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GENERALIZED FACTORIZABILITY OF CERTAIN IMPLICIT DEPENDENCE COPULAS

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เราเรียกคอปูลา $C_{X,Y}$ ของตัวแปรสุ่ม X และ Y ที่มีการแจกแจงเอกรูปบน $[0,1]$ ว่าคอปูลาการขึ้นต่อกันแบบปริยายถ้า $f(X) = g(Y)$ เกือบแน่นอน สำหรับบางฟังก์ชันบอเรล f และ g บน $[0,1]$ เราแสดงโดยใช้ผลคูณมาร์คอฟแบบทั่วไปว่าคอปูลาการขึ้นต่อกันโดยปริยายสมนัยแบบหนึ่งต่อหนึ่งกับคลาสอิงพารามิเตอร์ของคอปูลาย่อยบนโดเมนที่สอดคล้องกัน สำหรับกรณี $f = g = \Lambda_\theta$ ฟังก์ชันเต็นท์ที่มีจุดสูงสุดที่ $(\theta, 1)$ เราแสดงด้วยว่าในกรณี $f = g = \alpha$ ฟังก์ชันคงเมเชอร์อย่างง่าย คอปูลาการขึ้นต่อกันโดยปริยายสามารถแยกตัวประกอบแบบทั่วไปได้

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The copula $C_{X,Y}$ of random variables X and Y that are uniformly distributed on $[0, 1]$ is called an implicit dependence copula if $f(X) = g(Y)$ almost surely for some Borel functions f and g on $[0, 1]$. Via a generalized Markov product, we give a one-to-one correspondence between the implicit dependence copulas and the parametric classes of subcopulas on a corresponding domain for the cases that $f = g = \Lambda_\theta$, the tent function whose top is at $(\theta, 1)$. We also show in the case $f = g = \alpha$, a simple measure-preserving function, that implicit dependence copulas are generalized factorizable.

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CHAPTER I
INTRODUCTION

For random variables X and Y with continuous distribution functions F_X and F_Y, respectively, and joint distribution function $F_{X,Y}$, the copula $C_{X,Y}$ of X and Y is the function on $\mathbb{I}^2 := [0,1]^2$ for which

$$F_{X,Y}(x,y) = C_{X,Y}(F_X(x), F_Y(y)) \text{ for } x, y \in \mathbb{R}.$$

By the probability integral transform, it is evident that $C_{X,Y}$ captures marginal-free dependence structure between X and Y [9, 13].

Copulas can be constructed from measure-preserving transformations. For measure-preserving transformations f and g on Lebesgue measure space $(\mathbb{I}, \mathscr{B}(\mathbb{I}), \lambda)$, the copula $C_{f,g}$ is defined by [4]

$$C_{f,g}(x,y) = \lambda\left(f^{-1}([0,x]) \cap g^{-1}([0,y])\right).$$

In fact, every copula can be constructed in this way [4]. This form of copulas is quite well-suited for the Markov product, introduced in [5] as a tool to study Markov processes. The generalized Markov product was later introduced in an attempt to solve the compatibility problem. For a parametric class of copulas $\mathcal{A} = \{A_t\}_{t\in[0,1]}$, the generalized Markov product [17] is a binary operation on the set of bivariate copulas of C and D, defined by

$$C *_{\mathcal{A}} D(x,y) = \int_0^1 A_t(\partial_2 C(x,t), \partial_1 D(t,y))dt$$

for $x, y \in [0,1]$.

Complete dependence between X and Y happens when one is a Borel function of the other almost surely. If continuous random variables X and Y are completely dependent, then their copula is called a complete dependence copula. Every complete dependence copula can be written in the form $C_{e,f}$ or $C_{f,e}$ for some measure-preserving transformation f [9, 21]. Despite its simplistic and deterministic nature, complete dependence copulas are ubiquitous and useful in theoretical studies of copulas [19, 20, 21].

Less studied but more stochastic is the notion of implicit dependence which occurs when the two random variables are equal almost surely after applying a corresponding pair of Borel transformations. If continuous random variables X and Y are implicitly dependent, then their copula is called an implicit dependence copula. To the best of our knowledge, there have never been any characterizations of implicit dependence copulas. In this work, we prove that some implicit dependence copulas can be written as the product of some complete dependence copulas. More precisely, C is the copula of random variables $X, Y \sim \mathcal{U}(0,1)$ for which $\Lambda_\theta(X) = \Lambda_\theta(Y)$ almost surely if and only if $C = C_{e,\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta,e}$ for some class $\mathcal{A}$ of sub-copulas on $\{0,\theta,1\}^2$ which extend to copulas by Sklar's theorem. Here, $\Lambda_\theta(x) = \min\left(\frac{x}{\theta}, \frac{1-x}{1-\theta}\right)$ where $\theta \in (0,1)$. The only if part can be generalized from Λ_θ to measure-preserving transformations α on $[0,1]$ that can be partitioned by $P = \{0 = a_0, a_1, \ldots, a_n = 1\}$ into strictly increasing bijections from $(a_{i-1}, a_i]$ onto $[0,1]$. We call such measure-preserving functions α **simple**.

CHAPTER II
PRELIMINARIES

2.1 Copulas

First, we introduce the notions of 2-increasing and grounded which are used to define copula. Let $\mathbb{I}$ denote $[0,1]$.

Definition 2.1. ([13]) Let S_1 and S_2 be nonempty subsets of $\mathbb{I}$, and let H be a two-dimensional real-valued function whose domain is $S_1 \times S_2$. Let $B = [x_1, x_2] \times [y_1, y_2]$ be a rectangle all of whose vertices are in $S_1 \times S_2$. Then, the H**-volume** of B is given by

$$V_H(B) = H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1). \tag{2.1}$$

H is said to be 2**-increasing**, if $V_H(B) \geq 0$ for all rectangles B whose vertices lie in $S_1 \times S_2$.

Definition 2.2. ([13]) Let S_1 and S_2 be nonempty subsets of $\mathbb{I}$ such that S_1 has a least element a and S_2 has a least element b. A function $H : S_1 \times S_2 \to \mathbb{I}$ is called **grounded** if

$$H(x, b) = 0 = H(a, y) \tag{2.2}$$

for all $(x, y) \in S_1 \times S_2$.

Now, we define subcopulas and copulas.

Definition 2.3. ([13]) A **subcopula** is a function C' with the following properties.

1. Domain of C' is $S_1 \times S_2$ where $S_1, S_2 \subseteq \mathbb{I}$ contains 0 and 1.

2. C' is grounded and 2-increasing.

3. For any $x \in S_1, y \in S_2$, $C'(x, 1) = x$ and $C'(1, y) = y$.

Definition 2.4. A **copula** is a subcopula C whose domain is $\mathbb{I}^2$.

Definition 2.5. The copula C^t is called the **transpose** of a copula C if $C^t(x, y) = C(y, x)$ for $(x, y) \in \mathbb{I}^2$.

Definition 2.6. The **support** of a copula C is defined by

$$\operatorname{supp}(C) = \mathbb{I}^2 \setminus \bigcup \{R \equiv (a,b) \times (c,d) \subseteq \mathbb{I}^2 : V_C(R) = 0\}.$$

Example 2.7. Listed below are some important copulas.

1. $M(x,y) = \min\{x,y\}$ is called the **Fréchet-Hoeffding upper bound** (see Theorem 2.8). We know that $M(x,y) = x$ if $(x,y) \in \{(x,y) \in \mathbb{I}^2 : x < y\} := A_1$ and $M(x,y) = y$ if $(x,y) \in \{(x,y) \in \mathbb{I}^2 : y < x\} := A_2$. Then, $V_M(R) = 0$ for any rectangle $R := (a,b) \times (c,d) \subseteq A_1 \cup A_2$; otherwise, $V_M(R) \neq 0$. That is, $\operatorname{supp}(M) = \{(x,x) : x \in \mathbb{I}\}$.

2. $W(x,y) = \max\{x+y-1, 0\}$ is called the **Fréchet-Hoeffding lower bound** (see Theorem 2.8). Then, $W(x,y) = 0$ if $(x,y) \in \{(x,y) \in \mathbb{I}^2 : x+y < 1\} := B_1$ and $W(x,y) = x+y-1$ if $(x,y) \in \{(x,y) \in \mathbb{I}^2 : x+y-1 > 0\} := B_2$. Then $V_W(R) = 0$ for any rectangle $R := (a,b) \times (c,d) \subseteq B_1 \cup B_2$; otherwise, $V_W(R) \neq 0$. That is, $\operatorname{supp}(W) = \{(x, 1-x) : x \in \mathbb{I}\}$.

3. $\Pi(x,y) = xy$ is called the **independence copula** (see Theorem 2.13). For any nonempty rectangle $R := (a,b) \times (c,d) \subseteq \mathbb{I}^2$, $V_\Pi(R) = (a-b)(c-d) > 0$ which implies that $\operatorname{supp}(\Pi) = \mathbb{I}^2$.

4. $C_1(x,y) = pM(x,y) + (1-p)W(x,y)$ where $p \in (0,1)$. It can be shown by a similar argument as above that $\operatorname{supp}(C_1) = \operatorname{supp}(M) \cup \operatorname{supp}(W)$.

Next, we present some properties of copulas.

Theorem 2.8. ***(Fréchet-Hoeffding bounds)*** *([13]) For every copula C and $(x,y) \in \mathbb{I}^2$,*

$$W(x,y) \leq C(x,y) \leq M(x,y). \tag{2.3}$$

Theorem 2.9. *([13]) The first partial derivatives $\partial_1 C$ and $\partial_2 C$ of a copula C exist almost everywhere and are Borel-measurable. For any $x, y \in \mathbb{I}$,*

$$0 \leq \partial_1 C(t,y) \leq 1 \text{ and } 0 \leq \partial_2 C(x,t) \leq 1$$

for almost every $t \in \mathbb{I}$.

The next theorem demonstrates the significance of copulas in probability and statistics. It explains the relationships between a copula and a joint distribution function of random variables.

Theorem 2.10. ***(Sklar's Theorem)*** *([13]) Let X, Y be random variables, H be the joint distribution function of X, Y with margins F and G, respectively. Then there exists a copula C such that for all $x, y \in \mathbb{R}$,*

$$H(x, y) = C(F(x), G(y)).$$

If F and G are continuous, then C is unique and denoted by $C_{X,Y}$; otherwise, C is uniquely determined on $Ran(F) \times Ran(G)$.

Definition 2.11. ([9, 15]) A nonempty subset Γ of $\mathbb{R}^2$ is said to be **comonotonic** if for all $(x_1, y_1), (x_2, y_2) \in \Gamma$, $(x_1 - x_2)(y_1 - y_2) \geq 0$ and is said to be **countermonotonic** if for all $(x_1, y_1), (x_2, y_2) \in \Gamma$, $(x_1 - x_2)(y_1 - y_2) < 0$.

A random vector (X, Y) is called **comonotonic** if its support is comonotonic and is called **countermonotonic** if its support is countermonotonic.

Definition 2.12. Let X and Y be random variables. X and Y are **independent** if $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B)$ for any set $A, B \in \mathscr{B}(\mathbb{R})$.

Theorem 2.13. *([9, 15]) Let X, Y be random variables with copula $C_{X,Y}$ and continuous marginal distributions.*

1. *(X, Y) is comonotonic if and only if $C_{X,Y} = M$.*
2. *(X, Y) is countermonotonic if and only if $C_{X,Y} = W$.*
3. *X, Y are independent if and only if $C_{X,Y} = \Pi$.*

Theorem 2.14. *(Probability integral transformation)([9]) Let X be a random variable whose distribution function is given by F. If F is continuous, then $F \circ X$ is uniformly distributed on $\mathbb{I}$.*

Theorem 2.15. *([9]) A copula can be extended to a joint distribution function whose marginals are uniformly distributed on $\mathbb{I}$.*

2.2 Measure-preserving Transformations

In this section, we introduce a construction method of copulas from measure-preserving transformations.

Definition 2.16. Let $(\mathbb{I}, \mathscr{B}(\mathbb{I}), \lambda)$ be the Lebesgue measure space. A mapping $f : \mathbb{I} \to \mathbb{I}$ is said to be **measure-preserving** if $\lambda(f^{-1}(B)) = \lambda(B)$ for every set $B \in \mathscr{B}(\mathbb{I})$.

Example 2.17. 1. Let e be denotes the identity function on $\mathbb{I}$, i.e., $e(x) = x$ for all $x \in \mathbb{I}$. It is clear that e is measure-preserving.

2. For $\theta \in (0, 1)$, we define, the tent function, $\Lambda_\theta : [0, 1] \to [0, 1]$ by

$$\Lambda_\theta(x) := \begin{cases} \frac{x}{\theta} & \text{if } x \leq \theta, \\ \frac{1-x}{1-\theta} & \text{if } x > \theta. \end{cases} \tag{2.4}$$

For any interval $[a, b] \subseteq \mathbb{I}$, $\Lambda_\theta^{-1}([a, b]) = [\theta a, \theta b] \cup [1 - (1 - \theta)b, 1 - (1 - \theta)a]$, so $\lambda(\Lambda_\theta^{-1}([a, b])) = \theta(b - a) + (1 - \theta)(b - a) = b - a = \lambda([a, b])$ which implies that Λ_θ is a measure-preserving transformation.

Theorem 2.18. *([4]) If f, g are measure-preserving transformations on the space $(\mathbb{I}, \mathscr{B}(\mathbb{I}), \lambda)$, then the function $C_{f,g} : \mathbb{I}^2 \to \mathbb{I}$ defined by*

$$C_{f,g}(x, y) := \lambda\Big(f^{-1}([0, x]) \cap g^{-1}([0, y])\Big) \tag{2.5}$$

is a copula. Conversely, for every copula C, there exist measure-preserving transformations f, g such that $C = C_{f,g}$.

Example 2.19. The copula with measure-preserving transformations e and Λ_θ is

$$\begin{aligned} C_{e,\Lambda_\theta}(x, y) &= \lambda\left(e^{-1}([0, x]) \cap \Lambda_\theta^{-1}([0, y])\right) \\ &= \lambda\left([0, x] \cap ([0, \theta y] \cup [1 - (1 - \theta)y, 1])\right) \\ &= \begin{cases} y\theta & \text{if } \Lambda_\theta(x) > y, \\ x & \text{if } \Lambda_\theta(x) < y \text{ and } x < \theta, \\ x + y - 1 & \text{if } \Lambda_\theta(x) < y \text{ and } x > \theta. \end{cases} \end{aligned}$$

Hence,

$$\partial_2 C_{e,\Lambda_\theta}(x, y) = \begin{cases} \theta & \text{if } \Lambda_\theta(x) > y, \\ 0 & \text{if } \Lambda_\theta(x) < y \text{ and } x < \theta, \\ 1 & \text{if } \Lambda_\theta(x) < y \text{ and } x > \theta. \end{cases} \tag{2.6}$$

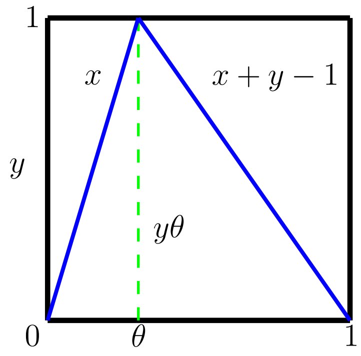


Figure 2.1: The value of $C_{e,\Lambda_\theta}(x, y)$.

Some properties of the copulas $C_{f,g}$ are given next.

Theorem 2.20. *([9]) Let f, g be measure-preserving transformations on $\mathbb{I}$. Then*

1. $C^t_{f,g} = C_{g,f}$.
2. $C_{f,g} = M$ *if and only if* $f = g$ *a.s. on* $\mathbb{I}$.
3. $C_{f,e} = C_{g,e}$ *if and only if* $f = g$ *a.s. on* $\mathbb{I}$.

Example 2.21. $C_{e,\Lambda_\theta}(x, y) = C^t_{e,\Lambda_\theta}(y, x) = C_{\Lambda_\theta,e}(y, x)$, so $\partial_2 C_{e,\Lambda_\theta}(x, y) = \partial_1 C_{\Lambda_\theta,e}(y, x)$.

Theorem 2.22. *([9]) If a copula $C = C_{f,g}$, then f, g are random variables on $(\mathbb{I}, \mathscr{B}(\mathbb{I}), \lambda)$ whose joint distribution function is given by C.*

Theorem 2.23. *([6]) Let f, g be measure-preserving transformations. Then, the following conditions are equivalent:*

1. $C_{f,g} = \Pi$;
2. *f and g, when regarded as random variables on the standard probability space $(\mathbb{I}, \mathscr{B}(\mathbb{I}), \lambda)$, are independent.*

2.3 Dependence Copulas

We will divide this section into two parts.

2.3.1 Complete Dependence Copulas

Definition 2.24. Random variables X and Y are said to be **completely dependent** if there exists a Borel function f such that $Y = f(X)$ a.s. or $X = f(Y)$ a.s.

Definition 2.25. Let $C = C_{X,Y}$ be the copula of random variables X, Y with continuous marginal distribution functions. C is called a **complete dependence copula** if X and Y are completely dependent.

Theorem 2.26. *([9, 21]) Let $C = C_{X,Y}$ be the copula of random variables X, Y with continuous marginal distribution functions. Then the following conditions are equivalent:*

1. *X and Y are completely dependent;*
2. *there exists a measure-preserving transformation ψ on $\mathbb{I}$ such that $C = C_{e,\psi}$ or $C = C_{\psi,e}$.*

Example 2.27.

1. $C_{e,e}(x, y) = \lambda([0, x] \cap [0, y]) = \min\{x, y\} = M(x, y)$. Then M is a complete dependence copula.
2. Let $g(x) = 1-x$. Then $C_{e,g}(x, y) = \lambda([0, x]\cap[1-y, 1]) = \max\{x+y-1, 0\} = W(x, y)$, so W is a complete dependence copula.
3. Since $\Lambda_\theta(x)$ is a measure-preserving transformation, C_{e,Λ_θ} is a complete dependence copula.

2.3.2 Implicit Dependence Copulas

Definition 2.28. Random variables X and Y are said to be **implicitly dependent** if there exist Borel functions f and g such that $f(X) = g(Y)$ a.s.

Definition 2.29. Let $C = C_{X,Y}$ be the copula of random variables X, Y with continuous marginal distribution functions. C is called an **implicit dependence copula** if X and Y are implicitly dependent.

Definition 2.30. We call the copula C is **symmetric implicit dependence copulas via function** f if there exist random variables X and Y are uniformly distributed on $\mathbb{I}$ such that $f(X) = f(Y)$ a.s. and $C = C_{X,Y}$.

Example 2.31. Let $X \sim \mathcal{U}(0, 1)$ and $Z \sim \text{Ber}(p)$, where $p \in (0, 1)$, be independence random variables and define $Y = \delta_1(Z)X+\delta_0(Z)(1-X)$. Then $Y \sim \mathcal{U}(0, 1)$, $\Lambda_{0.5}(X) = \Lambda_{0.5}(Y)$ and $C_{X,Y} = pM + (1 - p)W$.

Solution. First, we will show that $Y \sim \mathcal{U}(0,1)$.

$$\begin{aligned}\mathbb{P}(Y \le y) &= \mathbb{P}(\delta_1(Z)X + \delta_0(Z)(1-X) \le y) \\ &= \mathbb{P}(1 - X \le y \mid Z = 1)\,\mathbb{P}(Z = 1) + \mathbb{P}(X \le y \mid Z = 0)\,\mathbb{P}(Z = 0) \\ &= (1 - (1-y))p + y(1-p) = y.\end{aligned}$$

It is easy to see that $\Lambda_{0.5}(1-X) = \Lambda_{0.5}(X)$, so $\Lambda_{0.5}(Y) = \Lambda_{0.5}(X)$. Next, we will compute the copula of X, Y.

$$\begin{aligned}C_{X,Y}(x,y) &= \mathbb{P}(X \le x, Y \le y) \\ &= \mathbb{P}(X \le x, X \le y \mid Z = 1)\,\mathbb{P}(Z = 1) \\ &\quad + \mathbb{P}(X \le x, 1 - X \le y \mid Z = 0)\,\mathbb{P}(Z = 0) \\ &= p \cdot \min\{x,y\} + (1-p)\max\{x+y-1, 0\} \\ &= p \cdot M(x,y) + (1-p) \cdot W(x,y).\end{aligned}$$

■

Example 2.31 shows that $pM + (1-p)W$ is an implicit dependence copula for $0 < p < 1$. Evidently, their support is $\mathrm{supp}(M) \cup \mathrm{supp}(W)$. However, there are many other implicit dependence copulas with this support. We shall give thier characterizations in Chapter III.

2.4 The Markov Product

Let $\mathscr{C}$ be the class of all copulas. In [3, 5, 14], the Markov product, defined as a binary operation on $\mathscr{C}$, was studied in many aspects, especially its relationship with the Markov processes. It is then later called the Markov product in [9].

Definition 2.32. The Markov product is the binary operation on $\mathscr{C}$ defined, for $A, B \in \mathscr{C}$, by

$$A * B(x,y) = \int_0^1 \partial_2 A(x,t)\partial_1 B(t,y)dt \tag{2.7}$$

for all $x, y \in [0,1]$.

The next theorem says that the Markov product of copulas is a copula, as well as some properties of the Markov product.

Theorem 2.33. *([9]) Let A, B, C be copulas and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$. Then:*

1. *$A * B$ is copula.*

2. *$A * (\alpha B + \beta C) = \alpha(A * B) + \beta(A * C)$.*

3. $(\alpha B + \beta C) * A = \alpha(B * A) + \beta(C * A)$.

4. $M * C = C = C * M$.

5. $\Pi * C = \Pi = C * \Pi$.

6. $A * (B * C) = (A * B) * C$.

7. $W * W = M$.

In fact, the copula M is the identity of the Markov product.

Definition 2.34. A copula A is called a **left inverse** of a copula C if $A * C = M$ and is called a **right inverse** of a copula C if $C * A = M$.

Definition 2.35. Let A be a copula. A is **right invertible** if there exists a copula B such that $A * B = M$.

Definition 2.36. Let B be a copula. B is **left invertible** if there exists a copula A such that $A * B = M$.

Theorem 2.37. *([9]) Let C be a copula. The inverse of C (if exists) must be C^t.*

The copulas satisfy some nice properties under the Markov product. Recall that e is the identity on $\mathbb{I}$.

Theorem 2.38. *([9]) Let f, g, h be measure-preserving transformations on $\mathbb{I}$. Then*

1. $C_{f,g} = C_{f,e} * C_{e,g}$.

2. $C_{f,e} * C_{g,e} = C_{f \circ g,e}$ *and* $C_{e,g} * C_{e,f} = C_{e,f \circ g}$.

3. $C_{f,e}$ *is right invertible and* $C_{e,f}$ *is left invertible.*

Definition 2.39. Let $\mathcal{A} = \{A_t\}_{t \in [0,1]}$ be a parametric class of copulas. **The generalized Markov product** of copulas C and D with respect to $\mathcal{A}$ is defined as

$$C *_{\mathcal{A}} D(x, y) = \int_0^1 A_t(\partial_2 C(x, t), \partial_1 D(t, y))dt \tag{2.8}$$

for all $(x, y) \in [0, 1]^2$ at which the integral exists.

Notice that if $A_t = \Pi$ for all $t \in [0, 1]$, then the generalized markov product reduces to the Markov product. In general, the measurability of the integrand in 2.8 needs to be verified. See [9, 17].

Theorem 2.40. *([17]) If the map $(t, x, y) \to A_t(x, y)$ is Borel measurable, then*

(∗) *for all* $x, y \in [0,1]$ *and for all* $C, D \in \mathscr{C}$, $A_t(\partial_2 C(x,t), \partial_1 D(t,y))$ *is Lebesgue measurable in* $t \in [0,1]$

and hence $C *_{\mathcal{A}} D$ *is a well-defined function on* $\mathbb{I}^2$.

Let $\mathcal{M}$ denote the collection of families $\{A_t\}$ such that (∗) holds.

Theorem 2.41. *([17]) Let* $\mathcal{A} \in \mathcal{M}$. *For every copulas* C *and* D, $C *_{\mathcal{A}} D$ *is a copula.*

In this thesis, the word Markov will be omitted and we shall call the Markov product simply as the **product** and the generalized Markov product as the **generalized product** or the $\{A_t\}$**-product**.

2.5 Conditional Expectation

In the last section, we will review some properties of conditional expectation and some main tools used in next chapter. By the Radon-Nikodym theorem, we recall the definition of conditional expectation.

Definition 2.42. ([2, 7]) Let X be a random variable on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with finite expectation and let $\mathscr{G}$ be a sub-σ-algebra of $\mathscr{F}$. The **conditional expectation** of X given $\mathscr{G}$, written $\mathbb{E}[X \mid \mathscr{G}]$, is the random variable on $(\Omega, \mathscr{G})$ satisfying

$$\int_A X d\,\mathbb{P} = \int_A \mathbb{E}[X \mid \mathscr{G}] d\,\mathbb{P} \tag{2.9}$$

for all $A \in \mathscr{G}$.

In general, there are many random variables that satisfy the equation (2.9), all of which must, of course, be equal $\mathbb{P}$-a.s. Any one of them is called a **version** of the conditional expectation $\mathbb{E}[X \mid \mathscr{G}]$.

Definition 2.43. Let X be a random variable on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and let $\mathscr{G}$ be a sub-σ-algebra of $\mathscr{F}$. X and $\mathscr{G}$ are **independent** if $\mathbb{P}(X \in B \mid G) = \mathbb{P}(X \in B)$ for every $B \in \mathscr{B}(\mathbb{R}), G \in \mathscr{G}$.

Theorem 2.44. *([2, 7]) Let* X, Y *be random variables on a probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *with finite expectations and let* $\mathscr{G}$ *be a sub-*σ*-algebra of* $\mathscr{F}$. *Then*

1. *If* X *and* $\mathscr{G}$ *are independent, then* $\mathbb{E}[X \mid \mathscr{G}] = \mathbb{E}[X]$ *a.s.*

2. *If* X *and* Y *are independent, then* $\mathbb{E}[X \mid Y] = \mathbb{E}[X]$ *a.s.*

3. *For $a, b \in \mathbb{R}$, $\mathbb{E}[aX + bY \mid \mathscr{G}] = a\,\mathbb{E}[X \mid \mathscr{G}] + b\,\mathbb{E}[Y \mid \mathscr{G}]$ a.s.*

4. *If X is $\mathscr{G}$-measurable and $\mathbb{E}[|XY|] < \infty$, then $\mathbb{E}[XY \mid \mathscr{G}] = X\,\mathbb{E}[Y \mid \mathscr{G}]$ a.s.*

5. *$\mathbb{E}[\mathbb{E}[X \mid \mathscr{G}]] = \mathbb{E}[X]$ a.s.*

Definition 2.45. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, $\mathscr{G}$ a sub-σ-algebra of $\mathscr{F}$ and $B \in \mathscr{F}$. **The conditional probability** of B given by $\mathscr{G}$ is

$$\mathbb{P}(B \mid \mathscr{G}) = \mathbb{E}[\mathbb{1}_B \mid \mathscr{G}]. \tag{2.10}$$

Definition 2.46. ([10, 12]) Let $(\Omega_1, \mathscr{F}_1)$ and $(\Omega_2, \mathscr{F}_2)$ be measurable spaces. A mapping $K : \Omega_1 \times \mathscr{F}_2 \to \mathbb{R}$ is called a **Markov kernel** (from Ω_1 to $\mathscr{F}_2$) if $\omega_1 \mapsto K(\omega_1, B)$ is $\mathscr{F}_1$-measurable for every fixed $B \in \mathscr{F}_2$ and $B \mapsto K(\omega_1, B)$ is a probability measure for every fixed $\omega_1 \in \Omega_1$.

Theorem 2.47. *(Regular conditional distribution)([10, 12]) Let X, Y be real-valued random variables on a common probability space. Then there exists a Markov kernel, called a regular conditional distribution of Y given X, $K : \mathbb{R} \times \mathscr{B}(\mathbb{R}) \to [0, 1]$ satisfying*

$$K(X(\omega), B) = \mathbb{P}(Y \in B \mid X)(\omega) \ \mathbb{P}\text{-}a.s.$$

Remark 2.48. (see [10, 12])

1. For every random vector (X, Y), a regular conditional distribution $K(\cdot, \cdot)$ of Y given X exists.

2. $K(\cdot, \cdot)$ is unique $\mathbb{P}_X$-a.s. where $\mathbb{P}_X$ is the probability measure induced by X: $\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B))$.

3. $K(\cdot, \cdot)$ only depends on the distribution of (X, Y).

Theorem 2.49. *(Disintegration) ([10, 12]) Let X, Y be real-valued random variables such that $\mathbb{P}(Y \in \cdot \mid X) = K(X, \cdot)$ for some Markov kernel K and let f be a Borel function on $\mathscr{B}(\mathbb{R}^2)$ with $\mathbb{E}[|f(Y, X)|] < \infty$. Then*

$$\mathbb{E}[f(Y, X) \mid X] = \int_{\mathbb{R}} f(s, X) K(X, ds) \ a.s. \tag{2.11}$$

Let C be the copula of random variables X and Y uniformly distributed on $\mathbb{I}$. We denote by $K_C(\cdot, \cdot)$ a version of the regular conditional distribution of Y given X.

The next theorem shows a relationship between copulas and Markov kernels.

Theorem 2.50. *Let X, Y be random variables uniformly distributed on $\mathbb{I}$ with copula C. Then*

$$C(x,y) = \int_0^x K_C(s,[0,y])ds.$$

Proof. Using $f(Y,X) = \mathbb{1}_{A\times B}(X,Y)$ where $A = [0,x]$ and $B = [0,y]$. Then,

$$C(x,y) = \mathbb{P}(X \in A, Y \in B) = \mathbb{E}[f(Y,X)]$$

Since $\mathbb{E}[\mathbb{E}[f(Y,X) \mid X]] = \mathbb{E}[f(Y,X)]$, by disintegration, we have

$$\begin{aligned}
\mathbb{E}[f(Y,X)] &= \mathbb{E}\left[\int_{\mathbb{R}} f(s,X)K_C(X,ds)\right] \\
&= \mathbb{E}\left[\int_B \mathbb{1}_A(X)K_C(X,ds)\right] \\
&= \int_A \int_B K_C(t,ds)dt \\
&= \int_A K_C(t,[0,y])dt.
\end{aligned}$$

□

Next, we recall some theorems in approximating the conditional probability given $X = x$ by the conditional probability given $X \in E_j$ where E_j is a sequence that shrinks to x nicely.

Theorem 2.51. *([16]) Let X be a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ and let B be a set in $\mathscr{F}$. Then there exists a function $\mathbb{P}(B \mid X = x)$ such that for each $A \in \mathscr{B}(\mathbb{R})$,*

$$\mathbb{P}(B \cap \{X \in A\}) = \int_A \mathbb{P}(B \mid X = x)d\,\mathbb{P}_X(x).$$

Definition 2.52. ([18]) Suppose $x \in \mathbb{R}$. A sequence $\{E_j\}$ of Borel sets in $\mathbb{R}$ is said to **shrink to x nicely** if there is a number $\alpha > 0$ with the following property: There is a sequence of ball $B(x, r_i)$, with $\lim_{i\to\infty} r_i = 0$ such that $E_i \subset B(x, r_i)$ and

$$\lambda(E_i) \geq \alpha\lambda(B(x,r_i))$$

for $i = 1, 2, \ldots$.

Theorem 2.53. *([18]) For each x in $\mathbb{R}$, let a sequence $\{E_j(x)\}_{j=1}^{\infty}$ shrink to x nicely and let $f \in L^1(\mathbb{R})$. Then, at almost every x,*

$$f(x) = \lim_{j\to\infty} \frac{1}{\lambda(E_j(x))} \int_{E_j(x)} f d\lambda.$$

The next theorem is a consequence of Theorem 2.51 and 2.53. We use this theorem to show some properties in the next chapter (see Lemma 3.3).

Theorem 2.54. *Let X, Y be random variables and $A \in \mathscr{B}(\mathbb{R})$ and $\{E_j(x)\}_{j=1}^{\infty}$ be a sequence that shrink to x nicely. Then,*

$$\mathbb{P}(Y \in A \mid X = x) = \lim_{j \to \infty} \mathbb{P}(Y \in A \mid X \in E_j).$$

Proof. We apply Theorem 2.53 to the function $f = \mathbb{P}(Y \in A \mid X)$ and obtain

$$\mathbb{P}(Y \in A \mid X = x) = \lim_{j \to \infty} \frac{1}{\mathbb{P}(X \in E_j)} \int_{E_j} \mathbb{P}(Y \in A \mid X = t) d\,\mathbb{P}_X(t).$$

By Theorem 2.51, we have

$$\int_{E_j} \mathbb{P}(Y \in A \mid X = t) d\,\mathbb{P}_X(t) = \mathbb{P}(Y \in A, X \in E_j).$$

Hence,

$$\begin{aligned} \mathbb{P}(Y \in A \mid X = x) &= \lim_{j \to \infty} \frac{\mathbb{P}(Y \in A, X \in E_j)}{\mathbb{P}(X \in E_j)} \\ &= \lim_{j \to \infty} \mathbb{P}(Y \in A \mid X \in E_j). \end{aligned}$$

□

CHAPTER III
Symmetric Implicit Dependence Copulas Via Tent Functions

3.1 Generalized Products of C_{e,Λ_θ} and $C_{\Lambda_\theta,e}$ are implicit dependence copulas

In this section, we show that the generalized products of some complete dependence copulas are implicit dependence copulas. We shall consider only the complete dependence copulas C_{e,Λ_θ} and $C_{\Lambda_\theta,e}$ where $\Lambda_\theta(x) = \min\{\frac{x}{\theta}, \frac{1-x}{1-\theta}\}$ for $0 < \theta < 1$.

For $0 < \theta < 1$, we define injections $\Lambda_\theta^1, \Lambda_\theta^2 : \mathbb{I} \to \mathbb{R}$ by $\Lambda_\theta^1(x) := \dfrac{x}{\theta}$ and $\Lambda_\theta^2(x) := \dfrac{1-x}{1-\theta}$, so that $\Lambda_\theta = \Lambda_\theta^1 \mathbb{1}_{[0,\theta]} + \Lambda_\theta^2 \mathbb{1}_{(\theta,1]}$. Denote $\Lambda_\theta^{ij} := (\Lambda_\theta^i)^{-1} \circ \Lambda_\theta^j$. In particular, $\Lambda_\theta^{12}(x) = \frac{\theta}{1-\theta}(1-x)$ and $\Lambda_\theta^{21}(x) = 1 - \frac{1-\theta}{\theta}x$. Technically, Λ_θ^{12} and Λ_θ^{21} map $\mathbb{I}$ onto $\left[0, \frac{\theta}{1-\theta}\right]$ and $\left[\frac{2\theta-1}{\theta}, 1\right]$, respectively. But they are usually considered as

$$\Lambda_\theta^{12}([0,y]) = (\Lambda_\theta^1)^{-1}\left(\Lambda_\theta^2([0,y])\right) = (\Lambda_\theta^1)^{-1}\left(\left[\frac{1-y}{1-\theta}, \frac{1}{1-\theta}\right]\right) = \left[\frac{\theta}{1-\theta}(1-y), \frac{\theta}{1-\theta}\right]$$

and

$$\Lambda_\theta^{21}([0,y]) = (\Lambda_\theta^2)^{-1}\left(\Lambda_\theta^1([0,y])\right) = (\Lambda_\theta^2)^{-1}\left(\left[0, \frac{y}{\theta}\right]\right) = \left[1 - \frac{1-\theta}{\theta}y, 1\right].$$

Lemma 3.1. *Let $\mathcal{A} := \{A_t\}_{t\in[0,1]}$ be a class of copulas. If $A_t(\theta,\theta)$ is measurable in t, then, for every $x, y \in \mathbb{I}$, $A_t(\partial_2 C_{e,\Lambda_\theta}(x,t), \partial_2 C_{e,\Lambda_\theta}(y,t))$ is measurable in t, i.e., $C_{e,\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta,e}$ is a copula.*

Proof. Notice from Example 2.19 that

$$\partial_2 C_{e,\Lambda_\theta}(x,y) = \begin{cases} \theta & \text{if } \Lambda_\theta(x) > y, \\ 0 & \text{if } \Lambda_\theta(x) < y \text{ and } x < \theta, \\ 1 & \text{if } \Lambda_\theta(x) < y \text{ and } x > \theta. \end{cases}$$

Let $g(x,t) = \partial_2 C_{e,\Lambda_\theta}(x,t)$ and $f(x,y,t) = A_t(g(x,t), g(y,t))$. Then

$$f(x,y,t) = \begin{cases} 0 & \text{if } g(x,t) = 0 \text{ or } g(y,t) = 0, \\ A_t(\theta,\theta) & \text{if } g(x,t) = \theta = g(y,t), \\ \theta & \text{if } (g(x,t), g(y,t)) = (\theta, 1) \text{ or } (1,\theta), \\ 1 & \text{if } g(x,t) = 1 = g(y,t). \end{cases}$$

Let $\beta \in \mathbb{R}$ and consider the set $B := \{t : f(x,y,t) < \beta\}$.

Case 1: $\beta > 1$. Then, $B = [0,1]$ is measurable.

Case 2: $\theta < \beta \leq 1$. Then,

$$\begin{aligned} B &= [0,1] \setminus \{t : g(x,t) = 1 = g(y,t)\} \\ &= \begin{cases} [0,1] \setminus \{t : t > \max\{\Lambda_\theta(x), \Lambda_\theta(y)\}\} & \text{if } x > \theta \text{ and } y > \theta \\ [0,1] & \text{otherwise} \end{cases} \\ &= \begin{cases} [0, \max\{\Lambda_\theta(x), \Lambda_\theta(y)\}] & \text{if } x > \theta \text{ and } y > \theta \\ [0,1] & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, B is measurable.

Case 3: $0 < \beta \leq \theta$. Then, $B = \{t : f(x,y,t) = 0\} \cup \{t : 0 < f(x,y,t) < \beta\}$. Denote $B_1 = \{t : 0 < f(x,y,t) < \beta\}$. Then,

$$\begin{aligned} B_1 &= \{t : 0 < A_t(\theta,\theta) < \beta, t < \Lambda_\theta(x), t < \Lambda_\theta(y)\} \\ &= \{t : 0 < A_t(\theta,\theta) < \beta\} \cap [0, \min\{\Lambda_\theta(x), \Lambda_\theta(y)\}]\,. \end{aligned}$$

By assumption, we have that B_1 is measurable and

$$\begin{aligned} \{t : f(x,y,t) = 0\} &= \begin{cases} \{t : t > \min\{\Lambda_\theta(x), \Lambda_\theta(y)\}\} & \text{if } x < \theta \text{ or } y < \theta, \\ \varnothing & \text{otherwise} \end{cases} \\ &= \begin{cases} [\min\{\Lambda_\theta(x), \Lambda_\theta(y)\}, 1] & \text{if } x < \theta \text{ or } y < \theta, \\ \varnothing & \text{otherwise.} \end{cases} \end{aligned}$$

Then, B is measurable.

□

Theorem 3.2. *Let $\mathcal{A} := \{A_t\}_{t\in[0,1]}$ be a class of copulas such that $A_t(\theta,\theta)$ is measurable in t. Then, $C_{e,\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta,e}$ is an implicit dependence copula, i.e., there exist random variables X and Y uniformly distributed on $[0,1]$ such that $\Lambda_\theta(X) = \Lambda_\theta(Y)$ a.s. and $C_{e,\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta,e} = C_{X,Y}$.*

Proof. Denote $C' := C_{e,\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta,e}$. By Lemma 3.1, C' is a copula. Let $\mathbb{Q}$ be the Borel probability measure extension of $V_{C'}$ to $\mathscr{B}(\mathbb{I}^2)$. Define functions X and Y on $\mathbb{I}^2 \to \mathbb{I}$ by $X(x,y) := x$ and $Y(x,y) := y$, which are both random variables on the probability space $(\mathbb{I}^2, \mathscr{B}(\mathbb{I}^2), \mathbb{Q})$. Then,

$$\begin{aligned} C'(x,y) &= V_{C'}([0,x] \times [0,y]) \\ &= \mathbb{Q}\left(([0,x] \times [0,1]) \cap ([0,1] \times [0,y])\right) \\ &= \mathbb{Q}\left(X^{-1}([0,x]) \cap Y^{-1}([0,y])\right) \\ &= \mathbb{Q}(X \le x, Y \le y). \end{aligned}$$

Thus, C' is the joint distribution function of X and Y. Since

$$\begin{aligned} \mathbb{Q}(X \le x) &= \mathbb{Q}(X^{-1}([0,x])) \\ &= V_{C'}([0,x] \times [0,1]) \\ &= C'(x,1) = x \end{aligned}$$

and, similarly, $\mathbb{Q}(Y \le y) = y$, X and Y are uniformly distributed on $[0,1]$. Hence, $C' = C_{X,Y}$.

We will prove the final claim that $\Lambda_\theta(X) = \Lambda_\theta(Y)$ $\mathbb{Q}$-a.s. Since $\{\Lambda_\theta(X) = \Lambda_\theta(Y)\} = \{(x,y) \in \mathbb{I}^2 : \Lambda_\theta(x) = \Lambda_\theta(y)\} =: \Delta$, it suffices to show that $V_{C'}(B) = 0$ for all rectangles $B \subseteq \mathbb{I}^2 \setminus \Delta$. Note that

$$\Delta = \left\{(x,y) : y = x \text{ or } y = \left(1 - \frac{1-\theta}{\theta}x\right)\mathbb{1}_{[0,\theta]}(x) + \frac{\theta}{1-\theta}(1-x)\mathbb{1}_{(\theta,1]}(x)\right\}$$

(see the blue line in Fig 3.1). Observe that each rectangle in $\mathbb{I}^2 \setminus \Delta$ can be written as the difference between two rectangles in $\mathbb{I}^2 \setminus \Delta$ both of which have one side lying on the boundary S of $\mathbb{I}^2$. For instance, if all four corners of a rectangle $B := [x_1, x_2] \times [y_1, y_2]$ are in the triangular region 2 bounded above by the line $y = \Lambda_\theta^{21}(x)$ and bounded below by the line $y = x$, then $B = B_1 \setminus B_2$ where both $B_1 := [0, x_2] \times [y_1, y_2]$ and $B_2 := [0, x_1] \times [y_1, y_2]$ lie in the same region 2. So, it suffices to show that $V_{C'}(B) = 0$ for every rectangle $B \subseteq \mathbb{I}^2 \setminus \Delta$ whose one side lies in S. Our proof naturally splits into four cases, depending on the region that B lies in. The four regions are partitioned by the graphs of $y = x, y = \Lambda_\theta^{21}(x)$ and $y = \Lambda_\theta^{12}(x)$, illustrated in Figure 3.1.

Case 1: $B = [x_1, x_2] \times [0, y] \subseteq \mathbb{I}^2 \setminus \Delta$. Equivalently, both (x_i, y), $i = 1, 2$, lie below $y = x$ and $y = \Lambda_\theta^{12}(x)$. So, for $i = 1, 2$, $x_i > y$ and $\Lambda_\theta^{12}(x_i) > y$. Consequently, $y < \theta$ and $\Lambda_\theta^{21}(y) > x_i$. Thus

$$C'(x_i, y) = \int_0^1 A_t\big(\partial_2 C_{e,\Lambda_\theta}(x_i, t), \partial_1 C_{\Lambda_\theta,e}(t, y)\big)dt$$

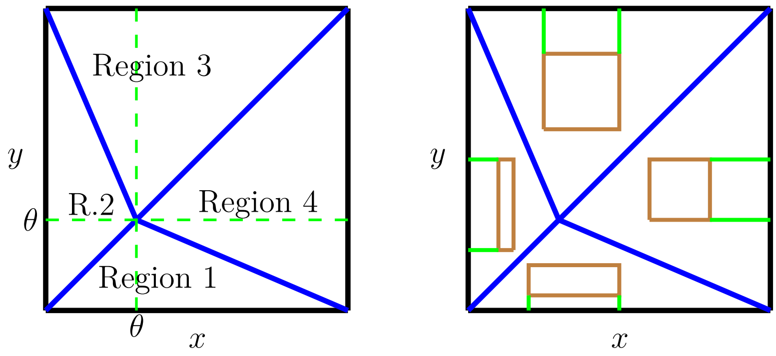


Figure 3.1: The four regions of $\mathbb{I}^2 \setminus \Delta$.

$$
\begin{aligned}
&= \int_0^1 A_t\big(\partial_2 C_{e,\Lambda_\theta}(x_i,t), \partial_2 C_{e,\Lambda_\theta}(y,t)\big)dt \\
&= \int_0^{\frac{y}{\theta}} A_t(\theta,\theta)dt + \int_{\frac{y}{\theta}}^1 A_t(\partial_2 C_{e,\Lambda_\theta}(x_i,t),0)dt \\
&= \int_0^{\frac{y}{\theta}} A_t(\theta,\theta)dt.
\end{aligned}
$$

In the third equality, we separate the interval $[0,1]$ into $\left[0,\frac{y}{\theta}\right]$ and $\left[\frac{y}{\theta},1\right]$ because, by equation (2.6), the value of $\partial_2 C_{e,\Lambda_\theta}(y,t)$ on intervals $\left[0,\frac{y}{\theta}\right]$ and $\left[\frac{y}{\theta},1\right]$ are different. Hence, $V_{C'}(B) = C'(x_1,0) - C'(x_2,0) - C'(x_1,y) + C'(x_2,y) = 0$.

Case 2: This case is similar to case 1. $B = [0,x] \times [y_1,y_2] \subseteq \mathbb{I}^2 \setminus \Delta$. Equivalently, both (x,y_i), $i = 1,2$, lie below $y = x$ and $y = \Lambda_\theta^{21}(x)$. So, for $i = 1,2$, $y_i > x, \Lambda_\theta^{21}(x) > y_i$. Consequently, $x < \theta$ and $\Lambda_\theta^{12}(y_i) > x$. Thus

$$
\begin{aligned}
C'(x,y_i) &= \int_0^1 A_t\big(\partial_2 C_{e,\Lambda_\theta}(x,t), \partial_2 C_{e,\Lambda_\theta}(y_i,t)\big)dt \\
&= \int_0^{\frac{x}{\theta}} A_t(\theta,\theta)dt + \int_{\frac{x}{\theta}}^1 A_t(0,\partial_2 C_{e,\Lambda_\theta}(y_i,t))dt \\
&= \int_0^{\frac{x}{\theta}} A_t(\theta,\theta)dt.
\end{aligned}
$$

Hence, $V_{C'}(B) = C'(0,y_1) - C'(x,y_1) - C'(0,y_2) + C'(x,y_2) = 0$.

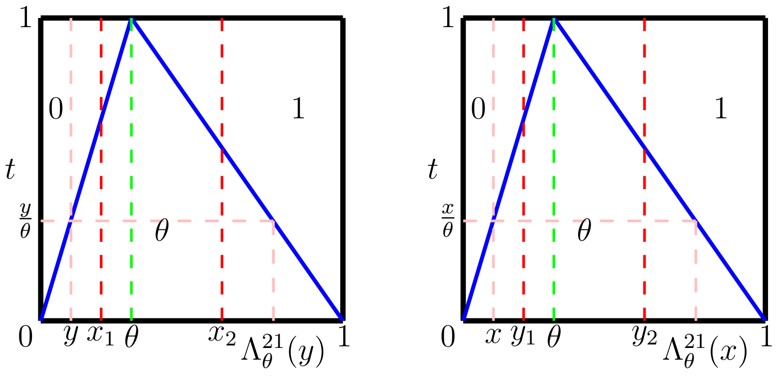


Figure 3.2: The value of $\partial_2 C_{e,\Lambda_\theta}(x,t)$ in case 1 (left) and 2 (right).

Case 3: $B = [x_1, x_2] \times [y, 1] \subseteq \mathbb{I}^2 \setminus \Delta$. Equivalently, both (x_i, y), $i = 1, 2$, lie below $y = x$ and $y = \Lambda_\theta^{21}(x)$. So, for $i = 1, 2$, $y > x_i, \Lambda_\theta^{21}(x_i) < y$. Consequently, $y > \theta$ and $\Lambda_\theta^{12}(y) < x_i$. Thus,

$$\begin{aligned} C'(x_i, y) &= \int_0^1 A_t\Big(\partial_2 C_{e,\Lambda_\theta}(x_i,t), \partial_2 C_{e,\Lambda_\theta}(y,t)\Big)dt \\ &= \int_0^{\frac{1-y}{1-\theta}} A_t(\theta,\theta)dt + \int_{\frac{1-y}{1-\theta}}^1 A_t(\partial_2 C_{e,\Lambda_\theta}(x_i,t), 1)dt \\ &= \int_0^{\frac{1-y}{\theta}} A_t(\theta,\theta)dt + \int_{\frac{1-y}{1-\theta}}^1 \partial_2 C_{e,\Lambda_\theta}(x_i,t)dt, \end{aligned}$$

where in the third equality, we separate the interval $[0,1]$ into $\left[0, \frac{1-y}{1-\theta}\right]$ and $\left[\frac{1-y}{1-\theta}, 1\right]$ because, by equation (2.6), the value of $\partial_2 C_{e,\Lambda_\theta}(y,t)$ on intervals $\left[0, \frac{1-y}{1-\theta}\right]$ and $\left[\frac{1-y}{1-\theta}, 1\right]$ are different. We have

$$\begin{aligned} C'(x_1, y) - C'(x_2, y) &= \int_{\frac{1-y}{1-\theta}}^1 \partial_2 C_{e,\Lambda_\theta}(x_1,t) - \partial_2 C_{e,\Lambda_\theta}(x_2,t)dt \\ &= C_{e,\Lambda_\theta}(x_1, 1) - C_{e,\Lambda_\theta}\left(x_1, \frac{1-y}{1-\theta}\right) \\ &\quad - C_{e,\Lambda_\theta}(x_2, 1) + C_{e,\Lambda_\theta}\left(x_2, \frac{1-y}{1-\theta}\right) \\ &= x_1 - x_2 + \int_{x_1}^{x_2} \partial_1 C_{e,\Lambda_\theta}\left(t, \frac{1-y}{1-\theta}\right) dt \\ &= x_1 - x_2 + \int_{x_1}^{x_2} 0 dt = x_1 - x_2. \end{aligned}$$

Hence, $V_{C'}(B) = C'(x_1, y) - C'(x_2, y) - C'(x_1, 1) + C'(x_2, 1) = 0$.

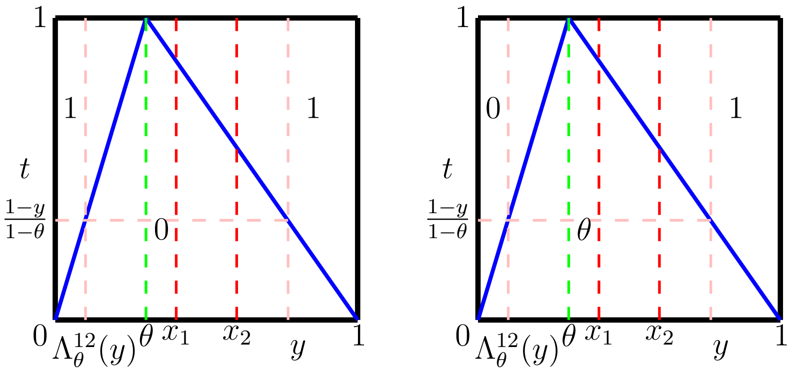


Figure 3.3: The value of $\partial_1 C_{e,\Lambda_\theta}(x,t)$ and $\partial_2 C_{e,\Lambda_\theta}(x,t)$, respectively, in case 3.

Case 4: This case is similar to case 3. $B = [x,1] \times [y_1, y_2] \subseteq \mathbb{I}^2 \setminus \Delta$. Equivalently, both (x, y_i), $i = 1,2$, lie below $y = x$ and $y = \Lambda_\theta^{12}(x)$. So, for $i = 1,2$, $x > y_i, \Lambda_\theta^{12}(x) > y_i$. Consequently, $x > \theta$ and $\Lambda_\theta^{21}(y_i) < x_i$. Thus,

$$\begin{aligned} C'(x,y_i) &= \int_0^1 A_t(\partial_2 C_{e,\Lambda_\theta}(x,t), \partial_2 C_{e,\Lambda_\theta}(y_i,t))dt \\ &= \int_0^{\frac{1-x}{1-\theta}} A_t(\theta,\theta)dt + \int_{\frac{1-x}{1-\theta}}^1 A_t(1, \partial_2 C_{e,\Lambda_\theta}(y_i,t))dt \\ &= \int_0^{\frac{1-x}{1-\theta}} A_t(\theta,\theta)dt + \int_{\frac{1-x}{1-\theta}}^1 \partial_2 C_{e,\Lambda_\theta}(y_i,t)dt. \end{aligned}$$

We have

$$\begin{aligned} C'(x,y_1) - C'(x,y_2) &= \int_{\frac{1-x}{1-\theta}}^1 \partial_2 C_{e,\Lambda_\theta}(y_1,t) - \partial_2 C_{e,\Lambda_\theta}(y_2,t)dt \\ &= C_{e,\Lambda_\theta}(y_1,1) - C_{e,\Lambda_\theta}\left(y_1, \frac{1-x}{1-\theta}\right) \\ &\quad - C_{e,\Lambda_\theta}(y_2,1) + C_{e,\Lambda_\theta}\left(y_2, \frac{1-x}{1-\theta}\right) \\ &= y_1 - y_2 + \int_{y_1}^{y_2} \partial_1 C_{e,\Lambda_\theta}\left(t, \frac{1-x}{1-\theta}\right) dt \\ &= y_1 - y_2 + \int_{y_1}^{y_2} 0\, dt = y_1 - y_2. \end{aligned}$$

Hence, $V_{C'}(B) = C'(x,y_1) - C'(x,y_2) - C'(1,y_1) + C'(1,y_2) = 0$.

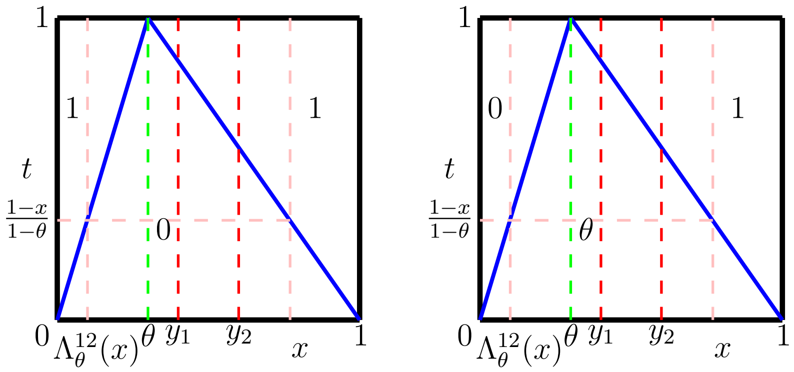


Figure 3.4: The value of $\partial_1 C_{e,\Lambda_\theta}(x,t)$ and $\partial_2 C_{e,\Lambda_\theta}(x,t)$, respectively, in case 4.

□

3.2 Generalized Factorizability of $C_{X,Y}$ where $\Lambda_\theta(X) = \Lambda_\theta(Y)$

In this section, we will write some implicit dependence copulas as the products of two complete dependence copulas. We will consider only the implicit dependence copulas of random variables X and Y where $\Lambda_\theta(X) = \Lambda_\theta(Y)$ a.s.

First, let random variables $X, Y \sim \mathcal{U}(0,1)$ be such that $\Lambda_\theta(X) = \Lambda_\theta(Y)$ a.s. where Λ_θ is defined in equation (2.4) and $\Lambda_\theta = \Lambda_\theta^1 \mathbb{1}_{[0,\theta]} + \Lambda_\theta^2 \mathbb{1}_{(\theta,1]}$.

Let $A_1 = \{X \le \theta, Y \le \theta\}$, $A_2 = \{X \le \theta, Y > \theta\}$, $A_3 = \{X > \theta, Y \le \theta\}$ and $A_4 = \{X > \theta, Y > \theta\}$. Since $\Lambda_\theta(X) = \Lambda_\theta(Y)$ a.s., we have that $X = Y$ a.s. in $A_1 \cup A_4$, $\Lambda_\theta^1(X) = \Lambda_\theta^2(Y)$ a.s. in A_2 and $\Lambda_\theta^2(X) = \Lambda_\theta^1(Y)$ a.s. in A_3. So,

$$Y = X\mathbb{1}_{A_1} + \Lambda_\theta^{21}(X)\mathbb{1}_{A_2} + \Lambda_\theta^{12}(X)\mathbb{1}_{A_3} + X\mathbb{1}_{A_4} \text{ a.s.} \tag{3.1}$$

By equation (2.6), for $s < \theta$ and using change of variable with $t = \Lambda_\theta(s)$, we have

$$\partial_2 C_{e,\Lambda_\theta}(x, \Lambda_\theta(s)) = \begin{cases} \theta & \text{if } \Lambda_\theta(x) > \Lambda_\theta(s), \\ 0 & \text{if } \Lambda_\theta(x) < \Lambda_\theta(s) \text{ and } x < \theta, \\ 1 & \text{if } \Lambda_\theta(x) < \Lambda_\theta(s) \text{ and } x > \theta. \end{cases} \tag{3.2}$$

Next, for a.e. s, we denote

$$\omega_1(s) = \mathbb{E}\left[\mathbb{1}_{\{Y \le \theta\}} \mid X = s\right] \text{ and } \omega_2(s) = \mathbb{E}\left[\mathbb{1}_{\{Y > \theta\}} \mid X = s\right].$$

Since $K(s, \cdot)$ is a probability measure, we have $\omega_1(s) + \omega_2(s) = 1$ a.s.

Lemma 3.3. *Let random variables X and Y uniformly distributed on $[0,1]$ be such that $\Lambda_\theta(X) = \Lambda_\theta(Y)$ a.s. For $s < \theta$, $\omega_1(s) + \frac{1-\theta}{\theta}\omega_1(1 - \frac{1-\theta}{\theta}s) = 1$ a.s.*

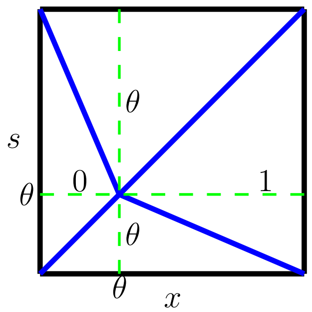


Figure 3.5: The value of $\partial_2 C_{e,\Lambda_\theta}(x, \Lambda_\theta(s))$.

Proof. Let $s < \theta, t = \Lambda_\theta(s)$ and $A_j(t) := \left(t - \frac{1}{j}, t + \frac{1}{j}\right) \cap [0,1]$ shrink to t nicely. By Theorem 2.54, we have

$$\mathbb{P}(Y \leq \theta \mid \Lambda_\theta(X) = t) = \lim_{j \to \infty} \mathbb{P}(Y \leq \theta \mid \Lambda_\theta(X) \in A_j(t)) \text{ a.e. t.}$$

Let the event $B_j(t) := \{\Lambda_\theta(X) \in A_j(t)\} = \{X \in (\Lambda_\theta^1)^{-1}(A_j(t))\} \cup \{X \in (\Lambda_\theta^2)^{-1}(A_j(t))\}$. Then the event $\{X \in (\Lambda_\theta^1)^{-1}(A_j(t))\}$ and $\{X \in (\Lambda_\theta^2)^{-1}(A_j(t))\}$ are disjoint as $j \to \infty$. Thus, by conditional probability,

$$\begin{aligned}
&\mathbb{P}(Y \leq \theta \mid \Lambda_\theta(X) = t) \\
&= \lim_{j \to \infty} \mathbb{P}(Y \leq \theta \mid B_j(t)) \\
&= \lim_{j \to \infty} \frac{\mathbb{P}(Y \leq \theta, B_j(t))}{\mathbb{P}(B_j(t))} \\
&= \lim_{j \to \infty} \left[\frac{\mathbb{P}(Y \leq \theta, X \in (\Lambda_\theta^1)^{-1}(A_j(t)))}{\mathbb{P}(B_j(t))} + \frac{\mathbb{P}(Y \leq \theta, X \in (\Lambda_\theta^2)^{-1}(A_j(t)))}{\mathbb{P}(B_j(t))} \right] \\
&= \lim_{j \to \infty} \left[\frac{\mathbb{P}(Y \leq \theta, X \in (\Lambda_\theta^1)^{-1}(A_j(t)))}{\mathbb{P}(X \in (\Lambda_\theta^1)^{-1}(A_j(t)))} \cdot \frac{\mathbb{P}(X \in (\Lambda_\theta^1)^{-1}(A_j(t)))}{\mathbb{P}(B_j(t))} \right. \\
&\quad \left. + \frac{\mathbb{P}(Y \leq \theta, X \in (\Lambda_\theta^2)^{-1}(A_j(t)))}{\mathbb{P}(X \in (\Lambda_\theta^2)^{-1}(A_j(t)))} \cdot \frac{\mathbb{P}(X \in (\Lambda_\theta^2)^{-1}(A_j(t)))}{\mathbb{P}(B_j(t))} \right] \\
&= \lim_{j \to \infty} \left[\mathbb{P}(Y \leq \theta \mid X \in (\Lambda_\theta^1)^{-1}(A_j(t))) \cdot \frac{\mathbb{P}(X \in (\Lambda_\theta^1)^{-1}(A_j(t)))}{\mathbb{P}(B_j(t))} \right. \\
&\quad \left. + \mathbb{P}(Y \leq \theta \mid X \in (\Lambda_\theta^2)^{-1}(A_j(t))) \cdot \frac{\mathbb{P}(X \in (\Lambda_\theta^2)^{-1}(A_j(t)))}{\mathbb{P}(B_j(t))} \right].
\end{aligned}$$

Since $X \sim \mathcal{U}(0,1)$ and Λ_θ is measure-preserving, we have

$$\mathbb{P}(X \in (\Lambda_\theta^i)^{-1}(A_j(t))) = \lambda((\Lambda_\theta^i)^{-1}(A_j(t)))$$

and

$$\mathbb{P}(B_j(t)) = \lambda((\Lambda_\theta)^{-1}(A_j(t))) = \lambda(A_j(t)).$$

By Theorem 2.53, for a.e. t,

$$\lim_{j\to\infty} \frac{\mathbb{P}(X \in \Lambda_\theta^1)^{-1}(A_j(t)))}{\mathbb{P}(B_j(t))} = (\Lambda_\theta^1)^{-1}(t) = \theta \text{ for } i = 1,2$$

and

$$\lim_{j\to\infty} \frac{\mathbb{P}(X \in \Lambda_\theta^1)^{-1}(A_j(t)))}{\mathbb{P}(B_j(t))} = (\Lambda_\theta^2)^{-1}(t) = 1-\theta.$$

The sequence $(\Lambda_\theta^1)^{-1}(A_j(t)) = \left(\theta t - \frac{\theta}{j}, \theta t + \frac{\theta}{j}\right)$ shrinks nicely to $\theta t := a$ and $(\Lambda_\theta^1)^{-1}(A_j(t)) = \left(1-(1-\theta)t - \frac{1-\theta}{j}, 1-(1-\theta)t + \frac{1-\theta}{j}\right)$ shrinks nicely to $b := 1-(1-\theta)t$. By Theorem 2.54 as $j \to \infty$,

$$\mathbb{P}(Y \le \theta, X \in (\Lambda_\theta^1)^{-1}(A_j(t))) \to \mathbb{P}(Y \le \theta, X = a)$$

and

$$\mathbb{P}(Y \le \theta, X \in (\Lambda_\theta^2)^{-1}(A_j(t))) \to \mathbb{P}(Y \le \theta, X = b).$$

Hence, for $s < \theta$,

$$\begin{aligned}\mathbb{P}(Y \le \theta \mid \Lambda_\theta(X) = t) &= \theta\,\mathbb{P}(Y \le \theta \mid X = a) + (1-\theta)\,\mathbb{P}(Y \le \theta \mid X = b)\\ &= \theta\,\mathbb{P}(Y \le \theta \mid X = s) + (1-\theta)\,\mathbb{P}\left(Y \le \theta \mid X = 1 - \frac{1-\theta}{\theta}s\right)\\ &= \theta\omega_1(s) + (1-\theta)\omega_1\left(1 - \frac{1-\theta}{\theta}s\right).\end{aligned}$$

Similarly, we have

$$\mathbb{P}(Y \le \theta \mid \Lambda_\theta(Y) = t) = \theta\,\mathbb{P}(Y \le \theta \mid Y = s) + (1-\theta)\,\mathbb{P}\left(Y \le \theta \mid Y = 1 - \frac{1-\theta}{\theta}s\right). \tag{3.3}$$

Next, we find the value of $\mathbb{P}(Y \le \theta \mid \Lambda_\theta(X) = t)$. Since $\Lambda_\theta(X) = \Lambda_\theta(Y)$ a.s., we have

$$\begin{aligned}\mathbb{P}(Y \le \theta \mid \Lambda_\theta(X) = t) &= \mathbb{P}(Y \le \theta \mid \Lambda_\theta(Y) = t)\\ &= \theta\,\mathbb{P}(Y \le \theta \mid Y = s) + (1-\theta)\,\mathbb{P}\left(Y \le \theta \mid Y = 1 - \frac{1-\theta}{\theta}s\right)\\ &= \theta.\end{aligned}$$

The second equality holds because of equation(3.3) and the last equality holds because for $s < \theta$, $1 - \frac{1-\theta}{\theta}s > \theta$. Hence, $\theta = \theta\omega_1(s) + (1-\theta)\omega_1\left(1 - \frac{1-\theta}{\theta}s\right)$. □

Lemma 3.4. *Let random variables $X, Y \sim \mathcal{U}(0,1)$ be such that $\Lambda_\theta(X) = \Lambda_\theta(Y)$ a.s. Then,*

$$\begin{aligned} K_C(s,[0,y]) &= \omega_1(s)\mathbb{1}_{[0,y]\cap[0,\theta]}(s) + \omega_2(s)\mathbb{1}_{\Lambda_\theta^{12}([0,y])\cap[0,\theta]}(s) \\ &\quad + \omega_1(s)\mathbb{1}_{\Lambda_\theta^{21}([0,y])\cap(\theta,1]}(s) + \omega_2(s)\mathbb{1}_{[0,y]\cap(\theta,1]}(s). \end{aligned}$$

Proof. By equation (3.1), we have $Y = X\mathbb{1}_{A_1} + \Lambda_\theta^{21}(X)\mathbb{1}_{A_2} + \Lambda_\theta^{12}(X)\mathbb{1}_{A_3} + X\mathbb{1}_{A_4}$ a.s. Since $0 \le \int_A \mathbb{1}_{\{Y=0\}}(\omega)d\,\mathbb{P}(\omega) \le \int_\Omega \mathbb{1}_{\{Y=0\}}(\omega)d\,\mathbb{P}(\omega) = \mathbb{P}(Y=0) = 0$ for all $A \in \sigma(X)$, we have $\mathbb{E}\left[\mathbb{1}_{\{Y=0\}} \mid X = s\right] = 0$. To prove the lemma, it is sufficient to consider $\mathbb{1}_{(0,y]} \circ Y$. Observe that for any $J \subseteq (0,1]$ and a random variable Z taking values in $[0,1]$, $\mathbb{1}_J \circ (Z\mathbb{1}_{A_i}) = (\mathbb{1}_J \circ Z)\mathbb{1}_{A_i}$ for every $i = 1,2,3,4$. Thus,

$$\begin{aligned} \mathbb{1}_{(0,y]} \circ (X\mathbb{1}_{A_1}) &= \left(\mathbb{1}_{(0,y]} \circ X\right)\mathbb{1}_{A_1}, \\ \mathbb{1}_{(0,y]} \circ \left(\Lambda_\theta^{21}(X)\mathbb{1}_{A_2}\right) &= \left(\mathbb{1}_{(0,y]} \circ \Lambda_\theta^{21}(X)\right)\mathbb{1}_{A_2}, \\ \mathbb{1}_{(0,y]} \circ \left(\Lambda_\theta^{12}(X)\mathbb{1}_{A_3}\right) &= \left(\mathbb{1}_{(0,y]} \circ \Lambda_\theta^{12}(X)\right)\mathbb{1}_{A_3} \end{aligned}$$

and

$$\mathbb{1}_{(0,y]} \circ (X\mathbb{1}_{A_4}) = \left(\mathbb{1}_{(0,y]} \circ X\right)\mathbb{1}_{A_4}.$$

We can derive that

$$\left(\mathbb{1}_{(0,y]} \circ \Lambda_\theta^{21}(X)\right)\mathbb{1}_{A_2} = \left(\mathbb{1}_{\Lambda_\theta^{12}((0,y])} \circ X\right)\mathbb{1}_{A_2}$$

and

$$\left(\mathbb{1}_{(0,y]} \circ \Lambda_\theta^{12}(X)\right)\mathbb{1}_{A_3} = \left(\mathbb{1}_{\Lambda_\theta^{21}((0,y])} \circ X\right)\mathbb{1}_{A_3}.$$

Thus,

$$\begin{aligned} \mathbb{E}\left[\mathbb{1}_{(0,y]} \circ Y \mid X\right] &= \mathbb{E}\left[\left(\mathbb{1}_{(0,y]} \circ X\right)\mathbb{1}_{A_1} \mid X\right] + \mathbb{E}\left[\left(\mathbb{1}_{\Lambda_\theta^{12}((0,y])} \circ X\right)\mathbb{1}_{A_2} \mid X\right] \\ &\quad + \mathbb{E}\left[\left(\mathbb{1}_{\Lambda_\theta^{21}((0,y])} \circ X\right)\mathbb{1}_{A_3} \mid X\right] + \mathbb{E}\left[\left(\mathbb{1}_{(0,y]} \circ X\right)\mathbb{1}_{A_4} \mid X\right]. \end{aligned}$$

Since $\mathbb{1}_{(0,y]} \circ X$, $\mathbb{1}_{\Lambda_\theta^{12}((0,y])} \circ X$ and $\mathbb{1}_{\Lambda_\theta^{21}((0,y])} \circ X$ are $\sigma(X)$-measurable, we have

$$\begin{aligned} \mathbb{E}\left[\left(\mathbb{1}_{(0,y]} \circ X\right)\mathbb{1}_{A_1} \mid X\right] &= \left(\mathbb{1}_{(0,y]} \circ X\right)\mathbb{E}\left[\mathbb{1}_{A_1} \mid X\right], \\ \mathbb{E}\left[\left(\mathbb{1}_{\Lambda_\theta^{12}((0,y])} \circ X\right)\mathbb{1}_{A_2} \mid X\right] &= \left(\mathbb{1}_{\Lambda_\theta^{12}((0,y])} \circ X\right)\mathbb{E}\left[\mathbb{1}_{A_2} \mid X\right], \\ \mathbb{E}\left[\left(\mathbb{1}_{\Lambda_\theta^{21}((0,y])} \circ X\right)\mathbb{1}_{A_3} \mid X\right] &= \left(\mathbb{1}_{\Lambda_\theta^{21}((0,y])} \circ X\right)\mathbb{E}\left[\mathbb{1}_{A_3} \mid X\right], \end{aligned}$$

and

$$\mathbb{E}\left[\left(\mathbb{1}_{(0,y]} \circ X\right)\mathbb{1}_{A_4} \mid X\right] = \left(\mathbb{1}_{(0,y]} \circ X\right)\mathbb{E}\left[\mathbb{1}_{A_4} \mid X\right].$$

We can show that

$$\mathbb{E}\left[\mathbb{1}_{A_1} \mid X\right] = \mathbb{E}\left[\mathbb{1}_{\{X\le\theta, Y\le\theta\}} \mid X\right] = \mathbb{1}_{\{X\le\theta\}}\mathbb{E}\left[\mathbb{1}_{\{Y\le\theta\}} \mid X\right],$$

$$\mathbb{E}\left[\mathbb{1}_{A_2} \mid X\right] = \mathbb{E}\left[\mathbb{1}_{\{X\le\theta, Y>\theta\}} \mid X\right] = \mathbb{1}_{\{X\le\theta\}}\mathbb{E}\left[\mathbb{1}_{\{Y>\theta\}} \mid X\right],$$

$$\mathbb{E}\left[\mathbb{1}_{A_3} \mid X\right] = \mathbb{E}\left[\mathbb{1}_{\{X>\theta, Y\le\theta\}} \mid X\right] = \mathbb{1}_{\{X>\theta\}}\mathbb{E}\left[\mathbb{1}_{\{Y\le\theta\}} \mid X\right],$$

and

$$\mathbb{E}\left[\mathbb{1}_{A_4} \mid X\right] = \mathbb{E}\left[\mathbb{1}_{\{X>\theta, Y>\theta\}} \mid X\right] = \mathbb{1}_{\{X>\theta\}}\mathbb{E}\left[\mathbb{1}_{\{Y>\theta\}} \mid X\right].$$

Hence,

$$\begin{aligned} K_C(s,[0,y]) &= \mathbb{E}\left[\mathbb{1}_{[0,y]}\circ Y \mid X = s\right] \\ &= \omega_1(s)\mathbb{1}_{[0,y]\cap[0,\theta]}(s) + \omega_2(s)\mathbb{1}_{\Lambda_\theta^{12}([0,y])\cap[0,\theta]}(s) \\ &\quad + \omega_1(s)\mathbb{1}_{\Lambda_\theta^{21}([0,y])\cap(\theta,1]}(s) + \omega_2(s)\mathbb{1}_{[0,y]\cap(\theta,1]}(s). \end{aligned}$$

□

Lemma 3.5. *Let random variables X, Y be uniformly distributed on $[0,1]$ with copula C such that $\Lambda_\theta(X) = \Lambda_\theta(Y)$ a.s. Then $K_C(s,[0,s]) = K_C(s,[0,y])$ for all $s \le \theta$ satisfying $\Lambda_\theta(s) \le \Lambda_\theta(y)$ where $y \in [0,1]$.*

Proof. Since $A_j := \left(s - \frac{1}{j}, s\right]$ shrinks nicely to s as $j \to \infty$ and the sets $\Lambda_\theta\left((s,y]\right) \subseteq (\Lambda_\theta(s),1]$ and $\Lambda_\theta\left(\left(s-\frac{1}{j},s\right]\right) = \left(\Lambda_\theta(s-\frac{1}{j}),\Lambda_\theta(s)\right]$, we have

$$\mathbb{P}\left(Y \in (s,y], X \in A_j(s)\right) \le \mathbb{P}\left(\Lambda_\theta(Y) \in \Lambda_\theta\left((s,y]\right), \Lambda_\theta(X) \in \Lambda_\theta\left(\left(s - \frac{1}{j}, s\right]\right)\right) = 0$$

for j large enough. Therefore, $\mathbb{P}\left(s < Y \le y \mid X = s\right) = 0$. □

Next, we will show that simple implicit dependence copulas can be written as generalized products of complete dependence copulas.

Theorem 3.6. *Let random variables $X, Y \sim \mathcal{U}(0,1)$ be such that $\Lambda_\theta(X) = \Lambda_\theta(Y)$ a.s. Then there exists $\mathcal{A} = \{A_t\}_{t\in[0,1]}$ such that $C_{X,Y} = C_{e,\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta,e}$.*

Proof. For each $s \in [0,\theta]$, let $A_{\Lambda_\theta(s)}$ be the subcopula on $\{0,\theta,1\}^2$ defined by

$$A_{\Lambda_\theta(s)}(\theta,\theta) = \theta \cdot K_C(s,[0,s]).$$

We can extend subcopula $A_{\Lambda_\theta(s)}$ to copula by Sklar's theorem. Since $s \le \theta, \Lambda_\theta(s) \le 1 = \Lambda_\theta(\theta)$ and $K_C(s,[0,\theta])$ is measurable in s, by Lemma 3.5, we have $K_C(s,[0,s]) = K_C(s,[0,\theta])$ is measurable in s. Thus, $A_t(\theta,\theta)$ is measurable in t where $t = \Lambda_\theta(s)$. By Lemma 3.1, $A_t(\partial_2 C_{e,\Lambda_\theta}(x,t), \partial_2 C_{e,\Lambda_\theta}(y,t))$ is measurable in t.

The generalized product of $C_{e\Lambda_\theta}$ and $C_{\Lambda_\theta e}$ with respect to $\mathcal{A} = \{A_t\}_{t\in[0,1]}$ is

$$C_{e,\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta,e}(x,y) = \int_0^1 A_t(\partial_2 C_{e,\Lambda_\theta}(x,t), \partial_2 C_{e,\Lambda_\theta}(y,t))\, dt \tag{3.4}$$

$$= \int_0^\theta \frac{1}{\theta} A_{\Lambda_\theta(s)}(\partial_2 C_{e,\Lambda_\theta}(x,\Lambda_\theta(s)), \partial_2 C_{e,\Lambda_\theta}(y,\Lambda_\theta(s)))\, ds. \tag{3.5}$$

The last equality uses the change of variable $t = \Lambda_\theta(s) = \dfrac{s}{\theta}$ for $s \in [0,\theta]$.

The proof of this theorem will be divided into four cases. Using the equation (3.5), the values of $\partial_2 C_{e,\Lambda_\theta}(x,\Lambda_\theta(s))$ in equation (3.2) and Lemma 3.5, at the end of each subcase, Theorem 2.50 gives $\int_0^x K_C(s,[0,y])ds = C_{X,Y}(x,y)$ which completes the proof.

Case 1: $x, y \le \theta$.

1. If $x < y$, then $C_{e,\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta,e}(x,y)$ is

$$\begin{aligned}
&= \int_0^x \frac{1}{\theta} A_{\Lambda_\theta(s)}(\theta,\theta)ds + \int_x^y \frac{1}{\theta} A_{\Lambda_\theta(s)}(0,\theta)ds + \int_y^\theta \frac{1}{\theta} A_{\Lambda_\theta(s)}(0,0)ds \\
&= \int_0^x K_C(s,[0,s])ds \\
&= \int_0^x K_C(s,[0,y])ds.
\end{aligned}$$

The first equality uses the equations (3.2) and (3.5) and the last equality uses Lemma 3.5.

2. If $y \le x$, then by Table 3.1 and Lemma 3.4, we have $\int_y^x K_C(s,[0,y])\, ds = 0$ and $C_{e,\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta,e}(x,y)$ is

$$\begin{aligned}
&= \int_0^y \frac{1}{\theta} A_{\Lambda_\theta(s)}(\theta,\theta)\, ds + \int_y^x \frac{1}{\theta} A_{\Lambda_\theta(s)}(\theta,0)\, ds + \int_x^\theta \frac{1}{\theta} A_{\Lambda_\theta(s)}(0,0)\, ds \\
&= \int_0^y K_C(s,[0,s])\, ds \\
&= \int_0^y K_C(s,[0,y])\, ds + \int_y^x K_C(s,[0,y])\, ds \\
&= \int_0^x K_C(s,[0,y])\, ds.
\end{aligned}$$

The first equality uses the equations (3.2) and (3.5) and the third equality uses Lemma 3.5.

B	$s \in (y, x]$
$[0, y] \cap [0, \theta]$	0
$\Lambda_\theta^{12}([0, y]) \cap [0, \theta]$	0
$\Lambda_\theta^{21}([0, y]) \cap (\theta, 1]$	0
$[0, y] \cap (\theta, 1]$	0

Table 3.1: The values of $\mathbb{1}_B(s)$ for given sets B at s in the interval in subcase 1.2.

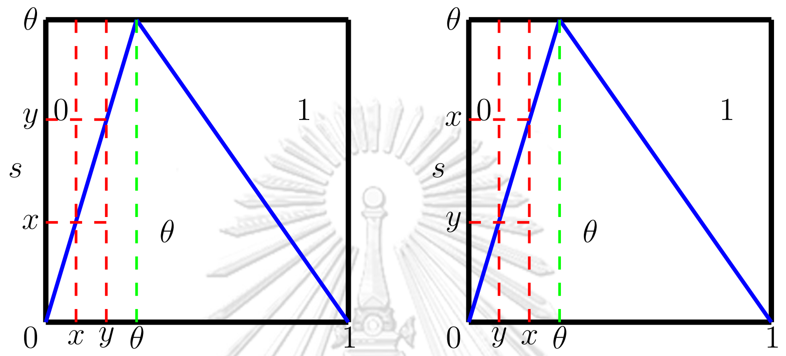


Figure 3.6: The value of $\partial_2 C_{e,\Lambda_\theta}(x, \Lambda_\theta(s))$ in subcase 1.1 (left) and 1.2 (right).

Case 2: $x \leq \theta, y > \theta$. Consider $b := \Lambda_\theta^{12}(y) = \frac{\theta}{1-\theta}(1-y) \in [0, \theta]$ and use the fact that $\Lambda_\theta(b) = \Lambda_\theta(y)$.

1. If $x < b$, then $C_{e,\Lambda_\theta} *_\mathcal{A} C_{\Lambda_\theta,e}(x, y)$ is

$$= \int_0^x \frac{1}{\theta} A_{\Lambda_\theta(s)}(\theta, \theta) ds + \int_x^b \frac{1}{\theta} A_{\Lambda_\theta(s)}(0, \theta) ds + \int_b^\theta \frac{1}{\theta} A_{\Lambda_\theta(s)}(0, 1) ds$$
$$= \int_0^x K_C(s, [0, s]) ds$$
$$= \int_0^x K_C(s, [0, y]) ds$$

The first equality uses the equations (3.2) and (3.5) and the last equality uses Lemma 3.5.

2. If $b < x$, then by Table 3.2 and Lemma 3.4, we have

$$\int_b^x K_C(s, [0, y]) ds = \int_b^x \omega_1(s) + \omega_2(s) ds = x - b$$

and $C_{e,\Lambda_\theta} *_\mathcal{A} C_{\Lambda_\theta,e}(x, y)$ is

$$= \int_0^b \frac{1}{\theta} A_{\Lambda_\theta(s)}(\theta, \theta) ds + \int_b^x \frac{1}{\theta} A_{\Lambda_\theta(s)}(\theta, 1) ds + \int_x^\theta \frac{1}{\theta} A_{\Lambda_\theta(s)}(0, 1) ds$$

$$
\begin{aligned}
&= \int_0^b K_C(s,[0,s])ds + (x-b) \\
&= \int_0^b K_C(s,[0,y])ds + \int_b^x K_C(s,[0,y])ds \\
&= \int_0^x K_C(s,[0,y])ds.
\end{aligned}
$$

The first equality uses the equations (3.2) and (3.5) and the third equality uses Lemma 3.5.

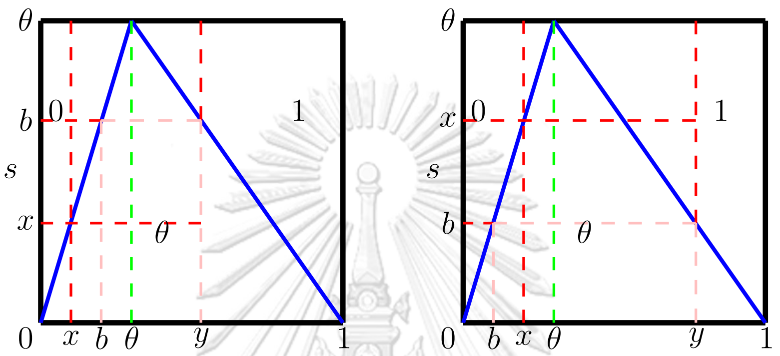


Figure 3.7: The value of $\partial_2 C_{e,\Lambda_\theta}(x, \Lambda_\theta(s))$ in subcase 2.1 (left) and 2.2 (right).

B	$s \in (b, x]$
$[0,y] \cap [0,\theta]$	1
$\Lambda_\theta^{12}([0,y]) \cap [0,\theta]$	1
$\Lambda_\theta^{21}([0,y]) \cap (\theta,1]$	0
$[0,y] \cap (\theta,1]$	0

Table 3.2: The values of $\mathbb{1}_B(s)$ for given sets B at s in the interval in subcase 2.2.

Case 3: $x > \theta, y \leq \theta$. Consider $a := \Lambda_\theta^{12}(x) = \frac{\theta}{1-\theta}(1-x), y \in [0,\theta]$ and use the fact that $\Lambda_\theta(a) = \Lambda_\theta(x)$.

1. If $y < a$, then by Table 3.3 and Lemma 3.4, we have $\int_y^x K_C(s,[0,y])ds = 0$ and $C_{e,\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta,e}(x,y)$ is

$$
\begin{aligned}
&= \int_0^y \frac{1}{\theta} A_{\Lambda_\theta(s)}(\theta,\theta)ds + \int_y^a \frac{1}{\theta} A_{\Lambda_\theta(s)}(\theta,0)ds + \int_a^\theta \frac{1}{\theta} A_{\Lambda_\theta(s)}(1,0)ds \\
&= \int_0^y K_C(s,[0,s])ds
\end{aligned}
$$

$$= \int_0^y K_C(s,[0,y])ds + \int_y^x K_C(s,[0,y])ds$$
$$= \int_0^x K_C(s,[0,y])ds.$$

The first equality uses the equations (3.2) and (3.5) and the third equality uses Lemma 3.5.

2. If $a < y$, then by Table 3.3 and Lemma 3.4, we have

$$\int_y^{1-\frac{1-\theta}{\theta}y} K_C(s,[0,y])ds = 0,$$

$$\begin{aligned}
&\int_a^y K_C(s,[0,y])ds + \int_{1-\frac{1-\theta}{\theta}y}^x K_C(t,[0,y])dt \\
&= \int_a^y \omega_1(s)ds + \int_{1-\frac{1-\theta}{\theta}y}^x \omega_1(t)dt \\
&= \int_a^y \omega_1(s) + \frac{1-\theta}{\theta}\omega_1\left(1 - \frac{1-\theta}{\theta}s\right) ds \\
&= y - a,
\end{aligned}$$

where we have made a change of variable $t = 1 - \frac{1-\theta}{\theta}s$ and

$$\begin{aligned}
C_{e,\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta,e}(x,y) &= \int_0^a \frac{1}{\theta} A_{\Lambda_\theta(s)}(\theta,\theta)ds + \int_a^y \frac{1}{\theta} A_{\Lambda_\theta(s)}(1,\theta)ds \\
&\quad + \int_y^\theta \frac{1}{\theta} A_{\Lambda_\theta(s)}(1,0)ds \\
&= \int_0^a K_C(s,[0,s])ds + (y-a) + 0 \\
&= \int_0^a K_C(s,[0,y])ds + \int_y^{1-\frac{1-\theta}{\theta}y} K_C(s,[0,y])ds \\
&\quad + \Big(\int_a^y K_C(s,[0,y])ds + \int_{1-\frac{1-\theta}{\theta}y}^x K_C(s,[0,y])ds \Big) \\
&= \int_0^x K_C(s,[0,y])ds.
\end{aligned}$$

The first equality uses the equations (3.2) and (3.5) and the third equality uses Lemma 3.5.

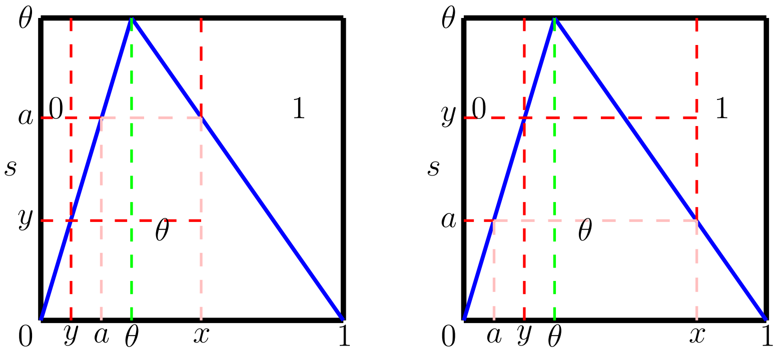


Figure 3.8: The value of $\partial_2 C_{e,\Lambda_\theta}(x, \Lambda_\theta(s))$ in subcase 3.1 (left) and 3.2 (right).

B	$s \in (y, x]$	$s \in (a, y]$	$s \in (y, \Lambda_\theta^{21}(y)]$	$s \in (\Lambda_\theta^{21}(y), x]$
$[0, y] \cap [0, \theta]$	0	1	0	0
$\Lambda_\theta^{12}([0, y]) \cap [0, \theta]$	0	0	0	0
$\Lambda_\theta^{21}([0, y]) \cap (\theta, 1]$	0	0	0	1
$[0, y] \cap (\theta, 1]$	0	0	0	0

Table 3.3: The values of $\mathbb{1}_B(s)$ for given sets B in subcases 3.1 (left) and 3.2 (right).

Case 4: $x, y > \theta$. Consider $a, b \in [0, \theta]$ and use the fact that $\Lambda_\theta(a) = \Lambda_\theta(x)$ and $\Lambda_\theta(b) = \Lambda_\theta(y)$. Notice that $x - a = \frac{x-\theta}{1-\theta}$ and $y - b = \frac{y-\theta}{1-\theta}$.

1. If $x < y$, then by Table 3.4 and Lemma 3.4, we have

$$\int_b^a K_C(s, [0, y])ds = \int_b^a (\omega_1(s) + \omega_2(s))\ ds = a - b,$$

$$\int_a^x K_C(s, [0, y])ds = \int_a^\theta (\omega_1(s)+\omega_2(s))\ ds + \int_\theta^x (\omega_1(s)+\omega_2(s))\ ds = x - a$$

and $C_{e,\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta,e}(x, y)$ is

$$\begin{aligned}
&= \int_0^b \frac{1}{\theta} A_{\Lambda_\theta(s)}(\theta, \theta)ds + \int_b^a \frac{1}{\theta} A_{\Lambda_\theta(s)}(\theta, 1)ds + \int_a^\theta \frac{1}{\theta} A_{\Lambda_\theta(s)}(1, 1)ds \\
&= \int_0^b K_C(s, [0, s])ds + (a - b) + \left(\frac{x - \theta}{1 - \theta}\right) \\
&= \int_0^b K_C(s, [0, y])ds + \int_b^a K_C(s, [0, y])ds + \int_a^x K_C(s, [0, y])ds \\
&= \int_0^x K_C(s, [0, y])ds.
\end{aligned}$$

The first equality uses the equations (3.2) and (3.5) and the third equality uses Lemma 3.5.

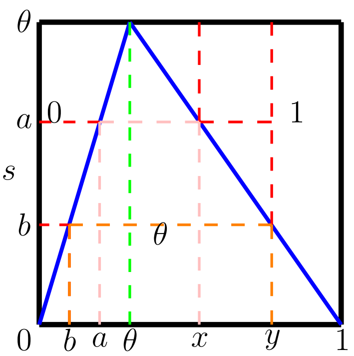
2. If $y < x$, then by Table 3.4 and Lemma 3.4, we have

$$\begin{aligned}\int_a^x K_C(s,[0,y])ds &= \int_a^b \omega_1(s)\ ds + \int_b^\theta (\omega_1(s)+\omega_2(s))\ ds \\ &\quad + \int_\theta^y (\omega_1(s)+\omega_2(s))\ ds + \int_y^x \omega_1(s)\ ds \\ &= \int_a^b \omega_1(s)\ ds + \int_y^x \omega_1(t)\ dt + (y-b) \\ &= \int_a^b \omega_1(s) + \frac{1-\theta}{\theta}\omega_1\left(1-\frac{1-\theta}{\theta}s\right)\ ds + \left(\frac{y-\theta}{1-\theta}\right) \\ &= (b-a) + \left(\frac{y-\theta}{1-\theta}\right),\end{aligned}$$

where we have made a change of variable $t = 1 - \frac{1-\theta}{\theta}s$ and

$$\begin{aligned}C_{e,\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta,e}(x,y) &= \int_0^a \frac{1}{\theta}A_{\Lambda_\theta(s)}(\theta,\theta)ds + \int_a^b \frac{1}{\theta}A_{\Lambda_\theta(s)}(1,\theta)ds \\ &\quad + \int_b^\theta \frac{1}{\theta}A_{\Lambda_\theta(s)}(1,1)ds \\ &= \int_0^a K_C(s,[0,s])ds + (b-a) + \left(\frac{y-\theta}{1-\theta}\right) \\ &= \int_0^a K_C(s,[0,y])ds + \int_a^x K_C(s,[0,y])ds \\ &= \int_0^x K_C(s,[0,y])ds.\end{aligned}$$

The first equality uses the equations (3.2) and (3.5) and the third equality uses Lemma 3.5.



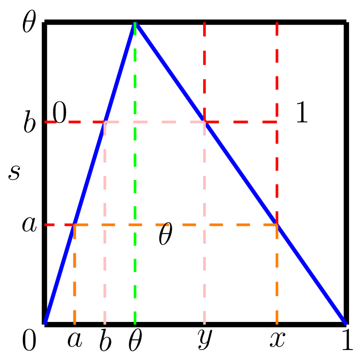


Figure 3.9: The value of $\partial_2 C_{e,\Lambda_\theta}(x, \Lambda_\theta(s))$ in subcase 4.1 (left) and 4.2 (right).

B	$s \in (b,a]$	$s \in (a,x]$	$s \in (a,b]$	$s \in (b,x]$
$[0,y] \cap [0,\theta]$	1	$\mathbb{1}_{(\frac{\theta}{1-\theta}(1-x),\theta]}$	1	$\mathbb{1}_{(\frac{\theta}{1-\theta}(1-y),\theta]}$
$\Lambda_\theta^{12}([0,y]) \cap [0,\theta]$	1	$\mathbb{1}_{(\frac{\theta}{1-\theta}(1-x),\theta]}$	0	$\mathbb{1}_{(\frac{\theta}{1-\theta}(1-y),\theta]}$
$\Lambda_\theta^{21}([0,y]) \cap (\theta,1]$	0	$\mathbb{1}_{(\theta,x]}$	0	$\mathbb{1}_{(\theta,x]}$
$[0,y] \cap (\theta,1]$	0	$\mathbb{1}_{(\theta,x]}$	0	$\mathbb{1}_{(\theta,y]}$

Table 3.4: The values of $\mathbb{1}_B(s)$ for given sets B in subcases 4.1 (left) and 4.2 (right).

□

CHAPTER IV
Symmetric Implicit Dependence Copulas Via Simple Functions

In this chapter, we will generalize the result in section 3.2 by giving a sufficient condition on the measure-preserving transformation α, replacing Λ_θ, under which the implicit dependence copulas $C_{X,Y}$, where $\alpha(X) = \alpha(Y)$, are generalized factorizable.

Let α be a measure-preserving (Borel) transformation on $[0, 1]$ for which there is a partition $P := \{0 = a_0, a_1, a_2, \dots, a_n = 1\}$ such that, for $i = 1, \dots, n$, $\alpha_i := \alpha|_{I_i}$ is one-to-one where $I_i := (a_{i-1}, a_i]$. By [11], α_i^{-1} is also Borel measurable and hence each $\alpha_{ij} := \alpha_i^{-1} \circ \alpha_j$ is an injective Borel functions from I_j into I_i. Note that α_{ii} is the identity on I_i; α_{ij} is onto I_i provided that $\alpha_j(I_j) = \alpha_i(I_i)$; and α_{ij} is an empty map if $\alpha_i(I_i) \cap \alpha_j(I_j) = \varnothing$. Clearly, $\alpha = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{I_i}$ on $(0, 1]$. In this chapter, we assume further that each α_i is strictly increasing and maps I_i onto $(0, 1]$. Under these additional assumptions, every α_{ij} is a bijection (in fact, a strictly increasing function) from I_j to I_i. Hence, all α_i's and α_{ij}'s are differentiable a.e. on their domains. Such a measure-preserving function α satisfying all above assumptions will be called **simple**.

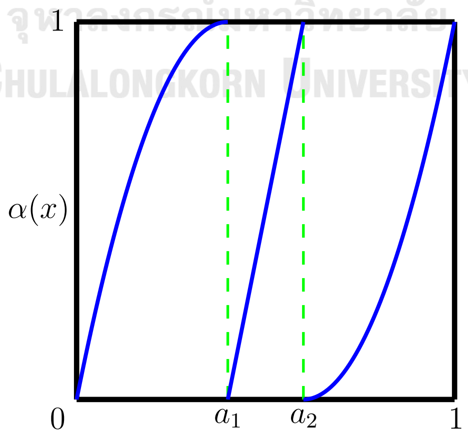


Figure 4.1: An example of simple functions α.

Remark 4.1. For $s \in (0,1]$ and $i = 1, 2, \ldots, n$, let $s_{(i)}$ denote the unique number in I_i such that $\alpha(s_{(i)}) = \alpha(s)$. Obviously, $s_{(1)} < s_{(2)} < \cdots < s_{(n)}$; and that $s = s_{(i)}$ if and only if $s \in I_i$. Consequently, if $s \in I_j$, then $s_{(i)} = \alpha_{ij}(s)$. Since every α_i is strictly increasing, it holds that $s_{(i)} < t_{(i)}$ if and only if $s_{(j)} < t_{(j)}$. It is also the case that $\alpha^{-1}\left((0, \alpha(s)]\right) = \bigcup_{i=1}^{n} (a_i, \alpha_i^{-1}(\alpha(s))] = \bigcup_{i=1}^{n} (a_i, s_{(i)}]$.

Let random variables $X, Y \sim \mathcal{U}(0,1)$ be such that $\alpha(X) = \alpha(Y)$ a.s. for some simple measure-preserving transformation α on $[0,1]$. For each $i = 1, \ldots, n$ and $j = 1, \ldots, n$, denote $A_{ij} := \{X \in I_i, Y \in I_j\}$ on which $\alpha_i(X) = \alpha_j(Y)$ a.s. Hence $Y = \alpha_{ji}(X)$ a.s. on A_{ij} and

$$Y = \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_{ji}(X) \mathbb{1}_{A_{ij}} \quad \text{a.s.} \tag{4.1}$$

For convenience, we denote $\omega_i(s) := K_{C_{X,Y}}(s, I_i)$ which is equal to $\mathbb{E}\left[\mathbb{1}_{\{Y \in I_i\}} \mid X = s\right]$ for almost every $s \in [0,1]$. Since $K_{C_{X,Y}}(s, \cdot)$ is a probability measure for all s and $\bigcup_{i=1}^{n} I_i = (0,1]$, we always have $\sum_{i=1}^{n} \omega_i(s) = 1$.

Next, we introduce Lemma 4.2 - 4.4 which are counterparts of Lemma 3.3 - 3.5 in section 3.2

Lemma 4.2. *Let random variables X and Y be uniformly distributed on $[0,1]$ such that $\alpha(X) = \alpha(Y)$ a.s. for some simple measure-preserving transformation α on $[0,1]$. Then, for $k = 1, 2, \ldots, n$ and a.e. $s \in I_1$,*

$$\sum_{i=1}^{n} \frac{1}{\alpha'(s_{(i)})} \mathbb{P}(Y \in I_k | X = s_{(i)}) = \frac{1}{\alpha'(s_{(k)})}. \tag{4.2}$$

Proof. Let $t = \alpha(s)$ and $A_j(t) = \left(t - \frac{1}{j}, t + \frac{1}{j}\right)$ shrinks to t nicely. By Theorem 2.54, we have

$$\mathbb{P}(Y \in I_k \mid \alpha(X) = t) = \lim_{j \to \infty} \mathbb{P}(Y \in I_k \mid \alpha(X) \in A_j(t)) \text{ a.e. t.}$$

Let the event $B_j := \{\alpha(X) \in A_j(t)\} = \dot{\bigcup}_i \{X \in \alpha_i^{-1}(A_j(t))\}$. Notation $\dot{\bigcup}$ mean disjoint union. Thus, by conditional probability, for a.e. t

$$\begin{aligned}
\mathbb{P}(Y \in I_k \mid \alpha(X) = t) &= \lim_{j \to \infty} \mathbb{P}(Y \in I_k \mid B_j) \\
&= \lim_{j \to \infty} \left[\sum_{i=1}^{n} \frac{\mathbb{P}(Y \in I_k, X \in \alpha_i^{-1}(A_j(t)))}{\mathbb{P}(B_j)} \right] \\
&= \lim_{j \to \infty} \left[\sum_{i=1}^{n} \frac{\mathbb{P}(Y \in I_k, X \in \alpha_i^{-1}(A_j(t)))}{\mathbb{P}(X \in \alpha_i^{-1}(A_j(t)))} \cdot \frac{\mathbb{P}(X \in \alpha_i^{-1}(A_j(t)))}{\mathbb{P}(B_j)} \right]
\end{aligned}$$

$$= \lim_{j\to\infty} \left[\sum_{i=1}^{n} \mathbb{P}(Y \in I_k \mid X \in \alpha_i^{-1}(A_j(t))) \cdot \frac{\mathbb{P}(X \in \alpha_i^{-1}(A_j(t)))}{\mathbb{P}(B_j)} \right].$$

Since $X \sim \mathcal{U}(0,1)$ and α is measure-preserving, $\mathbb{P}(X \in \alpha_i^{-1}(A_j(t))) = \lambda\left(\alpha_i^{-1}(A_j(t))\right)$ and $\mathbb{P}(B_j(t)) = \lambda\left(\alpha^{-1}(A_j(t))\right) = \lambda\left(A_j(t)\right)$. Therefore, by Theorem 2.53, for a.e. t,

$$\lim_{j\to\infty} \frac{\mathbb{P}(X \in \alpha_i^{-1}(A_j(t)))}{\mathbb{P}(B_j(t))} = \left(\alpha_i^{-1}\right)'(t) = \frac{1}{\alpha_i'(t)} =: \beta_i.$$

Since the sequence $\alpha_i^{-1}(A_j(t)) = \left(\alpha_i^{-1}(t-\frac{1}{j}), \alpha_i^{-1}(t+\frac{1}{j})\right)$ shrinks nicely to $s_{(i)}$ as $j \to \infty$, we have by Theorem 2.54 that

$$\mathbb{P}(Y \in I_k \mid X \in \alpha_i^{-1}(A_j(t))) \to \mathbb{P}(Y \in I_k \mid X = s_{(i)}).$$

Hence, $\mathbb{P}(Y \in I_k \mid \alpha(X) = t) = \sum_{i=1}^{n} \beta_i \, \mathbb{P}(Y \in I_k \mid X = s_{(i)})$.

Since $\alpha(X) = \alpha(Y)$ a.s., we have $\mathbb{P}(Y \in I_k \mid \alpha(X) = t) = \mathbb{P}(Y \in I_k \mid \alpha(Y) = t)$ which, by the same arguments as above, is equal to $\sum_{i=1}^{n} \beta_i \, \mathbb{P}(Y \in I_k \mid Y = s_{(i)})$. Finally, as it is clear that $\mathbb{P}(Y \in I_k \mid Y = s_{(i)}) = 1$ if $i = 1$ and 0 if $i \neq 1$, we obtain that

$$\sum_{i=1}^{n} \beta_i \, \mathbb{P}(Y \in I_k \mid X = s_{(i)}) = \beta_i.$$

□

Lemma 4.3. *Let α be a simple measure-preserving transformation on $[0,1]$ and random variables $X, Y \sim \mathcal{U}(0,1)$ with copula $C = C_{X,Y}$ and such that $\alpha(X) = \alpha(Y)$ a.s. Then, for $y \in I_k$, $k \in \{1,\dots,n\}$ and a.e. $s \in [0,1]$,*

$$K_C(s,[0,y]) = \begin{cases} \displaystyle\sum_{j=1}^{k} \omega_j(s) & \text{if } s \in (a_{i-1}, y_{(i)}] \text{ for some } i, \\ \displaystyle\sum_{j=1}^{k-1} \omega_j(s) & \text{if } s \in (y_{(i)}, a_i] \text{ and } k \geq 2 \text{ for some } i, \\ 0 & \text{if } s \in (y_{(i)}, a_i] \text{ and } k = 1 \text{ for some } i. \end{cases} \tag{4.3}$$

Proof. Recall from (4.1) that $Y = \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_{ji}(X) \mathbb{1}_{A_{ij}}$ a.s.

Since $0 \leq \int_A \mathbb{1}_{\{Y=0\}}(\omega) d\,\mathbb{P}(\omega) \leq \int_\Omega \mathbb{1}_{\{Y=0\}}(\omega) d\,\mathbb{P}(\omega) = \mathbb{P}(Y=0) = 0$ for all $A \in \sigma(X)$, we have $\mathbb{E}\left[\mathbb{1}_{\{Y=0\}} \mid X\right] = 0$ a.s. To prove the lemma, it is sufficient to consider $\mathbb{1}_{(0,y]} \circ Y$. Observe that for any $J \subseteq (0,1]$ and a random variable Z taking values in $[0,1]$, $\mathbb{1}_J \circ (Z\mathbb{1}_{A_{ij}}) = (\mathbb{1}_J \circ Z)\mathbb{1}_{A_{ij}}$ for every i,j. Thus, for every i,j,

$$\mathbb{1}_{(0,y]} \circ \left(\alpha_{ji}(X)\mathbb{1}_{A_{ij}}\right) = \left(\mathbb{1}_{(0,y]} \circ \alpha_{ji}(X)\right)\mathbb{1}_{A_{ij}} \quad \text{a.s.}$$

Then

$$\mathbb{1}_{(0,y]} \circ Y = \sum_{j=1}^{n} \sum_{i=1}^{n} \left(\mathbb{1}_{(0,y]} \circ \alpha_{ji}(X) \right) \mathbb{1}_{A_{ij}} \quad \text{a.s.}$$

Since α_i is bijective on I_i for each i, we have that $\alpha_{ji}(X) \in (0, y]$ if and only if $X \in \alpha_{ij}((0, y])$. Thus, $\mathbb{1}_{(0,y]} \circ Y = \sum_{j=1}^{n} \sum_{i=1}^{n} \left(\mathbb{1}_{\alpha_{ij}((0,y])}(X) \right) \mathbb{1}_{A_{ij}}$ a.s. Since $\mathbb{1}_{\alpha_{ij}((0,y])} \circ X$ and $\mathbb{1}_{I_i} \circ X$ are $\sigma(X)$-measurable, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\mathbb{1}_{\alpha_{ij}((0,y])} \circ X \right) \mathbb{1}_{A_{ij}} \mid X \right] &= \mathbb{E}\left[\left(\mathbb{1}_{\alpha_{ij}((0,y])} \circ X \right) \mathbb{1}_{I_i}(X) \mathbb{1}_{I_j}(Y) \mid X \right] \\
&= \left(\mathbb{1}_{\alpha_{ij}((0,y]) \cap I_i} \circ X \right) \mathbb{E}\left[\mathbb{1}_{Y \in I_j} \mid X \right] \\
&= \omega_j(X) \left(\mathbb{1}_{\alpha_{ij}((0,y]) \cap I_i} \circ X \right) \quad \text{a.s.}
\end{aligned}$$

Hence,

$$\begin{aligned}
K_C(s, [0, y]) &= \mathbb{E}\left[\mathbb{1}_{[0,y]} \circ Y \mid X = s \right] \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \omega_j(s) \mathbb{1}_{\alpha_{ij}((0,y]) \cap I_i}(s) \quad \text{a.s.}
\end{aligned}$$

Next, we will consider $s \in I_i$, so $K_C(s, [0, y]) = \sum_{j=1}^{n} \omega_j(s) \mathbb{1}_{\alpha_{ij}((0,y])}(s)$. If $j < k$, then $\alpha_{ij}((0, y]) = \alpha_i^{-1}([0, 1]) = I_i$, so $\mathbb{1}_{\alpha_{ij}((0,y])}(s) = 1$. For $j > k$, since α_j define on I_j, so $\alpha_j((0, y]) = \varnothing$, then $\alpha_{ij}((0, y]) = \alpha_i^{-1}(\varnothing) = \varnothing$, i.e., $\mathbb{1}_{\alpha_{ij}((0,y])}(s) = 0$. If $j = k$, then $\alpha_{ij}((0, y]) = \alpha_i^{-1}([0, \alpha(y)]) = (a_{i-1}, y_{(i)}]$, i.e. $\mathbb{1}_{\alpha_{ij}((0,y])}(s) = \mathbb{1}_{(a_{i-1}, y_{(i)}]}(s)$. This completes the proof of equation (4.3) □

Lemma 4.4. *Let α be a simple measure-preserving transformation on $[0, 1]$ and random variables X, Y uniformly distributed on $[0, 1]$ with copula $C = C_{X,Y}$ such that $\alpha(X) = \alpha(Y)$ a.s. Then, for $i, k = 1, \ldots, n$, $y \in I_k$, and a.e. $s \in [0, 1]$,*

1. *$K_C(s_{(i)}, [0, s_{(k)}]) = K_C(s_{(i)}, [0, y])$ if $s_{(k)} \leq y$; and*
2. *$K_C(s_{(i)}, [0, s_{(k-1)}]) = K_C(s_{(i)}, [0, y])$ if $s_{(k)} > y$.*

Proof. Equivalently, it suffices to show that $\mathbb{P}\left(s_{(k)} < Y \leq y \mid X = s \right) = 0$. Since $A_j := \left(s - \frac{1}{j}, s \right]$ shrinks nicely to s as $j \to \infty$ and the sets

$$\alpha\left((s_{(k)}, y] \right) \subseteq [0, \alpha(y)) \cup \left(\alpha(s_{(k)}), 1 \right]$$

and

$$\alpha\left(\left(s - \frac{1}{j}, s \right] \right) \subseteq \left(\alpha(s - \frac{1}{j}), \alpha(s) \right] = \left(\alpha(s - \frac{1}{j}), \alpha(s_{(k)}) \right]$$

are eventually disjoint. Note the use of the assumption that α can be partitioned into finitely many α_i's. Hence,

$$\begin{aligned}\mathbb{P}\left(Y \in \left(s_{(k)}, y\right], X \in A_j(s)\right) &\le \mathbb{P}\left(\alpha(Y) \in \alpha\left(\left(s_{(k)}, y\right]\right), \alpha(X) \in \alpha\left(\left(s - \frac{1}{j}, s\right]\right)\right)\\ &= 0,\end{aligned}$$

for j large enough. Therefore, $\mathbb{P}\left(s_{(k)} < Y \le y \mid X = s\right) = 0$ which proves the claim. □

Lemma 4.5. *Let α be a simple measure-preserving transformation on $[0,1]$. For $x \in I_j$ and a.e. $s \in I_1$,*

$$\partial_2 C_{e,\alpha}(x, \alpha(s)) = \begin{cases} \displaystyle\sum_{i=1}^{j-1} \frac{1}{\alpha'(s_{(i)})} = \beta_{j-1} & \text{if } x \le s_{(j)}, \\ \displaystyle\sum_{i=1}^{j} \frac{1}{\alpha'(s_{(i)})} = \beta_j & \text{if } x > s_{(j)}. \end{cases} \tag{4.4}$$

Proof. By definition, $C_{e,\alpha}(x,y) = \lambda\left(\bigcup_{i=1}^{n} [0,x] \cap (a_{i-1}, \alpha_i^{-1}(y)]\right)$. Let $x \in I_j$. Then, the intersection is empty for $i > j$ and

$$C_{e,\alpha}(x,y) = \begin{cases} \displaystyle\sum_{i=1}^{j-1} (\alpha_i^{-1}(y) - a_{i-1}) + (x - a_{j-1}) & \text{if } x \le s_{(j)}, \\ \displaystyle\sum_{i=1}^{j} (\alpha_i^{-1}(y) - a_{i-1}) & \text{if } x > s_{(j)}. \end{cases}$$

Let $s \in I_1$. For $x \le s_{(j)}$, a.e. s

$$\begin{aligned}\partial_2 C_{e,\alpha}(x, \alpha(s)) &= \lim_{h \to 0} \frac{C_{e,\alpha}(x, \alpha(s) + h) - C_{e,\alpha}(x, \alpha(s))}{h}\\ &= \sum_{i=1}^{j-1} \lim_{h \to 0} \frac{\alpha_i^{-1}(\alpha(s) + h) - \alpha_i^{-1}(\alpha(s))}{h}\\ &= \sum_{i=1}^{j-1} \left(\alpha_i^{-1}\right)' (\alpha(s))\\ &= \sum_{i=1}^{j-1} \frac{1}{\alpha_i'(s_{(i)})}.\end{aligned}$$

Also for $x > s_{(j)}$, a.e. s

$$\partial_2 C_{e,\alpha}(x, \alpha(s)) = \lim_{h \to 0} \frac{C_{e,\alpha}(x, \alpha(s) + h) - C_{e,\alpha}(x, \alpha(s))}{h}$$

$$= \sum_{i=1}^{j} \lim_{h\to 0} \frac{\alpha_i^{-1}(\alpha(s)+h) - \alpha_i^{-1}(\alpha(s))}{h}$$
$$= \sum_{i=1}^{j} \left(\alpha_i^{-1}\right)' (\alpha(s))$$
$$= \sum_{i=1}^{j} \frac{1}{\alpha_i'(s_{(i)})}.$$

□

Computing $\partial_2 C_{e,\alpha}(1, \alpha(s))$ in two different ways, i.e., by using (4.4) and by the boundary condition of copulas, gives $\beta_n = \sum_{i=1}^{n} \frac{1}{\alpha'(s_{(i)})} = 1$.

For less cumbersome notation, we denote $\beta_k := \sum_{i=1}^{k} \frac{1}{\alpha'(s_{(i)})}$ for $k = 1, \ldots, n$ and $B := \{0 = \beta_0, \beta_1, \ldots, \beta_{n-1}, \beta_n = 1\}$. For $s \in (0, a_1)$, let $A_{\alpha(s)}$ be defined on $B \times B$ by

$$A_{\alpha(s)}\left(\beta_k, \beta_\ell\right) := \frac{1}{\alpha'(s)} \sum_{i=1}^{k} \alpha_{i1}'(s) K_C(s_{(i)}, [0, s_{(\ell)}]) \quad \text{for } k, \ell \in \{1, \ldots, n-1\} \tag{4.5}$$

and $A_{\alpha(s)}\left(\beta_k, 1\right) = A_{\alpha(s)}\left(1, \beta_k\right) = \beta_k$ for $k = 0, 1, \ldots, n$.

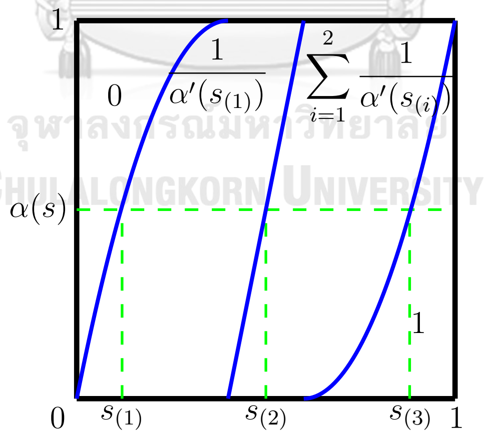


Figure 4.2: The value of $\partial_2 C_{e,\alpha}(x, \alpha(s))$.

Lemma 4.6. *For every $s \in (0, a_1)$, $A_{\alpha(s)}$ is a subcopula on $B \times B$.*

Proof. It is only left to show that $A_{\alpha(s)}$ has 2-increasing property. Let $\beta_p \le \beta_q$ and $\beta_k \le \beta_\ell$. For $j = k, \ell$,

$$A_{\alpha(s)}(\beta_q, \beta_j) - A_{\alpha(s)}(\beta_p, \beta_j) = \frac{1}{\alpha'(s)} \sum_{i=p+1}^{q} \alpha'_{i1}(s) K_C(s_{(i)}, [0, s_{(j)}]).$$

So $V_{A_{\alpha(s)}}([\beta_p, \beta_q] \times [\beta_k, \beta_\ell]) = \frac{1}{\alpha'(s)} \sum_{i=p+1}^{q} \alpha'_{i1}(s) K_C\left(s_{(i)}, (s_k, s_{(\ell)}]\right) \ge 0.$ □

Lemma 4.7. *$A_t(\partial_2 C_{e,\alpha}(x,t), \partial_2 C_{e,\alpha}(y,t))$ is measurable in $t \in [0,1]$.*

Proof. Let $s \in I_1$. For each $k, \ell \in \{1, \dots, n-1\}$, consider

$$A_{\alpha(s)}\left(\beta_k, \beta_\ell\right) = \frac{1}{\alpha'(s)} \sum_{i=1}^{k} \alpha'_{i1}(s) K_C(s_{(i)}, [0, s_{(\ell)}]).$$

Since $s_{(\ell)} \le a_\ell$ and $s_{(i)}$ is a linear function of s, by Lemma 4.4, $K_C(s_{(i)}, [0, s_{(\ell)}]) = K_C(s_{(i)}, [0, a_{(\ell)}])$ is measurable in s. Hence, $A_{\alpha(s)}\left(\beta_k, \beta_\ell\right)$ is measurable in s. Next, it is similar to Lemma 3.1 to show that $A_{\alpha(s)}(\partial_2 C_{e,\alpha}(x, \alpha(s), \partial_2 C_{e,\alpha}(y, \alpha(s))$ is measurable in s. Using a change of variable $t = \alpha_1(s)$, the proof is complete. □

Theorem 4.8. *Let random variables $X, Y \sim \mathcal{U}(0,1)$ be such that $\alpha(X) = \alpha(Y)$ a.s. for some simple measure-preserving transformation α on $[0,1]$. Then, there exists $\mathcal{A} = \{A_t\}_{t\in[0,1]} \subseteq \mathscr{C}$ such that $C_{X,Y} = C_{e,\alpha} *_{\mathcal{A}} C_{\alpha,e}$.*

Proof. For each $s \in (0, a_1)$, we extend the subcopula $A_{\alpha(s)}$ defined above to a copula, still denoted by $A_{\alpha(s)}$. A_0 and A_1 can be taken to be any copulas as they do not affect the $\mathcal{A}$-product. By Lemma 4.7, $A_t(\partial_2 C_{e,\alpha}(x,t), \partial_2 C_{e,\alpha}(y,t))$ is measurable in $t \in [0,1]$. Putting $\mathcal{A} := \{A_t\}_{t\in[0,1]}$, we have

$$\begin{aligned} C_{e,\alpha} *_{\mathcal{A}} C_{\alpha,e}(x,y) &= \int_0^1 A_t(\partial_2 C_{e,\alpha}(x,t), \partial_2 C_{e,\alpha}(y,t))\, dt \\ &= \int_0^{a_1} \alpha'(s) A_{\alpha(s)}\left(\partial_2 C_{e,\alpha}(x, \alpha(s)), \partial_2 C_{e,\alpha}(y, \alpha(s))\right) ds, \qquad (4.6) \end{aligned}$$

where the last equality uses the change of variable $t = \alpha(s)$. Denote $C := C_{X,Y}$ and let K_C be its Markov kernel. The rest of the proof is devoted to deriving that (4.6) equals $\int_0^x K_C\left(s, [0,y]\right) ds$ which, by Theorem 2.50, is equal to $C(x,y)$. The proof is divided into four cases according to where (x,y) is. By Lemma 4.5, if $(x,y) \in I_p \times I_q$ then $\partial_2 C_{e,\alpha}(x, \alpha(s)) = \beta_p \mathbb{1}_{(0,x_{(1)})}(s) + \beta_{p-1} \mathbb{1}_{[x_{(1)},a_1]}(s)$ and $\partial_2 C_{e,\alpha}(y, \alpha(s)) = \beta_q \mathbb{1}_{(0,y_{(1)})}(s) + \beta_{q-1} \mathbb{1}_{[y_{(1)},a_1]}(s)$ for a.e. $s \in (0, a_1]$.

Case 1: $x \in I_1, y \in I_q$ where $q = 1, \dots, n$.

1.1. If $x \le y_{(1)}$, then $C_{e,\alpha} *_{\mathcal{A}} C_{\alpha,e}(x,y)$ is

$$\begin{aligned} &= \int_0^x \alpha'(s) A_{\alpha(s)}\left(\beta_1, \beta_q\right) ds + \int_x^{y_{(1)}} \alpha'(s) A_{\alpha(s)}\left(0, \beta_q\right) ds \\ &\quad + \int_{y_{(1)}}^{a_1} \alpha'(s) A_{\alpha(s)}\left(0, \beta_{q-1}\right) ds \\ &= \begin{cases} \int_0^x K_C\left(s, [0, s_{(q)}]\right) ds & \text{if } q < n, \\ \int_0^x \alpha'(s)\beta_1\, ds = \int_0^x 1\, ds & \text{if } q = n. \end{cases} \end{aligned}$$

The case $q = n$ is done by noting from (4.3) that $K_C(s, [0, y]) = \sum_1^n \omega_j(s) = 1$. Consider $q < n$. For $s \le x \le y_{(1)}$, it follows from Remark 4.1 that $s_{(q)} \le y_{(q)}$. So, we have $K_C(s, [0, s_{(q)}]) = K_C(s, [0, y])$ by Lemma 4.4.

1.2. If $x > y_{(1)}$, then $C_{e,\alpha} *_{\mathcal{A}} C_{\alpha,e}(x,y)$ is

$$\begin{aligned} &= \int_0^{y_{(1)}} \alpha'(s) A_{\alpha(s)}\left(\beta_1, \beta_q\right) ds + \int_{y_{(1)}}^x \alpha'(s) A_{\alpha(s)}\left(\beta_1, \beta_{q-1}\right) ds \qquad (4.7) \\ &\quad + \int_x^{a_1} \alpha'(s) A_{\alpha(s)}\left(0, \beta_{q-1}\right) ds \\ &= \begin{cases} \int_0^{y_{(1)}} K_C(s, [0, s]) ds & \text{if } q = 1, \\ \int_0^{y_{(1)}} K_C(s, [0, s_{(q)}]) ds + \int_{y_{(1)}}^x K_C(s, [0, s_{(q-1)}]) ds & \text{if } 1 < q < n, \\ \int_0^{y_{(1)}} 1 ds + \int_{y_{(1)}}^x K_C(s, [0, s_{(q-1)}]) ds & \text{if } q = n. \end{cases} \qquad (4.8) \end{aligned}$$

For $1 < q < n$, it follows from Remark 4.1 and Lemma 4.4 that $K_C(s, [0, s_{(q)}]) = K_C(s, [0, y])$ if $s \le y_{(1)}$, and $K_C(s, [0, s_{(q-1)}]) = K_C(s, [0, y])$ if $y_{(1)} < s \le x$. Hence, (4.8) is equal to $\int_0^x K_C(s, [0, y]) ds$.

For $q = 1$, we obtain by using Lemma 4.4 that $K_C(s, [0, s]) = K_C(s, [0, y])$ if $s \le y_{(1)} = y$. For $y_{(1)} < s \le x$, we have $K_C(s, [0, y]) = 0$ by Lemma 4.3.

For $q = n$, we note from Lemma 4.3 that $K_C(s, [0, y]) = \sum_{j=1}^n \omega_j(s) = 1$ if $s \le y_{(1)}$, and $K_C(s, [0, s_{(q-1)}]) = K_C(s, [0, y])$ if $y_{(1)} < s \le x$.

Case 2: $x \in I_p, y \in I_q$ where $1 < p < n$ and $1 \le q < n$.

2.1. If $x_{(1)} \le y_{(1)}$, then $C_{e,\alpha} *_{\mathcal{A}} C_{\alpha,e}(x,y)$, for $q > 1$, is

$$\begin{aligned} &= \int_0^{x_{(1)}} \alpha'(s) A_{\alpha(s)}\left(\beta_p, \beta_q\right) ds + \int_{x_{(1)}}^{y_{(1)}} \alpha'(s) A_{\alpha(s)}\left(\beta_{p-1}, \beta_q\right) ds \\ &\quad + \int_{y_{(1)}}^{a_1} \alpha'(s) A_{\alpha(s)}\left(\beta_{p-1}, \beta_{q-1}\right) ds \end{aligned}$$

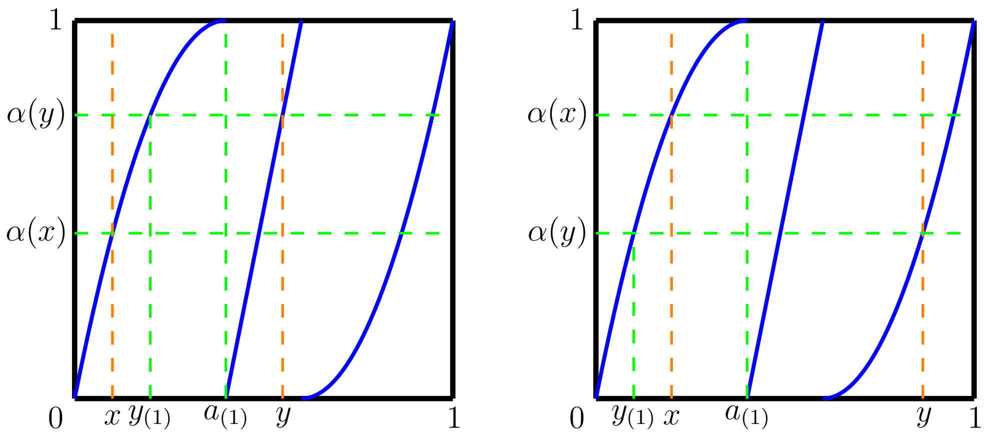


Figure 4.3: The value of $\partial_2 C_{e,\alpha}(x, \alpha(s))$ in subcase 1.1 (left) and 1.2 (right).

$$= \int_0^{x_{(1)}} \sum_{i=1}^{p} \alpha'_{i1}(s) K_C(s_{(i)}, [0, s_{(q)}]) ds + \int_{x_{(1)}}^{y_{(1)}} \sum_{i=1}^{p-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, s_{(q)}]) ds$$
$$+ \int_{y_{(1)}}^{a_1} \sum_{i=1}^{p-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, s_{(q-1)}]) ds$$
$$= \int_0^{x_{(1)}} \sum_{i=1}^{p} \alpha'_{i1}(s) K_C(s_{(i)}, [0, y]) ds + \int_{x_{(1)}}^{y_{(1)}} \sum_{i=1}^{p-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, y]) ds$$
$$+ \int_{y_{(1)}}^{a_1} \sum_{i=1}^{p-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, y]) ds$$
$$= \sum_{i=1}^{p} \int_{a_{i-1}}^{x_{(i)}} K_C(s, [0, y]) ds + \sum_{i=1}^{p-1} \int_{x_{(i)}}^{y_{(i)}} K_C(s, [0, y]) ds + \sum_{i=1}^{p-1} \int_{y_{(i)}}^{a_i} K_C(s, [0, y]) ds.$$

The third equality holds because, by Remark 4.1 and Lemma 4.4, $K_C(s, [0, s_{(q)}]) = K_C(s, [0, y])$ for $s \in [0, y_{(1)}]$ and $K_C(s, [0, s_{(q-1)}]) = K_C(s, [0, y])$ for $s \in (y_{(1)}, a_1]$, and the last equality uses the change of variable $s_{(1)} = \alpha_{i1}(s_{(1)})$ $[= s_{(i)}]$.

Case $q = 1$ is similar to case $q > 1$ except for that $A_{\alpha(s)}(\beta_{p-1}, \beta_{q-1}) = 0$. This case is proved because $\sum_{i=1}^{p-1} \int_{y_{(i)}}^{a_i} K_C(s, [0, y]) ds = 0$ by equation (4.3).

2.2. If $x_{(1)} > y_{(1)}$, then $C_{e,\alpha} *_{\mathcal{A}} C_{\alpha,e}(x, y)$, for $q > 1$, is

$$= \int_0^{y_{(1)}} \alpha'(s) A_{\alpha(s)}(\beta_p, \beta_q) \, ds + \int_{y_{(1)}}^{x_{(1)}} \alpha'(s) A_{\alpha(s)}(\beta_p, \beta_{q-1}) \, ds$$

$$
\begin{aligned}
&+\int_{x_{(1)}}^{a_1} \alpha'(s) A_{\alpha(s)}\left(\beta_{p-1}, \beta_{q-1}\right) ds \\
&= \int_0^{y_{(1)}} \sum_{i=1}^{p} \alpha'_{i1}(s) K_C(s_{(i)}, [0, s_{(q)}]) ds + \int_{y_{(1)}}^{x_{(1)}} \sum_{i=1}^{p} \alpha'_{i1}(s) K_C(s_{(i)}, [0, s_{(q-1)}]) ds \\
&\quad + \int_{x_{(1)}}^{a_1} \sum_{i=1}^{p-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, s_{(q-1)}]) ds \\
&= \int_0^{y_{(1)}} \sum_{i=1}^{p} \alpha'_{i1}(s) K_C(s_{(i)}, [0, y]) ds + \int_{y_{(1)}}^{x_{(1)}} \sum_{i=1}^{p} \alpha'_{i1}(s) K_C(s_{(i)}, [0, y]) ds \\
&\quad + \int_{x_{(1)}}^{a_1} \sum_{i=1}^{p-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, y]) ds \\
&= \sum_{i=1}^{p} \int_{a_{i-1}}^{y_{(i)}} K_C(s, [0, y]) ds + \sum_{i=1}^{p} \int_{y_{(i)}}^{x_{(i)}} K_C(s, [0, y]) ds + \sum_{i=1}^{p-1} \int_{x_{(i)}}^{a_i} K_C(s, [0, y]) ds.
\end{aligned}
$$

The third equality holds because, by Remark 4.1 and Lemma 4.4, $K_C(s, [0, s_{(q)}]) = K_C(s, [0, y])$ for $s \in [0, y_{(1)}]$ and $K_C(s, [0, s_{(q-1)}]) = K_C(s, [0, y])$ for $s \in (y_{(1)}, a_1]$, and the last equality uses the change of variable $s_{(1)} = \alpha_{i1}(s_{(1)})$ $[= s_{(i)}]$.

Case $q = 1$ is similar to case $q > 1$ except for that $A_{\alpha(s)}(\beta_p, \beta_0) = 0 = A_{\alpha(s)}(\beta_{p-1}, \beta_0)$. Using equation (4.3), we have $\sum_{i=1}^{p} \int_{y_{(i)}}^{x_{(i)}} K_C(s, [0, y]) ds = 0 = \sum_{i=1}^{p-1} \int_{x_{(i)}}^{a_i} K_C(s, [0, y]) ds$ to complete this case.

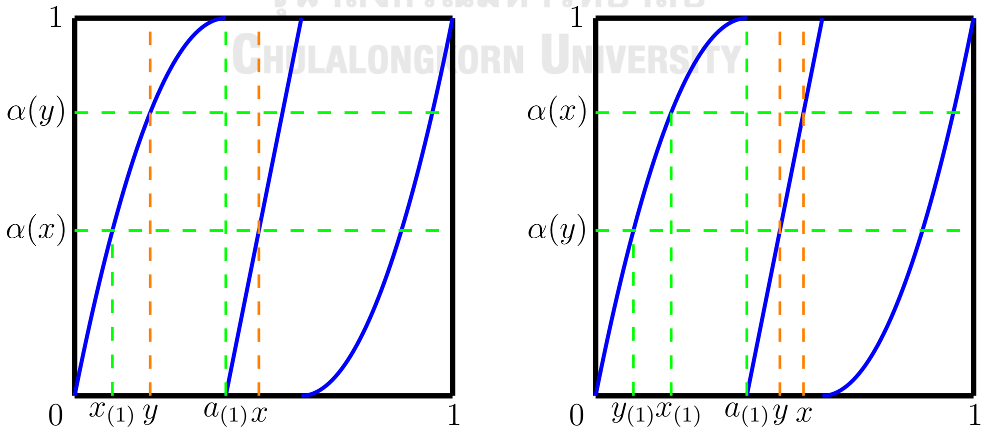


Figure 4.4: The value of $\partial_2 C_{e,\alpha}(x, \alpha(s))$ in subcase 2.1 (left) and 2.2 (right).

Case 3: $x \in I_p, y \in I_n$ where $1 < p \leq n$.

3.1. If $x_{(1)} \leq y_{(1)}$, then $C_{e,\alpha} *_{\mathcal{A}} C_{\alpha,e}(x,y)$ is

$$= \int_0^{x_{(1)}} \alpha'(s) A_{\alpha(s)}\left(\beta_p, \beta_n\right) ds + \int_{x_{(1)}}^{y_{(1)}} \alpha'(s) A_{\alpha(s)}\left(\beta_{p-1}, \beta_n\right) ds$$
$$+ \int_{y_{(1)}}^{a_1} \alpha'(s) A_{\alpha(s)}\left(\beta_{p-1}, \beta_{n-1}\right) ds$$
$$= \int_0^{x_{(1)}} \alpha'(s) \sum_{i=1}^{p} \frac{1}{\alpha'(s_{(i)})} ds + \int_{x_{(1)}}^{y_{(1)}} \alpha'(s) \sum_{i=1}^{p-1} \frac{1}{\alpha'(s_{(i)})} ds$$
$$+ \int_{y_{(1)}}^{a_1} \sum_{i=1}^{p-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, s_{(n-1)}]) ds$$
$$= \sum_{i=1}^{p} \int_0^{x_{(1)}} \frac{\alpha'(s)}{\alpha'(s_{(i)})} ds + \sum_{i=1}^{p-1} \int_{x_{(1)}}^{y_{(1)}} \frac{\alpha'(s)}{\alpha'(s_{(i)})} ds + \int_{y_{(1)}}^{a_1} \sum_{i=1}^{p-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, y]) ds$$
$$= \sum_{i=1}^{p} \int_{a_{i-1}}^{x_{(i)}} 1 ds + \sum_{i=1}^{p-1} \int_{x_{(i)}}^{y_{(i)}} 1 ds + \sum_{i=1}^{p-1} \int_{y_{(i)}}^{a_i} K_C(s, [0, s_{(n-1)}]) ds.$$

The third equality holds because, by Remark 4.1 and Lemma 4.4, $K_C(s, [0, s_{(n-1)}]) = K_C(s, [0, y])$ for $s \in (y_{(1)}, a_1]$, and the last equality uses the change of variable $s_{(1)} = \alpha_{i1}(s_{(1)})[= s_{(i)}]$. Note that $d\alpha_{i1}(s_{(1)}) = \frac{\alpha'(s)}{\alpha'(s_{(i)})} ds_{(1)}$.

3.2. If $x_{(1)} > y_{(1)}$, then $C_{e,\alpha} *_{\mathcal{A}} C_{\alpha,e}(x,y)$ is

$$= \int_0^{y_{(1)}} \alpha'(s) A_{\alpha(s)}\left(\beta_p, \beta_n\right) ds + \int_{y_{(1)}}^{x_{(1)}} \alpha'(s) A_{\alpha(s)}\left(\beta_p, \beta_{n-1}\right) ds$$
$$+ \int_{x_{(1)}}^{a_1} \alpha'(s) A_{\alpha(s)}\left(\beta_{p-1}, \beta_{n-1}\right) ds$$
$$= \int_0^{y_{(1)}} \alpha'(s) \sum_{i=1}^{p} \frac{1}{\alpha'(s_{(i)})} ds + \int_{y_{(1)}}^{x_{(1)}} \sum_{i=1}^{p} \alpha'_{i1}(s) K_C(s_{(i)}, [0, s_{(n-1)}]) ds$$
$$+ \int_{x_{(1)}}^{a_1} \sum_{i=1}^{p-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, s_{(n-1)}]) ds$$
$$= \sum_{i=1}^{p} \int_0^{y_{(1)}} \frac{\alpha'(s)}{\alpha'(s_{(i)})} ds + \int_{y_{(1)}}^{x_{(1)}} \sum_{i=1}^{p} \alpha'_{i1}(s) K_C(s_{(i)}, [0, y]) ds$$
$$+ \int_{x_{(1)}}^{a_1} \sum_{i=1}^{p-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, y]) ds$$
$$= \sum_{i=1}^{p} \int_{a_{i-1}}^{y_{(i)}} 1 ds + \sum_{i=1}^{p} \int_{y_{(i)}}^{x_{(i)}} K_C(s, [0, y]) ds + \sum_{i=1}^{p-1} \int_{x_{(i)}}^{a_i} K_C(s, [0, y]) ds.$$

The third equality holds because, by Remark 4.1 and Lemma 4.4, $K_C(s, [0, s_{(n-1)}]) = K_C(s, [0, y])$ for $s \in (y_{(1)}, a_1]$, and the last equality uses the change of variable

$s_{(1)} = \alpha_{i1}(s_{(1)})[= s_{(i)}]$. Note that $d\alpha_{i1}(s_{(1)}) = \frac{\alpha'(s)}{\alpha'(s_{(i)})} ds_{(1)}$.

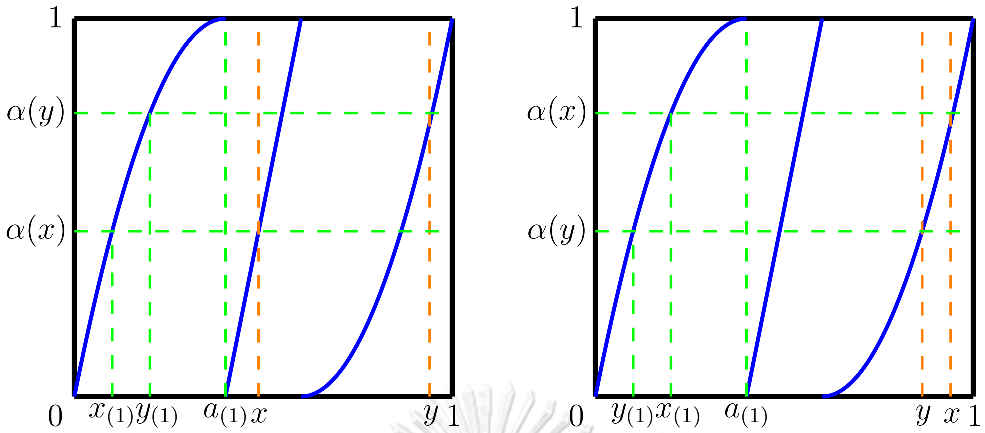


Figure 4.5: The value of $\partial_2 C_{e,\alpha}(x, \alpha(s))$ in subcase 3.1 (left) and 3.2 (right).

Case 4: $x \in I_n, y \in I_q$ where $1 \le q < n$.

4.1. If $x_{(1)} \le y_{(1)}$, then $C_{e,\alpha} *_{\mathcal{A}} C_{\alpha,e}(x, y)$, for $q > 1$, is

$$= \int_0^{x_{(1)}} \alpha'(s) A_{\alpha(s)}(\beta_n, \beta_q)\, ds + \int_{x_{(1)}}^{y_{(1)}} \alpha'(s) A_{\alpha(s)}(\beta_{n-1}, \beta_q)\, ds$$

$$+ \int_{y_{(1)}}^{a_1} \alpha'(s) A_{\alpha(s)}(\beta_{n-1}, \beta_{q-1})\, ds$$

$$= \int_0^{x_{(1)}} \alpha'(s) \sum_{i=1}^{q} \frac{1}{\alpha'(s_{(i)})} ds + \int_{x_{(1)}}^{y_{(1)}} \sum_{i=1}^{n-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, s_{(q)}]) ds$$

$$+ \int_{y_{(1)}}^{a_1} \sum_{i=1}^{n-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, s_{(q-1)}]) ds$$

$$= \int_0^{x_{(1)}} \alpha'(s) \sum_{i=1}^{q} \frac{1}{\alpha'(s_{(i)})} ds + \int_{x_{(1)}}^{y_{(1)}} \sum_{i=1}^{n-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, y]) ds$$

$$+ \int_{y_{(1)}}^{a_1} \sum_{i=1}^{n-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, y]) ds$$

$$= \int_0^{x_{(1)}} \alpha'(s) \sum_{i=1}^{q} \frac{1}{\alpha'(s_{(i)})} ds + \sum_{i=1}^{n-1} \int_{x_{(i)}}^{y_{(i)}} K_C(s, [0, y]) ds + \sum_{i=1}^{n-1} \int_{y_{(i)}}^{a_i} K_C(s, [0, y]) ds.$$

The third equality holds because, by Remark 4.1 and Lemma 4.4, $K_C(s, [0, s_{(q)}]) = K_C(s, [0, y])$ for $s \in (x_{(1)}, y_{(1)}]$, and $K_C(s, [0, s_{(q-1)}]) = K_C(s, [0, y])$ for $s \in (y_{(1)}, a_1]$ and the last equality uses the change of variable $s_{(1)} = \alpha_{i1}(s_{(1)})$ $[= s_{(i)}]$.

The first term in the last equality, by Lemma 4.2, becomes

$$\int_0^{x_{(1)}} \alpha'(s) \sum_{i=1}^{q} \frac{1}{\alpha'(s_{(i)})} ds = \int_0^{x_{(1)}} \alpha'(s) \sum_{i=1}^{q} \left(\sum_{j=1}^{n} \frac{1}{\alpha'(s_{(j)})} \omega_i(s_{(j)}) \right) ds.$$

Using the change of variable $s_{(1)} = \alpha_{j1}(s_{(1)})[= s_{(j)}]$, we have

$$\int_0^{x_{(1)}} \alpha'(s) \sum_{i=1}^{q} \sum_{j=1}^{n} \frac{1}{\alpha'(s_{(j)})} \omega_i(s_{(j)}) ds = \sum_{j=1}^{n} \int_{a_{j-1}}^{x_{(j)}} \sum_{i=1}^{q} \omega_i(s) ds.$$

By equation (4.3),

$$\sum_{j=1}^{n} \int_{a_{j-1}}^{x_{(j)}} \sum_{i=1}^{q} \omega_i(s) ds = \sum_{j=1}^{n} \int_{a_{j-1}}^{x_{(j)}} K_C(s, [0, y]) ds.$$

Case $q = 1$ is similar to case $q > 1$ except for that $A_{\alpha(s)}(\beta_{n-1}, \beta_{q-1}) = 0$. Using equation (4.3), we have $\sum_{i=1}^{n-1} \int_{y_{(i)}}^{a_i} K_C(s, [0, y]) ds = 0$. This completes the case.

4.2. If $x_{(1)} > y_{(1)}$, then $C_{e,\alpha} *_{\mathcal{A}} C_{\alpha,e}(x, y)$ is equal to

$$\begin{aligned}
&= \int_0^{y_{(1)}} \alpha'(s) A_{\alpha(s)}(\beta_n, \beta_q) ds + \int_{y_{(1)}}^{x_{(1)}} \alpha'(s) A_{\alpha(s)}(\beta_n, \beta_{q-1}) ds \\
&\quad + \int_{x_{(1)}}^{a_1} \alpha'(s) A_{\alpha(s)}(\beta_{n-1}, \beta_{q-1}) ds \\
&= \int_0^{y_{(1)}} \alpha'(s) \sum_{i=1}^{q} \frac{1}{\alpha'(s_{(i)})} ds + \int_{y_{(1)}}^{x_{(1)}} \alpha'(s) \sum_{i=1}^{q-1} \frac{1}{\alpha'(s_{(i)})} ds \\
&\quad + \int_{x_{(1)}}^{a_1} \sum_{i=1}^{n-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, s_{(q-1)}]) ds \\
&= \int_0^{y_{(1)}} \alpha'(s) \sum_{i=1}^{q} \frac{1}{\alpha'(s_{(i)})} ds + \int_{y_{(1)}}^{x_{(1)}} \alpha'(s) \sum_{i=1}^{q-1} \frac{1}{\alpha'(s_{(i)})} ds \\
&\quad + \int_{x_{(1)}}^{a_1} \sum_{i=1}^{n-1} \alpha'_{i1}(s) K_C(s_{(i)}, [0, y]) ds \\
&= \int_0^{y_{(1)}} \alpha'(s) \sum_{i=1}^{q} \frac{1}{\alpha'(s_{(i)})} ds + \int_{y_{(1)}}^{x_{(1)}} \alpha'(s) \sum_{i=1}^{q-1} \frac{1}{\alpha'(s_{(i)})} ds + \sum_{i=1}^{n-1} \int_{x_{(i)}}^{a_i} K_C(s, [0, y]) ds.
\end{aligned}$$

The third equality holds because, by Remark 4.1 and Lemma 4.4, $K_C(s, [0, s_{(q-1)}]) = K_C(s, [0, y])$ for $s \in (y_{(1)}, a_1]$, and the last equality uses the change of variable $s_{(1)} = \alpha_{i1}(s_{(1)})[= s_{(i)}]$.

The first term in the last equality, by Lemma 4.2, becomes

$$\int_0^{y_{(1)}} \alpha'(s) \sum_{i=1}^{q} \frac{1}{\alpha'(s_{(i)})} ds = \int_0^{y_{(1)}} \alpha'(s) \sum_{i=1}^{q} \left(\sum_{j=1}^{n} \frac{1}{\alpha'(s_{(j)})} \omega_i(s_{(j)}) \right) ds$$

and

$$\int_{y_{(1)}}^{x_{(1)}} \alpha'(s) \sum_{i=1}^{q-1} \frac{1}{\alpha'(s_{(i)})} ds = \int_{y_{(1)}}^{x_{(1)}} \alpha'(s) \sum_{i=1}^{q-1} \left(\sum_{j=1}^{n} \frac{1}{\alpha'(s_{(j)})} \omega_i(s_{(j)}) \right) ds.$$

Using the change of variable $s_{(1)} = \alpha_{j1}(s_{(1)})[= s_{(j)}]$, we have

$$\int_0^{y_{(1)}} \alpha'(s) \sum_{i=1}^{q} \sum_{j=1}^{n} \frac{1}{\alpha'(s_{(j)})} \omega_i(s_{(j)}) ds = \sum_{j=1}^{n} \int_{a_{j-1}}^{y_{(j)}} \sum_{i=1}^{q} \omega_i(s) ds$$

and

$$\int_{y_{(1)}}^{x_{(1)}} \alpha'(s) \sum_{i=1}^{q-1} \sum_{j=1}^{n} \frac{1}{\alpha'(s_{(j)})} \omega_i(s_{(j)}) ds = \sum_{j=1}^{n} \int_{y_{(j)}}^{x_{(j)}} \sum_{i=1}^{q-1} \omega_i(s) ds.$$

By equation (4.3),

$$\sum_{j=1}^{n} \int_{a_{j-1}}^{y_{(j)}} \sum_{i=1}^{q} \omega_i(s) ds = \sum_{j=1}^{n} \int_{a_{j-1}}^{y_{(j)}} K_C(s, [0, y]) ds$$

and

$$\sum_{j=1}^{n} \int_{y_{(j)}}^{x_{(j)}} \sum_{i=1}^{q-1} \omega_i(s) ds = \sum_{j=1}^{n} \int_{y_{(j)}}^{x_{(j)}} K_C(s, [0, y]) ds.$$

Case $q = 1$ is similar to case $q > 1$ except for that $A_{\alpha(s)}(\beta_n, \beta_{q-1}) = 0 = A_{\alpha(s)}(\beta_{n-1}, \beta_{q-1})$. Using equation (4.3), we have $\sum_{i=1}^{n} \int_{y_{(i)}}^{x_{(i)}} K_C(s, [0, y]) ds = 0 = \sum_{i=1}^{n-1} \int_{x_{(i)}}^{a_i} K_C(s, [0, y]) ds.$ □

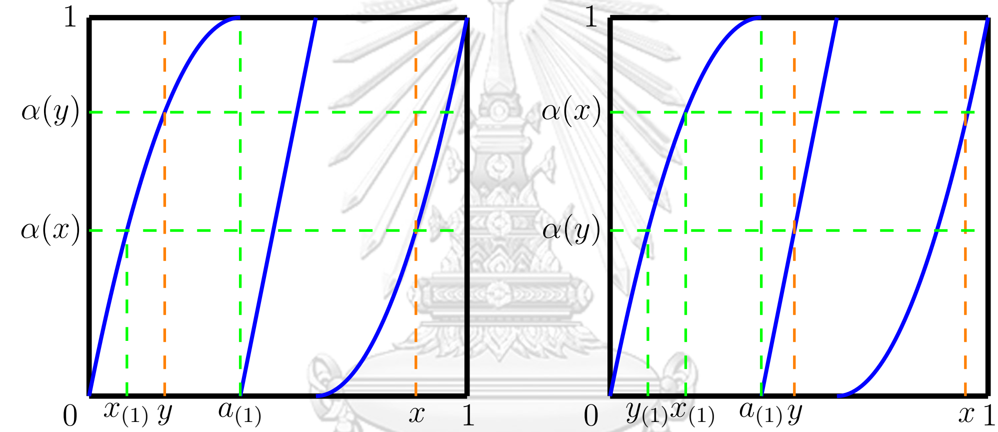


Figure 4.6: The value of $\partial_2 C_{e,\alpha}(x, \alpha(s))$ in subcase 4.1 (left) and 4.2 (right).

CHAPTER V
CONCLUSION

5.1 Our results

We started out trying to characterize implicit dependence copulas and finally found a relationship between implicit dependence copulas and products of complete dependence copula.

In section 3.1, we show that generalized products of complete dependence copulas C_{e,Λ_θ} and $C_{\Lambda_\theta,e}$ are implicit dependence copulas of some random variables X and Y uniformly distributed on $[0,1]$ with $\Lambda_\theta(X) = \Lambda_\theta(Y)$ a.s. Vice versa, we factor some implicit dependence copula into a generalized product of complete dependence copulas, i.e., for every random variables X and Y uniformly distributed on $[0,1]$ with $\Lambda_\theta(X) = \Lambda_\theta(Y)$ a.s. and with copula $C_{X,Y}$, there exists a class of copulas $\mathcal{A}$ such that $C_{X,Y} = C_{e\Lambda_\theta} *_{\mathcal{A}} C_{\Lambda_\theta e}$ (section 3.2). Moreover, in chapter IV, we generalize the result in section 3.2 from the function Λ_θ to the function α.

5.2 Further studies

Naturally, we conjecture that, for measure-preserving transformations f and g, C is the copula of implicitly dependent $\mathcal{U}(0,1)$-random variables X and Y with $f(X) = g(Y)$ a.s. if and only if $C = C_{e,f} *_{\mathcal{A}} C_{g,e}$ for some class of copulas $\mathcal{A} = \{A_t\}_{t\in[0,1]}$.

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