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W-DISTANCES ON A METRIC SPACE

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ในงานวิจัยนี้เราได้สืบสวนสมบัติต่างๆของระยะทางดับเบิลยูบนปริภูมิอิงระยะทางและได้ความสัมพันธ์ระหว่างระยะทางดับเบิลยู ระยะทาง ฟังก์ชันคงสภาพระยะทาง และสมบัติทางทอพอโลยีและได้สร้างแนวคิดของระยะทางดับเบิลยูเชิงคอชีและแถลงสมบัติบางประการของระยะทางดับเบิลยูเชิงคอชี

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In this thesis, we investigate properties of w-distances on metric spaces. We give a relationship among w-distances, metrics, metric-preserving functions, and topological properties. We introduce the notion of Cauchy w-distances and state some of their properties.

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CONTENTS

CHAPTER I

INTRODUCTION

A w-distance on a metric space (X, d) is a function $p : X \times X \to [0, \infty)$ satisfying the following properties :

(i) $p(x, z) \leq p(x, y) + p(y, z)$ for any x, y, $z \in X$;

(ii) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;

(iii) for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any x, y, $z \in X$, $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

In [5], Takahashi gives some examples and properties of w-distances. This concept is used as a tool in nonlinear functional analysis and fixed-point theory. For example, in [5], the concept of w-distance was used to generalize Caristi's fixed point theorem. Also, it was shown that if a mapping T from a complete metric space X into itself is p-contractive, then T has a unique fixed point $x_0 \in X$. This is a generalization of the Banach contraction principle [5]. However, the properties of w-distances were not so clearly study. Most of the study are concentrated on applying the concept of w-distances in nonlinear functional analysis and fixed point theory.

Our purpose is to study the properties of w-distances. We study some relationship among w-distances, metrics, metric-preserving functions, and topological properties. In particular, we give the notion of Cauchy w-distances and prove their properties.

This thesis is organized as follows : In section 2.1, the definition and some examples of w-distances are given. In section 2.2, we show that every metric equivalent to d is a w-distance on (X, d) if (X, d) is compact. An example is given to show that this is not true in a general metric space. In section 2.3, it is proved that $f \circ d$ is a w-distance on (X, d) if f is lower semicontinuous metric preserving.

In section 3.1, the notion of p-Cauchy sequences, Cauchy w-distances, and simple w-distances are introduced, and also some examples are given. In section 3.2, a relationship between Cauchy w-distances and metric-preserving functions is investigated. Theorem 3.23 shows that $f \circ d$ is a Cauchy w-distance on (X, d) if f is a strongly metric-preserving function or (X, d) is a uniformly discrete metric space, and $f \circ d$ is a w-distance which is not Cauchy if f is not strong and (X, d) is not discrete.

In section 4.1, it is shown that Cauchy w-distances are simple but the converse is not true. Furthermore, Theorem 4.8 shows that Cauchy w-distances are continuous. An example of discontinuous w-distance and continuous w-distance which is not Cauchy is provided. Furthermore, Theorem 4.12 shows that $f \circ d$ is uniformly continuous Cauchy w-distance on (X, d) if f is strongly metric-preserving. In section 4.2, the notion of w-distance topology is introduced and it is found that the w-distance topology coincides with the metric topology if the w-distance is Cauchy. Finally, section 4.3, we provides a characterization of Cauchy w-distances and its consequences.

CHAPTER II

W-DISTANCES ON A METRIC SPACE

In this chapter, the precise definition of w-distance is given and some related terminologies are stated. Also examples of w-distances which will be referred throughout our work are given.

2.1 Definitions and examples

First, let us recall the definition of metric spaces, topological spaces, and lower semicontinuous functions.

Definition 2.1. *A* **metric** *on a nonempty set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following conditions, for all x, y, $z \in X$,*

(i) $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

The real number $d(x, y)$ is called **the distance** *between x and y, and (X, d) is called a* **metric space**.

Definition 2.2. *Let $X \neq \varnothing$. $\tau \subseteq \mathcal{P}(X)$ is called a* **topology** *on X if it satisfies the following conditions :*

(i) $X \in \tau$ *and* $\varnothing \in \tau$.

(ii) If $\{G_\alpha \mid \alpha \in \Lambda\} \subseteq \tau$, *then* $\displaystyle\bigcup_{\alpha \in \Lambda} G_\alpha \in \tau$.

(iii) If $G_1, G_2 \in \tau$, then $G_1 \cap G_2 \in \tau$.

The sets in τ are called the **open** *sets in X and (X, τ) is called a* **topological space**.

Definition 2.3. *If X is a set, a* **basis** *for a topology on X is a collection $\mathcal{B}$ of subsets of X (called* **basis element***) such that*

(i) for each $x \in X$, there is at least one basis element B containing x

(ii) if x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

Definition 2.4. *Let X be a topological space and $f : X \to (-\infty, \infty]$. Then f is said to be* **lower semicontinuous** *on X if for every $a \in \mathbb{R}$, the set $f^{-1}(-\infty, a] = \{x \in X \mid f(x) \leq a\}$ is closed in X.*

Elementary properties of metric spaces, topological spaces, and lower semicontinuous functions can be found in standard texts of topology and analysis. In addition, some properties of lower semicontinuous functions are given in [5]. Now, the precise definition of w-distance on a metric space is given.

Definition 2.5 ([5], p.40). *Let (X, d) be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called a* **w-distance** *on X if the following are satisfied :*

(i) $p(x, z) \leq p(x, y) + p(y, z)$ for any x, y, $z \in X$;

(ii) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;

(iii) for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any x, y, $z \in X$, $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

The notion of w-distance was first introduced by Kada, Suzuki, and Takahashi [4].

The following are examples of w-distances which are provided in [5].

Examples 2.6. Let X be a metric space with metric d. Then $p = d$ is a w-distance on X.

Examples 2.7. Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ defined by $p(x, y) = c$ for every $x,\ y \in X$, where c is a nonnegative number, is a w-distance on X.

Examples 2.8. Let X be a normed linear space with norm $\|\cdot\|$. Then the function $p : X \times X \to [0, \infty)$ defined by

$$p(x, y) \;=\; \|x\| + \|y\| \quad \text{for every } x,\ y \in X$$

is a w-distance on X.

Examples 2.9. Let X be a metric space and T a continuous mapping from X into itself. Then a function $p : X \times X \to [0, \infty)$ defined by

$$p(x, y) \;=\; \max\{d(Tx, y),\, d(Tx, Ty)\} \quad \text{for every } x,\ y \in X$$

is a w-distance on X.

Examples 2.10. Let F be a bounded and closed subset of a metric space X. Assume that F contains at least two points and c is a constant with $c \geq \delta(F)$, where $\delta(F)$ is the diameter of F. Then a function $p : X \times X \to [0, \infty)$ defined by

$$p(x, y) \;=\; \begin{cases} d(x, y) & \text{if } x,\ y \in F, \\ c & \text{if } x \notin F \text{ or } y \notin F \end{cases}$$

is a w-distance on X.

Remark 2.11. The w-distance p defined in Example 2.10 is lower semicontinuous but may not be continuous.

There are some examples of w-distances provided as exercises in [5]. For the sake of completeness, the verification is given here. To do this, the following theorem will be helpful.

Theorem 2.12. *Let X be a topological space, f and g lower semicontinuous functions of X into $(-\infty, \infty]$, and α a nonnegative number. Then* $\sup\{f, g\}$, *$f + g$, and αf are lower semicontinuous.*

Examples 2.13. Let X be a normed linear space with norm $\|\cdot\|$. Then $p : X \times X \to [0, \infty)$ defined by

$$p(x, y) \;=\; \|y\| \quad \text{for all } x,\, y \in X,$$

is a w-distance on X.

Proof. Let x, y, $z \in X$. Then we have

$$p(x, z) \;=\; \|z\| \;\leq\; \|y\| + \|z\| \;=\; p(x, y) + p(y, z).$$

It is obvious that p satisfies condition (ii) of w-distance. Let $\varepsilon > 0$ be arbitrary and put $\delta = \frac{\varepsilon}{2}$. Then if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$, we have

$$d(x, y) \;=\; \|x - y\| \;\leq\; \|x\| + \|y\| \;=\; p(z, x) + p(z, y) \;\leq\; \delta + \delta \;=\; \varepsilon.$$

This implies that p satisfies condition (iii) of w-distance. □

Examples 2.14. Let (X, d) be a metric space, p_1 and p_2 w-distances on X. Then $\max\{p_1,\, p_2\}$, $p_1 + p_2$, and $\alpha p_1 (\alpha \geq 0)$ are w-distances on X.

Proof. First, we will prove that $\max\{p_1,\, p_2\}$ is a w-distance.
Let $p(x, y) = \max\{p_1(x, y),\, p_2(x, y)\}$ for every x and y in X. Let x, y, $z \in X$.

Let $i \in \{1, 2\}$.

$$\begin{aligned} p_i(x,y) &\leq p_i(x,z) + p_i(z,y) \\ &\leq \max\{p_1(x,z),\, p_2(x,z)\} + \max\{p_1(z,y),\, p_2(z,y)\} \\ &= p(x,z) + p(z,y). \end{aligned}$$

Then $p(x,y) = \max\{p_1(x,y),\, p_2(x,y)\} \leq p(x,z) + p(z,y)$, and condition (i) holds. Since p_1 and p_2 are lower semicontinous, we obtain from Theorem 2.12 that $\max\{p_1,\, p_2\}$ is lower semicontinuous. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $p_1(z,x) \leq \delta$ and $p_1(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$. If $x,\, y,\, z \in X$ are such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$, then

$$\begin{aligned} p_1(z,x) &\leq \max\{p_1(z,x),\, p_2(z,x)\} = p(z,x) \leq \delta, \\ p_1(z,y) &\leq \max\{p_1(z,y),\, p_2(z,y)\} = p(z,y) \leq \delta. \end{aligned}$$

Therefore $d(x,y) \leq \varepsilon$. Hence $\max\{p_1,\, p_2\}$ is a w-distance.

Next, let $q(x,y) = p_1(x,y) + p_2(x,y)$ for every x and y in X. Let $x,\, y,\, z \in X$. Then

$$\begin{aligned} q(x,z) &= p_1(x,z) + p_2(x,z) \\ &\leq p_1(x,y) + p_1(y,z) + p_2(x,y) + p_2(y,z) \\ &= q(x,y) + q(y,z). \end{aligned}$$

Since p_1 and p_2 are lower semicontinuous, we obtain from Theorem 2.12 that $p_1 + p_2$ is lower semicontinuous. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $p_1(z,x) \leq \delta$ and $p_1(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$. Let $x,\, y,\, z \in X$ be such that $q(z,x) \leq \delta$ and $q(z,y) \leq \delta$. Then

$$p_1(z,x) \;\leq\; p_1(z,x) + p_2(z,x) \;=\; q(z,x) \;\leq\; \delta$$
$$p_1(z,y) \;\leq\; p_1(z,y) + p_2(z,y) \;=\; q(z,y) \;\leq\; \delta.$$

This implies that $d(x,y) \leq \varepsilon$. Hence $p_1 + p_2$ is a w-distance.

Next, let $r(x,y) = \alpha p_1(x,y)$ for every x and y in X. Let $x,\ y,\ z \in X$. Then

$$\begin{aligned} r(x,z) \;=\; \alpha p_1(x,z) \;&\leq\; \alpha(p_1(x,y) + p_1(y,z)) \\ &=\; \alpha p_1(x,y) + \alpha p_1(y,z) \\ &= r(x,y) + r(y,z). \end{aligned}$$

Since p_1 is lower semicontinuous, Theorem 2.12 implies that αp_1 is lower semicontinuous. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $p_1(z,\, x) \leq \delta$ and $p_1(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$. Choose $\delta' = \alpha\delta$. Then $\delta' > 0$. Let $x,\ y,\ z \in X$ be such that $r(z,x) \leq \delta'$ and $r(z,y) \leq \delta'$. Then $\alpha p_1(z,x) \;\leq\; \alpha\delta$ and $\alpha p_1(z,y) \;\leq\; \alpha\delta$. Therefore

$$p_1(z,x) \;\leq\; \delta \quad \text{and} \quad p_1(z,y) \;\leq\; \delta.$$

Thus $d(x,y) \leq \varepsilon$. Hence αp_1 is a w-distance. This completes the proof. $\square$

Examples 2.15. Let X be a metric space, let p be a w-distance on X and let f be a function from X into $[0,\infty)$. Then a function $g : X \times X \to [0,\infty)$ defined by

$$g(x,y) = \max\{f(x),\, p(x,y)\} \quad \text{for all } x,\, y \in X,$$

is a w-distance on X.

Proof. Let $x,\ y,\ z \in X$. Then if $p(x,z) \geq f(x)$, we have

$$\begin{aligned} g(x,z) \;=\; p(x,z) \;&\leq\; p(x,y) + p(y,z) \\ &\leq\; \max\{f(x),\, p(x,y)\} + \max\{f(y),\, p(y,z)\} \\ &=\; g(x,y) + g(y,z). \end{aligned}$$

In the case $f(x) > p(x,z)$, we have

$$\begin{aligned} g(x,z) \;=\; f(x) \;&\leq\; f(x) + f(y) \\ &\leq\; \max\{f(x),\, p(x,y)\} + \max\{f(y),\, p(y,z)\} \\ &=\; g(x,y) + g(y,z). \end{aligned}$$

This proves condition (i). Let $x \in X$. Then $g(x,\cdot) = \max\{f(x),\, p(x,\cdot)\}$. Since $f(x)$ is a constant function and $p(x,\cdot)$ is lower semicontinous, we obtain from Theorem 2.12 that g is lower semicontinuous. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$. Let $x,\ y,\ z \in X$ be such that $g(z,x) \leq \delta$ and $g(z,y) \leq \delta$. Then

$$\begin{aligned} p(z,x) \;&\leq\; \max\{f(z),\, p(z,x)\} \;=\; g(z,x) \;\leq\; \delta, \\ p(z,y) \;&\leq\; \max\{f(z),\, p(z,y)\} \;=\; p(z,y) \;\leq\; \delta. \end{aligned}$$

This implies that $d(x,y) \leq \varepsilon$. Hence g is a w-distance. $\square$

Examples 2.16. Let X be a metric space with metric d, let p be a w-distance on X and let f be a function from X into $[0,\infty)$. Then a function q from $X \times X$ into $[0,\infty)$ given by

$$q(x,y) = f(x) + p(x,y) \quad \text{for each } (x,y) \in X \times X$$

is also a w-distance.

Proof. Let $x,\, y,\, z \in X$.

$$\begin{aligned} q(x,z) &= f(x) + p(x,z) \\ &\le f(x) + p(x,y) + f(y) + p(y,z) \\ &= q(x,y) + q(y,z). \end{aligned}$$

This proves condition (i). Let $x \in X$. Then $q(x,\cdot) = f(x) + p(x,\cdot)$. Since $f(x)$ is a constant function and $p(x,\cdot)$ is lower semicontinous, by Theorem 2.12 q is lower semicontinuous. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $p(z,x) \le \delta$ and $p(z,y) \le \delta$ imply $d(x,y) \le \varepsilon$. Let $x,\, y,\, z \in X$ be such that $q(z,x) \le \delta$ and $q(z,y) \le \delta$. Then

$$\begin{aligned} p(z,x) &\le f(z) + p(z,x) = q(z,x) \le \delta, \\ p(z,y) &\le f(z) + p(z,y) = q(z,y) \le \delta. \end{aligned}$$

Therefore $d(x,y) \le \varepsilon$. Hence q is a w-distance. □

From Example 2.14, it is noted that $p_1 + p_2$, $\alpha p_1 (\alpha \ge 0)$, $\max\{p_1,\, p_2\}$ are w-distances on $(X,\, d)$ whenever p_1 and p_2 are. However it is not true in general that if p_1 and p_2 are w-distances, then $p_1 p_2$ and $\min\{p_1,\, p_2\}$ are w-distances on (X, d). This is shown in the next example.

Examples 2.17. Let $X = \mathbb{R}$. Let p_1 be the usual metric on $\mathbb{R}$. Then p_1 is a w-distance. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = x^2$. Then T is continuous. Then as shown in Example 2.9, the function $p_2 : X \times X \to [0, \infty)$ given by

$$\begin{aligned} p_2(x,y) &= \max\{|Tx - y|,\, |Tx - Ty|\} \\ &= \max\{|x^2 - y|,\, |x^2 - y^2|\} \end{aligned}$$

is a w-distance. Let $p = \min\{p_1,\, p_2\}$. It is observed that p is not a w-distance. As it happens that

$$\begin{aligned} p(-1,1) &= \min\{p_1(-1,1),\, p_2(-1,1)\} = \min\{2,\, 0\} = 0 \\ p(1,2) &= \min\{p_1(1,2),\, p_2(1,2)\} = \min\{1,\, 3\} = 1 \\ p(-1,2) &= \min\{p_1(-1,2),\, p_2(-1,2)\} = \min\{3,\, 3\} = 3. \end{aligned}$$

Then $p(-1,2) \nleq p(-1,1) + p(1,2)$, and so p is not a w-distance.

Next, we observe that p_1^2 is not a w-distance. Since $p_1^2(1,3) = 4$, $p_1^2(1,2) = 1$, $p_1^2(2,3) = 1$, we have $p_1^2(1,3) \nleq p_1^2(1,2) + p_1^2(2,3)$. Hence p_1^2 is not a w-distance.

2.2 Equivalent metrics and w-distances

Let (X,d) be a metric space. Then the topology generated by the collection

$$\{B_d(x,\varepsilon) \mid x \in X,\, \varepsilon > 0\}$$

of all open balls is called the metric topology (induced by d), and is denoted by τ_d. We also say that d generates the topology τ_d. If d_1 and d_2 are metrics on X which generate the same topology on X, then it is said that d_1 and d_2 are **equivalent**.

In Example 2.6, we see that the metric d is a w-distance on (X,d). An interesting question arises from this example : Is it true that if p is a metric on X equivalent to d, then is p a w-distance on (X,d)?

Example 2.19 shows that the answer is negative. To clarify the assertion the following lemma is needed.

Lemma 2.18 ([2], p.122). *Let d and d' be two metrics on the set X, τ and τ' be the topologies induced by d and d', respectively. Then τ' is finer that τ if and only if for each x in X and each $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$B_{d'}(x,\delta) \subseteq B_d(x,\varepsilon).$$

Examples 2.19. Let $X = (0,1)$. Let d, $d' : X \times X \to [0,\infty)$ be defined by

$$\begin{aligned} d(x,y) &= \left|\frac{1}{x} - \frac{1}{y}\right|, \quad \text{and} \\ d'(x,y) &= |x-y| \quad \text{for every } x,\, y \in X. \end{aligned}$$

Then it is obvious that d and d' are metrics on X. Also the following statements hold.

(i) d' and d are equivalent.

(ii) d' is not a w-distance on (X,d).

Proof. We will prove (i). Let

$$\begin{aligned} \mathcal{B} &= \{B_d(x,\varepsilon) \mid x \in (0,1),\, 0 < \varepsilon \leq 1\} \quad \text{and} \\ \mathcal{B}' &= \{B_{d'}(x,\varepsilon) \mid x \in (0,1),\, \varepsilon > 0\}. \end{aligned}$$

Then $\mathcal{B}$ and $\mathcal{B}'$ are bases for the topologies generated by d and d', respectively. Let $x \in (0,1)$, and $0 < \varepsilon \leq 1$. We will identify the set $B_d(x,\varepsilon)$. Let $y \in (0,1)$.

$$\begin{aligned} \left|\frac{1}{y} - \frac{1}{x}\right| < \varepsilon &\leftrightarrow \left|\frac{x-y}{xy}\right| < \varepsilon \\ &\leftrightarrow \frac{|x-y|}{xy} < \varepsilon \\ &\leftrightarrow |x-y| < \varepsilon xy \\ &\leftrightarrow -\varepsilon xy < x-y < \varepsilon xy \\ &\leftrightarrow x-y > -\varepsilon xy \wedge x-y < \varepsilon xy \\ &\leftrightarrow x > y - \varepsilon xy \wedge x < y + \varepsilon xy \\ &\leftrightarrow x > y(1-\varepsilon x) \wedge x < y(1+\varepsilon x) \\ &\leftrightarrow y < \frac{x}{1-\varepsilon x} \wedge y > \frac{x}{1+\varepsilon x} \\ &\leftrightarrow \frac{x}{1+\varepsilon x} < y < \frac{x}{1-\varepsilon x}. \end{aligned}$$

This shows that, for each $x \in (0,1)$, $0 < \varepsilon \leq 1$, we have

$$B_d(x,\varepsilon) \;=\; \left(\frac{x}{1+\varepsilon x},\, \frac{x}{1-\varepsilon x}\right) \cap (0,1). \tag{2.1}$$

For each $x \in (0,1)$ and $\varepsilon > 0$, it is obvious that

$$B_{d'}(x,\varepsilon) \;=\; (x-\varepsilon,\, x+\varepsilon) \cap (0,1). \tag{2.2}$$

From (2.1) and (2.2), it is easy to see that if $B_d(x,\varepsilon) \in \mathcal{B}$ is given, choose $\delta = \min\left\{x - \frac{x}{1+\varepsilon x},\, x,\, \frac{x}{1-\varepsilon x} - x,\, 1-x\right\}$. So $\delta > 0$ and $B_{d'}(x,\delta) \subseteq B_d(x,\varepsilon)$. By Lemma 2.18, we obtain that $\mathcal{B} \subseteq \mathcal{B}'$. Next, let $B_{d'}(x,\varepsilon) \in \mathcal{B}'$ be given. Let $\varepsilon' = \frac{1}{2}\min\{x,\, 1-x,\, \varepsilon\}$. Then $\varepsilon' > 0$ and $B_{d'}(x,\varepsilon') \subseteq B_{d'}(x,\varepsilon)$. Let $\delta = \frac{\varepsilon'}{x}$. Then $0 < \delta \leq 1$. Therefore

$$\begin{aligned}
B_d(x,\delta) \;&=\; \left(\frac{x}{1+\delta x},\, \frac{x}{1-\delta x}\right) \cap (0,1) \;=\; \left(\frac{x}{1+\varepsilon'},\, \frac{x}{1-\varepsilon'}\right) \cap (0,1).\\
\frac{x}{1+\varepsilon'} - x \;&=\; \frac{x - x - \varepsilon' x}{1+\varepsilon'} \;=\; -\varepsilon'\left(\frac{x}{1+\varepsilon'}\right) \;>\; -\varepsilon', \quad \text{and}\\
\frac{x}{1-\varepsilon'} - x \;&=\; \frac{x - x + \varepsilon' x}{1-\varepsilon'} \;=\; \varepsilon'\left(\frac{x}{1-\varepsilon'}\right) \;<\; \varepsilon'.
\end{aligned}$$

Therefore $\dfrac{x}{1+\varepsilon'} > x - \varepsilon'$ and $\dfrac{x}{1-\varepsilon'} < x + \varepsilon'$. This shows that

$$\begin{aligned}
B_d(x,\delta) \;&=\; \left(\frac{x}{1+\varepsilon'},\, \frac{x}{1-\varepsilon'}\right) \cap (0,1)\\
&\subseteq\; (x-\varepsilon',\, x+\varepsilon') \cap (0,1)\\
&=\; B_{d'}(x,\varepsilon')\\
&\subseteq\; B_{d'}(x,\varepsilon).
\end{aligned}$$

By Lemma 2.18, we obtain that $\mathcal{B}' \subseteq \mathcal{B}$. Hence $\mathcal{B} = \mathcal{B}'$. That is d and d' are equivalent. Next, we will prove (ii). Choose $\varepsilon = 1$, and let $\delta > 0$. Then there exists an $n \in \mathbb{N}$, such that $n > 2$ and $\dfrac{1}{n-1} < \delta$. Let $x = \dfrac{1}{n+1}$, $y = \dfrac{1}{n-1}$, and $z = \dfrac{1}{n}$.

Then x, y, $z \in X$ and we obtain that

$$\begin{aligned}
d'(z,x) &= \left|\frac{1}{n} - \frac{1}{n+1}\right| = \left|\frac{1}{n(n+1)}\right| \leq \delta, \\
d'(z,y) &= \left|\frac{1}{n} - \frac{1}{n-1}\right| = \left|\frac{1}{n(n-1)}\right| \leq \delta, \quad \text{and} \\
d(x,y) &= \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{\frac{1}{n+1}} - \frac{1}{\frac{1}{n-1}}\right| = |(n+1)-(n-1)| = 2 > \varepsilon.
\end{aligned}$$

This shows that d' is not a w-distance on (X,d). □

Next, if the assumption that (X,d) is compact is added, then every equivalent metric d' of d is a w-distance on (X,d), as will be proved in Theorem 2.22. To do this, we need Lemma 2.20 and 2.21.

Lemma 2.20. *Let d and d' be equivalent metrics on X, $x \in X$ and (x_n) a sequence in X. Then (x_n) converges to x in (X,d) if and only if (x_n) converges to x in (X,d').*

Proof. It follows from the fact that the identity map from (X,d) onto (X,d') is a homeomorphism. □

Lemma 2.21 ([2], p.175 (Lebesgue number lemma)). *Let $\mathcal{A}$ be an open covering of a metric space (X,d). If X is compact, there is a $\delta > 0$ such that for each subset E of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing E.*

Theorem 2.22. *Let (X,d) be a compact metric space and d' a metric equivalent to d. Then d' is a w-distance on (X,d).*

Proof. Since d' is a metric, we have $d'(x,y) \geq 0$ and $d'(x,z) \leq d'(x,y) + d'(y,z)$. Let $x_0 \in X$. We will show that $d'(x_0,\cdot) : (X,d) \to [0,\infty)$ is continuous. Let (x_n) be a sequence in X converging to x (in (X,d)). By Lemma 2.20, (x_n) converge to x in (X,d'). Since $d' : (X,d') \times (X,d') \to [0,\infty)$ is continuous and (x_0,x_n)

converges to (x_0, x) in $(X, d') \times (X, d')$, we have $d'(x_0, x_n)$ converges to $d'(x_0, x)$. Hence $d'(x_0, \cdot)$ is continuous. Next, we will verify the condition (iii) of w-distance. Let $\varepsilon > 0$. Let $\mathcal{G} = \{B_d(x, \frac{\varepsilon}{2}) \mid x \in X\}$. Then $\mathcal{G}$ is an open cover of (X, d). Since $d' \sim d$, we obtain that (X, d') is compact and $\mathcal{G}$ is an open cover of (X, d'). By Lemma 2.21, there is a Lebesgue number $\delta > 0$ such that for every subset A of X

$$\text{if } \sup\{d'(x, y) \mid x, y \in A\} \leq \delta \text{ then there is an } l \in X,\ A \subseteq B_d(l, \tfrac{\varepsilon}{2}). \tag{2.3}$$

Let $\delta' = \dfrac{\delta}{3}$. Then $\delta' > 0$. Let $a, b, c \in X$ be such that $d'(c, a) \leq \delta'$ and $d'(c, b) \leq \delta'$. Since $\delta' = \dfrac{\delta}{3} < \dfrac{\delta}{2}$, we have

$$a \in B_{d'}(c, \tfrac{\delta}{2}) \text{ and } b \in B_{d'}(c, \tfrac{\delta}{2}). \tag{2.4}$$

Let $A = B_{d'}(c, \frac{\delta}{2}) \subseteq X$. Then $\sup\{d'(x, y) \mid x, y \in A\} \leq \delta$. By (2.3), there exists an $l \in X$, such that $B_{d'}(c, \frac{\delta}{2}) \subseteq B_d(l, \frac{\varepsilon}{2})$. From (2.4), we conclude that $a, b \in B_d(l, \frac{\varepsilon}{2})$. Hence $d(a, b) \leq d(a, l) + d(l, b) \leq \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$. □

2.3 Metric-preserving functions and w-distances

Let (X, d) be a metric space. Although an equivalent metric may not be a w-distance, there is a certain family of metrics on X which are w-distances on (X, d). In this section, it is shown in Theorem 2.27 that the metrics of the form $f \circ d$ are w-distances on (X, d) if f is a lower semicontinuous metric-preserving functions.

First, we recall the definition of metric-preserving functions and state some of their properties that will be useful in our investigation.

Definition 2.23 ([1], p.309). *A function $f : [0, \infty) \to [0, \infty)$ is called* **metric-preserving** *(respectively,* **strongly metric-preserving***) if for all metric spaces (X, d), $f \circ d$ is a metric on X (respectively, is a metric which is equivalent to d).*

Theorem 2.24 ([1], p.313). *Suppose f is a metric-preserving.*

(i) For each $x_0 > 0$, there is an $\varepsilon > 0$ such that $f(x) \geq \varepsilon$ for every $x \geq x_0$.

(ii) If f is discontinuous at 0, there is some $\varepsilon > 0$ such that $f(x) > \varepsilon$ for all $x > 0$.

Theorem 2.25 ([1], p.318). *Suppose f is metric-preserving. Then the following are equivalent :*

(i) f is strongly metric-preserving,

(ii) f is continuous at 0,

(iii) f is continuous on $[0, \infty)$,

(iv) for each $\varepsilon > 0$, there is an $x > 0$ with $f(x) < \varepsilon$.

Next, we will prove that $f \circ d$ is a w-distance on (X, d) if f is a lower semicontinuous metric-preserving function. To do this, we first give the next lemma.

Lemma 2.26. *Let X, Y be topological spaces, $f : X \to Y$, and $g : Y \to \mathbb{R}$. If f is continuous and g is lower semicontinuous, the $g \circ f : X \to \mathbb{R}$ is lower semicontinuous.*

Proof. Let f be continuous and g lower semicontinuous on X into Y. To show that $g \circ f$ is lower semicontinuous, let $a \in \mathbb{R}$. Then $(g \circ f)^{-1}(-\infty, a] = f^{-1}(g^{-1}(-\infty, a])$. Since g is lower semicontinuous, we have $g^{-1}(-\infty, a]$ is closed in Y. Since f is continuous and $g^{-1}(-\infty, a]$ is closed in Y, we obtain that $f^{-1}(g^{-1}(-\infty, a])$ is closed in X. That is $(g \circ f)^{-1}(-\infty, a]$ is closed in X. This shows that $g \circ f$ is lower semicontinuous. $\square$

Theorem 2.27. *Let (X, d) be a metric space. Then $f \circ d$ is a w-distance on (X, d) for every lower semicontinuous metric-preserving function f. In particular, if f is strongly metric-preserving, then $f \circ d$ is a w-distance.*

Proof. Let f be a lower semicontiuous metric-preserving function. Then $f \circ d$ is a metric on X. Therefore for every x, y, $z \in X$

$$f \circ d(x,y) \leq f \circ d(x,z) + f \circ d(z,y).$$

Since $d : X \times X \to [0,\infty)$ is continuous and $f : [0,\infty) \to \mathbb{R}$ is lower semicontinuous, we obtain from Lemma 2.26 that $f \circ d : X \times X \to \mathbb{R}$ is lower semicontinuous. In particular, $f \circ d(x_0, \cdot) : X \to \mathbb{R}$ is lower semicontinuous for each $x_0 \in X$. Next, we will prove that $f \circ d$ satisfies the condition (iii) in the definition of w-distance. Let $\varepsilon > 0$. Then, by Theorem 2.24, we obtain a $\delta > 0$ such that

$$\text{for every } x \geq \frac{\varepsilon}{2}, f(x) \geq \delta. \tag{2.5}$$

Let $\delta' = \dfrac{\delta}{2}$. Let x, y, $z \in X$ be such that $f(d(z,x)) \leq \delta'$ and $f(d(z,y)) \leq \delta'$. Then $f(d(z,x)) < \delta$ and $f(d(z,y)) < \delta$. By (2.5), $d(z,x) < \dfrac{\varepsilon}{2}$ and $d(z,y) < \dfrac{\varepsilon}{2}$. Thus $d(x,y) \leq d(x,z) + d(z,y) \leq \varepsilon$. Therefore $f \circ d$ is a w-distance. $\square$

In our investigation it is observed that the continuity of f is not enough to guarantee that $f \circ d$ is a w-distance on the space (X,d), as shown in the next example.

Examples 2.28. Let $X = \mathbb{R}$, and d the usual metric on $\mathbb{R}$. Define

$$f : [0,\infty) \to [0,\infty) \quad \text{by} \quad f(x) = x^2.$$

Then f is continuous and $f \circ d(x,y) = f(d(x,y)) = f(|x-y|) = |x-y|^2 = (x-y)^2$. Then $f \circ d(1,3) = 4$, $f \circ d(1,2) = 1$, $f \circ d(2,3) = 1$. Therefore

$$f \circ d(1,3) \nleq f \circ d(1,2) + f \circ d(2,3).$$

This shows that $f \circ d$ does not satisfy the condition (i) of w-distance.

CHAPTER III

CAUCHY W-DISTANCES

In this chapter, the notion of Cauchy w-distances is introduced and their properties are state and proved.

3.1 Definitions and examples

In this section, we first recall the definition of Cauchy sequences. After that we will define p-Cauchy sequences and compare them to Cauchy sequences. Then at the end of this section, we give the definition of complete w-distances.

Recall that a sequence (x_n) in a metric space (X, d) is said to be a ***Cauchy sequence*** if for each $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\text{for } m > n \geq N,\ d(x_n, x_m) < \varepsilon. \tag{3.1}$$

The following theorem give an equivalent definition of Cauchy sequence in (X, d).

Theorem 3.1. *Let (X, d) be a metric space, and (x_n) a sequence in X. Then (x_n) is a Cauchy sequence if and only if there exists a nonnegative sequence (α_n) converging to 0 such that $d(x_n, x_m) < \alpha_n$ for all $m > n$.*

Proof. First, we will prove the "if"part. Assume that there is a nonnegative sequence (α_n) converging to 0, such that for $m > n$, $d(x_n, x_m) \leq \alpha_n$. Let $\varepsilon > 0$. Since α_n converges to 0, we have $N \in \mathbb{N}$ such that for $n \geq N$, $\alpha_n < \varepsilon$. Then for $m > n \geq N$, $d(x_n, x_m) \leq \alpha_n < \varepsilon$. This shows that (x_n) is a Cauchy sequence.

Now, assume that (x_n) is a Cauchy sequence. Then

$$\forall\varepsilon > 0\,\exists N \in \mathbb{N}\,\forall m > n \geq N,\ d(x_n, x_m) < \varepsilon. \tag{3.2}$$

Let $\varepsilon = 1$. Then there exists an $N_1 > 2$ such that for $m > n \geq N_1$, $d(x_n, x_m) < 1$. Let $\alpha_1 = \max\{d(x_1, x_2),\ d(x_1, x_3),\ \ldots,\ d(x_1, x_{N_1}) + 1\}$. Then $\alpha_1 \geq d(x_1, x_l)$ for all $1 < l \leq N_1$. For $l > N_1$,

$$d(x_1, x_l) \leq d(x_1, x_{N_1}) + d(x_{N_1}, x_l) < d(x_1, x_{N_1}) + 1 \leq \alpha_1.$$

Thus $\alpha_1 \geq d(x_1, x_l)$ for all $l > 1$.
Let $\varepsilon = \frac{1}{2}$. Then there exists an $N_2 > 3$ such that for $m > n \geq N_2$, $d(x_n, x_m) < \frac{1}{2}$. Let $\alpha_2 = \max\{d(x_2, x_3),\ d(x_2, x_4),\ \ldots,\ d(x_2, x_{N_2}) + \frac{1}{2}\}$. Then $\alpha_2 \geq d(x_2, x_l)$ for all $2 < l \leq N_2$. For $l > N_2$,

$$d(x_2, x_l) \leq d(x_2, x_{N_2}) + d(x_{N_2}, x_l) < d(x_2, x_{N_2}) + \frac{1}{2} \leq \alpha_2.$$

Thus $\alpha_2 \geq d(x_2, x_l)$ for all $l > 2$.
For each $k \in \mathbb{N}$, there is an $N_k > k + 1$ such that for $m > n \geq N_k$, $d(x_n, x_m) < \frac{1}{k}$. Let $\alpha_k = \max\{d(x_k, x_{k+1}),\ d(x_k, x_{k+2}),\ \ldots,\ d(x_k, x_{N_k}) + \frac{1}{k}\}$. Then $\alpha_k \geq d(x_k, x_l)$ for all $k < l \leq N_k$. For $l > N_k$,

$$d(x_k, x_l) \leq d(x_k, x_{N_k}) + d(x_{N_k}, x_l) < d(x_k, x_{N_k}) + \frac{1}{k} \leq \alpha_k.$$

Thus $\alpha_k \geq d(x_k, x_l)$ for all $l > k$. Therefore, we obtain a nonnegative sequence (α_n) with the property that $\alpha_n \geq d(x_n, x_m)$ for all $m > n$. Next, we will show that α_n converges to 0. Let $\varepsilon > 0$. Then there exists an $N_1' \in \mathbb{N}$ such that for $m > n \geq N_1'$, $d(x_n, x_m) \leq \dfrac{\varepsilon}{2}$ and there is an $N_2' \in \mathbb{N}$ such that $\dfrac{1}{N_2'} < \dfrac{\varepsilon}{2}$. Let $N = \max\{N_1',\ N_2'\}$. Then $\dfrac{1}{N} < \dfrac{\varepsilon}{2}$ and for $m > n \geq N$, $d(x_n, x_m) \leq \dfrac{\varepsilon}{2}$. Let $n > N$.

We obtain that

$$d(x_n, x_{n+1}),\ d(x_n, x_{n+2}),\ \ldots,\ \text{and } d(x_n, x_{N_n-1}) < \frac{\varepsilon}{2}, \tag{3.3}$$

where N_n is the natural number in the construction of α_n. In addition,

$$d(x_n, x_{N_n}) + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{3.4}$$

Since $\alpha_n = \max\{d(x_n, x_{n+1}),\ \ldots,\ d(x_n, x_{N_n}) + \frac{1}{n}\}$, we obtain from (3.3) and (3.4) that $\alpha_n < \varepsilon$. This shows that α_n converges to 0. $\square$

The next lemma appears in the text of Takahashi [5].

Lemma 3.2 ([5]). *Let X be a metric space with metric d and let p be a w-distance on X. Let (x_n) and (y_n) be sequences in X. Let (α_n) and (β_n)be sequences in $[0, \infty)$ converging to 0, and let x, y, $z \in X$. Then the following hold :*

(i) If $p(x_n, y) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$,

(ii) if $p(x_n, y_n) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converge to z,

(iii) if $p(x_n, x_m) \le \alpha_n$ for any m, $n \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence,

(iv) if $p(y, x_n) \le \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.

Let (x_n) be a sequence in a metric space (X, d). From Lemma 3.2(iii), we obtain that if there exists a nonnegative sequence (α_n) converging to 0 and a w-distance p on (X, d) is such that

$$\text{for } m > n,\ p(x_n, x_m) \le \alpha_n \tag{3.5}$$

then (x_n) is a Cauchy sequence. If p in (3.5) is replaced by the metric d, then the result will be the same as equivalent definition of Cauchy sequence given in Thereom 3.1. Therefore, we can generalize the notion of Cauchy sequences in (X, d) via w-distances by replacing the metric d in Theorem 3.1 by any w-distance p. This motivates us to study a class of sequences called "p-Cauchy sequence".

Definition 3.3. *Let (X, d) be a metric space, and p a w-distance on (X, d). A sequence (x_n) in X is p-**Cauchy** if there exists a nonnegative sequence (α_n) converging to 0 such that for $m > n$, $p(x_n, x_m) \leq \alpha_n$.*

Let (X, d) be a metric space, and p a w-distance. If $p = d$, then Cauchy sequences and p-Cauchy sequences coincide. However, if $p \neq d$, then a p-Cauchy sequence is certainly a Cauchy sequence (Lemma 3.2(iii)) but a Cauchy sequence need not be a p-Cauchy sequence, as the next example shows.

Examples 3.4. Let $X = \mathbb{R}$ (equipped with the usual norm $|\cdot|$). Let $p : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be defined by

$$p(x, y) = |x| + |y|.$$

From Example 2.8, it is known that p is a w-distance. Let $x_n = 1$ for all $n \in \mathbb{N}$. Then (x_n) is a Cauchy sequence. Let (α_n) be a sequence in $[0, \infty)$ converging to 0. Therefore X, p, (x_n) and (α_n) satisfy the condition in Lemma 3.2(iii). However, since α_n converges to 0 and $p(x_n, x_m) = 2$ for every n, $m \in \mathbb{N}$, the sequence (x_n) does not satisfy the condition that " for $m > n$, $p(x_n, x_m) \leq \alpha_n$".

Here is an immediate property of p-Cauchy sequence.

Theorem 3.5. *Let (X, d) be a metric space, p a w-distance, and (x_n) a sequence in X. Then (x_n) is a p-Cauchy sequence if and only if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for $m > n \geq N$, $p(x_n, x_m) < \varepsilon$.*

Proof. The proof of this theorem is similar to that of Theorem 3.1. Note that we use only the triangle inequality property of d, so we can replace d by p and obtain a proof of this theorem. $\square$

From Lemma 3.2(iii) and Example 3.4, it is seen that the condition of p-Cauchy sequence is stronger than that of Cauchy sequence. It may be too strong that even a constant sequence may not be p-Cauchy. Furthermore, for a particular w-distance p on (X, d), it may happen that there is no p-Cauchy sequences in this space. So, we add some properties on w-distances p which make the condition of p-Cauchy sequence more interesting.

Definition 3.6. *Let (X, d) be a metric space and p a w-distance. Then p is said to be a* **Cauchy w-distance** *if every Cauchy sequence is a p-Cauchy sequence, and p is said to be a* **simple w-distance** *if $p(x, x) = 0$ for every $x \in X$.*

Next, we give some examples of p-Cauchy sequences.

Notation Let (X, d) be a metric space, p a w-distance on (X, d). Denote by $C_p(X)$, and $C(X)$ the set of all p-Cauchy sequences, and Cauchy sequences in X, respectively.

Examples 3.7. Let p be the w-distance given in Example 2.6. In this case, it is easy to see that p is a Cauchy w-distance, and

$$C_p(X) = C(X).$$

Examples 3.8. Let p be the w-distance given in Example 2.7 with $c > 0$. Then $p(x_n, x_m) = c$ for any sequence (x_n) in X. Therefore there is no p-Cauchy sequences in X. Hence $C_p(X) = \varnothing$ and every constant sequence is a Cauchy sequence. So $C_p(X) \neq C(X)$, and p is not a Cauchy w-distance.

Examples 3.9. Let p be the w-distance given in Example 2.8. We will determine $C_p(X)$. Let $(x_n) \in C_p(X)$. We will show that (x_n) converges to 0. Let $\varepsilon > 0$. Since (x_n) is a p-Cauchy, we obtain an $N \in \mathbb{N}$ such that

$$\text{for } m > n \geq N,\ p(x_n, x_m) < \varepsilon. \tag{3.6}$$

Let $n > N$. Then by (3.6), $p(x_N, x_n) < \varepsilon$. Therefore

$$\|x_n\| \leq \|x_N\| + \|x_n\| = p(x_N, x_n) < \varepsilon.$$

This shows that (x_n) converges to 0. Conversely, if (x_n) converges to 0, then $\|x_n\|$ converges to 0 and thus $p(x_n, x_m) = \|x_n\| + \|x_m\|$ converges to 0 as m, $n \to \infty$, and hence (x_n) is a p-Cauchy sequence. This shows that

$$C_p(X) = \{(x_n) \,|\, (x_n) \text{ converges to } 0\}.$$

So $C_p(X) \neq C(X)$, and p is not a Cauchy w-distance.

Examples 3.10. Let p be the w-distance given in Example 2.13. Let $(x_n) \in C_p(X)$. We will show that (x_n) converges to 0. Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that

$$\text{for } m > n \geq N,\ p(x_n, x_m) < \varepsilon. \tag{3.7}$$

Let $n > N$. Then, by (3.7), we obtain that

$$\|x_n\| = p(x_N, x_n) < \varepsilon.$$

Hence x_n converges to 0.

Conversely, if (x_n) converges to 0, then $p(x_n, x_m) = \|x_m\|$ converges to 0 and $(x_n) \in C_p(X)$. This shows that $C_p(X) = \{(x_n) \,|\, (x_n) \text{ converges to } 0\}$. Therefore $C_p(X) \neq C(X)$, and p is a not a Cauchy w-distance.

Examples 3.11. Let (X, d), F, c and p be as in Example 2.10. We will compute $C_p(X)$. Let $(x_n) \in C_p(X)$. Let $\varepsilon = \frac{c}{2}$. Then there exists an $N \in \mathbb{N}$ such that for $m > n \geq N$, $p(x_n, x_m) < \frac{c}{2}$. This implies that x_n and x_m are in F and

$$p(x_n, x_m) = d(x_n, x_m) \quad \text{for all} \quad m > n \geq N. \tag{3.8}$$

That is

$$x_n \in F \quad \text{for all} \quad n \geq N. \tag{3.9}$$

Next, we will show that (x_n) is a Cauchy sequence in $(X,\, d)$. Let $\varepsilon > 0$. Then there is an $N' > N$ such that

$$\text{for } \ m > n \geq N',\, p(x_n, x_m) < \varepsilon. \tag{3.10}$$

Let $m > n \geq N'$. Then $m > n \geq N$. From (3.8), we obtain that $d(x_n, x_m) = p(x_n, x_m)$. Also from (3.10), we have $p(x_n, x_m) < \varepsilon$. Hence $d(x_n, x_m) < \varepsilon$. This shows that

$$(x_n) \text{ is a Cauchy sequence in } (X, d). \tag{3.11}$$

From (3.9) and (3.11), we obtain that (x_n) is a Cauchy sequence in X which eventually lies in F. This shows that

$$C_p(X) \subseteq \{(x_n) \,|\, (x_n) \text{ is a Cauchy sequence which eventually lies in } F\}.$$

For the converse, let (x_n) be a Cauchy sequence which eventually lies in F. Let $N_1 \in \mathbb{N}$ be such that

$$x_n \in F \quad \text{for all} \quad n \geq N_1. \tag{3.12}$$

Let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence, there exists an $N_2 \in \mathbb{N}$ such that

$$\text{for } \ m > n \geq N_2,\, d(x_n, x_m) < \varepsilon. \tag{3.13}$$

Let $N = \max\{N_1,\, N_2\}$. Let $m > n \geq N$. Then $m,\, n \geq N_1$. From (3.12), we obtain that $x_n,\, x_m \in F$ and thus $p(x_n, x_m) = d(x_n, x_m)$. Also from (3.13), we obtain that

$p(x_n, x_m) < \varepsilon$. This show that $(x_n) \in C_p(X)$ and

$$C_p(X) = \{(x_n) \,|\, (x_n) \text{ is a Cauchy sequence which eventually lies in } F\}.$$

If $F \neq X$, we can choose $a \in X \smallsetminus F$ and the constant sequence $(a, a, a, \ldots)$ is a Cauchy sequence which does not eventually lie in F. Therefore $C_p(X) \neq C(X)$ and p is not a Cauchy w-distance.

From this example we have the next theorem.

Theorem 3.12. *Let (X, d) be a complete metric space. Then for every closed and bounded subset F of X, there is a w-distance p on X with the property that the set of all p-Cauchy sequences consisting of all sequences converging to a point in F.*

Proof. Let F be closed and bounded in (X, d). Define p as in Example 3.11. We obtain that

$$C_p(X) = \{(x_n) \,|\, (x_n) \text{ is a Cauchy sequence which eventually lies in } F\}.$$

Since X is complete, every Cauchy sequence is convergent. Since F is closed, the limit of a sequence in F is in F. Therefore

$$C_p(X) = \{(x_n) \,|\, (x_n) \text{ is a convergent sequence in } (X, d) \text{ whose limit lies in } F\}.$$

□

3.2 Sufficient conditions for Cauchy w-distance

In this section, we study relationship between Cauchy w-distances and metric-preserving functions. The study will also give us more examples of Cauchy w-distances. We also introduce the notion of uniformly discrete metric spaces and give some examples.

Theorem 3.13. *Let (X, d) be a metric space, and f a metric preserving function. If f is strongly metric preserving, then $f \circ d$ is a Cauchy w-distance.*

Proof. Assume that f is strongly metric preserving. By Theorem 2.25, f is continuous. Therefore by Theorem 2.27, $f \circ d$ is a w-distance on (X, d). Next, we will prove that $f \circ d$ is a Cauchy w-distance. Let (x_n) be a Cauchy sequence in (X, d). To show that (x_n) is $f \circ d$-Cauchy, let $\varepsilon > 0$ be arbitrary. Since f is continuous at 0, there exists a $\delta > 0$ such that for every $x \in \mathbb{R}$, $0 \leq x < \delta$ implies $|f(x) - f(0)| < \varepsilon$. Since $f(0) = 0$, for this δ,

$$\text{if } x \in \mathbb{R}, \text{ and } 0 \leq x < \delta \text{ then } f(x) < \varepsilon. \tag{3.14}$$

Since (x_n) is a Cauchy sequence, there exists an $N \in \mathbb{N}$ such that for $m > n \geq N$, $d(x_n, x_m) < \delta$. Let $m > n \geq N$. Then by (3.14)

$$f \circ d(x_n, x_m) \;=\; f(d(x_n, x_m)) \;<\; \varepsilon.$$

This shows that (x_n) is $f \circ d$-Cauchy and hence $f \circ d$ is a Cauchy w-distance. $\square$

In Theorem 3.13, the condition that f is strongly metric preserving guarantees that $f \circ d$ is a Cauchy w-distance. A natural question arise :

If f is metric-preserving function which is lower semicontinuous
then is $f \circ d$ a Cauchy w-distance on (X, d)? (3.15)

Recall that, if X is a nonempty set and $d : X \times X \to [0, \infty)$ is defined by

$$d(x,y) = \begin{cases} 1 & x \neq y, \\ 0 & x = y, \end{cases} \tag{3.16}$$

then d is a metric which generates the discrete topology on X. We would like to distinguish this metric from other metrics which generate the discrete topology in the next definition.

Definition 3.14. *Let d be a metric on a nonempty set X. If d is defined by (3.16), then we call d* **the discrete metric** *and $(X,\, d)$* **a discrete metric space***.*

We also call the topology τ on X in which every subset of X is in τ **the discrete topology***.*

Examples 3.15. Let $X = \mathbb{N}$ and d is the usual metric on $\mathbb{N}$. Then for each $x \in \mathbb{N}$, $\{x\} = B_d(x, \frac{1}{2})$. Therefore d generates the discrete topology on X. Hence d is a discrete metric which is not of the form in (3.16)

Definition 3.16. *Let (X, d) be a metric space. The metric d is said to be a* **uniformly discrete metric** *if there is an $\varepsilon > 0$ such that $B_d(x, \varepsilon) = \{x\}$ for all $x \in X$. In this case, we also say that (X, d) is a* **uniformly discrete metric space** *or (X, d) is* **uniformly discrete***.*

Examples 3.17. Let X be a nonempty set and $c > 0$. The metric d of the form

$$d(x,y) = \begin{cases} c & x \neq y, \\ 0 & x = y, \end{cases}$$

is a uniformly discrete metric.

The metric defined in Example 3.15 is also a uniformly discrete metric.

Theorem 3.18. *Let d be a uniformly discrete metric on X. Then*

(i) every Cauchy sequence (x_n) in (X, d) is eventually constant,

(ii) (X, d) is a complete metric space.

Proof. Since (X, d) is uniformly discrete, there is an $\varepsilon > 0$ such that $B_d(x, \varepsilon) = \{x\}$ for all $x \in X$. Let (x_n) be a Cauchy sequence in X. Then there exists an $N \in \mathbb{N}$ such that for $m > n \geq N$, $d(x_n, x_m) < \varepsilon$. Therefore for $m \geq N$, $d(x_N, x_m) < \varepsilon$. Thus for $m \geq N$, $x_m \in B(x_N, \varepsilon) = \{x_N\}$. That is for $m \geq N$, $x_m = x_N$. This proves (i). From (i), every Cauchy sequence in X is eventually constant and thus is convergent. Therefore (X, d) is complete. This proves (ii). $\square$

There are some discrete metric which is not uniform, as shown in the following example. And, a metric d generating the discrete topology need not assure that (X, d) is complete.

Examples 3.19. Let $X = \mathbb{N}$. Defines $d : \mathbb{N} \times \mathbb{N} \to [0, \infty)$ by $d(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right|$ for all m, $n \in \mathbb{N}$. It is easy to see that d is a metric on X. Claim that

(i) d generates the discrete topology,

(ii) $(n) = (1, 2, 3, \ldots)$ is a Cauchy sequence in (X, d) which is not convergent.

Proof. (i) It is easy to see that $B(1, \frac{1}{3}) = \{1\}$. Therefore $\{1\}$ is open. Next, let $n \in \mathbb{N} - \{1\}$. We will show that $\{n\}$ is open in (X, d). Let $\varepsilon = \dfrac{1}{2n(n+1)}$. Then $B_d(n, \varepsilon)$ is open in (X, d). Let $m \in B_d(n, \varepsilon)$. Then $\left|\dfrac{1}{m} - \dfrac{1}{n}\right| < \varepsilon$.

If $m \geq n + 1$, then $\dfrac{1}{m} \leq \dfrac{1}{n+1} < \dfrac{1}{n}$, and thus

$$\left|\frac{1}{m} - \frac{1}{n}\right| = \frac{1}{n} - \frac{1}{m} \geq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > \varepsilon.$$

If $1 \leq m \leq n-1$, then $\dfrac{1}{m} \geq \dfrac{1}{n-1} > \dfrac{1}{n}$, and therefore

$$\left|\frac{1}{m} - \frac{1}{n}\right| = \frac{1}{m} - \frac{1}{n} \geq \frac{1}{n-1} - \frac{1}{n} = \frac{1}{(n-1)n} > \varepsilon.$$

Since $\left|\dfrac{1}{m} - \dfrac{1}{n}\right| < \varepsilon$, we have $m = n$. This shows that $B_d(n, \varepsilon) = \{n\}$, and $\{n\}$ is open. This shows that d generates the discrete topology.

(ii) We can see that

$$d(n, m) = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \to 0 \quad \text{as } n,\, m \to \infty.$$

Therefore $(n) = (1,\, 2,\, 3,\, \ldots)$ is a Cauchy sequence. Next, we will show that this sequence does not converge in X. Let $l \in \mathbb{N}$. If $l = 1$, we can see that $d(m,\, l) \geq \dfrac{1}{2}$ for all $m \neq 1$. Therefore (n) does not converge to 1. Assume that $l \neq 1$. Let $\varepsilon = \dfrac{1}{2l(l+1)}$. As in the proof of (i), we see that

$$d(m, l) > \varepsilon \text{ for all } m \neq l.$$

Therefore (n) does not converge to l. This shows that (n) does not converge to any point of X. $\square$

This example shows that (X, d) is not complete and there is a Cauchy sequence in (X, d) which is not eventually constant. By Theorem 3.18, it can be concluded that d is **a discrete metric which is not uniform**.

It is obvious that if $X \neq \varnothing$, $c > 0$, and d which is defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = c$ if $x \neq y$, then d is a uniform discrete metric. Next is an example of a uniform discrete metric which is not of this form.

Examples 3.20. Let $X = \{1, 2, 3\}$. Let $d : X \times X \to [0, \infty)$ be defined by

$$\begin{aligned} d(1,1) &= d(2,2) = d(3,3) = 0 \\ d(1,2) &= d(2,1) = d(1,3) = d(3,1) = 1 \\ d(2,3) &= d(3,2) = 2. \end{aligned}$$

It is to see that $d(x, y) \geq 0$, $d(x, y) = 0$ if and only if $x = y$, and $d(x, y) = d(y, x)$, for all x, $y \in X$. Next, we prove the triangle inequality. Let x, y, $z \in X$. If $x = y$, then $d(x, y) = 0 \leq d(x, z) + d(z, y)$. Assume that $x \neq y$. If $z = x$ or $z = y$, we can see that $d(x, y) = d(x, z) + d(z, y)$. Assume that $z \neq x$ and $z \neq y$. Then $d(x, z) + d(z, y) \geq 1 + 1 = 2 \geq d(x, y)$. This shows that d satisfy the triangle inequality. Hence d is a metric on X. It is easy to see that $B(x, \frac{1}{2}) = \{x\}$ for all $x \in X$. Thus d is a strong discrete metric. However, there is no $c > 0$ such that $B_d(x, c) = \{x\}$ for all $x \in X$.

Next, we will answer the question in (3.15). To do this, we find give the next lemma is needed.

Lemma 3.21. *Let (X, d) be a metric space. If every Cauchy sequence in (X, d) is eventually constant, then (X, τ_d) is a discrete space.*

Proof. Let $x \in X$. Suppose on the contrary that for every $\varepsilon > 0$, $B_d(x, \varepsilon) - \{x\} \neq \varnothing$. Therefore, for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that

$$x_n \in B(x, \frac{1}{n}) \quad \text{and} \quad x_n \neq x. \tag{3.17}$$

Consider the sequence (x_n). Since $x_n \in B(x, \frac{1}{n})$ for every $n \in \mathbb{N}$, $d(x_n, x) < \dfrac{1}{n}$ for all $n \in \mathbb{N}$. Therefore (x_n) converges to x. Then (x_n) is a Cauchy sequence, and thus eventually constant. Let $x_0 \in X$, and $N \in \mathbb{N}$ be such that for $n \geq N$, $x_n = x_0$. Hence (x_n) converges to x_0. Since limit of convergent sequence in a metric space is unique, we have $x = x_0$. Therefore for $n \geq N$, $x_n = x$, which contradicts (3.17).

Therefore there exists an $\varepsilon > 0$, such that $B_d(x, \varepsilon) = \{x\}$. That is $\{x\}$ is open. This shows that (X, τ_d) is a discrete space. $\square$

Theorem 3.22. *Let (X, d) be a metric space and f a metric-preserving function.*

(i) If f is not strongly metric-preserving and $f \circ d$ is Cauchy, then (X, τ_d) is a discrete space.

(ii) If d is uniformly discrete, then $f \circ d$ is a Cauchy w-distance.

Proof. (i) Assume that f is not strongly metric-preserving and $f \circ d$ is a Cauchy w-distance. Then by Theorem 2.25, f is not continuous at 0. By Theorem 2.24, there is an $\varepsilon > 0$ such that $f(x) > \varepsilon$ for all $x > 0$. Then for each $x \in X$,

$$\begin{aligned} B_{f \circ d}(x, \varepsilon) &= \{y \in X \mid f(d(x, y)) < \varepsilon\} \\ &= \{y \in X \mid d(x, y) = 0\} \\ &= \{x\}. \end{aligned}$$

That is $f \circ d$ is a uniformly discrete metric. Since $f \circ d$ is Cauchy, every Cauchy sequence in (X, d) is $f \circ d$-Cauchy. Since $f \circ d$ is a uniformly discrete metric, a sequence in X is $f \circ d$-Cauchy if it is eventually constant. Therefore every Cauchy sequence in (X, d) is eventually constant. By Lemma 3.21, we conclude that (X, τ_d) is discrete. Next, assume that d is uniformly discrete. We will first prove that $f \circ d$ is a w-distance. We will prove condition (i) and (iii) by using the same arguments in Theorem 2.27. Also, for $x_0 \in X$, $f \circ d(X_0, \cdot) : (X, d) \to [0, \infty)$ is continuous since (X, d) is discrete. Hence $f \circ d$ is a w-distance. To show that $f \circ d$ is Cauchy, let (x_n) be a Cauchy sequence in (X, d). Then (x_n) is eventually constant. Therefore (x_n) is $f \circ d$-Cauchy. Hence $f \circ d$ is Cauchy. $\square$

Corollary 3.23. *Let (X, d) be a metric space and f a metric-preserving function. Then*

(i) If f is strongly metric-preserving or d is uniformly discrete, then $f \circ d$ is Cauchy, and

(ii) if f is not strongly metric-preserving and d is not discrete, then $f \circ d$ is not Cauchy.

Proof. This corollary follows from Theorem 3.13 and 3.22. □

CHAPTER IV

THEOREMS ON CAUCHY W-DISTANCE

Throughout this chapter, p is a w-distance on a metric space (X, d). A characterization of Cauchy w-distance is given.

4.1 Simple w-distances, continuous functions and Cauchy w-distances

Recall that a w-distance p is said to be simple if for each $x \in X$, $p(x, x) = 0$.

Theorem 4.1. *p is simple if and only if for any x, $y \in X$, $p(x, y) = 0$ if and only if $x = y$.*

Proof. By the definition, we have p is simple. Assume that p is simple. Then $p(x, x) = 0$ for all $x \in X$. Next, let x, $y \in X$ be such that $p(x, y) = 0$. Since $p(x, y) = 0$ and $p(x, x) = 0$, by Lemma 3.2(i), $y = x$. □

Theorem 4.2. *If p is a Cauchy w-distance, then p is simple.*

Proof. Assume that p is a Cauchy w-distance in (X, d). Then every Cauchy sequence in X is p-Cauchy. Therefore every convergent sequence is p-Cauchy. In particular, every constant sequence is p-Cauchy. To show that p is simple, let $x_0 \in X$. Consider the constant sequence (x_0). Then (x_0) is a p-Cauchy sequence. Thus there exists a nonnegative sequence (α_n) converging to 0 such that $0 \leq p(x_0, x_0) \leq \alpha_n$ for all n. This implies that $p(x_0, x_0) = 0$. □

The next example demonstrates a simple w-distances which are not Cauchy.

Examples 4.3. Let $X = \mathbb{R}$, and let d be the usual metric on $\mathbb{R}$. Let f be a metric-preserving function which is lower semicontinuous. Suppose that f is not continuous at 0. Then f is not a strongly metric-preserving function. Thus by Theorem 2.27, $f \circ d$ is a w-distance. In addition, $f \circ d$ is simple since $f \circ d$ is a metric. However by Corollary 3.23(ii), we obtain that $f \circ d$ is simple but not Cauchy.

We will prove that every Cauchy w-distances is continuous. Since the domain of a w-distance is a product space, we recall some theorems on product topology and give a criterion to assure continuity of w-distances.

Theorem 4.4 ([2], p.118). *Let $x_1, x_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Then this sequence converges to the point x if and only if the sequence $\pi_\alpha(x_1), \pi_\alpha(x_2), \ldots$ converges to $\pi_\alpha(x)$ for each α.*

Theorem 4.5 ([2], p.190). *Let X be a topological space.*

(a) Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is first-countable.

(b) Let $f : X \to Y$. If f is continuous, then for every convergent sequence (x_n) converges to x in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.

Theorem 4.6 ([2], p.191). *A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable. A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.*

Theorem 4.7. *Let (X, d) be a metric space and Z a topological space. Then $f : X \times X \to Z$ is continuous(in the product topology) if and only if for every sequence (x_n) and (y_n) in X, if x_n converges to x and y_n converges to y, then $f(x_n, y_n)$ converges to $f(x, y)$.*

Proof. Since every metric space is first countable, $X \times X$ is first countable. Assume that f is continuous, (x_n) and (y_n) are sequences in X converging to x and y, respectively. By Theorem 4.4, (x_n, y_n) converges to (x, y) in $X \times X$. Therefore, by Theorem 4.5, $f(x_n, y_n)$ converges to $f(x, y)$.

Conversely, assume that for every sequence (x_n) and (y_n) in X, if (x_n) converges to x and (y_n) converges to y, then $f(x_n, y_n)$ converges to $f(x, y)$. To show that f is continuous, we apply Theorem 4.5. Let (x_n, y_n) be sequence in $X \times X$ converging to $(x,\ y)$. Then by Theorem 4.4, (x_n) converges to x and (y_n) converges to y. By assumption, $f(x_n, y_n)$ converges to $f(x, y)$. Hence the theorem is proved. $\square$

Next, we will prove that Cauchy w-distances are continuous by applying Theorem 4.7.

Theorem 4.8. *If p is a Cauchy w-distance in (X, d), then p is continuous.*

Proof. Assume that p is a Cauchy w-distance. Let (x_n) and (y_n) be sequences in X converging to x and y, respectively. We will show that $p(x_n, y_n)$ converges to $p(x, y)$ (in $\mathbb{R}$). Let $n \in \mathbb{N}$. Since $p(x_n, y_n) \le p(x_n, x) + p(x, y) + p(y, y_n)$, we have

$$p(x_n, y_n) - p(x, y) \le p(x_n, x) + p(y, y_n).$$

Since $p(x, y) \le p(x, x_n) + p(x_n, y_n) + p(y_n, y)$, we have

$$p(x_n, y_n) - p(x, y) \ge -p(x, x_n) - p(y_n, y).$$

We obtain that

$$-p(x, x_n) - p(y_n, y) \le p(x_n, y_n) - p(x, y) \le p(x_n, x) + p(y, y_n) \tag{4.1}$$

for all $n \in \mathbb{N}$. Next, we will show that $p(x, x_n), p(x_n, x), p(y, y_n)$ and $p(y_n, y)$ all converge to 0 as $n \to \infty$. Let $\varepsilon > 0$. Let $(a_n) = (x_1,\ x,\ x_2,\ x,\ x_3,\ x, \ldots)$. Then (a_n) is a sequence in X such that $a_{2n} = x$ and $a_{2n-1} = x_n$ for all $n \in \mathbb{N}$. Since (x_n)

converges to x, we obtain that (a_n) converges to x. Therefore (a_n) is a Cauchy sequence. Since p is Cauchy, (a_n) is p-Cauchy. Thus there exists $N \geq 2$ such that

$$\text{for } m > n \geq N,\ p(a_n, a_m) < \varepsilon.$$

Let $n \geq N$. Then $p(x_n,\ x) = p(a_{2n-1}, a_{2n}) < \varepsilon$, and $p(x, x_n) = p(a_{2n-2}, a_{2n-1}) < \varepsilon$. Thus $p(x, x_n)$ and $p(x_n, x)$ converge to 0. Similarly, $p(y, y_n)$ and $p(y_n, y)$ converge to 0. Hence

$$p(x_n, x) + p(y, y_n) \text{ and } -p(x, x_n) - p(y_n, y) \text{ converge to } 0. \tag{4.2}$$

From (4.1) and (4.2), we obtain that $p(x_n, y_n)$ converges to $p(x, y)$. $\square$

The next example shows that, in general, a w-distance need not be continuous.

Examples 4.9. Let $X = \mathbb{R}, F = [0, 1]$, $c = 2$. Let $p : X \times X \to [0, \infty)$ be defined by

$$p(x, y) = \begin{cases} |x - y| & \text{if } x,\ y \in F, \\ \quad 2 & \text{if } x \notin F \text{ or } y \notin F. \end{cases}$$

From Example 2.10, it is seen that p is a w-distance. Let $(x_n) = (1+\frac{1}{n})$ and $(y_n) = (-\frac{1}{n})$ for each $n \in \mathbb{N}$. Then (x_n) converges to 1 and (y_n) converges to 0. Since for every $n \in \mathbb{N}$, $x_n \notin F$, we obtain that $p(x_n, y_n) = 2$ for all $n \in N$. Therefore $p(x_n, y_n)$ converges to 2. Since $1 \in F$ and $0 \in F$, we have $p(1, 0) = |1 - 0| = 1$. This shows that $p(x_n, y_n)$ does not converges to $p(1, 0)$. Hence p is not continuous.

The next example shows that a continuous w-distance need not be Cauchy.

Examples 4.10. Let p be defined as in Example 2.7. Then it is easily seen that p is continuous. However, p is not simple and thus is not Cauchy.

Theorem 4.11. *Let (X, d) be a metric space. Define $d' : (X \times X) \times (X \times X) \rightarrow [0, \infty)$ by*

$$d'((x, y),\, (a, b)) = d(x, a) + d(y, b),$$

for x, y, a, b in X. Then the following statements hold :

(i) d' is a metric on $X \times X$.

(ii) d' generates the product topology.

We will call this metric d' the "taxi-cab metric"on $X \times X$.

Proof. (i) We will show that d' is a metric on $X \times X$. Let (x, y), $(a, b) \in X \times X$.

1) $d'((x, y),\, (a, b)) = d(x, a) + d(b, y) \geq 0$.

2)
$$\begin{aligned} d'((x, y),\, (a, b)) = 0 &\leftrightarrow d(x, a) + d(a, b) = 0 \\ &\leftrightarrow d(x, a) = 0 \,\wedge\, d(y, b) = 0 \\ &\leftrightarrow x = a \,\wedge\, y = b \\ &\leftrightarrow (x, y) = (a, b). \end{aligned}$$

3)
$$\begin{aligned} d'((x, y),\, (a, b)) &= d(x, a) + d(y, b) \\ &= d(a, x) + d(b, y) \\ &= d'((a, b),\, (x, y)). \end{aligned}$$

4) Let $(m, n) \in X \times X$.

$$\begin{aligned} d'((x, y),\, (a, b)) &= d(x, a) + d(y, b) \\ &\leq d(x, m) + d(m, a) + d(y, n) + d(n, b) \\ &= (d(x, m) + d(y, n)) + (d(m, a) + d(n, b)) \\ &= d'((x, y), (m, n)) + d'((m, n),\, (a, b)). \end{aligned}$$

Hence d' is a metric on $X \times X$. Next, we will show that d' generates the product topology. Let

$$\mathcal{B}_1 = \{B_{d'}((x,y),\, \varepsilon)\,|\,(x,y) \in X \times X,\, \varepsilon > 0\} \quad \text{and}$$
$$\mathcal{B}_2 = \{B_d(m,\delta_1) \times B_d(n,\delta_2)\,|\,m,\, n \in X,\, \delta_1,\, \delta_2 > 0\}.$$

Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases of the topology generated by d' and the product topology, respectively. First, we will show that

$$\text{for each } B_{d'}((x,y),\varepsilon) \in \mathcal{B}_1 \text{ there are } m,\, n \in X, \text{ and } \delta_1,\, \delta_2 > 0$$
$$\text{such that } (x,y) \in B_d(m,\delta_1) \times B_d(n,\delta_2) \subseteq B_{d'}((x,y),\varepsilon). \tag{4.3}$$

Let $B_{d'}((x,y),\varepsilon) \in \mathcal{B}_1$ be given. Choose $m = x$, $n = y$, $\delta_1 = \frac{\varepsilon}{2}$, and $\delta_2 = \frac{\varepsilon}{2}$. Then $(x,y) \in B_d(m,\delta_1) \times B_d(n,\delta_2)$. Next, let $(a,b) \in B_d(m,\delta_1) \times B_d(n,\delta_2)$. Then $a \in B_d(x,\frac{\varepsilon}{2})$ and $b \in B_d(y,\frac{\varepsilon}{2})$. Thus $d(a,x) < \frac{\varepsilon}{2}$ and $d(b,y) < \frac{\varepsilon}{2}$. Hence $d'((a,b),\,(x,y)) = d(a,x) + d(b,y) < \varepsilon$. Therefore $(a,\,b) \in B_{d'}((x,\,y),\,\varepsilon)$. This shows that

$$(x,y) \in B_d(m,\delta_1) \times B_d(n,\delta_2) \subseteq B_{d'}((x,y),\varepsilon).$$

Next, we will prove that the product topology is finer than the topology generated by d'. To show this, it suffices to show that every set in $\mathcal{B}_1$ is open in the product topology. Let $B_{d'}((x,y),\,\varepsilon) \in \mathcal{B}_1$ be given. Let $(a,b) \in B_{d'}((x,y),\varepsilon)$. Since $B_{d'}((x,y),\varepsilon)$ is open in $(X \times X, d')$, there is an $\varepsilon_1 > 0$ such that $B_{d'}((a,b),\varepsilon_1) \subseteq B_{d'}((x,y),\varepsilon)$. Apply (4.3) to $B_{d'}((a,b),\varepsilon_1)$, we obtain $m,\, n \in X$, $\delta_1,\, \delta_2 > 0$ such that

$$(a,b) \in B_d(m,\delta_1) \times B_d(n,\delta_2) \subseteq B_{d'}((a,b),\varepsilon_1) \subseteq B_{d'}((x,y),\varepsilon).$$

This shows that $B_{d'}((x,y),\varepsilon)$ is open in the product topology. Next, we will prove that the topology generated by d' is finer than the product topology. To show this, we first prove that

$$\begin{aligned}&\text{for each } B_d(m,\delta_1)\times B_d(n,\delta_2)\in\mathcal{B}_2,\ \text{ there are } x,\ y\in X \text{ and } \varepsilon>0\\ &\text{such that } (m,n)\in B_{d'}((x,y),\varepsilon)\subseteq B_d(m,\delta_1)\times B_d(n,\delta_2).\end{aligned} \tag{4.4}$$

Let $B_d(m,\delta_1)\times B_d(n,\delta_2)\in\mathcal{B}_2$ be given. Choose $x=m,\ y=n$ and $\varepsilon=\min\{\delta_1,\ \delta_2\}$. Then $(m,n)\in B_{d'}((x,y),\varepsilon)$. Next, let $(a,b)\in B_{d'}((x,y),\varepsilon)$. Then $d(a,x)+d(b,y)<\varepsilon$. Thus $d(a,x)<\varepsilon\le\delta_1$ and $d(b,y)<\varepsilon\le\delta_2$. Hence $a\in B_d(x,\delta_1)$, $b\in B_d((y,\delta_2)$, and thus $(a,b)\in B_d(m,\delta_1)\times B_d(n,\delta_2)$. This shows that

$$(m,n)\in B_{d'}((x,y),\varepsilon)\subseteq B_d(m,\delta_1)\times B_d(n,\delta_2).$$

Next, let $B_d(m,\delta_1)\times B_d(n,\delta_2)\in\mathcal{B}_2$ be given. Let $(a,b)\in B_d(m,\delta_1)\times B_d(n,\delta_2)$. Then there exist $\delta_3,\ \delta_4>0$ such that

$$B_d(a,\delta_3)\subseteq B_d(m,\delta_1)\quad\text{and}\quad B_d(b,\delta_4)\subseteq B_d(n,\delta_2).$$

Therefore $B_d(a,\delta_3)\times B_d(b,\delta_4)\subseteq B_d(m,\delta_1)\times B_d(n,\delta_2)$. Apply (4.4) to $B_d(a,\delta_3)\times B_d(b,\delta_4)$, we obtain $x,\ y\in X$, and $\varepsilon>0$ such that

$$(a,b)\in B_{d'}((x,y),\varepsilon)\subseteq B_d(a,\delta_3)\times B_d(b,\delta_4)\subseteq B_d(m,\delta_1)\times B_d(n,\delta_2).$$

This shows that $B_d(m,\delta_1)\times B_d(n,\delta_2)$ is open in the topology generated by d'. Hence d' generates the product topology, as claimed. $\square$

Next, we show that Cauchy w-distances of the form $f\circ d$ are uniformly continuous if f is a strongly metric-preserving, as shown in the next theorem.

Theorem 4.12. *Let (X,d) be a metric space, and d' defined as in Theorem 4.11. If f is strongly metric-preserving, then $f\circ d$ is a Cauchy w-distance on (X,d) and uniformly continuous on $(X\times X,d')$.*

Proof. We need to prove only the uniform continuity of $f \circ d$. Let $\varepsilon > 0$. Since f is continuous at 0, there exists an $\delta > 0$ such that

$$f(x) < \frac{\varepsilon}{2}, \text{ for all } 0 \leq x < \delta. \tag{4.5}$$

Let (a, b), $(x, y) \in X \times X$ be such that $d'((x, y), (a, b)) < \delta$. That is $d(x, a) + d(y, b) < \delta$. Then $d(x, a) < \delta$ and $d(y, b) < \delta$. By (4.5), we obtain that

$$f(d(x, a)) < \frac{\varepsilon}{2} \quad \text{and} \quad f(d(y, b)) < \frac{\varepsilon}{2}. \tag{4.6}$$

Since $f \circ d(x, y) \leq f \circ d(x, a) + f \circ d(a, b) + f \circ d(b, y)$, we have

$$f \circ d(x, y) - f \circ d(a, b) \leq f \circ d(x, a) + f \circ d(b, y).$$

Also $f \circ d(a, b) \leq f \circ d(a, x) + f \circ d(x, y) + f \circ d(y, b)$, so we obtain that

$$f \circ d(x, y) - f \circ d(a, b) \geq -f \circ d(a, x) - f \circ d(y, b).$$

Hence

$$\begin{aligned} |f \circ d(x, y) - f \circ d(a, b)| &\leq f \circ d(x, a) + f \circ d(y, b) \\ &= f(d(x, a)) + f(d(y, b)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{by (4.6)} \\ &= \varepsilon. \end{aligned}$$

This shows that $f \circ d$ is uniformly continuous. $\square$

The next example shows that Cauchy w-distances need not be uniformly continuous.

Examples 4.13. Let $X = [0, \infty)$, $d(x, y) = |\sqrt{x} - \sqrt{y}|$, and $p(x, y) = |x - y|$, for all x, $y \in X$. Then

(i) d is a metric on X,

(ii) p is a symmetric Cauchy w-distance on (X, d),

(iii) $p(0, \cdot) : (X, d) \to (X, d)$ is not uniformly continuous, and

(iv) $p : (X \times X, d') \to [0, \infty)$ is not uniformly continuous where d' is defined as in Theorem 4.11.

Proof. We will show that d is a metric on X. Let x, $y \in X$.

1) $d(x, y) = |\sqrt{x} - \sqrt{y}| \geq 0$.

2)

$$\begin{aligned} d(x, y) = 0 &\leftrightarrow |\sqrt{x} - \sqrt{y}| = 0 \\ &\leftrightarrow \sqrt{x} = \sqrt{y} \\ &\leftrightarrow x = y. \end{aligned}$$

3) $d(x, y) = |\sqrt{x} - \sqrt{y}| = |\sqrt{y} - \sqrt{x}| = d(y, x)$.

4) Let $z \in X$.

$$\begin{aligned} d(x, z) &= |\sqrt{x} - \sqrt{z}| \leq |\sqrt{x} - \sqrt{y}| + |\sqrt{y} - \sqrt{z}| \\ &= d(x, y) + d(y, z). \end{aligned}$$

Hence d is a metric, which proves (i). Next, we will show that p is a w-distance. It is obvious that $p(x, z) \leq p(x, y) + p(y, z)$ for all x, y, $z \in X$. Let $x_0 \in X$. We will show that $p(x_0, \cdot) : X \to [0, \infty)$ is continuous. Let (x_n) be sequence in X such that (x_n) converge to x (in (X, d)). Then $d(x_n, x)$ converges to 0. That is $|\sqrt{x_n} - \sqrt{x}|$ converges to 0. Therefore $\sqrt{x_n}$ converges to $\sqrt{x}$. Thus (x_n) converges to x (in $\mathbb{R}$ with the usual metric). Then

$$\begin{aligned} |p(x_0, x_n) - p(x_0, x)| &= ||x_0 - x_n| - |x_0 - x|| \\ &\leq |(x_0 - x_n) - (x_0 - x)| \\ &= |x - x_n| \to 0. \end{aligned}$$

Hence $p(x_0, x_n)$ converges to $p(x_0, x)$. This shows that $p(x_0, \cdot)$ is continuous. We will prove condition (iii) of w-distances. Let $\varepsilon > 0$. Since $f : [0, \infty) \to [0, \infty)$ given by $f(x) = \sqrt{x}$ is uniformly continuous, there exists a $\delta > 0$ such that

$$\text{for all } x,\, y \in [0, \infty),\ |x - y| < \delta \text{ implies } |\sqrt{x} - \sqrt{y}| < \varepsilon. \tag{4.7}$$

Choose $\delta' = \dfrac{\delta}{2}$. Let $x,\, y,\, z \in X$ be such that $p(z, x) \leq \delta'$ and $p(z, y) \leq \delta'$. That is $|z - x| \leq \delta'$ and $|z - y| \leq \delta'$. Then $|x - y| \leq |x - z| + |z - y| \leq 2\delta' = \delta$. Hence, by (4.7), we obtain that $|\sqrt{x} - \sqrt{y}| < \varepsilon$. That is $d(x, y) \leq \varepsilon$. Thus p is a w-distance on (X, d). To show that p is Cauchy, let (x_n) be a Cauchy sequence in (X, d). We will show that there exists an $M > 0$ such that $\sqrt{x_n} \leq M$ for all $n \in \mathbb{N}$. Let $\varepsilon = 1$. Since (x_n) is a Cauchy sequence in (X, d), there is an $N > 1$ such that for $m > n \geq N$, $d(x_n, x_m) < 1$. That is for $m > n \geq N$, $|\sqrt{x_n} - \sqrt{x_m}| < 1$. Then for all $n \geq N$ $|\sqrt{x_N} - \sqrt{x_n}| < 1$. Therefore for all $n \geq N$, $\sqrt{x_n} < \sqrt{x_N} + 1$. Let

$$M = \max\{\sqrt{x_N} + 1,\, \sqrt{x_1},\, \ldots,\, \sqrt{x_{N-1}}\}.$$

Then for $n \in \mathbb{N}$, $\sqrt{x_n} \leq M$. Now, we will show that (x_n) is p-Cauchy. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2M} > 0$ and there exists an $N \in \mathbb{N}$ such that for $m > n \geq N$,

$$d(x_n, x_m) < \frac{\varepsilon}{2M}.$$

That is for $m > n \geq N$, $|\sqrt{x_n} - \sqrt{x_m}| < \frac{\varepsilon}{2M}$. Let $m > n \geq N$. Then

$$\begin{aligned}
p(x_n, x_m) &= |x_n - x_m| \\
&= |(\sqrt{x_n} - \sqrt{x_m})(\sqrt{x_n} + \sqrt{x_m})| \\
&= |\sqrt{x_n} - \sqrt{x_m}||\sqrt{x_n} + \sqrt{x_m}| \\
&< \left(\frac{\varepsilon}{2M}\right)(2M) \\
&= \varepsilon.
\end{aligned}$$

This shows that (x_n) is p-Cauchy. Therefore p is a Cauchy w-distance on $(X,\, d)$. The symmetric property of p is obvious. Hence (ii) is satisfied. Next, we will show that $p(0, \cdot) : (X, d) \to [0, \infty)$ is not uniformly continuous. That is there exists an $\varepsilon > 0$ such that for $\delta > 0$ there are $x,\, y \in X$, $d(x, y) < \delta$ and $|p(0, x) - p(0, y)| \geq \varepsilon$. In other words, for each $\varepsilon > 0$ such that for $\delta > 0$ there are $x,\, y \in [0, \infty)$,

$$|\sqrt{x} - \sqrt{y}| < \delta \text{ and } ||x| - |y|| \geq \varepsilon.$$

Let $\varepsilon = 1$, and $\delta > 0$. Choose $x = \left(\frac{\delta}{2} + \frac{1}{\delta}\right)^2$ and $y = \frac{1}{\delta^2}$. Then

$$|\sqrt{x} - \sqrt{y}| = \left|\frac{\delta}{2} + \frac{1}{\delta} - \frac{1}{\delta}\right| = \frac{\delta}{2} < \delta,$$

and

$$\begin{aligned}
||x| - |y|| &= \left|\left(\frac{\delta}{2} + \frac{1}{\delta}\right)^2 - \frac{1}{\delta^2}\right| \\
&= \left|\frac{\delta^2}{4} + 1 + \frac{1}{\delta^2} - \frac{1}{\delta^2}\right| \\
&= \left|\frac{\delta^2}{4} + 1\right| \;>\; 1.
\end{aligned}$$

Thus $p(0, \cdot)$ is not uniformly continuous. This proves (iii). Next, we will prove (iv). Let d' be defined as in Theorem 4.11. That is d' is given by $d'((x, y), (a, b)) =$

$|\sqrt{x}-\sqrt{a}|+|\sqrt{y}-\sqrt{b}|$. From (iii), we know that $p(0,\cdot):(X,d)\to[0,\infty)$ is not uniformly continuous. Therefore there exists an $\varepsilon>0$ such that for each $\delta>0$, there are y, $b\in X$ satisfying $d(y,b)<\delta$ and $|p(0,y)-p(0,b)|<\varepsilon$. That is for each $\delta>0$, there are y, $b\in X$ satisfying

$$|\sqrt{y}-\sqrt{b}|<\delta \quad \text{and} \quad ||y|-|b||\geq\varepsilon. \tag{4.8}$$

To show that $p:(X\times X,d')\to[0,\infty)$ is not uniformly continuous, let $\delta>0$ be given. Then from (4.8), there are y, $b\in X$ such that $|\sqrt{y}-\sqrt{b}|<\delta$ and $||y|-|b||\geq\varepsilon$. Let $x=a=0\in X$. Then $(x,y),(a,b)\in X\times X$ satisfying

$$\begin{aligned} d'((x,y),(a,b)) &= |\sqrt{x}-\sqrt{a}|+|\sqrt{y}-\sqrt{b}| = |\sqrt{y}-\sqrt{b}| < \delta \quad \text{and} \\ |p(x,y)-p(a,b)| &= ||x-y|-|a-b|| = ||y|-|b|| \geq \varepsilon. \end{aligned}$$

This shows that p is not uniformly continuous. □

4.2 W-distance topology

Recall that if (X,d) is a metric space, then the collection

$$\mathcal{B}=\{B_d(x,\varepsilon)\,|\,x\in X,\,\varepsilon>0\}$$

of all d-balls is a basis for a topology on X. This topology is called the **metric topology** or **topology generated by** d, and is denoted by τ_d. In this section, we define the notion of p-topology when p is a w-distance. In particular, we prove that p-topology and the metric topology coincide if p is a Cauchy w-distance. First, we give a theorem which will be used later.

Theorem 4.14. *Let (X, d) be a metric space, and p a continuous w-distance. Let (x_n) be a convergent sequence in X with $\lim x_n = x$. Then both $p(x_n, x)$ and $p(x, x_n)$ converge to $p(x, x)$ (in $\mathbb{R}$). Furthermore, if p is a simple w-distance, then $p(x_n, x)$ and $p(x, x_n)$ converge to 0.*

Proof. Assume that (x_n) is a sequence in X converging to x. By Theorem 4.4, we obtain that (x_n, x) and (x, x_n) converge to (x, x). Since p is simple and continuous, we have $p(x_n, x)$ and $p(x, x_n)$ converge to $p(x, x) = 0$. $\square$

Definition 4.15. *Let (X, d) be metric space and p a w-distance. the* ***p-ball of center x, and radius*** *ε is defined to be the set*

$$Bp(x, \varepsilon) = \{y \in X \mid p(x, y) < \varepsilon\}.$$

The next theorem shows that the collection of all p-balls is a basis for a topology whenever p is a simple w-distance. To do this, we first give a lemma.

Lemma 4.16. *Let (X, d) be a metric space, p a w-distance, and $Bp(x, \varepsilon) = \{y \in X \mid p(x, y) < \varepsilon\}$ the p-ball center at x and radius ε. Then for each $y \in Bp(x, \varepsilon)$ there exists a $\delta > 0$ such that $Bp(y, \delta) \subseteq Bp(x, \varepsilon)$.*

Proof. Let $y \in Bp(x, \varepsilon)$. Then $p(x, y) < \varepsilon$. Let $\delta = \varepsilon - p(x, y)$. Then $\delta > 0$. Claim $Bp(y, \delta) \subseteq Bp(x, \varepsilon)$. Let $z \in Bp(y, \delta)$. Then $p(y, z) < \delta$. Thus

$$p(x, z) \leq p(x, y) + p(y, z) < p(x, y) + \delta = \varepsilon.$$

Hence $z \in Bp(x, \varepsilon)$. $\square$

Theorem 4.17. *Let (X, d) be a metric space and p a simple w-distance. Let $\mathcal{B} = \{Bp(x, \varepsilon) \mid x \in X,\ \varepsilon > 0\}$. Then $\mathcal{B}$ is a basis for a topology on X.*

Proof. Let $x \in X$. Then $p(x, x) \geq 0$ and there is an $n \in \mathbb{N}$ such that $p(x, x) < n$. Therefore $x \in Bp(x, n)$. Let $Bp(x_1, \varepsilon_1)$ and $Bp(x_2, \varepsilon_2)$ be given, and $x \in$

$Bp(x_1, \varepsilon_1) \cap Bp(x_2, \varepsilon_2)$. Then $x \in Bp(x_1, \varepsilon_1)$ and $x \in Bp(x_2, \varepsilon_2)$. By Lemma 4.16, we obtain $\delta_1,\ \delta_2 > 0$ such that

$$Bp(x, \delta_1) \subseteq Bp(x_1, \varepsilon_1) \text{ and}$$
$$Bp(x, \delta_2) \subseteq Bp(x_2, \varepsilon_2).$$

Let $\delta = \min\{\delta_1,\ \delta_2\}$. Then

$$x \in Bp(x, \delta) \subseteq Bp(x, \delta_1) \cap Bp(x, \delta_2) \subseteq Bp(x_1, \varepsilon_1) \cap Bp(x_2, \varepsilon_2).$$

Hence $\mathcal{B}$ is a basis for a topology on X. $\square$

Definition 4.18. *Let (X, d) be a metric space, p a w-distance. If $\{Bp(x, \varepsilon) \mid x \in X,\ \varepsilon > 0\}$ is a basis for a topology, then the topology it generates is called* ***p-topology****.*

Examples 4.19. Let p_1 and p_2 be w-distances defined in Example 2.6, and 2.7, respectively. Then p_1 generated the metric topology and p_2 generated indiscrete topology.

Examples 4.20. Let p be the w-distance defined in Example 2.13. then for each $x \in X$, $\varepsilon > 0$, we have

$$\begin{aligned} Bp(x, \varepsilon) \ = \ \{y \in X \mid p(x, y) < \varepsilon\} \ &= \ \{y \in X \mid \ \|y\| < \varepsilon\} \\ &= \ B_d(0, \varepsilon) \end{aligned}$$

where d is the metric induced from $\|\cdot\|$. In particular when $X = \mathbb{R}$ and $p(x, y) = |y|$, $B_p(x, \varepsilon) = (-\varepsilon, \varepsilon)$ for any $x \in \mathbb{R}$, and $\varepsilon > 0$. Then

$$p - \text{topology} = \{\varnothing\} \cup \{(-\varepsilon, \varepsilon) \mid \varepsilon > 0\} \cup \{\mathbb{R}\}.$$

From Example 4.19 and 4.20, we can see that p-topology may finer, or coaser than the metric topology τ_d, or it may be the same as the metric topology τ_d. The next theorem assert that p-topology and topology generated by d coincide if p is a Cauchy w-distance on (X, d).

Theorem 4.21. *Let (X, d) be a metric space, and p a Cauchy w-distance on (X, d). Then p and d generate the same topology.*

Proof. First, we will prove the following statements :

(i) for all $x \in X$, $\varepsilon > 0$ there is a $\delta > 0$ such that $Bp(x, \delta) \subseteq B_d(x, \varepsilon)$, and

(ii) for all $x \in X$, $\varepsilon > 0$ there is a $\delta > 0$ such that $B_d(x, \delta) \subseteq Bp(x, \varepsilon)$.

(i) Let $x \in X$ and $\varepsilon > 0$. Since p is a w-distance, there is a $\delta > 0$ such that

$$\text{for } x,\ y,\ z \in X,\ p(z, x) \leq \delta \text{ and } p(z, y) \leq \delta \text{ imply } d(x, y) \leq \frac{\varepsilon}{2}. \tag{4.9}$$

To show $Bp(x, \delta) \subseteq B_d(x, \varepsilon)$, let $y \in Bp(x, \delta)$. Then $p(x, y) \leq \delta$. Also $p(x, x) = 0 \leq \delta$ since p is simple. By (4.9), we obtain that $d(x, y) \leq \frac{\varepsilon}{2} < \varepsilon$. Therefore $y \in B_d(x, \varepsilon)$.

(ii) Let $x \in X$ and $\varepsilon > 0$. Suppose that for $\delta > 0$, $B_d(x, \delta) \nsubseteq Bp(x, \varepsilon)$. Then for each $n \in \mathbb{N}$, there exists $y_n \in B_d(x, \frac{1}{n}) - Bp(x, \varepsilon)$. Then (y_n) is a sequence in X such that

$$d(x, y_n) \leq \frac{1}{n} \quad \text{and} \quad p(x, y_n) \geq \varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{4.10}$$

From (4.10), $d(x, y_n) \leq \frac{1}{n}$, so (y_n) converges to x. Since p is simple and continuous, by Theorem 4.14, we have $p(x, y_n)$ converges to 0. This is impossible, since $p(x, y_n) \geq \varepsilon$ for all $n \in \mathbb{N}$. Hence there is a $\delta > 0$ such that $B_d(x, \delta) \subseteq Bp(x, \varepsilon)$. Now we will show that $\tau_d = p$-topology. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. We will show that $B_d(x, \varepsilon)$ is open in the p-topology. That is there is a $\delta > 0$ such that $Bp(x, \delta) \subseteq B_d(x, \varepsilon)$. Let $y \in B_d(x, \varepsilon)$. Since $B_d(x, \varepsilon)$ is open in τ_d, there exists an

$\varepsilon_1 > 0$ such that $B_d(y, \varepsilon_1) \subseteq B_d(x, \varepsilon)$. Applying the condition (i) to $B_d(y, \varepsilon_1)$, we obtain a $\delta > 0$ such that $Bp(y, \delta) \subseteq B_d(y, \varepsilon_1)$. Since $p(y, y) = 0$, therefore

$$y \in Bp(y, \delta) \subseteq B_d(y, \varepsilon_1) \subseteq B_d(x, \varepsilon).$$

Thus $B_d(x, \varepsilon)$ is open in the p-topology. This implies that $\tau_d \subseteq p$-topology. Now, we will show that $Bp(x, \varepsilon)$ is open in τ_d. Let $y \in Bp(x, \varepsilon)$. By Lemma 4.16, there exists an $\varepsilon_1 > 0$ such that $Bp(y, \varepsilon_1) \subseteq Bp(x, \varepsilon)$. Applying (ii) to $Bp(y, \varepsilon_1)$, we obtain a $\delta > 0$ such that $B_d(y, \delta) \subseteq Bp(y, \varepsilon_1)$. Then

$$y \in B_d(y, \delta) \subseteq Bp(y, \varepsilon_1) \subseteq Bp(x, \varepsilon).$$

Thus $Bp(x, \varepsilon)$ is open in τ_d. This implies that p-topology is contained in τ_d. $\square$

4.3 Characterization of Cauchy w-distances

We give a characterization of Cauchy w-distances and its consequences. Recall that if p is a Cauchy w-distance on a metric space (X, d), then p is simple and continuous. Example 4.3 shows that a simple w-distance need not be Cauchy. In addition, Example 4.10 shows that a continuous w-distance may not be Cauchy. However, if a w-distance p is both continuous and simple and (X, d) is complete, then p is a Cauchy w-distance, as shown in the next theorem.

Theorem 4.22 (Characterization of Cauchy w-distances). *Let (X, d) be a complete metric space, p a w-distance on (X, d). Then p is Cauchy if and only if p is simple and continuous.*

Proof. By Theorem 4.2 and Theorem 4.8, it suffices to prove only the converse. Assume that p is simple and continuous. Let (x_n) be a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, x_n converges to a point $x \in X$. By Theorem 4.14, $p(x_n, x)$ converges to 0 and $p(x, x_n)$ converges to 0. To show that

(x_n) is a p-Cauchy sequence, let $\varepsilon > 0$. Since $p(x_n, x)$ and $p(x, x_n)$ converge to 0, we have an $N \in \mathbb{N}$ such that for $n \geq N$ $p(x_n, x) < \frac{\varepsilon}{2}$ and $p(x, x_n) < \frac{\varepsilon}{2}$. Let $m > n \geq N$,

$$p(x_n, x_m) \leq p(x_n, x) + p(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus (x_n) is a p-Cauchy sequence. This implies that p is Cauchy. □

Corollary 4.23. *Let (X, d) be a compact metric space. Then every metric d' equivalent to d, is a Cauchy w-distance on (X, d).*

Proof. From Theorem 2.22, we obtain that d' is a w-distance on (X, d). Since (X, d) is compact, (X, d) is complete. Therefore, we can apply Theorem 4.22 to d' and (X, d). It is clear that d' is a simple w-distance. Next, consider $d' : (X, d) \times (X, d) \to [0, \infty)$. We will use Theorem 4.7 to prove the continuity of d'. Let (x_n) and (y_n) be sequences in X converging to x and y in (X, d), respectively. By Lemma 2.20, (x_n) and (y_n) also converge to x and y in (X, d'). Therefore (x_n, y_n) converges to (x, y) in $(X, d') \times (X, d')$. Hence $d'(x_n, y_n)$ converges to $d'(x, y)$. This shows that d' is continuous. Thus by Theorem 4.22, d' is Cauchy. □

Theorem 4.24 ([2], p. 131). *If X is a topological space, and if f, $g : X \to \mathbb{R}$ are continuous functions, then $f + g$, $f - g$, and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all x, then $\frac{f}{g}$ is continuous.*

Corollary 4.25. *Let p_1 and p_2 are Cauchy w-distances on a complete metric space. Then $p_1 + p_2$, $\alpha p_1 (\alpha \geq 0)$, and $\max\{p_1, p_2\}$ are Cauchy w-distances.*

Proof. From Theorem 2.14, we know that $p_1 + p_2$, αp, and $\max\{p_1, p_2\}$ are w-distances. Since p_1 and p_2 are Cauchy, they are simple and continuous. It is easy to see that $p_1 + p_2$, αp, and $\max\{p_1, p_2\}$ are simple. By Theorem 4.24, we see that $p_1 + p_2$, αp_1, and $\max\{p_1, p_2\} = \frac{|p_1 + p_2| + |p_1 - p_2|}{2}$ are continuous. Therefore all of them are Cauchy. □

Corollary 4.26. *Let X be a complete metric space, let p be a Cauchy w-distance on X and let f be a function from X into $[0, \infty)$. Then a function g, $q : X \times X \to [0, \infty)$ defined by*

$$\begin{aligned} g(x, y) &= \max\{f(x), p(x, y)\} \quad \textit{for all } x, y \in X, \textit{ and} \\ q(x, y) &= f(x) + p(x, y) \qquad \textit{for all } x, y \in X. \end{aligned}$$

Then the following are equivalent.

(i) g *is Cauchy,*

(ii) g *is simple,*

(iii) $f(x) = 0$ *for all* $x \in X$

(iv) q *is Cauchy,*

(v) q *is simple.*

Proof. (i)→(ii) and (iv)→(v) are already proved in Theorem 4.2.

$$\begin{aligned} g \text{ is simple } &\to g(x, x) = 0 \quad \text{for all } x \in X \\ &\to \max\{f(x), p(x, x)\} = 0 \quad \text{for all } x \in X \\ &\to \max\{f(x), 0\} = 0 \quad \text{for all } x \in X \\ &\to f(x) = 0 \quad \text{for all } x \in X. \end{aligned}$$

This means (ii) implies (iii).

(iii)→(iv) is obvious.

Now, $$\begin{aligned} q \text{ is simple} &\rightarrow q(x,x) = 0 \quad \text{for all } x \in X \\ &\rightarrow f(x) + p(x,x) = 0 \quad \text{for all } x \in X \\ &\rightarrow f(x) = 0 \quad \text{for all } x \in X \\ &\rightarrow g = p \\ &\rightarrow g \text{ is Cauchy.} \end{aligned}$$

So (v) implies (i). $\square$

Conclusion

The following statements are the conclusion obtained from our investigation.

1. p_1, p_2 are w-distances $\Rightarrow$ p_1+p_2, $\max\{p_1, p_2\}$, and $\alpha p_1(\alpha \geq 0)$ are w-distances.
2. (X, d) is compact and $d' \sim d$ $\Rightarrow$ d' is a Cauchy w-distance on (X, d).
3. f is lower semicontinuous and metric-preserving $\Rightarrow$ $f \circ d$ is a w-distance on (X, d). In particular, f is strongly metric-preserving $\Rightarrow$ $f \circ d$ is a w-distance on (X, d).
4. f is strongly metric preserving $\Rightarrow$ $f \circ d$ is a Cauchy w-distance and uniformly continuous on $(X \times X, d')$.
5. If f is a metric-preserving function, then

   (i) f is not strongly metric-preserving and $f \circ d$ is Cauchy $\Rightarrow$ (X, τ_d) is a discrete space; and

   (ii) d is uniformly discrete $\Rightarrow$ $f \circ d$ is a Cauchy w-distance.

6. p is a Cauchy w-distance on (X, d) $\Rightarrow$ (i) p is simple,
   (ii) p is continuous, and
   (iii) p and d generate the same topology.
7. If (X, d) is complete, then (p is Cauchy $\Leftrightarrow$ p is simple and continuous).
8. p_1, p_2 are Cauchy w-distances and (X, d) is complete $\Rightarrow$ $p_1 + p_2, \max\{p_1, p_2\}$, and $\alpha p_1(\alpha \geq 0)$ are Cauchy w-distances.

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