# CHAPTER IV

## EMBEDDING THEOREMS

In this chapter we shall study embedding theorems involving ratio seminear-rings and seminear-fields.

Theorem 4.1. Every ratio seminear-ring can be embedded into a o-seminear-field.

Proof. Let D be a ratio seminear-ring. Let a be a symbol not representing any element of D. Extend + and $\cdot$ from D to $D \cup \{a\}$ by $ax = xa = a$ for all $x \in D \cup \{a\}$ and $a + x = x + a = x$ for all $x \in D \cup \{a\}$. It is easy to check that $(D \cup \{a\},+,\cdot)$ is a seminear-field. Define $f : D \to D \cup \{a\}$ by $f(x) = x$ for all $x \in D$. Then f is clearly a monomorphism. Hence D can be embedded into a o-seminear-field.

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Theorem 4.2. Every ratio seminear-ring can be embedded into an $\infty$-seminear-field.

Proof. Let D be a ratio seminear-ring. Let a be a symbol not representing any element of D. Extend + and $\cdot$ from D to $D \cup \{a\}$ by $ax = xa = a$ for all $x \in D \cup \{a\}$ and $a + x = x + a = a$ for all $x \in D \cup \{a\}$. It is easy to check that $(D \cup \{a\},+,\cdot)$ is a seminear-field. Define $g : D \to D \cup \{a\}$ by $g(x) = x$ for all $x \in D$. Then g is clearly a monomorphism. Hence D can be embedded into a $\infty$-seminear-field.

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Theorem 4.3. Let D be a ratio seminear-ring such that $(D,+)$ is a left zero semigroup. Then D can be embedded into an additive left zero seminear-field with a category I special element.

Proof. Let a be a symbol not representing any element of D. Extend + and • from D to D $\cup$ {a} by ax = xa = a for all x $\varepsilon$ D $\cup$ {a} and a + x = a and x + a = x for all x $\varepsilon$ D $\cup$ {a}. Claim that x + y = x for all x,y $\varepsilon$ D $\cup$ {a}. Let x,y $\varepsilon$ D $\cup$ {a}. If x,y $\varepsilon$ D then x + y = x since (D,+) is a left zero semigroup. Suppose that x = a or y = a. By extending +, we get that x + y = x. Hence x + y = x for all x,y $\varepsilon$ D $\cup$ {a}. Therefore (D $\cup$ {a},+,•) is a seminear-field. Define f : D$\longrightarrow$D $\cup$ {a} by f(x) = x for all x $\varepsilon$ D. Then f is clearly a monomorphism. Hence we obtain this Theorem. #

Theorem 4.4. Let D be a ratio seminear-ring such that (D,+) is a right zero semigroup. Then D can be embedded into an additive right zero seminear-field with a category I special element.

Proof. This proof is similar to the proof of Theorem 4.3. #

From Theorem 2.12 and Theorem 3.24 we get the following two theroems.

Theorem 4.5. Every ratio seminear-ring can be embedded into a seminear-field with a category II special element.

Theorem 4.6. Every ratio seminear-ring can be embedded into a seminear-field with a category VI special element.

Theorem 4.7. Let K be a seminear-field with a category I special element. Then K can be embedded into a seminear-field with a category II special element if and only if $|K| = 2$.

Proof. Let a be the category I special element of K and e the identity of (K$\setminus${a},•). Assume that that K can be embedded into a seminear-field $K'$ with a category II special element. Let $a'$ be the category II special element of $K'$ and let $e'$ be the identity of $(K'\setminus\{a'\},\cdot)$. Then, up to isomorphism, we can consider that $K \subseteq K'$. Since $K'$ has two multiplicative idempotents, $\{e,a\} = \{e',a'\}$. If $e = e'$ then $a = a'$. Thus $a' = a = a \cdot e = a' \cdot e' = e'$, a contradiction. Hence $e = a'$ and $a = e'$. Suppose that $|K| > 2$. Let $x \in K\setminus\{e,a\}$. Then $x = e'x = ax = a$, a contradiction. Therefore $|K| = 2$.

Conversely, assume that $|K| = 2$. Then $K = \{a,e\}$. By Theorem 1.31, K must have one of structures given below :

(1)

| · | e | a |
|---|---|---|
| e | e | a |
| a | a | a |

and

| · | e | a |
|---|---|---|
| e | e | a |
| a | a | a |

or

(2)

| · | e | a |
|---|---|---|
| e | e | a |
| a | a | a |

and

| + | e | a |
|---|---|---|
| e | e | e |
| a | e | a |

or

(3)

| · | e | a |
|---|---|---|
| e | e | a |
| a | a | a |

and

| + | e | a |
|---|---|---|
| e | e | e |
| a | a | a |

or

(4)

| · | e | a |
|---|---|---|
| e | e | a |
| a | a | a |

and

| + | e | a |
|---|---|---|
| e | e | a |
| a | e | a |

or

(5)

| · | e | a |
|---|---|---|
| e | e | a |
| a | a | a |

and

| + | e | a |
|---|---|---|
| e | a | a |
| a | a | a |

.

Let $K' = \{a',e'\}$ be a set. Define + and · on $K'$ as follows :

(i)

| · | $e'$ | $a'$ |
|---|---|---|
| $e'$ | $e'$ | $e'$ |
| $a'$ | $e'$ | $a'$ |

and

| + | $e'$ | $a'$ |
|---|---|---|
| $e'$ | $e'$ | $e'$ |
| $a'$ | $e'$ | $a'$ |

or

(ii)

| $\cdot$ | $e'$ | $a'$ |
|---|---|---|
| $e'$ | $e'$ | $e'$ |
| $a'$ | $e'$ | $a'$ |

and

| $+$ | $e'$ | $a'$ |
|---|---|---|
| $e'$ | $e'$ | $a'$ |
| $a'$ | $a'$ | $a'$ |

or

(iii)

| $\cdot$ | $e'$ | $a'$ |
|---|---|---|
| $e'$ | $e'$ | $e'$ |
| $a'$ | $e'$ | $a'$ |

and

| $+$ | $e'$ | $a'$ |
|---|---|---|
| $e'$ | $e'$ | $e'$ |
| $a'$ | $a'$ | $a'$ |

or

(iv)

| $\cdot$ | $e'$ | $a'$ |
|---|---|---|
| $e'$ | $e'$ | $e'$ |
| $a'$ | $e'$ | $a'$ |

and

| $+$ | $e'$ | $a'$ |
|---|---|---|
| $e'$ | $e'$ | $a'$ |
| $a'$ | $e'$ | $a'$ |

or

(v)

| $\cdot$ | $e'$ | $a'$ |
|---|---|---|
| $e'$ | $e'$ | $e'$ |
| $a'$ | $e'$ | $a'$ |

and

| $+$ | $e'$ | $a'$ |
|---|---|---|
| $e'$ | $e'$ | $e'$ |
| $a'$ | $e'$ | $e'$ |

.

It is easy to verify that K with stureture (1),(2),(3),(4) and (5) are isomorphic to $K'$ with structure (i),(ii),(iii),(iv) and (v) respectively.

Theorem 4.8. Let K be a seminear-field with a category I or II special element. Then K cannot be embedded into a seminear-field with a category VI special element.

Proof. By hypothesis, K has two multiplicative idempotents. Since a seminear-field with a category VI special element contains exactly one multiplicative idempotent, we are done.

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Corollary 4.9. Let K be a 0-seminear-field or an $\infty$-seminear-field. Then K can not be embedded into a seminear-field with a category VI special element.

Proof. Follows directly from Theorem 4.9.

<u>Theorem 4.10</u>. Let K be a seminear-field with a category VI special element. Then K cannot be embedded into a seminear-field with a category I special element.

<u>Proof</u>. Let a be a category VI special element of K and e the identity of $(K\setminus\{a\},\cdot)$. Suppose that K can be embedded into a seminear-field $K'$ with a category I special element $a'$.

Let $e'$ be the identity of $(K'\setminus\{a'\},\cdot)$. Consider, up to isomorphism, $K \subseteq K'$. Note that $a \neq a'$. Since $e^2 = e$, $e = e'$ or $e = a'$. If $e = e'$ then $a = e'a = ea \neq a$, a contradiction. Hence $e = a'$. Since $a^2 \neq a$, $a' = a^2a' = a^2e = a^2$. Since $a \in K'\setminus\{a'\}$, there is a $y \in K'\setminus\{a'\}$ such that $ay = e'$. Thus $a = ae' = a(ay) = a^2y = a'y = a'$, a contradiction.

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<u>Theorem 4.11</u>. Let K be a seminear-field with a category VI special element. Then K cannot be embedded into a seminear-field with a category II special element.

<u>Proof</u>. Let a be a category VI special element of K and e the identity of $(K\setminus\{a\},\cdot)$. Suppose that K can be embedded into a seminear-field $K'$ with $a'$ as a category II special element. Let $e'$ be the identity of $(K'\setminus\{a'\},\cdot)$. Consider, up to isomorphism, $K \subseteq K'$. Note that $a \neq a'$. Since $e^2 = e$, $e = e'$ or $e = a'$. If $e = e'$ then $ae = ae' = a$ contradicting the fact that $ae \neq a$. If $e = a'$ then $ae = aa' = a$ contradicting the fact that $ae \neq a$.

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<u>Remark 4.12</u>. Let K be a seminear-field with a category III or IV special element. Then K cannot be embedded into a seminear-field with a category V special element.

Proof. Since K has two multiplicative idempotents but a seminear-field with a category V special element contains exactly one multiplicative idempotent, we obtain Remark 4.3.

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Using a similar proof as in Remark 4.12, we obtain the following remarks :

Remark 4.13. Let K be a seminear-field with a category III or IV special element. Then K cannot be embedded into a seminear-field with a category VI special element.

Remark 4.14. Let K be a seminear-field with a category V special element. Then K cannot be embedded into a seminear-field with a category III and IV special element.

Remark 4.15. Let K be a seminear-field of order 2 with a category VI special element. Then K cannot be embedded into a seminear-field with a category III and IV special element.

Remark 4.16. Let K be a seminear-field of order 2 containing a category I or II special element. Then K cannot be embedded into a seminear-field with a category V special element.

From page 64 we see that, up to isomorphism, there are 3 seminear-fields with a category V special element. Let $K = \{a,e\}$ with structure given in page 64.

Let $(G,\cdot)$ be a group containing an element $x_o$ of order 2. Let a be a symbol not representing any element of G and let $d \in G$. Let $K' = G \cup \{a\}$. Define + on $K'$ and extend $\cdot$ to $K'$ by

$ax = dx$ and $xa = xd$ for all $x \in G$, $a^2 = d^2$ and either

(i) $x + y = y$ for all $x,y \in K'$ or

(ii) $x + y = x$ for all $x,y \in K'$.

Then $(K',+,\cdot)$ is a seminear-field with a as a category VI special element. Let $e'$ be the identity of $(G,\cdot)$

Define $f : K \longrightarrow K'$ by $f(e) = e'$ and $f(a) = x_o$. It is easy to check that K with structure (3) can be embedded into a seminear-field $K'$ with statement (i) and K with structure (4) can be embedded into a seminear-field $K'$ with statement (ii).

Conjecture : $K'$ with structure (1) can be embedded into a seminear-field with a category VI special element.

Definition 4.17. Let K be a seminear-field and $L \subseteq K$. L is said to be a subseminear-field of K iff L forms a seminear-field with respect to the same operations on K.

Definition 4.18. Let K be a seminear-field. Let L be a subseminear-field of K. L is called the prime seminear-field of K iff L is the smallest subseminear-fied of K.

Let K be a seminear-field and let a be a special element of K. If $a^2 = a$ (i.e. a is a category I,II,III or IV special element of K) then, by Theorem 3.9 in [1] page 27, there exist a smallest subseminear-field contained in K.

For examples, see page 47 - 59 in [1].

Now we shall give an examples of a seminear-field K with a as a category VI special element such that the prime seminear-field does not exist.

Example 4.19. Let $K = \{e,b,a\}$ with the structure :

| $\cdot$ | e | b | a |
|---|---|---|---|
| e | e | b | e |
| b | b | e | b |
| a | e | b | e |

| + | e | b | a |
|---|---|---|---|
| e | e | b | a |
| b | e | b | a |
| a | e | b | a |

It is easy to check that K is a seminear-field with a as a category VI special element.

Let $K_1 = \{e,b\}$ . Then $(K_1,+,\cdot)$ is a subseminear-field of K. Note that $K_1$ is a seminear-field with b as a category V special element.

Let $K_2 = \{e,a\}$. Then $(K_2,+,\cdot)$ is a subseminear-field of K. Note that $K_2$ is a seminear-field with a as a category VI special element. Thus $K_1 \cap K_2 = \{e\}$ which is not seminear-field.

Hence a prime seminear-field does not exist.

Remark : We shall state open questions for future research.

(1) Can a seminear-field with a category V special element be embedded into a seminear-field with a category II special element?

(2) Can a seminear-field with a category V special element be embedded into a seminear-field with a category VI special element?

(3) Under what conditions can a seminear-field with a category I special element be embedded into a near-field.