CHAPTER IV
SHAFT($n, 1$)

In this chapter, we show that a prism-like graph constructed from two copies of wheel graphs W_n joining at the middle, which is denoted by $\text{Shaft}(n, 1)$, is an edge-odd graceful graph whenever $n \geq 3$ is an odd integer.

Definition 4.1 Let $n \geq 3$ and W_n be a wheel graph with $V(W_n) = \{u_1, u_2, u_3, \ldots, u_n, u\}$ and W_n' be a copy of W_n with the corresponding $V(W_n') = \{u_1', u_2', u_3', \ldots, u_n', u'\}$. Define $\text{Shaft}(n, 1)$ by joining u of W_n to the corresponding vertex u' of W_n'. That is,

$$E(\text{Shaft}(n, 1)) = E(W_n) \cup E(W_n') \cup \{uu' \mid u \in W_n \text{ and } u' \in W_n'\}.$$

Example 4.1 From Definition 4.1, $\text{Shaft}(3,1)$ is shown in Figure 4.1.

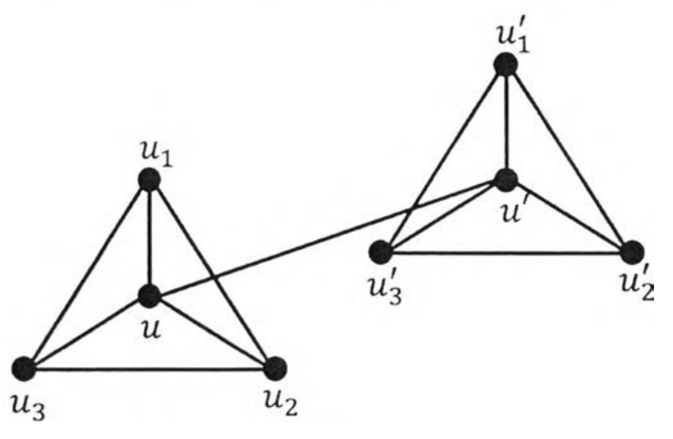

Figure 4.1 $\text{Shaft}(3,1)$.

Next, we give an algorithm for labeling the edges of $\text{Shaft}(n, 1)$, where $n \geq 3$ is an odd integer.

Algorithm 4.1

Let $n \geq 3$ be an odd integer and G denote Shaft$(n, 1)$. Then, $q = 4n + 1$. Define $f: E(G) \to \{1, 3, 5, \dots, 8n + 1\}$ by

1.1 $f(uu') = 8n + 1$;

1.2 $f(u_1u_n) = 2n - 1$;

1.3 $f(u_iu_{i+1}) = 2i - 1$, for $i \in \{1, 2, 3, \dots, n - 1\}$;

1.4 $f(u_1u) = 2n + 1$;

1.5 $f(u_iu) = 4n - 2i + 3$, for $i \in \{2, 3, 4, \dots, n\}$;

1.6 $f(u_1'u_n') = 6n - 1$;

1.7 $f(u_i'u_{i+1}') = 4n + 2i - 1$, for $i \in \{1, 2, 3, \dots, n - 1\}$;

1.8 $f(u_1'u') = 6n + 1$;

1.9 $f(u_i'u') = 8n - 2i + 3$, for $i \in \{2, 3, 4, \dots, n\}$.

Example 4.2 From Algorithm 4.1, we can label each edge of Shaft(5,1) as shown in Figure 4.2.

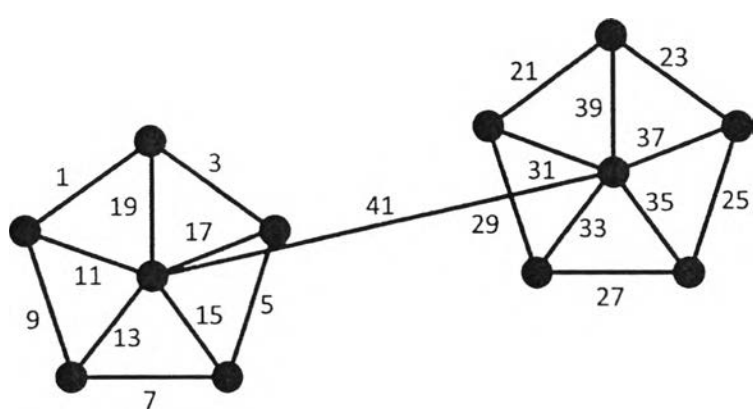

Figure 4.2 Edge-labeling for Shaft(5,1).

Next, we show that if $n \geq 3$ and n is an odd integer, then $\text{Shaft}(n,1)$ is an edge-odd graceful graph.

Theorem 4.1 *$Shaft(n,1)$ is an edge-odd graceful graph whenever $n \geq 3$ is odd.*

Proof. To prove that f in Algorithm 4.1 is a bijection from $E(G)$ to $\{1,3,5,\dots,8n+1\}$, we consider the followings. From Algorithm 4.1(1.1), we have

$$A = \{f(uu')\} = \{8n+1\}.$$

From Algorithm 4.1(1.2 and 1.3), we have

$$B = \{f(u_iu_{i+1}), f(u_1u_n) \mid i \in \{1,2,3,\dots,n-1\}\}$$
$$= \{1,3,5,\dots,2n-3,2n-1\}.$$

From Algorithm 4.1(1.4 and 1.5), we have

$$C = \{f(u_1u), f(u_iu) \mid i \in \{2,3,4,\dots,n\}\}$$
$$= \{2n+1,2n+3,2n+5,\dots,4n-1\}.$$

From Algorithm 4.1(1.6 and 1.7), we have

$$D = \{f(u_i'u_{i+1}'), f(u_1'u_n') \mid i \in \{1,2,3,\dots,n-1\}\}$$
$$= \{4n+1,4n+3,4n+5,\dots,6n-3,6n-1\}.$$

From Algorithm 4.1(1.8 and 1.9), we have

$$E = \{f(u_i'u'), f(u_1'u') \mid i \in \{2,3,4,\dots,n\}\}$$
$$= \{6n+1,6n+3,6n+5,6n+7,\dots,8n-1\}.$$

We can see clearly that A, B, C, D and E are disjoint and

$$f\left(E(\text{Shaft}(n,1))\right) = A \cup B \cup C \cup D \cup E = \{1,3,5,\dots,8n+1\}.$$

Next, we will show that the induced vertex-labels from edge-labels using Algorithm 4.1 are in $\{0, 1, 2, \dots, 8n+1\}$ and all distinct. From Algorithm 4.1, we have

$$
\begin{aligned}
f^+(u) &= \left(\textstyle\sum_{i=2}^{n} f(u_i u) + f(u_1 u) + f(uu')\right) (\text{mod } 8\text{n} + 2) \\
&= \left(\textstyle\sum_{i=2}^{n} (4n - 2i + 3) + (2n + 1) + (8n + 1)\right) (\text{mod } 8\text{n} + 2) \\
&= \left((4n^2 - (n^2 + n) + 3n - 4n + 2 - 3) + 2n + 1 + 8n + 1\right) \\
&\quad (\text{mod } 8\text{n} + 2) \\
&= (3n^2 - 2n - 1 + 2n + 1 + 8n + 1) (\text{mod } 8\text{n} + 2) \\
&= (3n^2 + 8n + 1) (\text{mod } 8\text{n} + 2) \\
&= (3n^2 - 1) (\text{mod } 8\text{n} + 2);
\end{aligned}
$$

$$
\begin{aligned}
f^+(u') &= \left(\textstyle\sum_{i=2}^{n} f(u_i' u') + f(u_1' u') + f(uu')\right) (\text{mod } 8\text{n} + 2) \\
&= \left(\textstyle\sum_{i=2}^{n} (8n - 2i + 3) + (6n + 1) + (8n + 1)\right) (\text{mod } 8\text{n} + 2) \\
&= \left((8n^2 - (n^2 + n) + 3n - 8n + 2 - 3) + 6n + 1 + 8n + 1\right) \\
&\quad (\text{mod } 8\text{n} + 2) \\
&= (7n^2 - 6n - 1 + 6n + 1 + 8n + 1) (\text{mod } 8\text{n} + 2) \\
&= (7n^2 + 8n + 1) (\text{mod } 8\text{n} + 2) \\
&= (7n^2 - 1) (\text{mod } 8n + 2);
\end{aligned}
$$

$$
\begin{aligned}
f^+(u_1) &= \left(f(u_1 u_n) + f(u_1 u_2) + f(u_1 u)\right) (\text{mod } 8n + 2) \\
&= \left((2n - 1) + 1 + (2n + 1)\right) (\text{mod } 8n + 2) \\
&= 4n + 1;
\end{aligned}
$$

$$
\begin{aligned}
f^+(u_i) &= \left(f(u_{i-1} u_i) + f(u_i u_{i+1}) + f(u_i u)\right) (\text{mod } 8n + 2) \\
&= \left((2i - 3) + (2i - 1) + (4n - 2i + 3)\right) (\text{mod } 8n + 2)
\end{aligned}
$$

$$= 4n + 2i - 1, \text{ for } i \in \{2, 3, 4, \dots, n-1\};$$

$$f^+(u_n) = \big(f(u_1u_n) + f(u_{n-1}u_n) + f(u_nu)\big) \pmod{8n+2}$$
$$= \big((2n-1) + (2n-3) + (2n+3)\big) \pmod{8n+2}$$
$$= 6n - 1;$$

$$f^+(u'_1) = \big(f(u'_1u'_n) + f(u'_1u'_2) + f(u'_1u')\big) \pmod{8n+2}$$
$$= \big((6n-1) + (4n+1) + (6n+1)\big) \pmod{8n+2}$$
$$= (16n+1) \pmod{8n+2}$$
$$= 8n - 1;$$

$$f^+(u'_2) = \big(f(u'_1u'_2) + f(u'_2u'_3) + f(u'_2u')\big) \pmod{8n+2}$$
$$= \big((4n+1) + (4n+3) + (8n-1)\big) \pmod{8n+2}$$
$$= (16n+3) \pmod{8n+2}$$
$$= 8n + 1;$$

$$f^+(u'_i) = \big(f(u'_{i-1}u'_i) + f(u'_iu'_{i+1}) + f(u'_iu')\big) \pmod{8n+2}$$
$$= \big((4n+2i-3) + (4n+2i-1) + (8n-2i+3)\big)$$
$$\pmod{8n+2}$$
$$= (16n+2i-1) \pmod{8n+2}$$
$$= 2i - 5, \text{ for } i \in \{3, 4, 5, \dots, n-1\};$$

$$f^+(u'_n) = \big(f(u'_1u'_n) + f(u'_{n-1}u'_n) + f(u'_nu')\big) \pmod{8n+2}$$
$$= \big((6n-1) + (6n-3) + (6n+3)\big) \pmod{8n+2}$$
$$= (18n-1) \pmod{8n+2}$$

$$= 2n - 5.$$

We can see that

$\{f^+(u_i) \mid i \in \{1, 2, 3, \dots, n\}\}$

$$= \{4n + 1\} \cup \{4n + 3, 4n + 5, 4n + 7, \dots, 6n - 3\} \cup \{6n - 1\}$$

$$= \{4n + 1, 4n + 3, 4n + 5, \dots, 6n - 1\}$$

and

$\{f^+(u_i') \mid i \in \{1, 2, 3, \dots, n\}\}$

$$= \{8n - 1\} \cup \{8n + 1\} \cup \{1, 3, 5, \dots, 2n - 7\} \cup \{2n - 5\}$$

$$= \{1, 3, 5, \dots, 2n - 5\} \cup \{8n - 1\} \cup \{8n + 1\}.$$

It is clear that if n is an odd integer and $n \geq 3$, these two sets are disjoint and both are subsets of $\{0, 1, 2, \dots, 8n + 1\}$. We can see that the vertex-labeling of vertices $\{u_1, u_2, u_3, \dots, u_n\}$ and $\{u_1', u_2', u_3', \dots, u_n'\}$ are odd integers. However, the vertex-labeling of vertices $\{u, u'\}$ are even integers. Thus they are disjoint from the vertex-labeling of $\{u_1, u_2, u_3, \dots, u_n\}$ and $\{u_1', u_2', u_3', \dots, u_n'\}$. Finally, we need to show that $f^+(u)$ and $f^+(u')$ are distinct under modulo $8n + 2$. Suppose in a contrary that

$$f^+(u) \equiv f^+(u') \pmod{8n + 2}.$$

Then, $3n^2 - 1 \equiv 7n^2 - 1 \pmod{8n + 2}$. That is, $2n^2 \equiv 0 \pmod{4n + 1}$. Thus, there exists an integer k such that $2n^2 = (4n + 1)k$. Since $2n^2$ is even and $4n + 1$ is odd, k is even and $2n^2 - 4nk - k = 0$.

By the quadratic formula, we have

$$n = \frac{4k \pm \sqrt{16k^2 + 8k}}{4} = k \pm \frac{\sqrt{4k^2 + 2k}}{2}$$

Since k is even, there exists an integer l such that $k = 2l$. Then,

$$n = 2l \pm \frac{\sqrt{16l^2 + 4l}}{2}$$

Since n is an odd integer and $n \geq 3$, $16l^2 + 4l = 4l(4l + 1)$ must be square. This is a contradiction since g.c.d.$(4l, 4l + 1) = 1$. Hence, $f^+(u)$ and $f^+(u')$ are distinct.

Thus, the function f defined in Algorithm 4.1 is an edge-odd graceful labeling and $\text{Shaft}(n, 1)$ is an edge-odd graceful graph for n is odd and $n \geq 3$. ■

Example 4.3 From the edge-labeling in Example 4.2, the induced vertex-labeling of $\text{Shaft}(5, 1)$ is shown in Figure 4.3.

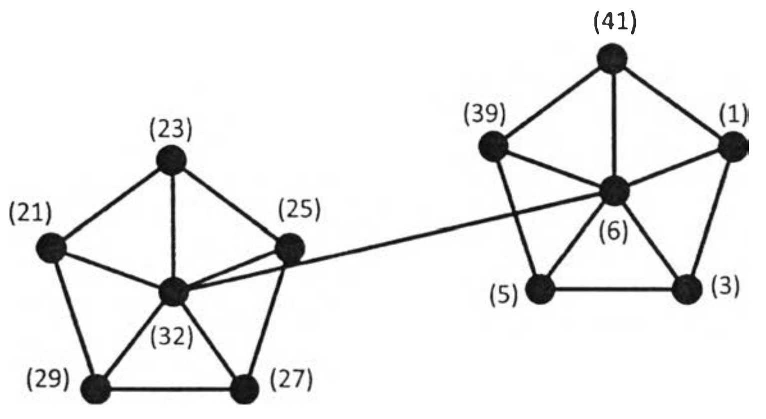

Figure 4.3 The vertex-labeling is induced from the edge-labeling in Example 4.2.

Conjecture We may extend the investigation by trying to find an algorithm for edge-labeling that makes $\text{Shaft}(n, 1)$ to be an edge-odd graceful graph for even integer n with $n \geq 4$. However, we still cannot construct a general algorithm for labeling such graph. Figures 4.4 and 4.5 show some examples of labeling that make $\text{Shaft}(4, 1)$ and $\text{Shaft}(6, 1)$ become edge-odd graceful graphs. Here, we make a conjecture that $\text{Shaft}(n, 1)$ is an edge-odd graceful graph for every integer $n \geq 3$.

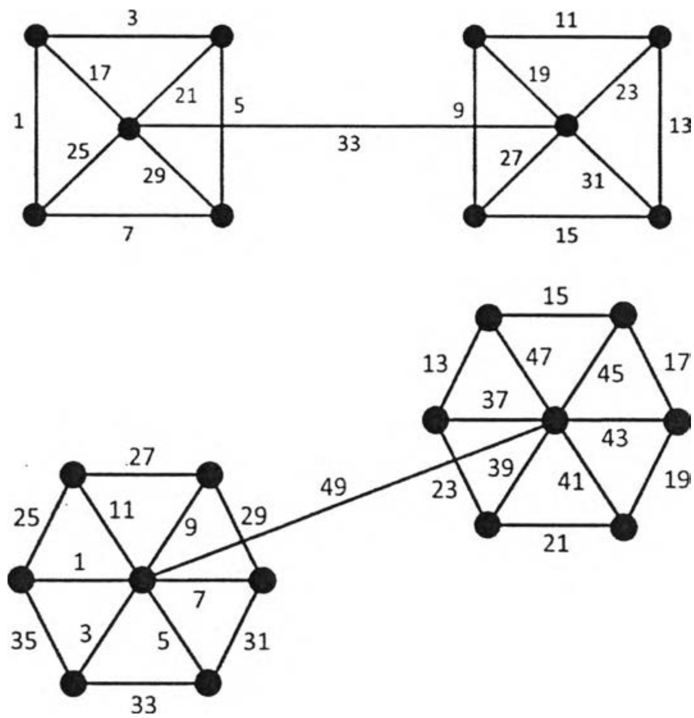

Figure 4.4 Edge-labelings for Shaft(4,1) and Shaft(6,1).

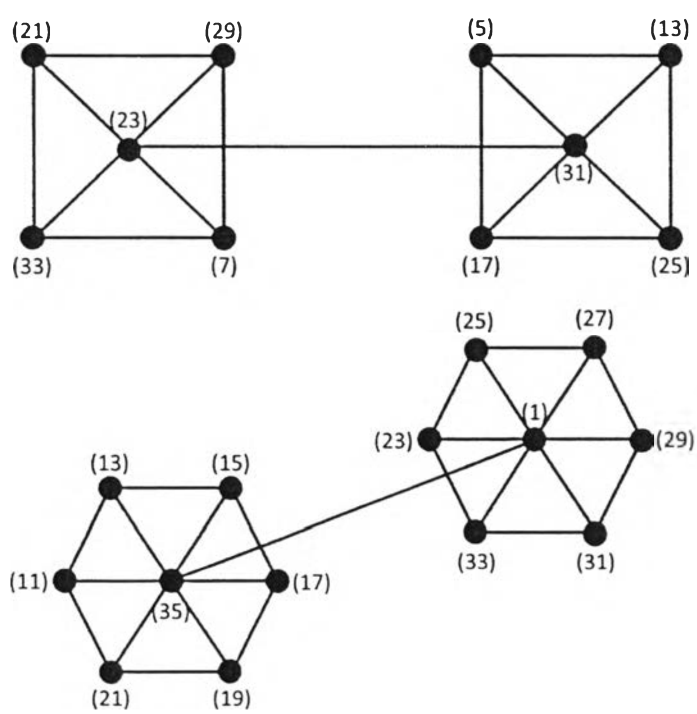

Figure 4.5 The vertex-labelings are induced from the edge-labelings in Figure 4.4.