F_R$, $a + x = e + x$ for all $x \in D \setminus F_L$ and

$$(3)\quad a + a = \begin{cases} a & \text{if } (D,+) \text{ is a band, } e \in F_R, \\ e & \text{if } (D,+) \text{ is a band, } e \in D \setminus F_R, \\ e + e & \text{if } (D,+) \text{ is not a band.} \end{cases}$$

To show that K is a seminear-field, we shall show that (b_3) $x + (y+z) = (x+y) + z$ for all $x,y,z \in K$ and (c_3) $(x+y)z = xz + yz$ for all $x,y,z \in K$.

To prove (b_3), let $x,y,z \in K$. Consider the following cases :

<u>Case 1</u>. $x = y = z = a$

<u>Subcase 1.1</u>. $(D,+)$ is a band, $e \in F_R$. Then $a + a = a$. $x + (y+z) = a + (a+a) = a + a = (a+a) + a = (x+y) + z$.

<u>Subcase 1.2</u>. $(D,+)$ is a band, $e \in D \setminus F_R$. Then $a + a = e$. $x + (y+z) = a + (a+a) = a + e = e + e = e + a = (a+a) + a = (x+y) + z$.

<u>Subcase 1.3</u>. $(D,+)$ is not a band. Then $a + a = e + e$ and $e \in D \setminus F_R$.

$x + (y+z) = a + (a+a) = a + (e+e) = e + (e+e) = (e+e) + e = (e+e) + a = (a+a) + a = (x+y) + z$.

<u>Case 2</u>. $x = y = a$, $z \neq a$.

<u>Subcase 2.1</u>. $a + a = a$. Then $(D,+)$ is a band .

If $z \in F_R$ then $x + (y+z) = a + (a+z) = a + a = a = a + z = (a+a) + z = (x+y) + z$. If $z \in D \setminus F_R$ then $e + z \in D \setminus F_R$. Thus

$x + (y+z) = a + (a+z) = a + (e+z) = e + (e+z) = (e+e) + z =$
$e + z = a + z = (a+a) + z = (x+y) + z.$

<u>Subcase 2.2.</u> $a + a = e$. Then $(D,+)$ is a band.

If $z \in F_R$ then $e + z = e$ since $F_R \subseteq RI_D(e)$. Thus $x + (y+z) =$
$a + (a+z) = a + a = e = e + z = (a+a) + z = (x+y) + z)$.

If $z \in D \setminus F_R$ then $e + z \in D \setminus F_R$. Thus $x + (y+z) = a + (a+z) =$
$a + (e+z) = e + (e+z) = (e+e) + z = e + z = (a+a) + z = (x+y) + z.$

<u>Subcase 2.3.</u> $a + a = e + e$.

If $z \in F_R$ then $e + z = e$ since $F_R \subseteq RI_D(e)$. Thus $x + (y+z) =$
$a + (a+z) = a + a = e + e = e + (e+z) = (e+e) + z = (a+a) + z =$
$(x+y) + z.$

If $z \in D \setminus F_R$ then $e + z \in D \setminus F_R$. Thus $x + (y+z) = a + (a+z) =$
$a + (e+z) = e + (e+z) = (e+e) + z = (a+a) + z = (x+y) + z.$

<u>Case 3.</u> $x = z = a$, $y \neq a$.

<u>Subcase 3.1.</u> $(D,+)$ is a band, $e \in F_R$. Then $a + a = a$.

<u>Subcase 3.1.1.</u> $y + e \in F_R$. Since F_R is a filter in $(D,+)$, $y \in F_R$.

$x + (y+z) = a + (y+a) = a + (y+e) = a = a + a = (a+y) + a = (x+y) + z.$

<u>Subcase 3.1.2.</u> $y + e \in D \setminus F_R$. Since $e \in F_R$, $y \in D \setminus F_R$.

$x + (y+z) = a+(y+a) = a + (y+e) = e + (y+e) = (e+y)+ e = (e+y) + a$
$= (a+y) + a = (x+y) + z.$

<u>Subcase 3.2.</u> $(D,+)$ is a band, $e \in D \setminus F_R$. Then $a + a = e$ and $y + e \in D \setminus F_R$.

If $y \in F_R$ then $e + y = e$. Thus $x + (y+z) = a + (y+a) = a + (y+e)$
$= e + (y+e) = (e+y) + e = e + e = e$, $(x+y) + z = (a+y) + a = a + a = e.$

If $y \in D \setminus F_R$ then $x + (y+z) = a + (y+a) = a + (y+e) = e + (y+e) =$
$(e+y) + e = (e+y) + a = (a+y) + a = (x+y) + z.$

<u>Subcase 3.3.</u> $(D,+)$ is not a band. Then $a + a = e + e$ and $e \in D \setminus F_R$.

If $y \in F_R$ then $e + y = e$. Thus $x + (y+z) = a + (y+a) = a + (y+e)$ $= e + (y+e) = (e+y) + e = e + e = a + a = (a+y) + a = (x+y) + z$.

If $y \in D \setminus F_R$ then $x + (y+z) = a + (y+a) = a + (y+e) = e + (y+e) =$ $(e+y) + e = (e+y) + a = (a+y) + a = (x+y) + z$.

Case 4. $x \neq a$, $y = z = a$.

Subcase 4.1. $a + a = a$. Then $(D,+)$ is a band.

$x + (y+z) = x + (a+a) = x + a = x + e$, $(x+y) + z = (x+a) + a =$ $(x+e) + a = (x+e) + e = x + (e+e) = x + e$.

Subcase 4.2. $a + a = e$. Then $(D,+)$ is a band.

$x + (y+z) = x + (a+a) = x + e$, $(x+y) + z = (x+a) + a = (x+e) + a$ $(x+e) + e = x + (e+e) = x + e$.

Subcase 4.3. $a + a = e + e$.

$x + (y+z) = x + (a+a) = x+(e+e) = (x+e) + e = (x+e) + a = (x+a) + a$ $= (x+y) + z$.

Case 5. $x \neq a$, $y \neq a$, $z = a$.

$x + (y+z) = x + (y+a) = x + (y+e) = (x+y) + e = (x+y) + a = (x+y) + z$.

Case 6. $x \neq a$, $y = a$, $z \neq a$.

If $z \in F_R$ then $e + z = e$. Thus $x + (y+z) = x + (a+z) = x + a = x + e$ $= x + (e+z) = (x+e) + z = (x+a) + z = (x+y) + z$.

If $z \in D \setminus F_R$ then $x + (y+z) = x + (a+z) = x + (e+z) = (x+e) + z =$ $(x+a) + z = (x+y) + z$.

Case 7. $x = a$, $y \neq a$, $z \neq a$.

Subcase 7.1. $y + z \in F_R$. Since F_R is a filter in $(D,+)$, $y,z \in F_R$.

$x + (y+z) = a + (y+z) = a = a + z = (a+y) + z = (x+y) + z$.

Subcase 7.2. $y + z \in D \setminus F_R$.

If $y \in F_R$ then $z \in D \setminus F_R$ and $e + y = e$. Thus $x + (y+z) = a + (y+z)$ $= e + (y+z) = (e+y) + z = e + z = a + z = (a+y) + z = (x+y) + z$.

If $y \in D \setminus F_R$ then $x + (y+z) = a + (y+z) = e + (y+z) = (e+y) + z$ $= (a+y) + z = (x+y) + z$.

<u>Case 8</u>. $x \neq a$, $y \neq a$, $z \neq a$.

$x + (y+z) = (x+y) + z$.

To prove (c_3), let $x,y,z \in K$. Consider the following cases :

<u>Case 1</u>. $z = a$.

$(x+y)z = (x+y)a = x + y = xa + ya = xz + yz$.

<u>Case 2</u>. $z \neq a$.

<u>Subcase 2.1</u>. $x = y = a$.

This proof is the same as the proof of Subcase 2.1 in (c_1).

<u>Subcase 2.2</u>. $x = a$, $y \neq a$.

If $y \in F_R$ then $e + y = e$. Thus $(x+y)z = (a+y)z = az = z = ez =$ $(e+y)z = z + yz = az + yz = xz + yz$.

If $y \in D \setminus F_R$ then $(x+y)z = (a+y)z = (e+y)z = z + yz = az + yz =$ $xz + yz$.

<u>Subcase 2.3</u>. $x \neq a$, $y = a$.

$(x+y)z = (x+a)z = (x+e)z = xz + z = xz + az = xz + yz$.

<u>Subcase 2.4</u>. $x \neq a$, $y \neq a$.

$(x+y)z = xz + yz$.

Hence K is a seminear-field and we obtain (1) - (4).

Suppose that $F_L = F_R = D$. Then $D = I_D(e) =$ $\{x \in D \mid x + e = e + x = e\}$. Now e is an additive zero of D. By Corollory 1.20, $D = \{e\}$. Thus $(D,+)$ is a band. Extend + and $\cdot$ from D to K by (1) $ea = ae = e$, $a^2 = a$, (2) $e + a = a + e = a$ and (3) $a + a = a$ or e. So K has the structure

$\cdot$	e	a
e	e	e
a	e	a

and

+	e	a
e	e	a
a	a	a

or

$\cdot$	e	a
e	e	e
a	e	a

and

+	e	a
e	e	a
a	a	e

.

It is easy to check that K is a seminear-field. And we obtain (1),(2) and (4).

The proof of the cases ($F_L = D$ and $F_R = \emptyset$) and (F_L is a proper filter in $(D,+)$ and $F_R = \emptyset$) are similar to the proof of the cases ($F_L = \emptyset$ and $F_R = D$) and ($F_L = \emptyset$ and F_R is a proper filter in $(D,+)$), respectively.

Suppose that $F_L = D$ and F_R is a proper filter in $(D,+)$. Now we have that $LI_D(e) = D$ and F_R is a filter in $(D,+)$. By Proposition 1.25 (4.5), $F_R = D = \{e\}$, a contradiction. Hence this case cannot occur.

Similarly, we can show that the case F_L is a proper filter in $(D,+)$ and $F_R = D$ cannot occur.

Suppose that F_L and F_R are proper filters in $(D,+)$. Then $D\setminus F_L$ and $D\setminus F_R$ are ideals of $(D,+)$. Consider the following cases :

<u>Case I</u> $F_L = F_R$

Extend + and $\cdot$ from D to K by

(1) $ax = xa = x$ for all $x \in K$,

(2) $x + a = a + x = a$ for all $x \in F_L$,

$x + a = x + e$ and $a + x = e + x$ for all $x \in D\setminus F_L$ and

(3) $a + a = \begin{cases} a \text{ or } e & \text{if } (D,+) \text{ is a band ,} \\ e + e & \text{if } (D,+) \text{ is not a band .} \end{cases}$

To show that K is a seminear-field, we shall show that (b_4) $x + (y+z) = (x+y) + z$ for all $x,y,z \in K$ and (c_4) $(x+y)z = xz + yz$ for all $x,y,z \in K$.

To prove (b_4), let $x,y,z \in K$. Consider the following cases :

<u>Case 1</u>. $x = y = z = a$.

<u>Subcase 1.1</u>. $a + a = a$.

$x + (y+z) = a + (a+a) = a + a = (a+a) + a = (x+y) + z.$

<u>Subcase 1.2</u>. $a + a = e$.

If $e \in F_L$ then $x + (y+z) = a + (a+a) = a + e = e + a = (a+a) + a = (x+y) + z$.

If $e \in D\setminus F_L$ then $x + (y+z) = a + (a+a) = a + e = e + e = e + a = (a+a) + a = (x+y) + z$.

<u>Subcase 1.3</u>. $a + a = e + e$.

If $e \in F_L$ then $e + e \in F_L$. Thus $x + (y+z) = a + (a+a) = a + (e+e) = (e+e) + a = (a+a) + a = (x+y) + z$.

If $e \in D\setminus F_L$ then $e + e \in D\setminus F_L$. Thus $x + (y+z) = a + (a+a) = a + (e+e) = (e+e) + a = (a+a) + a = (x+y) + z$.

<u>Case 2</u>. $x = y = a$, $z \neq a$.

<u>Subcase 2.1</u>. $a + a = a$. Then $(D,+)$ is a band.

If $z \in F_R$ then $x + (y+z) = a + (a+z) = a + a = a = a + z = (a+a) + z = (x+y) + z$.

If $z \in D\setminus F_R$ then $e + z \in D\setminus F_R$. Thus $x + (y+z) = a + (a+z) = a + (e+z) = e + (e+z) = (e+e) + z = e + z = a + z = (a+a) + z = (x+y) + z$.

<u>Subcase 2.2</u>. $a + a = e$. Then $(D,+)$ is a band.

If $z \in F_R$ then $e + z = e$. Thus $x + (y+z) = a + (a+z) = a + a = e = e + z = (a+a) + z = (x+y) + z$.

If $z \in D\setminus F_R$ then $e + z \in D\setminus F_R$. Thus $x + (y+z) = a + (a+z) = a + (e+z) = e + (e+z) = (e+e) + z = e + z = (a+a) + z = (x+y) + z$.

<u>Subcase 2.3</u>. $a + a = e + e$.

If $z \in F_R$ then $e + z = e$. Thus $x + (y+z) = a + (a+z) = a + a =$

$e + e = e + (e+z) = (e+e) + z = (a+a) + z = (x+y) + z.$

If $z \in D \setminus F_R$ then $e + z \in D \setminus F_R$. Thus $x + (y+z) = a + (a+z) = a + (e+z) = e + (e+z) = (e+e) + z = (a+a) + z = (x+y) + z.$

<u>Case 3</u>. $x = z = a,\ y \neq a.$

If $y \in F_L$ then $x + (y+z) = a + (y+a) = a + a = (a+y) + a = (x+y) + z.$

If $y \in D \setminus F_L$ then $y + e,\ e + y \in D \setminus F_L$. Thus $x + (y+z) = a + (y+a) = a + (y+e) = e + (y+e) = (e+y) + e = (e+y) + a = (a+y) + a = (x+y) + z.$

<u>Case 4</u>. $x \neq a,\ y = z = a.$

This proof is similar to Case 2.

<u>Case 5</u>. $x \neq a,\ y \neq a,\ z = a.$

<u>Subcase 5.1</u>. $x + y \in F_L$. Since F_L is a filter in $(D,+)$, $x,y \in F_L$.

$x + (y+z) = x + (y+a) = x + a = a = (x+y) + a = (x+y) + z.$

<u>Subcase 5.2</u>. $x + y \in D \setminus F_L$.

If $y \in F_L$ then $x \in D \setminus F_L$ and $y + e = e$. Thus $x + (y+z) = x + (y+a) = x + a = x + e = x + (y+e) = (x+y) + e = (x+y) + a = (x+y) + z.$

If $y \in D \setminus F_L$ then $x + (y+z) = x + (y+a) = x + (y+e) = (x+y) + e = (x+y) + a = (x+y) + z.$

<u>Case 6</u>. $x \neq a,\ y = a,\ z \neq a.$

<u>Subcase 6.1</u>. $x,z \in F_L$.

$x + (y+z) = x + (a+z) = x + a = a = a + z = (x+a) + z = (x+y) + z.$

<u>Subcase 6.2</u>. $x \in F_L,\ z \in D \setminus F_L$. Since $F_L \subseteq LI_D(e)$, $x + e = e$.

$x + (y+z) = x + (a+z) = x + (e+z) = (x+e) + z = e + z = a + z = (x+a) + z = (x+y) + z.$

<u>Subcase 6.3</u>. $x \in D \setminus F_L,\ z \in F_L$.

This proof is similar to Subcase 6.2.

Subcase 6.4. $x, z \in D \setminus F_L$.

$x + (y+z) = x + (a+z) = x + (e+z) = (x+e) + z = (x+a) + z = (x+y) + z$.

Case 7. $x = a$, $y \neq a$, $z \neq a$

This proof is similar to Case 5.

Case 8. $x \neq a$, $y \neq a$, $z \neq a$.

$x + (y+z) = (x+y) + z$.

To prove (c_4), let $x, y, z \in K$. Consider the following cases :

Case 1. $z = a$.

It is clear that $(x+y)z = xz + yz$.

Case 2. $z \neq a$.

Subcase 2.1. $x = y = a$.

This proof is the same as the proof of Subcase 2.1 in (c_1).

Subcase 2.2. $x = a$, $y \neq a$.

If $y \in F_R$ then $e + y = e$. Thus $(x+y)z = (a+y)z = az = z = ez = (e+y)z = z + yz = az + yz = xz + yz$.

If $y \in D \setminus F_R$ then $(x+y)z = (a+y)z = (e+y)z = z + yz = az + yz = xz + yz$.

Subcase 2.3. $x \neq a$, $y = a$.

This proof is similar to Subcase 2.2.

Subcase 2.4. $x \neq a$, $y \neq a$.

$(x+y)z = xz + yz$.

Hence K is a seminear-field and we obtain (1) - (4).

Case II. Either $F_L \subset F_R$ or $F_R \subset F_L$. We may assume that $F_L \subset F_R$. Extend $+$ and $\cdot$ from D to K by

(1) $ax = xa = x$ for all $x \in K$,

(2) $x + a = a$ for all $x \in F_L$, $x + a = x + e$ for all $x \in D \setminus F_L$,

$a + x = a$ for all $x \in F_R$, $a + x = e + x$ for all $x \in D \setminus F_R$ and

$$(3)\quad a + a = \begin{cases} a & \text{if } (D,+) \text{ is a band, } e \in F_R, \\ e & \text{if } (D,+) \text{ is a band, } e \in D\setminus F_R, \\ e + e & \text{if } (D,+) \text{ is not a band.} \end{cases}$$

We shall first show that $x + (y+a) = (x+y) + a$ for all $x,y \in D$. Let $x,y \in D$.

<u>Case i</u> $x + y \in F_L$. Since F_L is a filter in $(D,+)$, $x,y \in F_L$. $x + (y+a) = x + a = a = (x+y) + a$.

<u>Case ii</u> $x + y \in D\setminus F_L$.

If $y \in F_L$ then $x \in D\setminus F_L$ and $y + e = e$. Thus $x + (y+a) = x + a = x + e = x + (y+e) = (x+y) + e = (x+y) + a$.

If $y \in D\setminus F_L$ then $x + (y+a) = x + (y+e) = (x+y) + e = (x+y) + a$.

Claim that $e \in D\setminus F_L$. Since $F_L \subset F_R$, there is an element t in $F_R\setminus F_L$. Thus $a + t = a$, $e + t = e$ and $t + a = t + e$. So $e + a = (e+t) + a = e + (t+a) = e + (t+e) = (e+t) + e = e + e \neq a$. Hence $e \in D\setminus F_L$.

To show that K is a seminear-field, we shall show that (b_5) $x + (y+z) = (x+y) + z$ for all $x,y,z \in K$ and (c_5) $(x+y)z = xz + yz$ for all $x,y,z \in K$.

To prove (b_5), let $x,y,z \in K$. Consider the following cases :

<u>Case 1</u>. $x = y = z = a$.

<u>Subcase 1.1</u>. $(D,+)$ is a band, $e \in F_R$. Then $a + a = a$. It is clear that $x + (y+z) = (x+y) + z$.

<u>Subcase 1.2</u>. $(D,+)$ is a band, $e \in D\setminus F_R$. Then $a + a = e$ and $e \in (D\setminus F_L) \cap (D\setminus F_R)$.

$x + (y+z) = a + (a+a) = a + e = e + e = e + a = (a+a) + a = (x+y) + z$.

Subcase 1.3. (D,+) is not a band. Then $a + a = e + e$ and $e \in (D \setminus F_L) \cap (D \setminus F_R)$.

$x + (y+z) = a + (a+a) = a + (e+e) = e + (e+e) = (e+e) + e = (e+e) + a = (a+a) + a = (x+y) + z$.

Case 2. $x = y = a$, $z \neq a$.

Subcase 2.1. $a + a = a$. Then (D,+) is a band.

If $z \in F_R$ then $x + (y+z) = a+(a+z) = a + a = a = a + z = (a+a) + z = (x+y) + z$.

If $z \in D \setminus F_R$ then $e + z \in D \setminus F_R$. Thus $x + (y+z) = a + (a+z) = a + (e+z) = e + (e+z) = (e+e) + z = e + z = a + z = (a+a) + z = (x+y) + z$.

Subcase 2.2. $a + a = e$. Then (D,+) is a band.

If $z \in F_R$ then $e = e + z$ since $F_R \subseteq RI_D(e)$. Thus $x + (y+z) = a + (a+z) = a + a = e = e + z = (a+a) + z = (x+y) + z$.

If $z \in D \setminus F_R$ then $e + z \in D \setminus F_R$. Thus $x + (y+z) = a + (a+z) = a + (e+z) = e + (e+z) = (e+e) + z = e + z = (a+a) + z = (x+y) + z$.

Subcase 2.3. $a + a = e + e$.

If $z \in F_R$ then $e + z = e$. Thus $x + (y+z) = a + (a+z) = a + a = e + e = e + (e+z) = (e+e) + z = (a+a) + z = (x+y) + z$.

If $z \in D \setminus F_R$ then $e + z \in D \setminus F_R$. Thus $x + (y+z) = a + (a+z) = a + (e+z) = e + (e+z) = (e+e) + z = (a+a) + z = (x+y) + z$.

Case 3. $x = z = a$, $y \neq a$.

Subcase 3.1. $y \in F_L$. Then $y \in F_R$, so $a + y = y + a = a$.
$x + (y+z) = a + (y+a) = a + a = (a+y) + a = (x+y) + z$.

Subcase 3.2. $y \in F_R \setminus F_L$. Since $F_R \subseteq FI_D(e)$, $e + y = e$.

Subcase 3.2.1. (D,+) is a band, $e \in F_R$. Then $a + a = a$ and $y + e \in F_R$.

$x + (y+z) = a + (y+a) = a + (y+e) = a = a + a = (a+y) + a = (x+y) + z$.

Subcase 3.2.2. $(D,+)$ is a band, $e \in D \setminus F_R$. Then $a + a = e$ and $y + e \in D \setminus F_R$.

$x + (y+z) = a + (y+a) = a + (y+e) = e + (y+e) = (e+y) + e = e + e$
$= e = a + a = (a+y) + a = (x+y) + z$.

Subcase 3.2.3. $(D,+)$ is not a band. Then $a + a = e + e$ and $e \in D \setminus F_R$. Thus $y + e \in D \setminus F_R$, so $x + (y+z) = a + (y+a) = a + (y+e) = e + (y+e) = (e+y) + e = e + e = a + a = (a+y) + a = (x+y) + z$.

Subcase 3.3. $y \in D \setminus F_R$. Since $D \setminus F_R \subset D \setminus F_L$, $y \in D \setminus F_L$. Thus $y + e \in D \setminus F_R$ and $e + y \in D \setminus F_L$. So $x + (y+z) = a + (y+a) = a + (y+e) = e + (y+e) = (e+y) + e = (e+y) + a = (a+y) + a = (x+y) + z$.

Case 4. $x \neq a$, $y = z = a$.

This proof is similar to Case 2.

Case 5. $x \neq a$, $y \neq a$, $z = a$.

We first showed that $x + (y+a) = (x+y) + a$.

Case 6. $x \neq a$, $y = a$, $z \neq a$.

Subcase 6.1. $x \in F_L$, $z \in F_R$.

$x + (y+z) = x + (a+z) = x + a = a = a + z = (x+a) + z = (x+y) + z$.

Subcase 6.2. $x \in F_L$, $z \in D \setminus F_R$. Since $F_L \subseteq LI_D(e)$, $x + e = e$.

$x + (y+z) = x + (a+z) = x + (e+z) = (x+e) + z = e + z = a + z = (x+a) + z = (x+y) + z$.

Subcase 6.3. $x \in D \setminus F_L$, $z \in F_R$. Since $F_R \subseteq RI_D(e)$, $e + z = e$.

$x + (y+z) = x + (a+z) = x + a = x + e = x + (e+z) = (x+e) + z = (x+a) + z = (x+y) + z$.

Subcase 6.4. $x \in D \setminus F_L$, $z \in D \setminus F_R$.

$x + (y+z) = x + (a+z) = x + (e+z) = (x+e) + z = (x+a) + z = (x+y) + z$.

Case 7. $x = a$, $y \neq a$, $z \neq a$.

Subcase 7.1. $y + z \in F_R$. Since F_R is a filter in $(D,+)$, $y,z \in F_R$.

$x + (y+z) = a + (y+z) = a = a + z = (a+y) + z = (x+y) + z$.

Subcase 7.2. $y + z \in D\setminus F_R$.

If $y \in F_R$ then $z \in D\setminus F_R$. Since $F_R \subseteq RI_D(e)$, $e + y = e$. Thus $x + (y+z) = a + (y+z) = e + (y+z) = (e+y) + z = e + z = a + z = (a+y) + z = (x+y) + z$.

If $y \in D\setminus F_R$ then $x + (y+z) = a + (y+z) = e + (y+z) = (e+y) + z = (a+y) + z = (x+y) + z$.

Case 8. $x \neq a$, $y \neq a$, $z \neq a$.

$x + (y+z) = (x+y) + z$.

To prove (c_5), let $x,y,z \in K$. Consider the following cases :

Case 1. $z = a$.

It is clear that $(x+y)z = xz + yz$.

Case 2. $z \neq a$.

Subcase 2.1. $x = y = a$.

This proof is the same as the proof of Subcase 2.1 in (c_1).

Subcase 2.2. $x = a$, $y \neq a$.

This proof is the same as the proof of Subcase 2.2 in (c_4)

Subcase 2.3. $x \neq a$, $y = a$.

This proof is similar to the proof of Subcase 2.2 in (c_4).

Subcase 2.4. $x \neq a$, $y \neq a$.

$(x+y)z = xz + yz$.

Hence K is a seminear-field and we obtain (1) - (4).

Case III $F_L \nsubseteq F_R$ and $F_R \nsubseteq F_L$.

Extend + and $\cdot$ from D to K by

(1) $ax = xa = x$ for all $x \in K$,

(2) $x + a = a$ for all $x \in F_L$, $x + a = x + e$ for all $x \in D\setminus F_L$,

$a + x = a$ for all $x \in F_R$, $a + x = e + x$ for all $x \in D\setminus F_R$ and

(3) $a + a = e + e$.

We shall first show that $x + (y+a) = (x+y) + a$ for all $x,y \in D$. Let $x,y \in D$.

Case i $x + y \in F_L$. Since F_L is a filter in $(D,+)$, $x,y \in F_L$. $x + (y+a) = x + a = a = (x+y) + a$.

Case ii $x + y \in D\setminus F_L$.

If $y \in F_L$ then $x \in D\setminus F_L$ and $y + e = e$. Thus $x + (y+a) = x + a = x + e = x + (y+e) = (x+y) + e = (x+y) + a$.

If $y \in D\setminus F_L$ then $x + (y+a) = x + (y+e) = (x+y) + e = (x+y) + a$.

Similarly, we can show that $a + (y+z) = (a+y) + z$ for all $y,z \in D$.

Claim that $e \in (D\setminus F_L) \cap (D\setminus F_R)$. Since $F_L \not\subseteq F_R$ and $F_R \not\subseteq F_L$, there are elements x_o and y_o in D such that $x_o \in F_L\setminus F_R$ and $y_o \in F_R\setminus F_L$. Thus $x_o + a = a$, $x_o + e = e$, $a + x_o = e + x_o$, $a + y_o = a$, $e + y_o = e$ and $y_o + a = y_o + e$. So $a + e = a + (x_o+e) = (a+x_o) + e = (e+x_o) + e = e + (x_o+e) = e + e \neq a$. Hence $e \in D\setminus F_R$. And $e + a = (e+y_o) + a = e + (y_o+a) = e + (y_o+e) = (e+y_o) + e = e + e \neq a$. Hence $e \in D\setminus F_L$. Therefore $e \in (D\setminus F_L) \cap (D\setminus F_R)$.

To show that K is a seminear-field, we shall show that (b_6) $x + (y+z) = (x+y) + z$ for all $x,y,z \in K$ and (c_6) $(x+y)z = xz + yz$ for all $x,y,z \in K$.

To prove (b_6), let $x,y,z \in K$. Consider the following cases :

Case 1. $x = y = z = a$. Since $e \in (D\setminus F_L) \cap (D\setminus F_R)$, $e + e \in (D\setminus F_L) \cap (D\setminus F_R)$.

$x + (y+z) = a + (a+a) = a + (e+e) = e + (e+e) = (e+e) + e =$
$(e+e) + a = (a+a) + a = (x+y) + z.$

Case 2. $x = y = a,\ z \neq a.$

If $z \in F_R$ then $e + z = e$. Thus $x + (y+z) = a + (a+z) = a + a =$
$e + e = e + (e+z) = (e+e) + z = (a+a) + z = (x+y) + z.$

If $z \in D \setminus F_R$ then $e + z \in D \setminus F_R$. Thus $x + (y+z) = a + (a+z) =$
$a + (e+z) = e + (e+z) = (e+e) + z = (a+a) + z = (x+y) + z.$

Case 3. $x = z = a,\ y \neq a.$

Subcase 3.1. $F_L \cap F_R = \emptyset$.

Subcase 3.1.1. $y \in F_L$. Then $y \in D \setminus F_R$, $e + y \in D \setminus F_L$ and $y + e = e$.

$x + (y+z) = a + (y+a) = a + a = e + e = e + (y+e) = (e+y) + e =$
$(e+y) + a = (a+y) + a = (x+y) + z.$

Subcase 3.1.2. $y \in D \setminus F_L$. Then $y + e \in D \setminus F_R$ and $e + y \in D \setminus F_L$.

If $y \in F_R$ then $e + y = e$. Thus $x + (y+z) = a + (y+a) = a + (y+e)$
$= e + (y+e) = (e+y) + e = e + e = a + a = (a+y) + a = (x+y) + z.$

If $y \in D \setminus F_R$ then $x + (y+z) = a + (y+a) = a + (y+e) = e + (y+e) =$
$(e+y) + e = (e+y) + a = (a+y) + a = (x+y) + z.$

Subcase 3.2. $F_L \cap F_R \neq \emptyset$.

Subcase 3.2.1. $y \in F_L \cap F_R$.

$x + (y+z) = a + (y+a) = a + a = (a+y) + a = (x+y) + z.$

Subcase 3.2.2. $y \in F_L \cap (D \setminus F_R)$

This proof is the same as the proof of Subcase 3.1.1.

Subcase 3.2.3. $y \in (D \setminus F_L) \cap F_R$. Then $y + e \in D \setminus F_R$, $e + y \in D \setminus F_L$ and $e + y = e$.

$x + (y+z) = a + (y+a) = a + (y+e) = e + (y+e) = (e+y) + e = e + e =$
$a + a = (a+y) + a = (x+y) + z.$

Subcase 3.2.4. $y \in (D \setminus F_L) \cap (D \setminus F_R)$. Then $y + e \in D \setminus F_R$ and $e + y \in D \setminus F_L$.

$x + (y+z) = a + (y+a) = a + (y+e) = e + (y+e) = (e+y) + e =$

$(e+y) + a = (a+y) + a = (x+y) + z$.

Case 4. $x \neq a$, $y = z = a$.

This proof is similar to Case 2.

Case 5. $x \neq a$, $y \neq a$, $z = a$.

By the first proof, we showed that $x + (y+a) = (x+y) + a$.

Case 6. $x \neq a$, $y = a$, $z \neq a$.

This proof is the same as Case 6 in (b_5).

Case 7. $x = a$, $y \neq a$, $z \neq a$.

We showed that $a + (y+z) = (a+y) + z$.

Case 8. $x \neq a$, $y \neq a$, $z \neq a$.

$x + (y+z) = (x+y) + z$.

The proof of (c_6) is the same as the proof of (c_5)

Hence K is a seminear-field and we obtain (1) - (4).

By Theorem 1.33 and Proposition 2.12, if extensions exist with (1) and (2) holding then these are the only possible extensions of the binary operations on D to K.

#

We shall give an example where (D,+) is a band and $F_R \subset F_L$.

Example 2.14. $\mathbb{Q}^+$ with the usual multiplication is a group. Define + on $\mathbb{Q}^+$ by $x + y = \min\{x,y\}$ for all $x,y \in \mathbb{Q}^+$. Then $(\mathbb{Q}^+,+,\cdot)$ is a ratio seminear-ring and $LI_{\mathbb{Q}^+}(1) = \{x \in \mathbb{Q}^+ | x \geq 1\} = RI_{\mathbb{Q}^+}(1)$.

Let $F_L = \{x \in \mathbb{Q}^+ | x > 2\}$ and $F_R = \{x \in \mathbb{Q}^+ | x > 4\}$. Clearly, F_L and F_R are filters in $(\mathbb{Q}^+,+)$.

Let a be a symbol not representing any element of $\mathbb{Q}^+$. Extend + and $\cdot$ from $\mathbb{Q}^+$ to $\mathbb{Q}^+ \cup \{a\}$ by

(1) $ax = xa = x$ for all $x \in Q^+ \cup \{a\}$,

(2) $x + a = a$ for all $x \in F_L$, $x + a = x + 1$ for all $x \in Q^+ \setminus F_L$

$a + x = a$ for all $x \in F_R$, $a + x = 1 + x$ for all $x \in Q^+ \setminus F_R$ and

(3) $a + a = 1$.

By Theorem 2.13, $(Q^+ \cup \{a\}, +, \cdot)$ is a seminear-field with a as a category II special element.

We shall now give an example where $F_L \not\subseteq F_R$ and $F_R \not\subseteq F_L$.

Example 2.15. $(Q^+, \min, \cdot)$ and $(Q^+, \max, \cdot)$ are ratio seminear-rings where $\cdot$ is the usual multiplication.

Let Q_m^+ and Q_M^+ denote the ratio seminear-rings $(Q^+, \min, \cdot)$ and $(Q^+, \max, \cdot)$, respectively. Then $Q_m^+ \times Q_M^+$ is a ratio seminear-ring. Now

$$LI_{Q_m^+ \times Q_M^+}(1,1) = \{(x,y) \in Q^+ \times Q^+ \mid x \geq 1,\ y \leq 1\}.$$

$$= RI_{Q_m^+ \times Q_M^+}(1,1).$$

Let $F_L = \{(x,y) \in Q^+ \times Q^+ \mid x \geq 1,\ y \leq \frac{1}{2}\}$ and

$F_R = \{(x,y) \in Q^+ \times Q^+ \mid x \geq 2,\ y \leq 1\}$.

Then $F_L \not\subseteq F_R$ and $F_R \not\subseteq F_L$. To show that F_L is a filter in $(Q_m^+ \times Q_M^+, +)$. It is clear that F_L is a subsemigroup of $(Q_m^+ \times Q_M^+, +)$. Let $(x_1,y_1),(x_2,y_2) \in Q^+ \times Q^+$ be such that $(x_1,y_1) + (x_2,y_2) \in F_L$. Then $x_1 + x_2 \geq 1$ and $y_1 + y_2 \leq \frac{1}{2}$, so $x_1, x_2 \geq 1$ and $y_1, y_2 \leq \frac{1}{2}$. Hence $(x_1,y_1),(x_2,y_2) \in F_L$. Therefore F_L is filter in $(Q_m^+ \times Q_M^+, +)$.

Similarly, we can show that F_R is a filter in $(Q_m^+ \times Q_M^+, +)$.

Let a be a symbol not representing any element of $Q^+ \times Q^+$. Extend + and $\cdot$ from $Q_m^+ \times Q_M^+$ to $(Q_m^+ \times Q_M^+) \cup \{a\}$ by

(1) $az = za = z$ for all $z \in (\mathbb{Q}^+ \times \mathbb{Q}^+) \cup \{a\}$,

(2) $z + a = a$ for all $z \in F_L$, $z + a = z + (1,1)$ for all $z \in (\mathbb{Q}^+ \times \mathbb{Q}^+) \setminus F_L$,

$a + z = a$ for all $z \in F_R$, $a + z = (1,1) + z$ for all $z \in (\mathbb{Q}^+ \times \mathbb{Q}^+) \setminus F_R$ and

(3) $a + a = (1,1)$

By Theorem 2.13, $(\mathbb{Q}^+_m \times \mathbb{Q}^+_M) \cup \{a\}$ is a seminear-field with a as a category II special element.

We shall give an example where $(D,+)$ is not a band.

Example 2.16. Define $\oplus$ and $\odot$ on $\mathbb{Q}^+ \times \mathbb{Z}$ by

$$(x,n) \oplus (y,m) = \begin{cases} (x,n) & \text{if } n < m, \\ (x+y,n) & \text{if } n = m, \\ (y,m) & \text{if } n > m, \end{cases}$$

$$(x,n) \odot (y,m) = (x\cdot y,\ n+m).$$

Using the same proof as in Remark 4.13 in [2] page 64 - 66, we obtain that $(\mathbb{Q}^+ \times \mathbb{Z},\oplus,\odot)$ is a ratio seminear-ring.

$LI_{\mathbb{Q}^+\times\mathbb{Z}}(1,0) = \{(x,n) \in \mathbb{Q}^+\times \mathbb{Z} \mid n > 0\} = RI_{\mathbb{Q}^+\times\mathbb{Z}}(1,0)$. Let $F_L = \{(x,n) \in \mathbb{Q}^+\times \mathbb{Z} \mid n > 2\}$ and $F_R = \{(x,n) \in \mathbb{Q}^+\times \mathbb{Z} \mid n > 3\}$. We must show that F_L is a filter in $(\mathbb{Q}^+\times \mathbb{Z},+)$. It is clear that F_L is a subsemigroup of $(\mathbb{Q}^+\times \mathbb{Z},+)$. Let $(x,m),(y,n) \in \mathbb{Q}^+\times \mathbb{Z}$ be such that $(x,m) + (y,n) \in F_L$. If $m < n$ then $(x,m) \oplus (y,n) = (x,m) \in F_L$. Thus $n > m > 2$. Hence $(x,m),(y,n) \in F_L$. If $m = n$ then $(x,m) \oplus (y,n) = (x+y,m) \in F_L$. Thus $m = n > 2$. Hence $(x,m),(y,n) \in F_L$. If $m > n$ then $(x,m) \oplus (y,n) = (y,m) \in F_L$. Thus $m > n > 2$. Hence $(x,m),(y,n) \in F_L$. Therefore F_L is a filter in $(\mathbb{Q}^+\times \mathbb{Z},+)$.

Similarly, we can show that F_R is a filter in $(\mathbb{Q}^+\times \mathbb{Z},\oplus)$.

Let a be a symbol not representing any element of $Q^+\times Z$. Extend $\oplus$ and $\odot$ from $Q^+\times Z$ to $(Q^+\times Z) \cup \{a\}$ by

(1) $z \odot a = a \odot z = z$ for all $z \in (Q^+\times Z) \cup \{a\}$,

(2) $z \oplus a = a$ for all $z \in F_L$, $z \oplus a = z \oplus (1,0)$ for all $z \in (Q^+\times Z)\setminus F_L$,

$a \oplus z = a$ for all $z \in F_R$, $a \oplus z = (1,0) \oplus z$ for all $z \in (Q^+\times Z)\setminus F_R$ and

(3) $a \oplus a = (1,0) \oplus (1,0)$.

By Theorem 2.13, $((Q^+\times Z) \cup \{a\},+,\cdot)$ is a seminear-field with a as a category II special element.

<u>Corollary 2.17</u>. If K is a seminear-field with a category II special element then ($S_L = \emptyset$ or $(S_L,\cdot)$ is a semigroup) and ($S_R = \emptyset$ or $(S_R,\cdot)$ is a semigroup).

<u>Proof</u>. Let a be a category II special element of K. Suppose that $S_L \neq \emptyset$. To show that $(S_L,\cdot)$ is a semigroup, let $x,y \in S_L$. Then $y + a = a$ and $x + e = e$. Thus $xy + a = xy + (y+a) = (xy+y) + a = (x+e)y + a = ey + a = y + a = a$. Hence $xy \in S_L$. Therefore $(S_L,\cdot)$ is a semigroup.

Similarly, we can show that $S_R = \emptyset$ or $(S_R,\cdot)$ is a semigroup. #

<u>Corollary 2.18</u>. Let D be a ratio seminear-ring and let a be a symbol not representing any element of D. Let F_L and F_R have the properties stated in Theorem 2.13. Then $K = D \cup \{a\}$ is a distributive seminear-field with a as a category II special element if and only if D is a distributive ratio seminear-ring.

<u>Proof</u>. By Theorem 2.13, we can construct K so that K is a seminear-field with a as a category II special element, F_L is the left fundamental set of a in K and F_R is the right fundamental set

of a in K. It is clear that if K is a distributive seminear-field with a as a category II special element then D is a left ratio seminear-ring.

Conversely, assume that D is a distributive ratio seminear-ring. It is sufficient to show that $x(y+z) = xz + yz$ for all $x,y,z \in K$. Let $x,y,z \in K$. By Theorem 1.33 (2), $a + a = a$ or $a + a = e + e$.

Case 1. $x = a$.

$x(y+z) = a(y+z) = y + z = ay + az = xy + xz$.

Case 2. $x \neq a$.

Subcase 2.1. $y = z = a$.

Subcase 2.1.1. $a + a = a$. Then $(D,+)$ is a band.

$x(y+z) = x(a+a) = xa = x = x + x = xa + xa = xy + xz$.

Subcase 2.1.2. $a + a = e + e$.

$x(y+z) = x(a+a) = x(e+e) = x + x = xa + xa = xy + xz$.

Subcase 2.2. $y = a$, $z \neq a$.

If $z \in F_R$ then $e + z = e$. Thus $x(y+z) = x(a+z) = xa = x = xe = x(e+z) = x + xz = xa + xz = xy + xz$.

If $z \in D \setminus F_R$ then $x(y+z) = x(a+z) = x(e+z) = x + xz = xa + xz = xy + xz$.

Subcase 2.3. $y \neq a$, $z = a$.

This proof is similar to Subcase 2.2.

Subcase 2.4. $y \neq a$, $z \neq a$.

$x(y+z) = xy + xz$.

Hence K is a distributive seminear-field.

#

Theorem 2.19. Let K and K' be seminear-fields with a and a' as category II special elements, respectively. Let $D = K \setminus \{a\}$ and $D' = K' \setminus \{a'\}$ with e and e' as their multiplicative identities respectively. Let S_L and S_R be the left and right fundamental sets of a in K respectively. Let S_L' and S_R' be the left and right fundamental sets of a' in K, respectively. Suppose that there exists an isomorphism $\eta : K \to K'$. Let $\varphi = \eta|_{S_L}$, $\psi = \eta|_{D \setminus S_L}$, $\varphi' = \eta|_{S_R}$ and $\psi' = \eta|_{D \setminus S_R}$. Then the following statements hold :

(1) $\eta(e) = e'$ and $\eta(a) = a'$.

(2) $S_L = \emptyset$ if and only if $S_L' = \emptyset$ and if $S_L \neq \emptyset$ then $S_L \cong S_L'$ as additive and multiplicative semigroups.

(3) $D \setminus F_L = \emptyset$ if and only if $D \setminus F_L' = \emptyset$ and if $D \setminus F_L \neq \emptyset$ then $D \setminus F_L \cong D \setminus F_L'$ as additive semigroups.

(4) $S_R = \emptyset$ if and only if $S_R' = \emptyset$ and if $S_R \neq \emptyset$ then $S_R \cong S_R'$ as additive and multiplicative semigroups.

(5) $D \setminus S_R = \emptyset$ if and only if $D \setminus S_R' = \emptyset$ and if $D \setminus S_R \neq \emptyset$ then $D \setminus S_R \cong D \setminus S_R'$ as additive semigroups.

(6) If $a + a = a$ then $a' + a' = a'$.

(7) If $a + a = e + e$ then $a' + a' = e' + e'$.

(8) If $x \in S_L \cap S_R$ then $\varphi(x) = \varphi'(x)$.

(9) If $x \in S_L \cap (D \setminus S_R)$ then $\varphi(x) = \psi'(x)$.

(10) If $x \in (D \setminus S_L) \cap S_R$ then $\psi(x) = \varphi'(x)$.

(11) If $x \in (D \setminus S_L) \cap (D \setminus S_R)$ then $\psi(x) = \psi'(x)$.

(12) If $x \in S_L$ and $y \in D \setminus S_L$ then $\psi(x+y) = \varphi(x) + \psi(y)$.

(13) If $x \in D \setminus S_L$ and $y \in S_L$ then $\psi(x+y) = \psi(x) + \varphi(y)$.

(14) If $x \in S_L$, $y \in D \setminus S_L$ and $xy \in S_L$ then $\varphi(xy) = \varphi(x)\psi(y)$.

(15) If $x \in D \setminus S_L$, $y \in S_L$ and $xy \in S_L$ then $\varphi(xy) = \psi(x)\varphi(y)$.

(16) If $x,y \in D \setminus S_L$ and $xy \in S_L$ then $\varphi(xy) = \psi(x)\psi(y)$.

(17) If $x \in S_L$, $y \in D \setminus S_L$ and $xy \in D \setminus S_L$ then $\psi(xy) = \varphi(x)\psi(y)$.

(18) If $x \in D \setminus S_L$, $y \in S_L$ and $xy \in D \setminus S_L$ then $\psi(xy) = \psi(x)\varphi(y)$.

(19) If $x, y, xy \in D \setminus S_L$ then $\psi(xy) = \psi(x)\psi(y)$.

Proof. (1) Since $[\eta(a)]^2 = \eta(a^2) = \eta(a)$, $\eta(a) = e'$ or $\eta(a) = a'$. Suppose that $\eta(a) = e'$. Since $[\eta(e)]^2 = \eta(e)$, $\eta(e) = a'$. Thus $a' = \eta(e) = \eta(e \cdot a) = \eta(e) \cdot \eta(a) = a' \cdot e' = e'$, a contradiction. Hence $\eta(a) = a'$. Consequently, $\eta(e) = e'$.

(2) Assume that $S_L = \emptyset$. Suppose that $S_L' \neq \emptyset$. Let $y \in S_L'$. Since η is onto, there exists an element x in D such that $\eta(x) = y$. Now $x + a = x + e$. $a' = y + a' = \eta(x) + \eta(a) = \eta(x+a) = \eta(x+e) = \eta(x) + \eta(e) = y + e' \in D'$, a contradiction. Hence $S_L' = \emptyset$.

Assume that $S_L \neq \emptyset$. Claim that $\varphi : S_L \longrightarrow S_L'$. Let $x \in S_L$. Then $x + a = a$, so $\varphi(x) + a' = \eta(x) + \eta(a) = \eta(x+a) = \eta(a) = a'$. Thus $\varphi(x) \in S_L'$. Hence $S_L' \neq \emptyset$. It is clear that φ is a monomorphism. To show φ is onto, let $y \in S_L'$. Then $y + a' = a'$. Since η is onto, there exists an element x in K such that $\eta(x) = y$. Now $x \neq a$ so $x \in D$. Claim that $x \in S_L$. Suppose that $x \in D \setminus S_L$. Then $x + a \neq a$, so $\eta(x+a) \in D'$. $a' = y + a' = \eta(x) + \eta(a) = \eta(x+a) \in D'$, a contradiction. Hence $x \in S_L$. So we get that $\varphi(x) = \eta(x) = y$. Thus φ is onto. Hence $S_L \cong S_L'$ as additive and multiplicative semigroups. Therefore we obtain (2).

(3) Assume that $D \setminus S_L = \emptyset$. Then $S_L = D$. To show that $S_L' = D'$, let $y \in D'$. Since η is onto, there exists an element x in S_L such that $\eta(x) = y$. Now $x + a = a$ so $y + a' = \eta(x) + \eta(a) = \eta(x+a) = \eta(a) = a'$. Hence $y \in S_L'$. Therefore $D' \setminus S_L' = \emptyset$.

Assume that $D \setminus S_L \neq \emptyset$. Claim that $\psi : D \setminus S_L \longrightarrow D' \setminus S_L'$. Let $x \in D \setminus S_L$. Then $\psi(x) + a' = \eta(x) + \eta(a) = \eta(x+a) = \eta(x+e) = \eta(x) + \eta(e) = \psi(x) + e'$ so $\psi(x) \in D' \setminus S_L'$. Thus $D' \setminus S_L' \neq \emptyset$. It is clear that ψ is a monomorphism. To show that ψ is onto, let

$y \in D' \setminus S'_L$. Then $y + a' = y + e'$. Since η is onto, there exists an an element x in K such that $\eta(x) = y$. Now $x \neq a$. If $x \in S_L$ then $x + a = a$. So $y + e' = y + a' = \eta(x) + \eta(a) = \eta(x+a) = \eta(a) = a'$, a contradiction. Hence $x \in D \setminus S_L$. Therefore ψ is onto and so $D \setminus S_L \cong D' \setminus S'_L$ as additive semigroups. We obtain (3).

The proofs of (4) and (5) are similar to the proofs of (2) and (3), respectively.

The proofs of (6) - (19) are easily shown.

#

<u>Theorem 2.20</u>. Let D and D' be ratio seminear-rings with e and e' as their multiplicative identities respectively. Let a and a' be symbols not representing any element of D or D'. Let $F_L \subseteq LI_D(e)$ be either $\emptyset$ or a filter in (D,+) and let $F_R \subseteq RI_D(e)$ be either $\emptyset$ or a filter in (D,+). Let $F'_L \subseteq LI_{D'}(e')$ be either $\emptyset$ or a filter in $(D',+)$ and let $F'_R \subseteq RI_{D'}(e')$ be either $\emptyset$ or a filter in $(D',+)$. Suppose that there are bijections $\varphi : F_L \to F'_L$ and $\psi : D \setminus F_L \to D' \setminus F'_L$ such that $\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x,y \in F_L$, $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x,y \in F_L$, $\psi(x+y) = \psi(x) + \psi(y)$ for all $x,y \in D \setminus F_L$ and $\psi(e) = e'$ if $e \in D \setminus F_L$. Suppose that there are bijections $\varphi' : F_R \to F'_R$ and $\psi' : D \setminus F_R \to D' \setminus F'_R$ such that $\varphi'(x+y) = \varphi'(x) + \varphi'(y)$ for all $x,y \in F_R$, $\varphi'(xy) = \varphi'(x)\varphi'(y)$ for all $x,y \in D \setminus F_R$, $\psi'(x+y) = \psi'(x) + \psi'(y)$ for all $x,y \in D \setminus F_R$ and $\psi'(e) = e'$ if $e \in D \setminus F_R$.

Suppose that the following conditions are satisfied :

(1) $F_L = \emptyset$ iff $F'_L = \emptyset$.

(2) $F_L = D$ iff $F'_L = D'$.

(3) $\emptyset \neq F_L \subset D$ iff $\emptyset \neq F'_L \subset D'$.

(4) $F_R = \emptyset$ iff $F'_R = \emptyset$.

(5) $F_R = D$ iff $F'_R = D$.

(6) $\emptyset \neq F_R \subset D$ iff $\emptyset \neq F_R' \subset D$.

(7) If $a + a = a$ then $a' + a' = a'$.

(8) If $a + a = e + e$ then $a' + a' = e' + e'$.

(9) If $x \in F_L \cap F_R$ then $\varphi(x) = \varphi'(x)$.

(10) If $x \in F_L \cap (D \setminus F_R)$ then $\varphi(x) = \psi'(x)$.

(11) If $x \in (D \setminus F_L) \cap F_R$ then $\psi(x) = \varphi'(x)$.

(12) If $x \in (D \setminus F_L) \cap (D \setminus F_R)$ then $\psi(x) = \psi'(x)$.

(13) If $x \in F_L$ and $y \in D \setminus F_L$ then $\psi(x+y) = \varphi(x) + \psi(y)$.

(14) If $x \in D \setminus F_L$ and $y \in F_L$ then $\psi(x+y) = \psi(x) + \varphi(y)$.

(15) If $x \in F_L$, $y \in D \setminus F_L$ and $xy \in F_L$ then $\varphi(xy) = \varphi(x)\psi(y)$.

(16) If $x \in D \setminus F_L$, $y \in F_L$ and $xy \in F_L$ then $\varphi(xy) = \psi(x)\varphi(y)$.

(17) If $x, y \in D \setminus F_L$ and $xy \in F_L$ then $\varphi(xy) = \psi(x)\psi(y)$.

(18) If $x \in F_L$, $y \in D \setminus F_L$ and $xy \in D \setminus F_L$ then $\psi(xy) = \varphi(x)\psi(y)$.

(19) If $x \in D \setminus F_L$, $y \in F_L$ and $xy \in D \setminus F_L$ then $\psi(xy) = \psi(x)\varphi(y)$.

(20) If $x, y, xy \in D \setminus F_L$ then $\psi(xy) = \psi(x)\psi(y)$.

Then $\eta : K \longrightarrow K'$ defined by

$$\eta(x) = \begin{cases} \varphi(x) & \text{if } x \in F_L, \\ \psi(x) & \text{if } x \in D \setminus F_L, \\ a' & \text{if } x = a, \end{cases}$$

is an isomorphism between K and K' where $K = D \cup \{a\}$ and $K' = D' \cup \{a'\}$ are seminear-fields with a and a' as category II special elements, respectively.

<u>Proof</u>. By Theorem 2.13, we can construct K and K' so that K and K' are seminear-fields and a and a' are category II special elements of K and K', respectively.

<u>Case I</u> $F_L = \emptyset$. Then $S_L' = \emptyset$. Define $\eta : K \longrightarrow K'$ by

$$\eta(x) = \begin{cases} \psi(x) & \text{if } x \in D, \\ a' & \text{if } x = a. \end{cases}$$

It is clear that η is a bijection. We need only show that (a_1) $\eta(xy) = \eta(x)\eta(y)$ for all $x,y \in K$ and (b_1) $\eta(x+y) = \eta(x) + \eta(y)$ for all $x,y \in K$.

To prove (a_1), let $x,y \in K$.

Case 1. $x = y = a$.

$\eta(xy) = \eta(a^2) = \eta(a) = a' = a'^2 = \eta(a)\eta(a) = \eta(x)\eta(y)$.

Case 2. $x = a$, $y \neq a$.

$\eta(xy) = \eta(ay) = \eta(y) = a'\eta(y) = \eta(a)\eta(y) = \eta(x)\eta(y)$.

Case 3. $x \neq a$, $y = a$.

This proof is similar to Case 2.

Case 4. $x \neq a$, $y \neq a$. Then $xy \neq a$. By (20), we obtain

$\eta(xy) = \psi(xy) = \psi(x)\psi(y) = \eta(x)\eta(y)$.

To prove (b_1), let $x,y \in K$.

Case 1. $x = y = a$.

Subcase 1.1. $a + a = a$. Then $a' + a' = a'$.

$\eta(x+y) = \eta(a+a) = \eta(a) = a' = a' + a' = \eta(a) + \eta(a) = \eta(x) + \eta(y)$.

Subcase 1.2. $a + a = e + e$. Then $a' + a' = e' + e'$ and $e + e \in D$.

$\eta(x+y) = \eta(a+a) = \eta(e+e) = \psi(e+e) = \psi(e) + \psi(e) = e' + e' = a' + a'$
$= \eta(a) + \eta(a) = \eta(x) + \eta(y)$.

Case 2. $x = a$, $y \neq a$.

If $y \in F_R$ then $y \in F_R \cap (D \setminus F_L)$. Thus $\varphi'(y) = \psi(y)$. Now $a + y = a$
$a' + \varphi'(y) = a'$.

$\eta(x+y) = \eta(a+y) = \eta(a) = a' = a' + \varphi'(y) = \eta(a) + \psi(y) = \eta(x) + \eta(y)$

If $y \in D \setminus F_R$ then $y \in (D \setminus F_R) \cap (D \setminus F_L)$. Thus $\psi(y) = \psi'(y)$. Now $a + y = e + y$ and $a' + \psi'(y) = e' + \psi'(y)$.

$\eta(x+y) = \eta(a+y) = \eta(e+y) = \psi(e+y) = \psi(e) + \psi(y) = e' + \psi'(y) =$
$e' + \psi'(y) = a' + \psi'(y) = \eta(a) + \psi(y) = \eta(x) + \eta(y)$.

Case 3. $x \neq a$, $y = a$. Then $x + a = x + e$ and $\psi(x) + a' = \psi(x) + e'$.

$\eta(x+y) = \eta(x+a) = \eta(x+e) = \psi(x+e) = \psi(x) + \psi(e) = \psi(x) + e' = \psi(x) + a'$
$= \eta(x) + \eta(a) = \eta(x) + \eta(y)$.

Case 4. $x \neq a$, $y \neq a$. Then $x + y \in D$ which is an additive semigroup.

$\eta(x+y) = \psi(x+y) = \psi(x) + \psi(y) = \eta(x) + \eta(y)$.

Hence η is an isomorphism.

Case II $F_L = D$. Then $F_L' = D'$. Define $\eta : K \longrightarrow K'$ by

$$\eta(x) = \begin{cases} \varphi(x) & \text{if } x \in D, \\ a & \text{if } x = a. \end{cases}$$

It is clear that η is a bijection. We need only show that (a_2) $\eta(xy) = \eta(x)\eta(y)$ for all $x,y \in K$ and (b_2) $\eta(x+y) = \eta(x) + \eta(y)$ for all $x,y \in K$.

The proof of (a_2) is the same as the proof of (a_1).

To prove (b_2), let $x,y \in K$.

Case 1. $x = y = a$.

This proof is the same as the proof of Case 1 in (b_1).

Case 2. $x = a$, $y \neq a$.

If $y \in F_R$ then $y \in F_L \cap F_R$. Thus $\varphi(y) = \varphi'(y)$, $a + y = a$ and $a' + \varphi'(y) = a'$.

$\eta(x+y) = \eta(a+y) = \eta(a) = a' = a' + \varphi'(y) = \eta(a) + \varphi(y) = \eta(x) + \eta(y)$.

If $y \in D\setminus F_R$ then $y \in F_L \cap (D\setminus F_R)$. Thus $\varphi(y) = \psi'(y)$, $a + y = e + y$ and $a' + \psi'(y) = e' + \psi'(y)$.

$\eta(x+y) = \eta(a+y) = \eta(e+y) = \varphi(e+y) = \varphi(e) + \varphi(y) = e' + \varphi(y) = e' + \psi'(y)$
$a' + \psi'(y) = \eta(a) + \varphi(y) = \eta(x) + \eta(y)$.

Case 3. $x \neq a$, $y = a$. Then $x + a = a$ and $\varphi(x) + a' = a'$.

$\eta(x+y) = \eta(x+a) = \eta(a) = a' = \varphi(x) + a' = \eta(x) + \eta(a) = \eta(x) + \eta(y)$.

Case 4. $x \neq a$, $y \neq a$. Then $x + y \in D$ which is an additive semigroup.

$\eta(x+y) = \varphi(x+y) = \varphi(x) + \varphi(y) = \eta(x) + \eta(y)$.

Hence η is an isomorphism.

Case III $\emptyset \neq F_L \subset D$. Then $\emptyset \neq F_L' \subset D$. Define $\eta : K \longrightarrow K'$ by

$$\eta(x) = \begin{cases} \varphi(x) & \text{if } x \in F_L, \\ \psi(x) & \text{if } x \in D\setminus F_L, \\ a' & \text{if } x = a. \end{cases}$$

It is clear that η is a bijection. We need only show that (a_3) $\eta(xy) = \eta(x)\eta(y)$ for all $x,y \in K$ and (b_3) $\eta(x+y) = \eta(x) + \eta(y)$ for all $x,y \in K$.

To prove (a_3), let $x,y \in K$.

Case 1. $x = y = a$.

This proof is the same as the proof of Case 1 in (a_1).

Case 2. $x = a$, $y \neq a$.

This proof is the same as the proof of Case 2 in (a_1).

Case 3. $x \neq a$, $y = a$.

This proof is similar to Case 2 in (a_1).

Case 4. $x \neq a$, $y \neq a$. By (15) - (20) and $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x,y \in K$, we can show that $\eta(xy) = \eta(x)\eta(y)$.

To prove (b_3), let $x,y \in K$. Note that $a + a = a$ or $a + a = e + e$.

Case 1. $x = y = a$

Subcase 1.1. $a + a = a$. Then $a' + a' = a'$.

This proof is the same as the proof of Subcase 1.1 in (a_1).

Subcase 1.2. $a + a = e + e$. Then $a' + a' = e' + e'$.

If $e \in F_L$ then $e + e \in F_L$. Thus $\eta(x+y) = \eta(a+a) = \eta(e+e) = \varphi(e+e) = \varphi(e) + \varphi(e) = e' + e' = a' + a' = \eta(a) + \eta(a) = \eta(x) + \eta(y)$.

If $e \in D \setminus F_L$ then $e + e \in F_L$. Thus $\eta(x+y) = \eta(a+a) = \eta(e+e) = \psi(e+e) = \psi(e) + \psi(e) = e' + e' = a' + a' = \eta(a) + \eta(a) = \eta(x) + \eta(y)$.

Case 2. $x = a$, $y \neq a$.

Subcase 2.1. $F_L \cap F_R = \emptyset$.

Subcase 2.1.1. $y \in F_R$. Then $y \in D \setminus F_L$, so $\psi(y) = \varphi'(y)$.

$\eta(x+y) = \eta(a+y) = \eta(a) = a' = a' + \varphi'(y) = \eta(a) + \psi(y) = \eta(x) + \eta(y)$.

Subcase 2.1.2. $y \in D \setminus F_R$, $y \in F_L$. Then $\varphi(y) = \psi'(y)$.

If $e \in F_L$ then $e + y \in F_L$. Thus $\eta(x+y) = \eta(a+y) = \eta(e+y) = \varphi(e+y) = \varphi(e) + \varphi(y) = e' + \psi'(y) = a' + \psi'(y) = \eta(a) + \varphi(y) = \eta(x) + \eta(y)$.

If $e \in D \setminus F_L$ then $e + y \in D \setminus F_L$. Thus $\eta(x+y) = \eta(a+y) = \eta(e+y) =$

$\psi(e+y) = \psi(e) + \varphi(y) = e' + \psi'(y) = a' + \psi'(y) = \eta(a) + \varphi(y) =$
$\eta(x) + \eta(y).$

Subcase 2.1.3. $y \in D \setminus F_R$, $y \in D \setminus F_L$. Then
$\psi(y) = \psi'(y).$

If $e \in F_L$ then $e + y \in D \setminus F_L$. Thus $\eta(x+y) = \eta(a+y) = \eta(e+y) =$
$\psi(e+y) \overset{\text{by(13)}}{=} \varphi(e) + \psi(y) = e' + \psi'(y) = a' + \psi'(y) = \eta(a) + \psi(y) =$
$\eta(x) + \eta(y).$

If $e \in D \setminus F_L$ then $e + y \in D \setminus F_L$. Thus $\eta(x+y) = \eta(a+y) = \eta(e+y) =$
$\psi(e+y) = \psi(e) + \psi(y) = e' + \psi'(y) = a' + \psi'(y) = \eta(a) + \psi(y) =$
$\eta(x) + \eta(y).$

Subcase 2.2. $F_L \cap F_R \neq \emptyset$.

Subcase 2.2.1. $y \in F_L \cap F_R$. Then $\varphi(y) = \varphi'(y)$
and $a' = a' + \varphi'(y) = a' + \varphi(y).$

$\eta(x+y) = \eta(a+y) = \eta(a) = a' = a' + \varphi(y) = \eta(a) + \eta(y) = \eta(x) + \eta(y).$

Subcase 2.2.2. $y \in F_L \cap (D \setminus F_R)$.

This proof is the same as the proof of Subcase 2.1.2.

Subcase 2.2.3. $y \in (D \setminus F_L) \cap F_R$.

This proof is the same as the proof of Subcase 2.1.1.

Subcase 2.2.4. $y \in (D \setminus F_L) \cap (D \setminus F_R)$.

This proof is the same as the proof of Subcase 2.1.3.

Case 3. $x \neq a$, $y = a$.

Subcase 3.1. $x \in F_L$. Then $a' = \varphi(x) + a'$.

$\eta(x+y) = \eta(x+a) = \eta(a) = a' = \varphi(x) + a' = \eta(x) + \eta(a) = \eta(x) + \eta(y).$

Subcase 3.2. $x \in D \setminus F_L$. Then $x + e \in D \setminus F_L$ and $\psi(x) + a' =$
$\psi(x) + e'$.

If $e \in F_L$ then $\eta(x+y) = \eta(x+a) = \eta(x+e) = \psi(x+e) \overset{\text{by(14)}}{=} \psi(x) + \varphi(e)$
$= \psi(x) + e' = \psi(x) + a' = \eta(x) + \eta(a) = \eta(x) + \eta(y)$. If $e \in D \setminus F_L$ then
$\eta(x+y) = \eta(x+a) = \eta(x+e) = \psi(x+e) = \psi(x) + \psi(e) = \psi(x) + e' = \psi(x) + a'$
$= \eta(x) + \eta(a) = \eta(x) + \eta(y).$

Case 4. $x \neq a$, $y \neq a$.

Subcase 4.1. $x + y \in F_L$. Since F_L is a filter in $(D,+)$, $x,y \in F_L$.

$\eta(x+y) = \varphi(x+y) = \varphi(x) + \varphi(y) = \eta(x) + \eta(y)$.

Subcase 4.2. $x + y \in D \setminus F_L$.

Subcase 4.2.1. $x \in F_L$. Then $y \in D \setminus F_L$. By (13), $\psi(x+y) = \varphi(x) + \psi(y)$.

$\eta(x+y) = \psi(x+y) = \varphi(x) + \psi(y) = \eta(x) + \eta(y)$.

Subcase 4.2.2. $x \in D \setminus F_L$.

If $y \in F_L$ then $\eta(x+y) = \psi(x+y) \overset{\text{by}(14)}{=} \psi(x) + \varphi(y) = \eta(x) + \eta(y)$.

If $y \in D \setminus F_L$ then $\eta(x+y) = \psi(x+y) = \psi(x) + \psi(y) = \eta(x) + \eta(y)$.

Hence η is an isomorphism.

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Remark. $\eta : K \to K'$ may be defined by

$$\eta(x) = \begin{cases} \varphi'(x) & \text{if } x \in F_R, \\ \psi'(x) & \text{if } x \in D \setminus F_R, \\ a' & \text{if } x = a. \end{cases}$$

The proof is straightforward but very long.

We shall give an open problem for future research.

In Theorem 2.20, suppose that we drop conditions (9) - (20).

Question : Is K isomorphic to K' ?

Now we shall compute all finite seminear-field with a category II special element up to isomorphism.

In [1], all finite seminear-field containing a category II special element of order 2 were computed (see page 55).

We shall compute a finite seminear-field of order greater than 2 which contain a category II special element.

Theorem 2.21. Let K be a finite seminear-field of order greater than 2 containing a category II special element a. Let $D = K\setminus\{a\}$ and let e be the identity of $(D,\cdot)$. Then

(1) $x + a = x + e$, $a + x = e + x$ for all $x \in D$ and $a + a = a$ or

(2) $x + a = x + e$, $a + x = e + x$ for all $x \in D$ and $a + a = e$ or

(3) $x + a = a$, $a + x = e + x$ for all $x \in D$ and $a + a = a$ or

(4) $x + a = x + e$, $a + x = a$ for all $x \in D$ and $a + a = a$.

Proof. By Theorem 1.32, D is a ratio seminear-ring. By Theorem 1.15, $D_1 = \{x \in D \mid x + e = x\}$ and $D_2 = \{x \in D \mid x + e = e\}$ are the unique ratio subseminear-rings of D such that (1) $x + y = x$ for all $x,y \in D_1$, (2) $x + y = y$ for all $x,y \in D_2$, (3) $(D,+) \cong (D_1,+) \times (D_2,+)$ and (4) $D_2 + D_1 = \{e\}$. By definition, $D_2 = LI_D(e)$. Claim that $D_1 = RI_D(e)$. Let $x \in D_1$. Then $x^{-1} \in D_1$, so $x^{-1} + e = x^{-1}$. Thus $e = x^{-1}x = (x^{-1}+e)x = x^{-1}x + x = e + x$. Hence $x \in RI_D(e)$. Therefore $D_1 \subseteq RI_D(e)$. Let $y \in RI_D(e)$. Then $e + y = e$, so $y^{-1} = ey^{-1} = (e+y)y^{-1} = y^{-1} + e$. Thus $y^{-1} \in D_1$. Since $(D_1,\cdot)$ is a group, $y \in D_1$. Hence $RI_D(e) \subseteq D_1$. Therefore $D_1 = RI_D(e)$.

Let $S_L = \{x \in D \mid x + a = a\}$ and $S_R = \{x \in D \mid a + x = a\}$. By Proposition 2.9 (1), $S_L \subseteq LI_D(e)$ and $S_R \subseteq RI_D(e)$.

Claim that (1) if S_L is a filter in $(D,+)$ then $S_L = D_2$,

(2) if S_R is a filter in $(D,+)$ then $S_R = D_1$.

To prove claim (1), assume that S_L is a filter in $(D,+)$. To show that $S_L = D_2$, it is sufficient to show that $D_2 \subseteq S_L$. Let $x \in D_2$. Since $S_L \neq \emptyset$, there exists an element y in S_L. Then $y \in D_2$. Thus $x + y = y \in S_L$. Since S_L is a filter in $(D,+)$, $x \in S_L$.

Hence $D_2 \subseteq S_L$. Therefore $D_2 = S_L$.

The proof of claim (2) is similar to the proof of Claim (1).

Consider D_1 and D_2.

Case 1. $D_1 \neq \{e\}$, $D_2 \neq \{e\}$. Claim that $S_L = S_R = \emptyset$.

Suppose that S_L is a filter in $(D,+)$. Then, by Claim (1), $S_L = D_2$. Let $d_1 \in D_1 \setminus \{e\}$, $d_2 \in D_2 \setminus \{e\}$. Then $(d_1+d_2) + e = d_1 + (d_2+e) = d_1 + e = d_1 \neq e$, so $d_1 + d_2 \notin LI_D(e) = D_2$. Now $d_2 + (d_1+d_2) = (d_2+d_1) + d_2 = e + d_2 = d_2 \in D_2 = S_L$. Since $D_2 = S_L$ is a filter in $(D,+)$, $d_1 + d_2 \in D_2$, a contradiction. Hence $S_L = \emptyset$.

Similarly, if S_R is a filter in $(D,+)$ then we get a contradiction. Hence $S_R = \emptyset$. Therefore $x + a = x + e$ and $a + x = e + x$ for all $x \in D$. By Theorem 1.33, $a + a = a$ or $a + a = e + e$. By Lemma 1.12, $(D,+)$ is a band. Hence we obtain (1) and (2).

Case 2. $D_1 = \{e\}$, $D_2 \neq \{e\}$. Claim that (3) $S_R = \emptyset$, (4) $S_L = \emptyset$ or $S_L = D$.

To prove Claim (3), suppose that S_R is a filter in $(D,+)$. By Claim (1), $S_R = D_1 = \{e\}$. Let $d_2 \in D_2 \setminus \{e\}$. Then $d_2 + e = e \in S_R$. Since S_R is a filter in $(D,+)$, $d_2 = e$, which is a contradiction. Hence $S_R = \emptyset$.

To prove Claim (4), suppose that $S_L \neq \emptyset$. Then S_L is a filter in $(D,+)$. We must show that $S_L = D$. Clearly $S_L \subseteq D$. Let $d \in D$. Since $(D,+) \cong (D_1,+) \times (D_2,+)$, there exists an element d_2 in D_2 such that $d = e + d_2$. So $d = e + d_2 = d_2 \in D_2$. Let $y \in S_L$. Since $S_L \subseteq LI_D(e) = D_2$, $y \in D_2$. Thus $d + y = y \in S_L$. Since S_L is a filter in $(D,+)$, $d \in S_L$. Hence $D = S_L$. Therefore $S_L = \emptyset$ or $S_L = D$. If $S_L = S_R = \emptyset$ then we obtain (1) or (2). If $S_L = D$ and $S_R = \emptyset$ then $x + a = a$ and $a + x = e + x$ for all $x \in D$.

By Proposition 2.12 (2), we get that $a + a = a$. Hence we obtain (3).

Case 3. $D_1 \neq \{e\}$, $D_2 = \{e\}$.

Using a proof similar proof to the one in Case 2, we can show that $S_L = \emptyset$ and ($S_R = \emptyset$ or $S_R = D$). If $S_L = S_R = \emptyset$ then we obtain (1) or (2). If $S_L = \emptyset$ and $S_R = D$ then $x + a = x + e$ and $a + x = a$ for all $x \in D$. By Proposition 2.12 (1), we get that $a + a = a$. Hence we obtain (4).

Case 4. $D_1 = D_2 = \{e\}$. Since $(D,+) \cong (D_1,+) \times (D_2,+)$, $D = \{e\}$. Hence $|K| = 2$ which is a contradiction. Therefore this case cannot occur.

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