CHAPTER II
PRELIMINARIES AND LITERATURE REVIEWS

In this chapter, we give several definitions which will be used in our thesis and also literature reviews are cited here.

2.1 PRELIMINARIES

Definition 2.1 [3] A **simple graph** $G = (V, E)$ consists of V, a nonempty set of **vertices**, and E, a set of unordered pairs of distinct elements of V called **edges**.

Example 2.1 A simple graph G with $V(G) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ and $E(G) = \{u_1u_2, u_2u_3, u_3u_4, u_1u_4, u_1u_5, u_2u_6, u_3u_7, u_4u_8\}$ is shown in Figure 2.1.

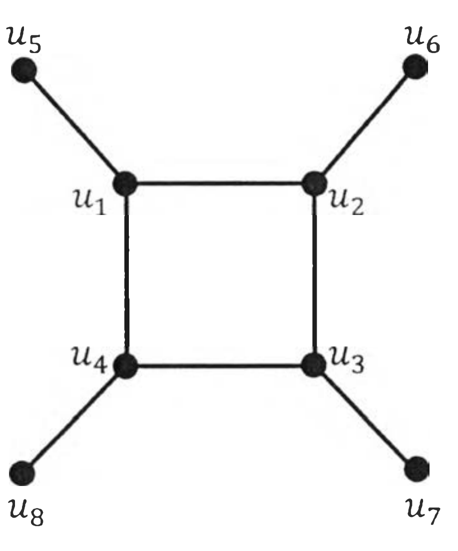


Figure 2.1 A simple graph G.

Definition 2.2 [6] **A path** P_n is a simple graph whose n-vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. The path P_4 is shown in Figure 2.2.

Figure 2.2 The path P_4.

Definition 2.3 [6] A **cycle** C_n is a simple graph with an equal number of n-vertices and n-edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the cycle.

In this thesis, we usually write a cycle C_n as $u_1u_2u_3\cdots u_nu_1$ and we name the vertices in the clockwise direction. The cycle $C_6: u_1u_2u_3\cdots u_6u_1$, is shown in Figure 2.5.

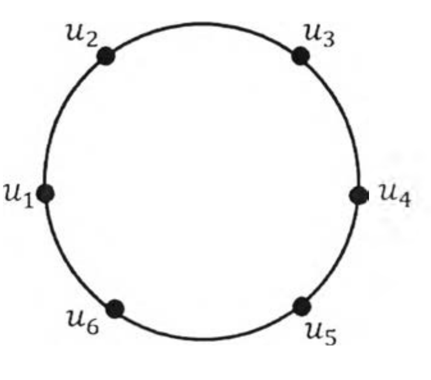


Figure 2.3 The cycle C_6.

Definition 2.4 [6] The **cartesian product** of G and H, denoted by $G\square H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$.

Example 2.2 The cartesian product of P_2 and P_3 is shown in Figure 2.4.

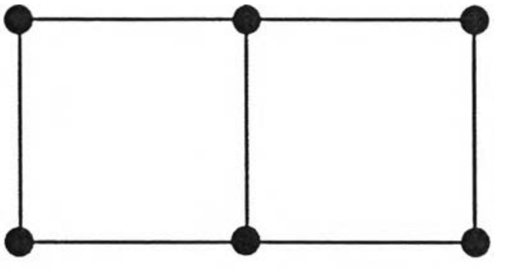


Figure 2.4 The Cartesian product $P_2 \square P_3$.

Definition 2.5 [3] A simple graph G is called **bipartite** if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2).

Example 2.3 A bipartite graph which each partite set contains 3 vertices is shown in Figure 2.5.

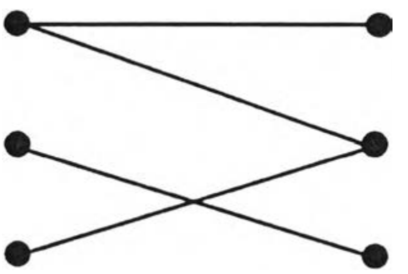


Figure 2.5 A bipartite graph.

Definition 2.6 [3] A **complete bipartite graph** $K_{m,n}$ is the graph that has its vertex set partitioned into two subsets of m and n vertices, respectively. There is an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.

Example 2.4 A complete bipartite graph $K_{2,3}$ is shown in Figure 2.6.

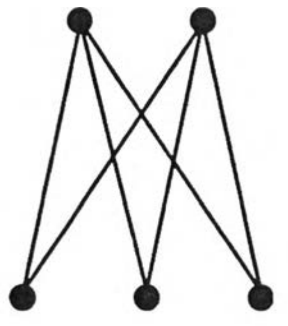


Figure 2.6 A complete bipartite $K_{2,3}$.

Definition 2.7 [6] A complete bipartite $K_{1,n}$ is called **star**, denoted by S_n. In this thesis, we usually let the first partite with one element be $\{u\}$ and the second partite with n elements be $\{u_1, u_2, u_3, \dots, u_n\}$.

Example 2.5 The star S_4 is shown in Figure 2.7.

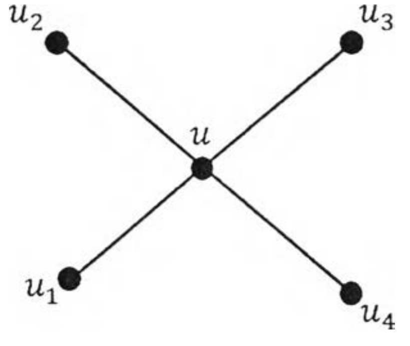
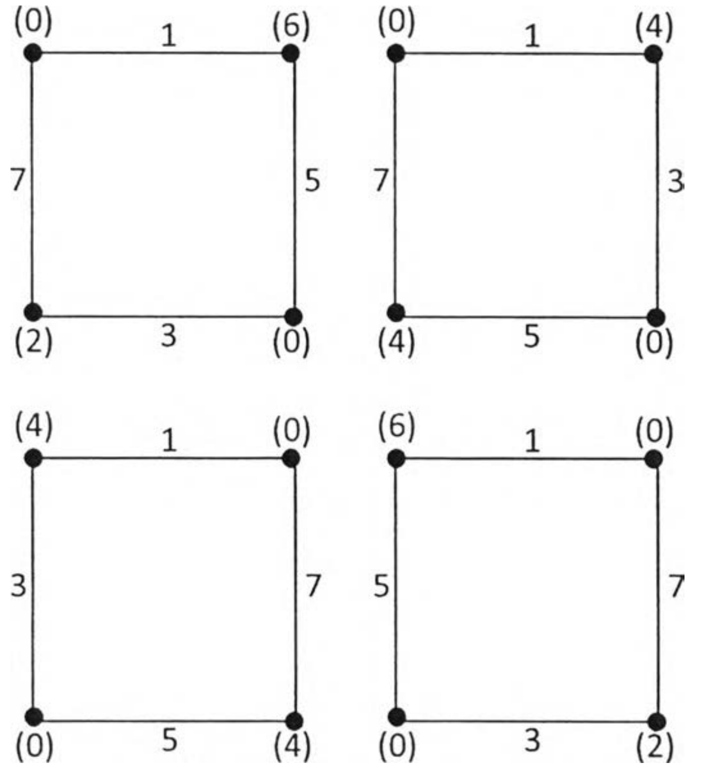


Figure 2.7 The star S_4.

2.2 LITERATURE REVIEWS

Definition 2.8 [5] Let G be a simple graph with q edges. An **edge-odd graceful labeling** of G is a bijection f from $E(G)$ to the set $\{1,3,5,\dots,2q-1\}$ so that the induced mapping f^+ from $V(G)$ to the set $\{0,1,2,\dots,2q-1\}$ given by $f^+(x) = \left(\sum_{xy\in E(G)} f(xy)\right) \pmod{2q}$. The edge labels and vertex labels are distinct. A graph that admitted an edge-odd graceful labeling is called **edge-odd graceful graph**.

One can see that some graphs are not edge-odd graceful. For example, we can show that C_4 is not an edge-odd graceful graph. Since we are considering the cycle C_4, we can construct only 6 one to one functions from $E(C_4)$ onto $f: E(G) \to \{1,3,5,7\}$. Hence, we use case-by-case analysis to illustrate that C_4 is not edge-odd graceful graph shown below. Note that the induced mapping f^+ shown in Figure 2.8 are calculated under modulo 8.



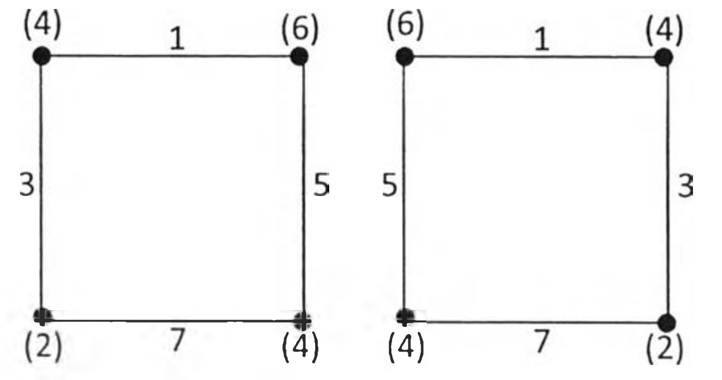


Figure 2.8 C_4 is not an edge-odd graceful graph.

Remark 2.1 Though out this thesis, the plain integers are the numbers labeled for each edge and the integers in parenthesis are the numbers labeled for the vertex induced from its adjacent edges.

In 2009, Solairaju and Chithra [5] showed edge-odd graceful labelings of graphs related to paths.

Definition 2.9 [5] **The Hoffman tree** P_n^+ is the graph obtained from a path P_n by attaching pendent edge at each vertex of the path.

Theorem 2.1 [5] *P_n^+ is an edge-odd graceful graph for all $n \geq 2$.*

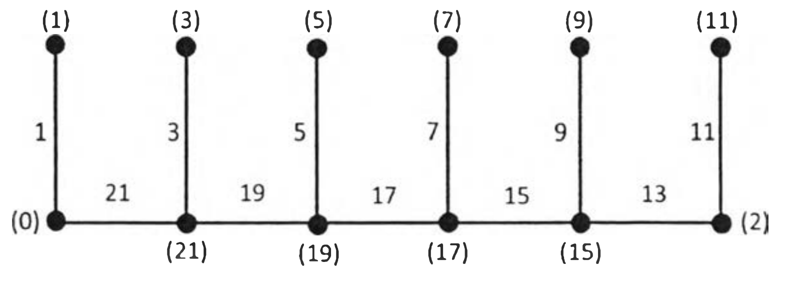


Figure 2.9 P_6^+ is an edge-odd graceful graph.

Definition 2.10 [5] K_2 with n pendent edges attached at each vertex is called a **bistar**, denoted by $B_{n,n}$.

Theorem 2.2 [5] *If n is odd, then the bistar $B_{n,n}$ is an edge-odd graceful graph.*

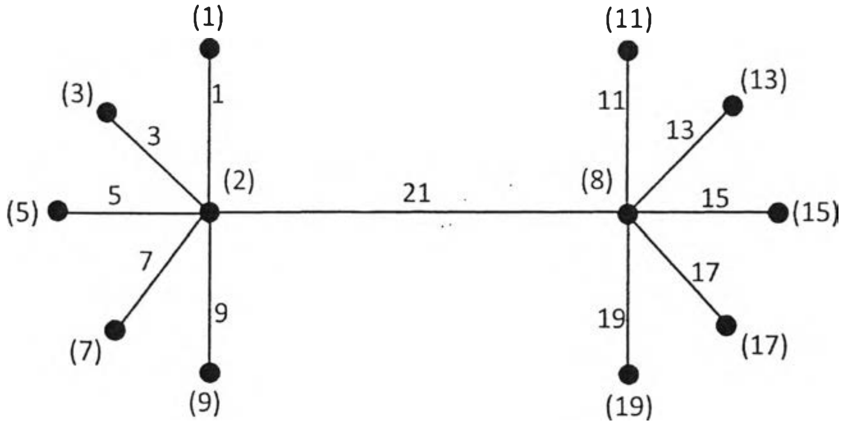


Figure 2.10 $B_{5,5}$ is an edge-odd graceful graph.

Definition 2.11 [5] A graph $\langle K_{1,n}{:}\,2\rangle$ is obtained from the n-bistars by subdividing the middle edge uv with a new vertex w.

Theorem 2.3 [5] *If n is odd, then the graph $\langle K_{1,n}{:}\,2\rangle$ is an edge-odd graceful graph.*

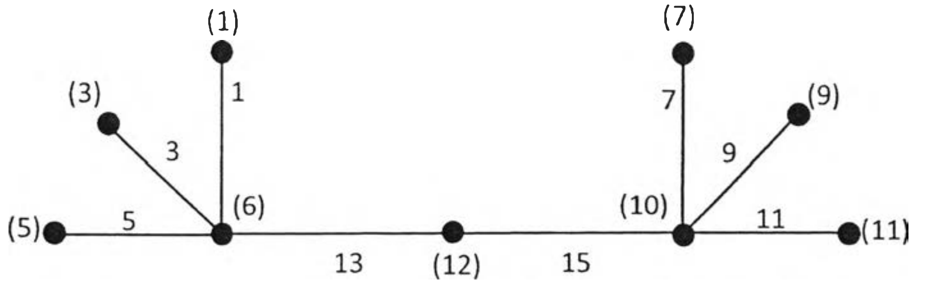


Figure 2.11 $\langle K_{1,3}{:}\,2\rangle$ is an edge-odd graceful graph.

Definition 2.12 [5] A **double star** $K_{1,n,n}$ is a tree obtained from the star S_n by adding a new pendent edge to each of the existing n pendent vertices.

Theorem 2.4 [5] *If n is even, then the graph $K_{1,n,n}$ is an edge-odd graceful graph for all $n \geq 2$.*

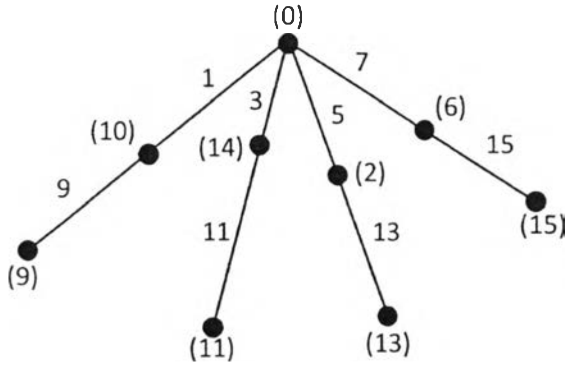


Figure 2.12 $K_{1,4,4}$ is an edge-odd graceful graph.

In 2013, Singhun [4] showed edge-odd graceful labeling of graphs related to cycles.

Definition 2.13 [4] Let $v_1, v_2, v_3, \dots, v_n$ be vertices on the cycle of $SF(n,m)$ and for each $j = 1,2,3,\dots,n$, the vertices $v_j^1, v_j^2, v_j^3, \dots, v_j^m$ be vertices joining v_j. That is, the vertex set of $SF(n,m)$ is the set

$$\left\{v_j \mid j \in \{1,2,3,\dots,n\}\right\} \cup \left\{v_j^i \mid j \in \{1,2,3,\dots,n\} \text{ and } i \in \{1,2,3,\dots,m\}\right\}.$$

The edge set is the set

$$\left\{v_j v_j^i \mid j \in \{1,2,3,\dots,n\} \text{ and } i \in \{1,2,3,\dots,m\}\right\} \cup$$

$$\left\{v_j v_{j+1} \mid j \in \{1,2,3,\dots,n-1\}\right\} \cup \{v_1 v_n\}.$$

Then, the number of edges is $n + nm$.

Theorem 2.5 [4] *The graph $SF(n,1)$ is an edge-odd graceful graph.*

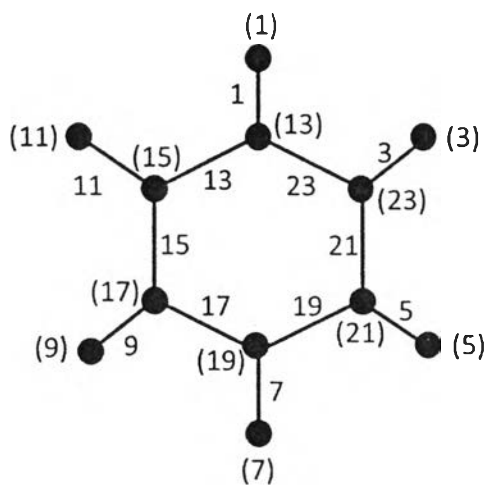


Figure 2.13 $SF(6,1)$ is an edge-odd graceful graph.

Theorem 2.6 [4] *The graph $SF(n,m)$ is an edge-odd graceful graph, when n is odd, m is even and $n|m$.*

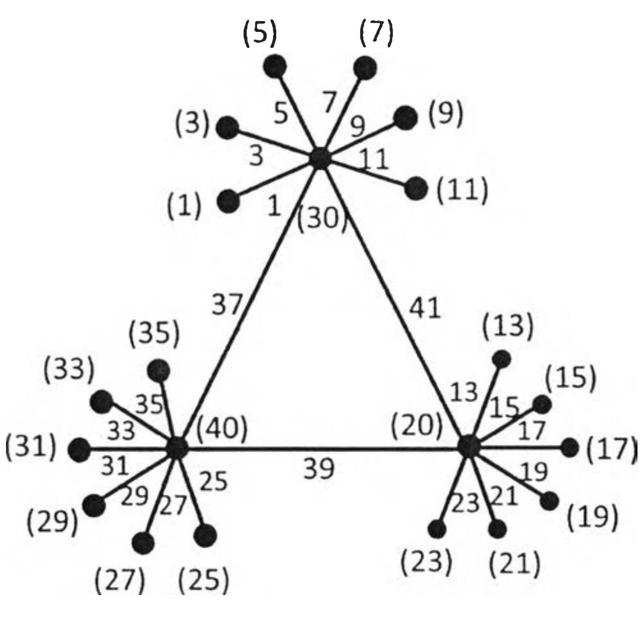


Figure 2.14 $SF(3,6)$ is an edge-odd graceful graph.

Another result from Singhun [4] is about the wheel graph. As we use our notation, the following are the definition of wheel graph and its result.

Definition 2.14 [4] A **wheel graph** W_n is a graph with $n+1$ vertices obtained by connecting a single vertex u to all vertices of a cycle $u_1u_2u_3 \cdots u_nu_1$. Then, the vertex set of W_n is the set $\{u, u_1, u_2, u_3, \ldots, u_n\}$ and the edge set of W_n is the set

$$\{uu_i \mid i \in \{1,2,3,\ldots,n\}\} \cup \{u_iu_{i+1} \mid i \in \{1,2,3,\ldots,n-1\}\} \cup \{u_1u_n\}.$$

Then, the number of edges is $2n$.

Theorem 2.7 [4] *The wheel graph W_n is edge-odd graceful when n is even.*

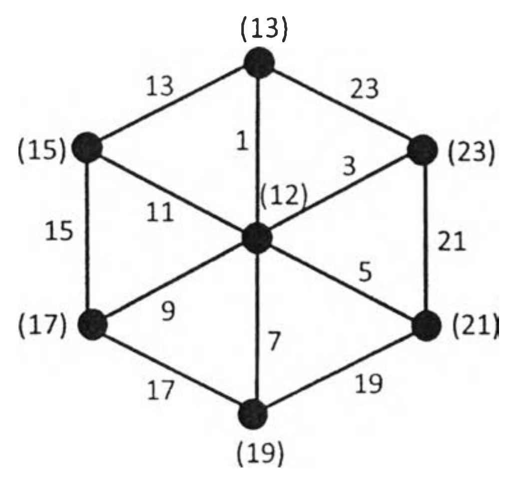


Figure 2.15 W_6 is an edge-odd graceful graph.