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ALGEBRAIC DEGREE OF SPECTRA OF CAYLEY HYPERGRAPHS

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ให้ $(G, \cdot)$ เป็นกรุปจำกัดที่มีสมาชิกเอกลักษณ์ e และ S เป็นสับเซตของ $G \setminus \{e\}$ ที่ $S = S^{-1}$ สำหรับ $t \in \mathbb{N}$ และ $2 \leq t \leq \max\{o(x) : x \in S\}$ เรานิยาม t-ไฮเพอร์กราฟเคย์เลย์ ของ G บน S ว่าเป็นไฮเพอร์กราฟที่มีเซตของจุดยอดคือ G และเซตของเส้นเชื่อมคือ $\{\{yx^i : 0 \leq i \leq t-1\} : x \in S$ และ $y \in G\}$ ในวิทยานิพนธ์นี้ เราศึกษาสมบัติเชิงสเปกตรัมบางประการของไฮเพอร์กราฟนี้ เราให้ลักษณะเฉพาะของ 2-ไฮเพอร์กราฟเคย์เลย์ของ G เมื่อ G เป็นกรุปสลับที่จำกัด นอกเหนือจากนี้ เรายังได้วิธีการคำนวณดีกรีเชิงพีชคณิตของสเปกตรัมของ t-ไฮเพอร์กราฟเคย์เลย์ของ $\mathbb{Z}_n$

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....
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Let $(G, \cdot)$ be a finite group with the identity e and S a subset of $G \smallsetminus \{e\}$ such that $S = S^{-1}$. For $t \in \mathbb{N}$ and $2 \leq t \leq \max\{o(x) : x \in S\}$, the t-Cayley hypergraph of G over S is the hypergraph whose vertex set is G and edge set is $\{\{yx^i : 0 \leq i \leq t-1\} : x \in S \text{ and } y \in G\}$. In this thesis, we study spectral properties of this hypergraph. We characterize integral 2-Cayley hypergraphs of G when G is abelian. In addition, we obtain the algebraic degree of t-Cayley hypergraphs of $\mathbb{Z}_n$.

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CONTENTS

page

ABSTRACT IN THAI iv

ABSTRACT IN ENGLISH v

ACKNOWLEDGEMENTS vi

CONTENTS vii

LIST OF FIGURES viii

CHAPTER

I PRELIMINARIES 1

1.1 Cayley graphs 1

1.2 Spectra of circulant matrices 3

1.3 Hypergraphs 5

1.4 Background in algebra 11

1.5 Objectives 14

II t-CAYLEY HYPERGRAPHS 16

2.1 t-Cayley hypergraphs 16

2.2 Integral Cayley graphs 19

III ALGEBRAIC DEGREE OF SPECTRA OF t-CAYLEY HYPERGRAPHS 21

3.1 Integral t-Cayley hypergraphs of $\mathbb{Z}_n$ 21

3.2 Algebraic degree of spectra of t-Cayley hypergraphs of $\mathbb{Z}_n$ 28

3.3 Algebraic degree of spectra of Cayley graphs of $\mathbb{Z}_n$ 33

REFERENCES 37

VITA 39

LIST OF FIGURES

Figure		page
1.1	A hypergraph H	6
1.2	Hypergraphs H_1 and H_2	10
1.3	A hypergraph $\mathrm{H}_1 \,\square\, \mathrm{H}_2$	11
2.1	$\mathrm{H}_1 = 2\text{-Cay}(\mathbb{Z}_6, S)$	17
2.2	$\mathrm{H}_2 = 3\text{-Cay}(\mathbb{Z}_6, S)$	17

CHAPTER I
PRELIMINARIES

This chapter contains some terminologies and backgrounds from algebraic graph and hypergraph theory, linear algebra, finite abelian groups, and field extensions. We also discuss many elementary results on hypergraphs.

1.1 Cayley graphs

We recall some terminologies of spectra of graphs. Let G be a graph with the vertex set $\{v_1, \ldots, v_n\}$. The *adjacency matrix* of G, denoted by $A(\mathrm{G})$, is the $n \times n$ matrix whose entry $a_{ij} = 1$ when v_i and v_j are adjacent and $a_{ij} = 0$ otherwise for all $1 \leq i, j \leq n$. The *spectrum* of a graph G, denoted by Spec(G), is the multi-set of all eigenvalues of its adjacency matrix including multiplicity. A graph is called *integral* if all eigenvalues are integers.

Let $(G, \cdot)$ be a finite group with the identity e and S a subset of $G \setminus \{e\}$ such that $S = S^{-1}$. The *Cayley graph of G over S* is the graph whose vertex set is G and for any $x, y \in G$, x and y are adjacent if and only if $y^{-1}x \in S$. Next, we discuss spectra of Cayley graphs.

Cayley graphs, as known as Cayley color graphs or Cayley color diagrams, were first introduced by Cayley [4] in 1878. They have been regularly studied and have many applications. Harary and Schwenk [10] asked "Which graphs have integral spectra?". From this question, the integral Cayley graphs have been widely studied, e.g., [15], [12], [13], [14] and [22]. For a finite commutative ring $(R, +, \cdot)$, a well-studied Cayley graph of $(R, +)$ over S is to set $S = R^{\times}$ where $R^{\times}$ denoted the set of all units in R and is called the *unitary Cayley graph of R*. This graph has the integral spectrum. Klotz and Sander [15] studied combinatorial properties of the unitary Cayley graph of $\mathbb{Z}_n$. They explored the chromatic number, the clique number, the independence number, the diameter and the vertex connectivity of this graph. In addition, they showed that the gcd-graphs are integral (gcd-graphs are introduced in Section 2.1). A few year later, Ilić [12] determined the energy of the unitary Cayley graph of $\mathbb{Z}_n$ which is the sum of absolute values of its eigenvalues. He also provided that the energy of the unitary Cayley graph of $\mathbb{Z}_n$ is greater than $2n-2$. Kiani et al. [14] worked on the eigenvalues of the unitary Cayley graph of finite local rings and extended the result to finite commutative rings. So [22] completely characterized integral Cayley graphs of $(\mathbb{Z}_n, +)$. He showed that the Cayley graph of $\mathbb{Z}_n$ over S is integral if and only if S is a union of some $G_n(d)$'s, where $d \mid n$ and $G_n(d) = \{k \in \{1, 2, \ldots, n-1\} : \gcd(k, n) = d\}$. This result is important on spectra of Cayley graphs and there are many works from So which study spectra of Cayley graphs in other approaches.

For non-integral graphs, Mönius et al. [17] defined the *algebraic degree* of a graph G to be the degree extension of the splitting field of the characteristic polynomial of its adjacency matrix $A(\mathrm{G})$ over $\mathbb{Q}$. They studied a relation between the diameter of arbitrary graph and its algebraic degree (the diameter is defined in Section 1.3). They showed that a graph with large diameter has large algebraic degree. Later, Mönius [18] determined the algebraic degree of Cayley graphs of $\mathbb{Z}_p$ where p is a prime number. He showed that the algebraic degree of the Cayley graphs of $\mathbb{Z}_p$ over S is $\frac{p-1}{m}$ where m is the maximum number of $M \in \{1, 2, \ldots, |S|\}$ such that M divides $\gcd(|S|, p-1)$ and $S = \bigcup_{l=1}^{|S|/M} S_l$ where $|S_l| = M$ and for each $l \in \{1, 2, \ldots, |S|/M\}$, $k^M = (k')^M \mod p$ for all $k, k' \in S_l$ by using Galois theory. Recently, Mönius [19] extended his work to Cayley graphs of $\mathbb{Z}_n$. He studied other properties of spectra of Cayley graphs and provided a deep connection between Schur rings and the splitting fields of Cayley graphs of $\mathbb{Z}_n$. By using this connection, the algebraic degree of Cayley graphs of $\mathbb{Z}_n$ is demonstrated (see more details in Section 3.3).

For a generalization of the Cayley graphs, Buratti [3] extended the notion of Cayley graphs to Cayley hypergraphs in 1994 as mentioned in Section 2.1. Since

Cayley hypergraphs are generalizations of Cayley graphs and we have known from the above discussion that the integrality and the algebraic degree of Cayley graphs are well-studied, these reasons motivate us to attempt results on the integral Cayley hypergraphs and their algebraic degree.

1.2 Spectra of circulant matrices

Throughout this section, we let $n \in \mathbb{N}$ and a matrix $A = [a_{ij}]_{n\times n}$. A matrix A is called a *symmetric matrix* if $a_{ij} = a_{ji}$ for all $1 \leq i, j \leq n$. The *spectrum* of A, denoted by $\mathrm{Spec}(A)$, is a multi-set of all eigenvalues of A including multiplicities. It is well-known that all eigenvalues of a real symmetric matrix are real, see [9]. Hence, its spectrum contains only real eigenvalues recorded in the following theorem.

Theorem 1.2.1. *The spectrum of a real symmetric matrix contains only real eigenvalues.*

Example 1.2.2. Let $A = \begin{bmatrix} 0 & 1 & 3 & 0 & 5 \\ 1 & -1 & 2 & 0 & 0 \\ 3 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 5 & 0 & 1 & 2 & 0 \end{bmatrix}$. By computing its eigenvalues, we have $\mathrm{Spec}(A) = \{7.46, 1.42, -0.04, -2.11, -5.74\}$. Note that all of the eigenvalues are approximated by rounding these numbers to two decimal places.

A *circulant matrix* is a square matrix in which each row is obtained by a right cyclic shift of the preceding row. In other word, a matrix is circulant if and only if it is in the following form

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}.$$

Note that if A is a circulant matrix, then it suffices to know only the first row of A. From now on, we shall write only the first row of a circulant matrix A. We now find the spectrum of a circulant matrix A. For any $j \in \{0, 1, \ldots, n-1\}$, we let $\mathbf{v}_j = \begin{bmatrix} 1 & e^{2\pi ji/n} & (e^{2\pi ji/n})^2 & \cdots & (e^{2\pi ji/n})^{n-1} \end{bmatrix}^t$ where i is the imaginary unit defined by $i^2 = -1$. Then

$$\begin{aligned} A\mathbf{v}_j &= \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix} \begin{bmatrix} 1 \\ e^{2\pi ji/n} \\ (e^{2\pi ji/n})^2 \\ \cdots \\ (e^{2\pi ji/n})^{n-1} \end{bmatrix} \\ &= \sum_{k=0}^{n-1} a_k (e^{2\pi ji/n})^k \begin{bmatrix} 1 \\ e^{2\pi ji/n} \\ (e^{2\pi ji/n})^2 \\ \cdots \\ (e^{2\pi ji/n})^{n-1} \end{bmatrix} \\ &= \lambda_j \mathbf{v}_j \end{aligned}$$

where $\lambda_j = \sum_{k=0}^{n-1} a_k (e^{2\pi ji/n})^k$ for all $j \in \{0, 1, \ldots, n-1\}$. We conclude this result in the theorem below.

Theorem 1.2.3. *The spectrum of a circulant matrix with the first row*

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix}$$

is the multi-set $\{\lambda_j : j \in \{0,1,\dots,n-1\}\}$ *where* $\lambda_j = \sum_{k=0}^{n-1} a_k(e^{2\pi ji/n})^k$ *for all* $j \in \{0,1,\dots,n-1\}$.

Example 1.2.4. Let A be a circulant matrix with the first row $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \end{bmatrix}$. This means $a_0 = 0, a_1 = 1, a_2 = 0, a_3 = 0, a_4 = 1$ and hence

$$\begin{aligned}
\lambda_0 &= \sum_{k=0}^{4} a_k(e^{2\pi(0)i/5})^k = \sum_{k=0}^{4} a_k = 2\\
\lambda_1 &= \sum_{k=0}^{4} a_k(e^{2\pi(1)i/5})^k = \sum_{k=0}^{4} a_k(e^{2\pi i/5})^k = e^{2\pi i/5} + (e^{2\pi i/5})^4 = 2\cos\left(\frac{2\pi}{5}\right)\\
\lambda_2 &= \sum_{k=0}^{4} a_k(e^{2\pi(2)i/5})^k = \sum_{k=0}^{4} a_k(e^{4\pi i/5})^k = e^{4\pi i/5} + (e^{4\pi i/5})^4 = 2\cos\left(\frac{4\pi}{5}\right)\\
\lambda_3 &= \sum_{k=0}^{4} a_k(e^{2\pi(3)i/5})^k = \sum_{k=0}^{4} a_k(e^{6\pi i/5})^k = e^{6\pi i/5} + (e^{6\pi i/5})^4 = 2\cos\left(\frac{6\pi}{5}\right)\\
\lambda_4 &= \sum_{k=0}^{4} a_k(e^{2\pi(4)i/5})^k = \sum_{k=0}^{4} a_k(e^{8\pi i/5})^k = e^{8\pi i/5} + (e^{8\pi i/5})^4 = 2\cos\left(\frac{8\pi}{5}\right).
\end{aligned}$$

By Theorem 1.2.3,

$$\operatorname{Spec}(A) = \left\{2, 2\cos\left(\frac{2\pi}{5}\right), 2\cos\left(\frac{4\pi}{5}\right), 2\cos\left(\frac{6\pi}{5}\right), 2\cos\left(\frac{8\pi}{5}\right)\right\}.$$

1.3 Hypergraphs

This section contains terminologies about hypergraphs following [1]. This includes the adjacency, Laplacian and distance matrix of a hypergraph. We discuss spectra, L-spectra and D-spectra of hypergraphs. In addition, the spectra of product hypergraphs are presented at the end of this section.

A *hypergraph* H is a pair $(V(\mathrm{H}), E(\mathrm{H}))$, where $V(\mathrm{H})$ is a finite set, called the *vertex set* of H, and $E(\mathrm{H})$ is a family of subsets of $V(\mathrm{H})$, called the *edge set* of H. The elements in $V(\mathrm{H})$ are called *vertices* and the elements in $E(\mathrm{H})$ are called *hyperedges*. In particular, if $E(\mathrm{H})$ consists only of 2-subsets of $V(\mathrm{H})$, then H is a simple graph. For $v \in V(\mathrm{H})$, we write $\mathfrak{D}(v)$ for the set of all hyperedges containing the vertex v and the number of elements in $\mathfrak{D}(v)$ is the *degree* of the vertex v, denoted by $\deg v$. A hypergraph in which all vertices have the same degree $k \geq 0$ is called *k-regular* and it is said to be *regular* if it is k-regular for some $k \geq 0$. A hypergraph in which all hyperedges have the same cardinality $l \geq 0$ is an *l-uniform* hypergraph. A *path* of length s in H is an alternating sequence $v_1E_1v_2E_2v_3 \ldots v_sE_sv_{s+1}$ of distinct vertices $v_1, v_2, \ldots, v_{s+1} \in V(\mathrm{H})$ and distinct hyperedges $E_1, E_2, \ldots, E_s \in E(\mathrm{H})$ satisfying $v_i, v_{i+1} \in E_i$ for any $i \in \{1, 2, \ldots, s\}$. The *distance* between two vertices v and w, denoted by $d(v, w)$, is the smallest length of a path from v to w. If there is no path from v to w, we define $d(v, w) = \infty$. The *diameter* of H is $\mathrm{diam}(\mathrm{H}) = \max\{d(v, w) : v, w \in V(\mathrm{H})\}$. A hypergraph H is *connected* if $\mathrm{diam}(\mathrm{H}) < \infty$.

Example 1.3.1. An example of a hypergraph H is shown in the following figure. The vertex set of H is $\{v_1, v_2, \ldots, v_6\}$ and the edge set of H is $\{E_1, E_2, E_3, E_4\}$ where $E_1 = \{v_1, v_2, v_3\}, E_2 = \{v_4, v_5\}, E_3 = \{v_3, v_4, v_5\}$ and $E_4 = \{v_5, v_6\}$. Since $|E_1| = |E_3| = 3$ and $|E_2| = |E_4| = 2$, we have that H is not uniform. Note that $\deg v_1 = \deg v_2 = \deg v_6 = 1$, $\deg v_3 = \deg v_4 = 2$ and $\deg v_5 = 3$. Then H is not regular. Moreover, it is easy to check that $\mathrm{diam}(\mathrm{H}) = 3$ and hence H is connected.

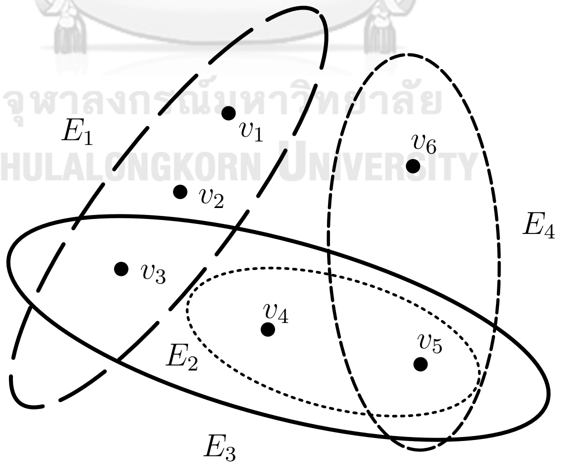

Figure 1.1: A hypergraph H

From the above discussion, we have known some structural definitions of hypergraphs, we shall move to spectral properties of hypergraphs. We start with the spectrum of a hypergraph as follows.

For a hypergraph H with vertex set $\{v_1, \ldots, v_n\}$, the *adjacency matrix* of H, denoted by $A(\mathrm{H})$, is the $n \times n$ matrix whose entry a_{ij}, $i \neq j$, is the number of hyperedges that contain both of v_i and v_j and $a_{ii} = 0$ for all $1 \leq i, j \leq n$.

This concept was investigated by Bretto [2]. Evidently, it is a generalization of the adjacency matrix of a graph. An equivalent definition of the adjacency matrix is given in [8] by using the bipartite graph associated to H which is the graph whose vertex set is the union of two independent sets $V(\mathrm{H})$ and $E(\mathrm{H})$ and for any $v \in V(\mathrm{H})$ and $E \in E(\mathrm{H})$, they are adjacent whenever $v \in E$. In particular, if H is an l-uniform hypergraph, there is another way to define an adjacency matrix by using hypermatrix, see [5] and [11]. In this work, our hypergraphs may not be l-uniform, so we follow Bretto's.

The adjacency matrix is one of matrices represented by a hypergraph. There are other matrices that can be used to explain some properties of a hypergraph e.g., Laplacian matrix and distance matrix. They are also related to spectral properties of a hypergraph. This version of Laplacian matrix was introduced by Rodríguez [21].

For a hypergraph H with vertex set $\{v_1, \ldots, v_n\}$, the *Laplacian matrix* of H, denoted by $L(\mathrm{H})$, is the $n \times n$ matrix defined by $L(\mathrm{H}) = \mathcal{D}(\mathrm{H}) - A(\mathrm{H})$ where $\mathcal{D}(\mathrm{H})$ is the diagonal matrix $\left[\deg v_i\right]_{1 \leq i \leq n}$. Moreover, if H is connected, the *distance matrix* of H, denoted by $D(\mathrm{H})$, is the $n \times n$ matrix in which entry $d_{ij} = d(v_i, v_j)$ for all $1 \leq i, j \leq n$.

Example 1.3.2. Let H be a hypergraph defined in Figure 1.1. Then

$$A(\mathrm{H}) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, L(\mathrm{H}) = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & -1 & -2 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

and $$D(\mathrm{H}) = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 2 & 2 & 3 \\ 1 & 1 & 0 & 1 & 1 & 2 \\ 2 & 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 1 & 0 & 1 \\ 3 & 3 & 2 & 2 & 1 & 0 \end{bmatrix}.$$

The *spectrum* of H, denoted by Spec(H), is the multi-set of all eigenvalues of $A(\mathrm{H})$ including multiplicity. Similarly, we can define Lspec(H) and Dspec(H) as the multi-sets of all eigenvalues of $L(\mathrm{H})$ and $D(\mathrm{H})$, respectively.

Observe that $A(\mathrm{H})$ is a real symmetric matrix, so Spec(H) contains only real eigenvalues by Theorem 1.2.1. By the definition of the adjacency matrix $A(\mathrm{H})$, we have known that the diagonal entries of $A(\mathrm{H})$ are zero. Then the characteristic polynomial of $A(\mathrm{H})$ is monic with integral coefficients, so its rational roots are integers. From this fact, a hypergraph which its spectrum contains only integral eigenvalues is defined to be an integral hypergraph.

A hypergraph is *integral* if all eigenvalues of this hypergraph are integers. Also, an L-*integral hypergraph* is a hypergraph with integral Laplacian eigenvalues and a D-*integral hypergraph* is a hypergraph with integral distance eigenvalues.

Example 1.3.3. Let H be the hypergraph defined in Example 1.3.2. We have the following results by computing the eigenvalues of $A(\mathrm{H})$, $L(\mathrm{H})$ and $D(\mathrm{H})$, respectively.

1. $\mathrm{Spec}(\mathrm{H}) = \{3.10, 1.52, 0.07, -1, -1.44, -2.24\}$

2. $\mathrm{Lspec}(\mathrm{H}) = \{4.76, 3.29, 2, 1.11, 0, -1.15\}$

3. $\mathrm{Dspec}(\mathrm{H}) = \{8.60, -0.57, -0.83, -1, -1.88, -4.31\}$

Hence, H is not integral, not L-integral and not D-integral.

Example 1.3.4. Let H be a hypergraph with a vertex set $V(\mathrm{H}) = \{v_1, v_2, v_3, v_4\}$ and $E(\mathrm{H}) = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}$. Then

$$A(\mathrm{H}) = \begin{bmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{bmatrix}, L(\mathrm{H}) = \begin{bmatrix} 3 & -2 & -2 & -2 \\ -2 & 3 & -2 & -2 \\ -2 & -2 & 3 & -2 \\ -2 & -2 & -2 & 3 \end{bmatrix}, \text{ and } D(\mathrm{H}) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

By computing the eigenvalues of $A(\mathrm{H}), L(\mathrm{H})$ and $D(\mathrm{H})$, we have

1. $\mathrm{Spec}(\mathrm{H}) = \{6, -2, -2, -2\}$,
2. $\mathrm{Lspec}(\mathrm{H}) = \{-3, 5, 5, 5\}$, and
3. $\mathrm{Dspec}(\mathrm{H}) = \{3, -1, -1, -1\}$.

Hence, H is integral, L-integral and D-integral.

Several properties of hypergraphs have been studied such as diameter, connectivity and chromatic number. Spectral and combinatorial properties of hypergraphs are widely related (see for example [6], [8], [15] and [21]). Feng and Li [8] showed the relation between the diameter of H and its eigenvalues. They proved that if $\{\mathrm{H}_n\}_{n\in\mathbb{N}}$ is a collection of k-regular and l-uniform hypergraphs with $\lim_{n\to\infty} |V(\mathrm{H}_n)| = \infty$, then $\lim_{n\to\infty} \mathrm{diam}(\mathrm{H}_n) = \infty$ by using the second largest eigenvalue of H_n. Later, Rodríguez [21] showed that if $b + 1$ is the number of distinct Laplacian eigenvalues of a connected hypergraph H, then $\mathrm{diam}(\mathrm{H}) \leq b$.

Now, we have known the way to compute spectrum of hypergraphs and some related works. We next give the spectrum of some products of hypergraphs. In this thesis, we focus only Cartesian and tensor products of hypergraphs. These two products will be used to classify integral Cayley graphs in Theorem 2.2.5.

For hypergraphs H_1 and H_2, the *Cartesian product* of H_1 and H_2, denoted by $\mathrm{H}_1 \,\square\, \mathrm{H}_2$, is the hypergraph with $V(\mathrm{H}_1 \,\square\, \mathrm{H}_2) = V(\mathrm{H}_1) \times V(\mathrm{H}_2)$ and $E(\mathrm{H}_1 \,\square\, \mathrm{H}_2) = \{\{x\} \times E' : x \in V(\mathrm{H}_1), E' \in E(\mathrm{H}_2)\} \cup \{E \times \{y\} : E \in E(\mathrm{H}_1) \text{ and } y \in V(\mathrm{H}_2)\}$.

Observe that $A(\mathrm{H}_1 \square \mathrm{H}_2) = (A(\mathrm{H}_1) \otimes I_{|V(\mathrm{H}_2)|}) + (I_{|V(\mathrm{H}_1)|} \otimes A(\mathrm{H}_2))$ where $A \otimes B$ denotes the Kronecker product of matrices A and B. Therefore,

$$\mathrm{Spec}(\mathrm{H}_1 \square \mathrm{H}_2) = \{\lambda + \beta : \lambda \in \mathrm{Spec}(\mathrm{H}_1) \text{ and } \beta \in \mathrm{Spec}(\mathrm{H}_2)\}. \tag{1.1}$$

Let H_1 and H_2 be t-uniform hypergraphs. Following Pearson [20], the *tensor product* of H_1 and H_2, denoted by $\mathrm{H}_1 \otimes \mathrm{H}_2$, is the t-uniform hypergraph with $V(\mathrm{H}_1 \otimes \mathrm{H}_2) = V(\mathrm{H}_1) \times V(\mathrm{H}_2)$ and $E(\mathrm{H}_1 \otimes \mathrm{H}_2) = \{\{(x_{i_1}, y_{j_1}), \ldots, (x_{i_t}, y_{j_t})\} : \{x_{i_1}, \ldots, x_{i_t}\} \in E(\mathrm{H}_1), \{y_{j_1}, \ldots, y_{j_t}\} \in E(\mathrm{H}_2)\}$. It follows that the number of hyperedges containing both of two vertices (x_i, y_l) and (x_j, y_m) in $\mathrm{H}_1 \otimes \mathrm{H}_2$ is $(t-2)!a_{ij}b_{lm}$ where a_{ij} is the number of hyperedges containing both of x_i and x_j and b_{lm} is the number of hyperedges containing both of y_l and y_m. Hence, $A(\mathrm{H}_1 \otimes \mathrm{H}_2) = (t-2)!A(\mathrm{H}_1) \otimes A(\mathrm{H}_2)$. Consequently,

$$\mathrm{Spec}(\mathrm{H}_1 \otimes \mathrm{H}_2) = \{(t-2)!\lambda\beta : \lambda \in \mathrm{Spec}(\mathrm{H}_1) \text{ and } \beta \in \mathrm{Spec}(\mathrm{H}_2)\}. \tag{1.2}$$

Example 1.3.5. Let H_1 and H_2 be hypergraphs with $V(\mathrm{H}_1) = \{v_1, v_2, v_3, v_4\}$, and $E(\mathrm{H}_1) = \{\{v_1, v_2, v_3\}, \{v_2, v_4\}\}, V(\mathrm{H}_2) = \{w_1, w_2\}$ and $E(\mathrm{H}_2) = \{\{w_1, w_2\}\}$. Then $\mathrm{H}_1 \square \mathrm{H}_2$ is a hypergraph with $V(\mathrm{H}_1 \square \mathrm{H}_2) = V(\mathrm{H}_1) \times V(\mathrm{H}_2)$ and

$$\begin{aligned} E(\mathrm{H}_1 \square \mathrm{H}_2) = &\{\{(v_1, w_1), (v_1, w_2)\}, \{(v_2, w_1), (v_2, w_2)\}, \{(v_3, w_1), (v_3, w_2)\}, \\ &\{(v_4, w_1), (v_4, w_2)\}\} \cup \{\{(v_1, w_1), (v_2, w_1), (v_3, w_1)\}, \\ &\{(v_1, w_2), (v_2, w_2), (v_3, w_2)\}\}. \end{aligned}$$

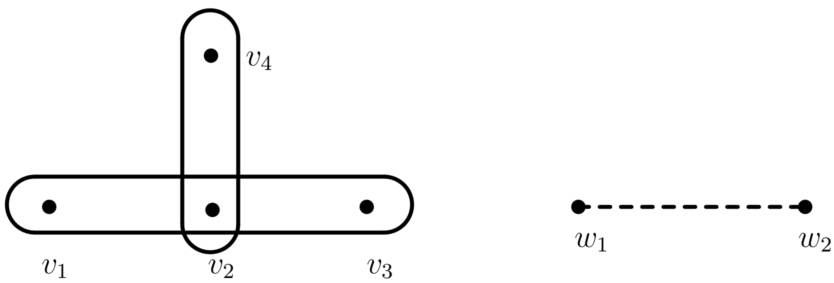

Figure 1.2: Hypergraphs H_1 (left) and H_2 (right)

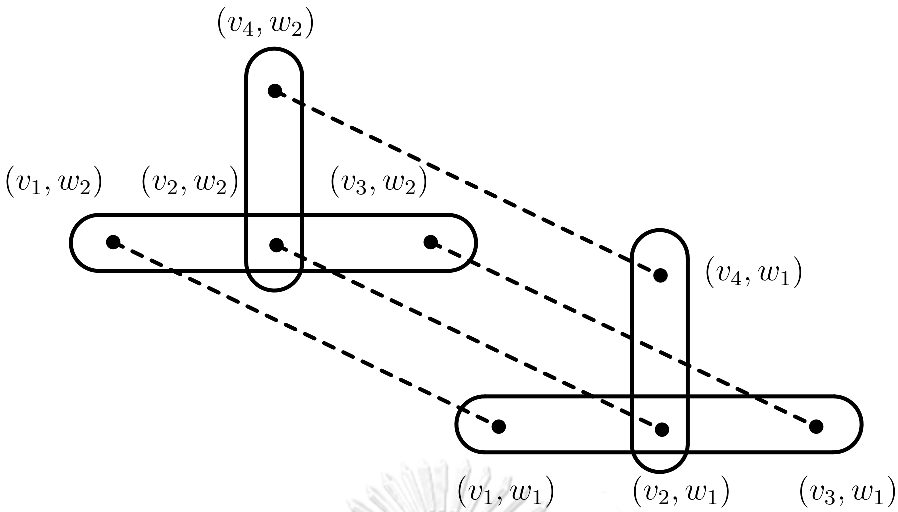

Figure 1.3: A hypergraph $\mathrm{H}_1 \,\square\, \mathrm{H}_2$

Example 1.3.6. Let H be a 3-uniform hypergraph with $V(\mathrm{H}) = \{v_1, v_2, v_3\}$ and $E(\mathrm{H}) = \{\{v_1, v_2, v_3\}\}$. Then $\mathrm{H} \otimes \mathrm{H}$ is a hypergraph with $V(\mathrm{H} \otimes \mathrm{H}) = V(\mathrm{H}) \times V(\mathrm{H})$ and

$$\begin{aligned} E(\mathrm{H} \otimes \mathrm{H}) = &\{\{(v_1, v_1), (v_2, v_2), (v_3, v_3)\}, \{(v_1, v_1), (v_2, v_3), (v_{3,2}\,)\}, \\ &\{(v_1, v_2), (v_2, v_1), (v_3, v_3)\}, \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}, \\ &\{(v_1, v_3), (v_2, v_2), (v_3, v_1)\}, \{(v_1, v_3), (v_2, v_1), (v_3, v_2)\}. \end{aligned}$$

1.4 Background in algebra

We recall some useful properties from algebra quoted from [7] and [16]. This section contains the structure theorem for finite abelian groups, field extensions and Galois theory.

Let G be a finite abelian group. We have known that G is isomorphic to a direct product of its Sylow p-subgroups (a maximal subgroup of G in which the order of every element is a power of p) where p is a prime number dividing $|G|$. Since any abelian Sylow p-subgroup is a direct product of cyclic groups of p-power order, we have that G is a direct product cyclic groups of p-power order. By this fact, we can prove that G is a direct product of cyclic groups as follows.

Theorem 1.4.1 (Structure Theorem for Finite Abelian Groups)**.** *Let G be a finite abelian group. Then there exist integers $n_1, \ldots, n_r > 1$ such that $n_1 \mid n_2, n_2 \mid n_3, \ldots, n_{r-1} \mid n_r$ and*

$$G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r},$$

where these integers are uniquely defined by G. More precisely, if $m_1, m_2, \ldots, m_s$ are positive integers greater than 1 such that $m_1 \mid m_2, m_2 \mid m_3, \ldots, m_{s-1} \mid m_s$, and

$$G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r} \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_s},$$

then $r = s$, and $n_1 = m_1, n_2 = m_2, \ldots, n_r = m_r$.

Example 1.4.2. Let G be a finite abelian group of order 36. Note that $36 = 2^2 \cdot 3^2$. By Theorem 1.4.1, G is isomorphic to one of the following groups:

1. $\mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2} \cong \mathbb{Z}_{36}$
2. $\mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_{12}$
3. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \cong \mathbb{Z}_2 \times \mathbb{Z}_{18}$
4. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \cong \mathbb{Z}_6 \times \mathbb{Z}_6$.

Next, we recall the definition of an extension field and some important properties as follows.

Let F and K be fields. A field K is said to be an *extension* of F if F is a subfield of K. If K is an extension of F, we can consider K as a vector space over F. The *degree* of K over F, denoted by $[K : F]$, is the dimension of K as a vector space over F. An extension is called *finite* if its degree is finite, and *infinite* otherwise. The theorem below shows one important property of a finite field extension.

Theorem 1.4.3. *Let L, K and F be fields such that $F \subseteq K \subseteq L$. If $[L : K]$ and $[K : F]$ are finite, then $[L : F]$ is finite and*

$$[L : F] = [L : K][K : F].$$

Let F be a field and $f(x)$ a monic polynomial in $F[x]$. An extension field E of F is a *splitting field* of $f(x)$ over F if

$$f(x) = (x - r_1) \cdots (x - r_n)$$

in $E[x]$ and

$$E = F(r_1, \ldots, r_n),$$

that is, E is generated by the roots of $f(x)$.

We recall the existence and uniqueness of the splitting field in the following theorems.

Theorem 1.4.4 (Existence of Splitting Fields)**.** *Let $f(x)$ be a monic polynomial of degree $n \geq 1$. Then there exists an extension field E of F such that $[E : F] \leq n!$ and E contains n roots of $f(x)$ counting multiplicities. Hence, in $E[x]$, $f(x) = c(t - r_1) \cdots (t - r_n)$ for some $c \in F$ and $r_1, \ldots, r_n \in E$, so that $r_1, \ldots, r_n$ are n roots of $f(x)$ in E.*

Theorem 1.4.5 (Uniqueness of Splitting Fields)**.** *Let $f(x)$ be a monic polynomial of degree $n \geq 1$. If K an E are splitting fields of $f(x)$ over F, then there is an isomorphism $\eta : K \to E$ extending the identity map of F.*

Example 1.4.6. The following examples show some splitting fields over $\mathbb{Q}$.

1. Let $F = \mathbb{Q}$ and $f(x) = x^4 - 1$. Note that $f(x) = (x - 1)(x + 1)(x^2 + 1)$. A field $\mathbb{Q}(i)$ is a splitting field of F over $\mathbb{Q}$ with degree 2.

2. Let $F = \mathbb{Q}$ and $f(x) = (x^2 - 2)(x^2 - 3)$. A field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is a splitting field of $f(x)$ over $\mathbb{Q}$ with degree 4.

Let E be an extension field of a field F. The *Galois group* of E over F denoted by $\mathrm{Gal}(E/F)$ is the group

$$\{\varphi \in \operatorname{Aut} E : \varphi(a) = a \text{ for all } a \in F\}$$

where $\operatorname{Aut} E$ denotes the set of all automorphisms of E.

Let G be a subgroup of $\operatorname{Aut} E$ where E is a field. Then the *field of G-invaraint* of E or the *fixed field of G* on E is the field

$$\{a \in E : \varphi(a) = a \text{ for all } \varphi \in G\}.$$

It is denoted by E^G.

Now, we recall the Fundamental Theorem of Galois Theory and the Galois group of $\mathbb{Q}(\omega)$ over $\mathbb{Q}$ when $\omega = e^{2\pi i/n}$ as follows.

Theorem 1.4.7 (Fundamental Theorem of Galois Theory)**.** *Let E be a finite dimensional Galois extension of a field F and let $G = \operatorname{Gal}(E/F)$. Let $\Gamma = \{H\}$, the set of subgroups of G, and Σ, the set of intermediate fields between E and F. Then the map $H \mapsto E^H$ and $K \mapsto \operatorname{Gal}(E/K)$, $H \in \Gamma, K \in \Sigma$, are inverse of each other. In particular, they are one-to-one correspondences between Γ and Σ.*

Theorem 1.4.8 (Galois Group of $\mathbb{Q}(\omega)$)**.** *Let $\omega = e^{2\pi i/n}$. The Galois group of $\mathbb{Q}(\omega)$ over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}_n^\times$. Explicitly, the elements of the Galois group are the automorphisms σ_y for $y \in \mathbb{Z}_n^\times$ defined by $\sigma_y(\omega) = \omega^y$.*

1.5 Objectives

In this thesis, we study the algebraic degree of spectra of t-Cayley hypergraphs. In Chapter 2, we present t-Cayley hypergraphs of G over S when G is a finite abelian group and $t \geq 2$. We show combinatorial properties of t-Cayley hypergraphs, i.e., conectivity, size of hyperedges and regularity. Integral Cayley graphs are determined in Section 2.2. We recall criteria on S of a Cayley graph of $\mathbb{Z}_n$ to be integral. By using facts on integral Cayley graphs of $\mathbb{Z}_n$, spectra of product graphs and Theorem 1.4.1, we explore integral Cayley graphs of G. In Chapter 3, we study the t-Cayley hypergraphs of G over S when $t \geq 2$. We specify criteria on S of this hypergraph to be integral, L-integral and D-integral by considering its adjacency, Laplacian and distance matrix, respectively. For t-Cayley hypergraphs, we show a condition on S for integral t-Cayley hypergraphs of $\mathbb{Z}_n$ generalized So's result. The gcd-hypergraphs of $\mathbb{Z}_n$ are defined to be the t-Cayley hypergraphs of $\mathbb{Z}_n$ over S where $S = \bigcup_{d \in D} G_n(d)$ and D is a set of divisors of n. We show that gcd-hypergraphs of $\mathbb{Z}_n$ are integral, L-integral and D-integral by clarifying the first row of its adjacency matrix. In addition, we see that the well-known unitary Cayley hypergraph of $\mathbb{Z}_n$ is associated with gcd-hypergraphs. In Section 3.2, non-integral hypergraphs are discussed. We compute the algebraic degree of t-Cayley hypergraphs of $\mathbb{Z}_n$ for all $n \geq 3$ which generalizes Mönius' results [18] and provides an answer to his outlook. Our combinatorial approach is different from him and presented in Lemma 3.2.1. The results have been published in Discrete Applied Mathematics [23]. Moreover, we focus on the algebraic degree of Cayley graphs of $\mathbb{Z}_n$ by comparing two approaches which are by using Corollary 3.2.3 and by using a Schur ring from Mönius' result [19] presented in Section 3.3.

CHAPTER II

t-CAYLEY HYPERGRAPHS

In this chapter, we introduce t-Cayley hypergraphs of a finite group. Some combinatorial properties of this hypergraph are presented in the first section. Next, in Section 2.2, we study integral Cayley graphs of a finite abelian group.

2.1 t-Cayley hypergraphs

We start this section with the definition of the t-Cayley hypergraph. We recall some well-known properties of this hypergraph. In addition, we show that the t-Cayley hypergraph is regular. Moreover, we classify integral Cayley graphs of finite abelian groups in the last theorem of this section.

Throughout this section, we let $(G, \cdot)$ be a finite group with the identity e and a subset S of $G \smallsetminus \{e\}$ such that $S = S^{-1}$.

For $t \in \mathbb{N}$ and $2 \leq t \leq \max\{o(x) : x \in S\}$, the *$t$-Cayley hypergraph* $\mathrm{H} = t$-$\mathrm{Cay}(G, S)$ of G over S is a hypergraph with vertex set $V(\mathrm{H}) = G$ and $E(\mathrm{H}) = \{\{yx^i : 0 \leq i \leq t-1\} : x \in S \text{ and } y \in G\}$. Here, $o(x)$ denotes the order of x in G. The 2-Cayley hypergraph of G over S is a Cayley graph of G over S.

Example 2.1.1. Consider a finite group $(\mathbb{Z}_6, +)$ and a subset $S = \{1, 3, 5\}$. Then $\max\{o(x) : x \in S\} = \max\{6, 2\} = 6$. The following hypergraphs are the t-Cayley hypergraphs for all $t \in \mathbb{N}$ with $2 \leq t \leq \max\{o(x) : x \in S\}$.

1. A hypergraph $\mathrm{H}_1 = 2$-$\mathrm{Cay}(\mathbb{Z}_6, S)$ has a vertex set $V(\mathrm{H}_1) = \{0, 1, 2, 3, 4, 5\}$ and $E(\mathrm{H}_1) = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 0\}, \{0, 3\}, \{1, 4\}, \{2, 5\}\}$.

2. A hypergraph $\mathrm{H}_2 = 3$-$\mathrm{Cay}(\mathbb{Z}_6, S)$ has a vertex set $V(\mathrm{H}_2) = \{0, 1, 2, 3, 4, 5\}$ and $E(\mathrm{H}_2) = \{\{0, 1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 0\}, \{5, 0, 1\}, \{0, 3\}, \{1, 4\}, \{2, 5\}\}$.

3. A hypergraph $\mathrm{H}_3 = 4\text{-Cay}(\mathbb{Z}_6, S)$ has a vertex set $V(\mathrm{H}_3) = \{0,1,2,3,4,5\}$ and $E(\mathrm{H}_3) = \{\{0,1,2,3\},\{1,2,3,4\},\{2,3,4,5\},\{3,4,5,0\},\{4,5,0,1\},$ $\{5,0,1,2\},\{0,3\},\{1,4\},\{2,5\}\}$.

4. A hypergraph $\mathrm{H}_4 = 5\text{-Cay}(\mathbb{Z}_6, S)$ has a vertex set $V(\mathrm{H}_4) = \{0,1,2,3,4,5\}$ and $E(\mathrm{H}_4) = \{\{0,1,2,3,4\},\{1,2,3,4,5\},\{2,3,4,5,0\},\{3,4,5,0,1\},$ $\{4,5,0,1,2\},\{5,0,1,2,3\},\{0,3\},\{1,4\},\{2,5\}\}$.

5. A hypergraph $\mathrm{H}_5 = 6\text{-Cay}(\mathbb{Z}_6, S)$ has a vertex set $V(\mathrm{H}_5) = \{0,1,2,3,4,5\}$ and $E(\mathrm{H}_5) = \{\{0,1,2,3,4,5\},\{0,3\},\{1,4\},\{2,5\}\}$.

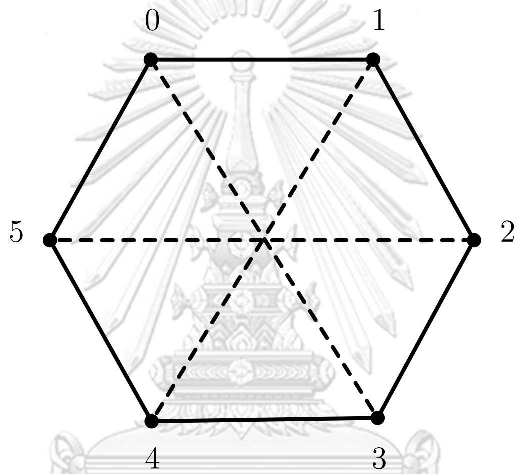

Figure 2.1: $\mathrm{H}_1 = 2\text{-Cay}(\mathbb{Z}_6, S)$

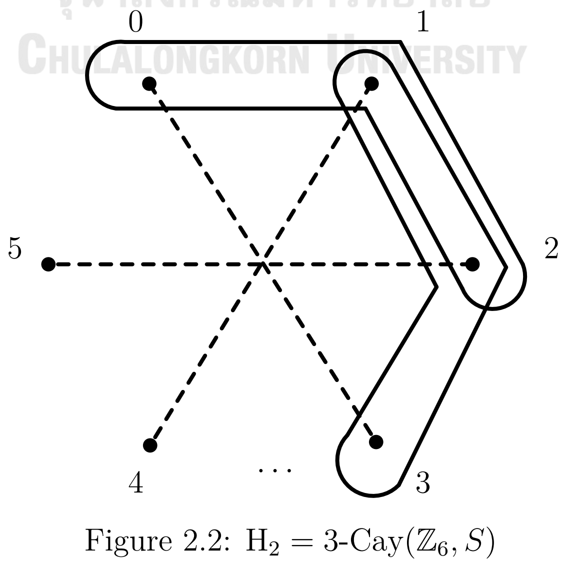

Figure 2.2: $\mathrm{H}_2 = 3\text{-Cay}(\mathbb{Z}_6, S)$

Example 2.1.2. For $\mathbf{m} = (m_1, \ldots, m_r)$ and $\mathbf{n} = (n_1, \ldots, n_r)$ in $\mathbb{Z}^r$, we define the greatest common divisor of $\mathbf{m}$ and $\mathbf{n}$ to be the vector $\mathbf{d} = (d_1, \ldots, d_r)$ where $d_i = \gcd(m_i, n_i)$ for all $i \in \{1, \ldots, r\}$. Now, let $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$ and a divisor tuple $\mathbf{d} = (d_1, \ldots, d_r) \in \mathbb{Z}_n^r$ of $\mathbf{n}$, i.e., $d_i \mid n_i$ for all $i \in \{1, \ldots, r\}$. Define

$$G_{\mathbf{n}}(\mathbf{d}) = \{\mathbf{x} = (x_1, \ldots, x_r) \in \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r} : \gcd(\mathbf{x}, \mathbf{n}) = \mathbf{d}\}.$$

Let D be a set of divisor tuples of $\mathbf{n}$ not containing the zero vector of $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ and $S = \bigcup_{\mathbf{d} \in D} G_{\mathbf{n}}(\mathbf{d})$. For $t \in \mathbb{N}$ and $2 \leq t \leq \max\{o(\mathbf{x}) : \mathbf{x} \in S\}$, the t-Cayley hypergraph of $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ over S is called a gcd-*hypergraph* and the 2-Cayley hypergraph of $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ over S is called a gcd-*graph*.

Some properties of t-Cayley hypergraphs quoted from [3] are as follows.

Proposition 2.1.3. *Let* $\mathrm{H} = t\text{-}\mathrm{Cay}(G, S)$.

1. H *is connected if and only if* $\langle S \rangle = G$.

2. *For any* $x \in S, y \in G$, $|\{yx^i : 0 \leq i \leq t-1\}| = \begin{cases} t & \text{if } t \leq o(x), \\ o(x) & \text{if } t > o(x). \end{cases}$

3. H *is* t*-uniform if and only if* $t \leq o(x)$ *for any* $x \in S$.

Clearly, a Cayley graph $2\text{-}\mathrm{Cay}(G, S)$ is $|S|$-regular. We study a Cayley hypergraph $t\text{-}\mathrm{Cay}(G, S)$. For any $y \in G$, we have that all hyperedges (may not be distinct) containing y are

$$\{yx^{-(t-1)}, yx^{-(t-2)}, \ldots, yx^{-1}, y\}, \{yx^{-(t-2)}, yx^{-(t-3)}, \ldots, y, yx\}, \ldots, \\ \{y, yx, \ldots, yx^{t-2}, yx^{t-1}\}$$

where $x \in S$. This implies

$$\begin{aligned} \deg y &= \left|\{\{yx^{i-j} : 0 \leq i \leq t-1\} : 0 \leq j \leq t-1, x \in S\}\right| \\ &= \left|\{\{x^{i-j} : 0 \leq i \leq t-1\} : 0 \leq j \leq t-1, x \in S\}\right|. \end{aligned}$$

for all $y \in G$. Hence, we have shown

Proposition 2.1.4. *A t-Cayley hypergraph of G over S is regular of degree equal to the number of distinct subsets $\{x^{i-j} : 0 \leq i \leq t-1\}$ where $0 \leq j \leq t-1$ and $x \in S$.*

2.2 Integral Cayley graphs

The main purpose of this section is to classify integral Cayley graphs of finite abelian groups. We first recall So's result [22] on integral Cayley graphs of $\mathbb{Z}_n$ as follows.

Theorem 2.2.1. *The Cayley graph 2-Cay$(\mathbb{Z}_n, S)$ is integral if and only if S is a union of some $G_n(d)$'s, where $d \mid n$ and $G_n(d) = \{k \in \{1, 2, \ldots, n-1\} : \gcd(k, n) = d\}$.*

Remark 2.2.2. From the above theorem, the Cayley graph 2-Cay$(\mathbb{Z}_n, S)$ is integral if and only if it is a gcd-graph.

Example 2.2.3. Consider a finite group $(\mathbb{Z}_6, +)$ and a subset $S = \{1, 3, 5\}$. Let $\mathrm{H} =$ 2-Cay$(\mathbb{Z}_6, S)$ a Cayley graph of $\mathbb{Z}_6$ over S. Since $S = \{1, 3, 5\} = G_6(1) \cup G_6(3)$, by Theorem 2.2.1 we can conclude that H is integral. In fact, $V(\mathrm{H}) = \{0, 1, 2, 3, 4, 5\}$ and $E(\mathrm{H}) = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 0\}, \{0, 3\}, \{1, 4\}, \{2, 5\}\}$. Then $A(\mathrm{H})$ is a circulant matrix with the first row $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$. By computation its eigenvalues, we have $\mathrm{Spec}(\mathrm{H}) = \{3, 0, 0, 0, 0, -3\}$.

Example 2.2.4. Consider a finite group $(\mathbb{Z}_9, +)$ and a subset $S = \{1, 3, 6, 8\}$. Let $\mathrm{H} =$ 2-Cay$(\mathbb{Z}_9, S)$ a Cayley graph of $\mathbb{Z}_9$ over S. Since $S = \{1, 3, 6, 8\}$ cannot be written as a union of $G_9(d)$ where d is a proper divisor of 9, by Theorem 2.2.1 we can conclude that H is not integral. In fact, $V(\mathrm{H}) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $E(\mathrm{H}) = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{8, 0\}, \{0, 3\}, \{1, 4\}, \{2, 5\}\}, \{3, 6\}, \{4, 7\}, \{5, 8\}, \{6, 0\}\}$. Then $A(\mathrm{H})$ is a circulant matrix with the first row $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$. By computation its eigenvalues, we have $\mathrm{Spec}(\mathrm{H}) = \{4, 1, 1, 0.53, 0.53, -0.65, -0.65, -2.88, -2.88\}$.

To characterize integral Cayley graphs of finite abelian groups, we first discuss the Cayley graph of the group $(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, +)$. Let $S = S_1 \times S_2$ be a subset of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \setminus \{(0,0)\}$ such that $S = -S$. The Cayley graph 2-Cay$(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S)$ can be distinguished into three cases.

1. $\overline{K}_{n_1} \square$ 2-Cay$(\mathbb{Z}_{n_2}, S_2)$ if $S_1 = \{0\}$ and $S_2 \neq \{0\}$, where $\overline{K}_n$ denotes the empty graph on n vertices.
2. 2-Cay$(\mathbb{Z}_{n_1}, S_1) \square\, \overline{K}_{n_2}$ if $S_1 \neq \{0\}$ and $S_2 = \{0\}$.
3. 2-Cay$(\mathbb{Z}_{n_1}, S_1) \otimes$ 2-Cay$(\mathbb{Z}_{n_2}, S_2)$ if $S_1 \neq \{0\}$ and $S_2 \neq \{0\}$.

It is clear that the eigenvalues of an empty graph are zero. By Equations (1.1), (1.2) and a Cayley graph always has an integral eigenvalue, the Cayley graph 2-Cay$(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S)$ is integral if and only if for any $i \in \{1, 2\}$ such that $S_i \neq \{0\}$, the 2-Cay$(\mathbb{Z}_{n_i}, S_i)$ is integral. By the fundamental theorem of finite abelian groups, a finite abelian group is a direct product of finite cyclic groups. We can obtain a characterization of the integral Cayley graphs of finite abelian groups similar to the above discussion.

Theorem 2.2.5. *Let G be a finite abelian group and S a subset of $G \setminus \{e\}$ such that $S = S^{-1}$. Suppose $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ and $S = S_1 \times \cdots \times S_r$. The Cayley graph 2-Cay$(G, S)$ is integral if and only if for any $i \in \{1, \ldots, r\}$ such that $S_i \neq \{0\}$, the 2-Cay$(\mathbb{Z}_{n_i}, S_i)$ is integral.*

Example 2.2.6. Consider a finite abelian group $(\mathbb{Z}_3 \times \mathbb{Z}_6, +)$ and a subset $S = \{(1,3), (3,3)\}$. Let H $=$ 2-Cay$(\mathbb{Z}_3 \times \mathbb{Z}_6, S)$ a Cayley graph of $\mathbb{Z}_3 \times \mathbb{Z}_6$ over S. Note that $S = \{1,3\} \times \{3\} = S_1 \times S_2$. From the above discussion, we observe that $H =$ 2-Cay$(\mathbb{Z}_3, S_1) \otimes$ 2-Cay$(\mathbb{Z}_6, S_2)$. Since $S_1 = G_3(1)$ and $S_2 = G_6(3)$, by Theorem 2.2.1 we can conclude that 2-Cay$(\mathbb{Z}_3, S_1)$ and 2-Cay$(\mathbb{Z}_6, S_2)$ are integral. By Theorem 2.2.5, we have 2-Cay$(\mathbb{Z}_3 \times \mathbb{Z}_6, S)$ is integral.

CHAPTER III
ALGEBRAIC DEGREE OF SPECTRA OF t-CAYLEY HYPERGRAPHS

In this chapter, we determine algebraic degree of spectra of t-Cayley hypergraphs. The main purpose of Section 3.1 is Theorem 3.1.2. This theorem shows all t-Cayley hypergraphs of algebraic degree one. We also study gcd-hypergraphs of $\mathbb{Z}_n$ in this section. We find the first row of its adjacency matrix mentioned in Theorem 3.1.5. Next, in Section 3.2 we compute the algebraic degree of spectra of t-Cayley hypergraphs of $\mathbb{Z}_n$ when $t \geq 2$ referred in Theorem 3.2.2. In addition, we focus the algebraic degree of Cayley graphs of $\mathbb{Z}_n$ in Section 3.3.

3.1 Integral t-Cayley hypergraphs of $\mathbb{Z}_n$

From Section 2.1, we classify integral Cayley graphs over finite abelian groups. In this section, we give a criterion for integral t-Cayley hypergraphs where $t \geq 2$ over $\mathbb{Z}_n$. In addition, we also discuss the first row of adjacency matrix of a gcd-hypergraph of $\mathbb{Z}_n$. Moreover, we prove that a gcd-hypergraph of $\mathbb{Z}_n$ is integral. Recall that a circulant matrix is a square matrix in which each row is obtained by a right cyclic shift of the preceding row. From now on, we let $n \geq 2$ and $\mathrm{H} = t$-$\mathrm{Cay}(\mathbb{Z}_n, S)$. By the natural labeling $\{0, 1, \ldots, n-1\}$ of $\mathbb{Z}_n$, it is easy to see that $A(\mathrm{H}) = [a_{ij}]_{0 \leq i,j \leq n-1}$ is circulant. To work on the adjacency matrix $A(\mathrm{H})$, it suffices to compute the first row of $A(\mathrm{H})$. Let C be the set of vertices adjacent to the vertex 0. Since all hyperedges containing 0 are of the form $\{(i-j)x : 0 \leq i \leq t-1\}$ where $x \in S$ and $0 \leq j \leq t-1$, and $S = -S$, we have the union of all hyperedges containing 0 is

$$\bigcup_{0\le i,j\le t-1}(i-j)S=\bigcup_{-(t-1)\le k\le t-1}kS=S\cup 2S\cup\cdots\cup(t-1)S.$$

It follows that $C=S\cup 2S\cup\cdots\cup(t-1)S\setminus\{0\}$. Since $A(\mathrm{H})$ is circulant, by Theorem 1.2.3, the eigenvalues of H are

$$\lambda_j=\sum_{k\in C}a_{0,k}(e^{2\pi ji/n})^k$$

where $0\le j\le n-1$. We recall some useful properties taken from [22].

Proposition 3.1.1. *1. If d is a proper divisor of n and x is an nth root of unity, then $\sum_{k\in G_n(d)}x^k$ is an integer.*

2. Let $\omega=e^{2\pi i/n}$ and

$$F=\begin{bmatrix}\omega^{1\cdot 1} & \omega^{1\cdot 2} & \cdots & \omega^{1\cdot(n-1)}\\ \omega^{2\cdot 1} & \omega^{2\cdot 2} & \cdots & \omega^{2\cdot(n-1)}\\ \vdots & \vdots & \ddots & \vdots\\ \omega^{(n-1)\cdot 1} & \omega^{(n-1)\cdot 2} & \cdots & \omega^{(n-1)\cdot(n-1)}\end{bmatrix}.$$

If $\mathcal{A}=\{\mathbf{v}\in\mathbb{Q}^{n-1}:F\mathbf{v}\in\mathbb{Q}^{n-1}\}$, then $\mathcal{A}$ is a vector space over $\mathbb{Q}$. Moreover, $\mathcal{A}=\mathrm{Span}\{\mathbf{v}_d : d\mid n \text{ and } d<n\}$ where $\mathbf{v}_d$ is the $(n-1)$-vector with 1 at the kth entry for all $k\in G_n(d)$ and 0 elsewhere.

Now, we prove a criterion for integral t-Cayley hypergraphs.

Theorem 3.1.2. *Let $\mathrm{H}=t\text{-}\mathrm{Cay}(\mathbb{Z}_n,S)$. Then H is integral if and only if C is a union of some $G_n(d)$'s where for each d, there is $c_d\in\left\{1,2,\ldots,\binom{n}{t-2}\right\}$ such that $a_{0,k}=c_d$ for all $k\in G_n(d)$.*

Proof. Let $d_1,\ldots,d_s$ be all proper divisors of n. Without loss of generality, we assume that $C=G_n(d_1)\cup\cdots\cup G_n(d_l)$ for some $l\in\{1,\ldots,s\}$. Clearly, $\lambda_0=\sum_{k\in C}a_{0,k}\in\mathbb{Z}$. For any $1\le j\le n-1$, by the assumption and Proposition

3.1.1 (1),

$$\begin{aligned}\lambda_j &= \sum_{k\in C} a_{0,k}(e^{2\pi ji/n})^k \\ &= \sum_{k\in G_n(d_1)} a_{0,k}(e^{2\pi ji/n})^k + \cdots + \sum_{k\in G_n(d_l)} a_{0,k}(e^{2\pi ji/n})^k \\ &= c_{d_1}\sum_{k\in G_n(d_1)} (e^{2\pi ji/n})^k + \cdots + c_{d_l}\sum_{k\in G_n(d_l)} (e^{2\pi ji/n})^k \in \mathbb{Z}.\end{aligned}$$

Conversely, suppose that H is integral. Then $\lambda_j \in \mathbb{Z}$ for any $0 \leq j \leq n-1$. We consider the vector $\mathbf{v} \in \mathbb{Q}^{n-1}$ with $a_{0,k}$ for the kth entry for any $k \in C$ and 0 elsewhere. Then

$$\begin{aligned} F\mathbf{v} &= \begin{bmatrix} \omega^{1\cdot 1} & \omega^{1\cdot 2} & \cdots & \omega^{1\cdot(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{(n-1)\cdot 1} & \omega^{(n-1)\cdot 2} & \cdots & \omega^{(n-1)\cdot(n-1)} \end{bmatrix} \begin{bmatrix} a_{0,1} \\ a_{0,2} \\ \vdots \\ a_{0,n-1} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k\in C} a_{0,k}\omega^{1\cdot k} \\ \sum_{k\in C} a_{0,k}\omega^{2\cdot k} \\ \vdots \\ \sum_{k\in C} a_{0,k}\omega^{(n-1)\cdot k} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{bmatrix} \in \mathbb{Z}^{n-1}. \end{aligned}$$

It follows that $\mathbf{v} \in \mathcal{A}$ in Proposition 3.1.1 (2), and hence $\mathbf{v} = \sum_{d|n, d<n} c_d \mathbf{v}_d$ for some rational coefficients c_d's. The definition of $\mathbf{v}$ implies that the coefficient $c_d \in \left\{0, 1, \ldots, \binom{n}{t-2}\right\}$. Therefore, C is a union of some $G_n(d)$'s where for each such d, we have $a_{0,k} = c_d$ for all $k \in G_n(d)$. $\square$

Remark 3.1.3. In particular, for $t = 2$, we have $S = C$. Theorem 3.1.2 implies that $\mathrm{H} = 2\text{-Cay}(\mathbb{Z}_n, S)$ is integral if and only if S is a union of some $G_n(d)$'s and for which d, $a_{0,k} = 1$ for all $k \in G_n(d)$. This coincides So's result recalled in Theorem 2.2.1.

Example 3.1.4. Consider a finite group $(\mathbb{Z}_6, +)$ and a subset $S = \{1, 3, 5\}$. Let $\mathrm{H} = 3\text{-Cay}(\mathbb{Z}_6, S)$ a 3-Cayley hypergraph of $\mathbb{Z}_6$ over S. We note that $V(\mathrm{H}) = \{0, 1, 2, 3, 4, 5\}$ and $E(\mathrm{H}) = \{\{0, 1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 0\}, \{5, 0, 1\}, \{0, 3\}, \{1, 4\}, \{2, 5\}\}$. Then $C = \{1, 2, 3, 4, 5\} = \{1, 5\} \cup \{2, 4\} \cup \{3\} = G_6(1) \cup G_6(2) \cup G_6(3)$. This implies $a_{0,1} = a_{0,5} = 2, a_{0,2} = a_{0,4} = 1$ and $a_{0,3} = 1$. By Theorem 3.1.2, we can conclude that H is integral. In fact, $A(\mathrm{H})$ is a circulant matrix with the first row $\begin{bmatrix} 0 & 2 & 1 & 1 & 1 & 2 \end{bmatrix}$ and we have $\mathrm{Spec}(\mathrm{H}) = \{7, 0, 0, -2, -2, -3\}$.

Let $\mathrm{H} = t\text{-Cay}(\mathbb{Z}_n, S)$ be a gcd-hypergraph. We shall use Theorem 3.1.2 to show that H is integral. By Example 2.1.2, $S = \bigcup_{e \in D} G_n(e)$ for some set D of proper divisors of n. Since $lG_n(e) = G_n(\gcd(le, n))$ for any $l \in \{1, 2, \ldots, t-1\}$, we have $C = S \cup 2S \cup \cdots \cup (t-1)S \smallsetminus \{0\}$ equals $\bigcup_{d \in D'} G_n(d)$ for some set D' of proper divisors of n and $D \subseteq D'$. For each $d \in D'$, we aim to show that $a_{0,k}$'s are identical for all $k \in G_n(d)$. Let $d \in D'$ and $k, k' \in G_n(d)$. There is $u \in G_n(1)$ such that $k' = uk$. Since hyperedges containing 0 are $\{(i-j)x : 0 \leq i \leq t-1\}$ where $x \in S$ and $0 \leq j \leq t-1$, we count such hyperedges containing k. For each $e \in D$, let $N_{d,k}(e)$ be the number of hyperedges containing 0 and k of the form $\{(i-j)x : 0 \leq i \leq t-1\}$ with $x \in G_n(e)$. For any $e, f \in D$ with $e \neq f$, such hyperedges with $x \in G_n(e)$ and $x \in G_n(f)$ are distinct, so

$$a_{0,k} = \sum_{e \in D} N_{d,k}(e).$$

Let $S_k = \{l : 1 \leq l \leq t-1 \text{ and } k \in lG_n(e)\}$. Since $G_n(d) = lG_n(e)$ for all $l \in S_k$ and $k' = uk$, it follows that $N_{d,k}(e) = N_{d,k'}(e)$. Hence,

$$a_{0,k} = \sum_{e \in D} N_{d,k}(e) = \sum_{e \in D} N_{d,k'}(e) = a_{0,k'}.$$

Therefore, we can conclude that H is integral by Theorem 3.1.2. We record this result in the following theorem.

Theorem 3.1.5. *Let* $\mathrm{H} = t\text{-Cay}(\mathbb{Z}_n, S)$ *be a* gcd*-hypergraph of* $\mathbb{Z}_n$ *where* $S = \bigcup_{e\in D} G_n(e)$ *for some set* D *of proper divisors of* n *and* $C = S \cup 2S \cup \cdots \cup (t-1)S \smallsetminus \{0\} = \bigcup_{d\in D'} G_n(d)$ *for some set* D' *of proper divisors of* n *and* $D \subseteq D'$*. Let* $d \in D'$ *and* $k \in G_n(d)$*. For each* $e \in D$*, let* $N_{d,k}(e)$ *be the number of hyperedges containing* 0 *and* k *of the form* $\{(i-j)x : 0 \le i \le t-1\}$ *with* $x \in G_n(e)$*. Then*

$$a_{0,k} = \sum_{e\in D} N_{d,k}(e).$$

Moreover, $a_{0,k}$*'s are identical for all* $k \in G_n(d)$ *and* H *is an integral hypergraph.*

For $d \in D'$, $k \in G_n(d)$ and $e \in D$ in Theorem 3.1.5, we can compute $N_{d,k}(e)$ as the remark below.

Remark 3.1.6. Let $d \in D'$, $k \in G_n(d)$ and $e \in D$. If $k \notin lG_n(e)$ for all $l \in \{1, 2, \ldots, t-1\}$, then hyperedges containing 0 of the form $\{(i-j)x : 0 \le i \le t-1\}$ where $x \in G_n(e)$ and $0 \le j \le t-1$ do not contain k, so $N_{d,k}(e) = 0$. Assume that $S_k = \{l : 1 \le l \le t-1 \text{ and } k \in lG_n(e)\} \ne \varnothing$. Note that $o(x) = \frac{n}{e}$ for all $x \in G_n(e)$. If $\frac{n}{e} \le t$, then $\{(i-j)x : 0 \le i \le t-1\} = \langle x\rangle = e\mathbb{Z}_n$ for all $x \in G_n(e)$ and $0 \le j \le t-1$, so we have only one hyperedge containing 0 and k and $N_{d,k}(e) = 1$.

Suppose that $\frac{n}{e} > t$. Let $l \in S_k$. Since $k \in lG_n(e)$, there is $x \in G_n(e)$ such that $k = lx$. We wish to find the number of elements y in $G_n(e)$ such that $k = ly$. Since $k \in lG_n(e)$, we have $d = \gcd(le, n)$, so

$$G_n(d) = G_n(\gcd(le, n)) = lG_n(e) = leG_{\frac{n}{e}}(1).$$

Suppose that $k = lx = leu$ for some $u \in G_{\frac{n}{e}}(1)$. To find the number of such y's in $G_n(e)$, it is equivalent to find the number of elements v in $G_{\frac{n}{e}}(1)$ such that $k = lev$. Now, we count such v's. For any $v \in G_{\frac{n}{e}}(1)$ with $k = lev$, we have $lev \equiv leu \mod n$, so $l(v-u) \equiv 0 \mod \frac{n}{e}$. If $v - u \equiv 0 \mod \frac{n}{e}$, then $l \cdot 0 \equiv 0 \mod \frac{n}{e}$, and if $v - u \not\equiv 0 \mod \frac{n}{e}$, then there are $q \in \mathbb{Z}$ and $r \in \{1, 2, \ldots, \frac{n}{e} - 1\}$ such that $v = u + \frac{n}{e}q + r$. Consequently, $l(v-u) \equiv 0 \mod \frac{n}{e}$ if and only if

$lr \equiv 0 \mod \frac{n}{e}$. Thus, the number of v in $G_{\frac{n}{e}}(1)$ such that $k \equiv lev \mod n$ equals to the number of r in $\{0, 1, \ldots, \frac{n}{e} - 1\}$ such that $lr \equiv 0 \mod \frac{n}{e}$. Note that $|G_n(e)| = \phi\left(\frac{n}{e}\right)$ if e is a divisor of n. Since this number is independent of k, there are exactly $\frac{\phi(n/e)}{\phi(n/d)}$ elements, say $v_1, v_2, \ldots, v_{\frac{\phi(n/e)}{\phi(n/d)}}$, in $G_{\frac{n}{e}}(1)$ such that $k = lev_i$ for all $i \in \left\{1, 2, \ldots, \frac{\phi(n/e)}{\phi(n/d)}\right\}$. Let $y_i = ev_i$ for all $i \in \left\{1, 2, \ldots, \frac{\phi(n/e)}{\phi(n/d)}\right\}$. Since $o(y_i) = \frac{n}{e} > t$, the sets

$$\begin{aligned}
&\{(l-t+1)y_i, (l-t+2)y_i, \ldots, 0, \ldots, ly_i\},\\
&\{(l-t+2)y_i, (l-t+3)y_i, \ldots, 0, \ldots, ly_i, (l+1)y_i\}, \ldots,\\
&\{0, \ldots, ly_i, (l+1)y_i, \ldots, (t-1)y_i\}
\end{aligned}$$

are hyperedges of H containing 0 and k for all $i \in \left\{1, 2, \ldots, \frac{\phi(n/e)}{\phi(n/d)}\right\}$. Thus,

$$N_{d,k}(e) = \left|\bigcup_{l \in S_k} \left\{\{(i-j)y_m : 0 \leq i \leq t-1\} : 0 \leq j \leq t-1-l, 1 \leq m \leq \frac{\phi(n/e)}{\phi(n/d)}\right\}\right|$$

if $\frac{n}{e} > t$. However, these hyperedges may not be distinct, it follows that $N_{d,k}(e) \leq \sum_{l \in S_k} (t-l) \cdot \frac{\phi(n/e)}{\phi(n/d)}$.

From Theorem 3.1.5 and the above discussion, we have $a_{0,k}$ for all $k \in C$. It can be used in computing the spectrum of gcd-hypergraphs of $\mathbb{Z}_n$ as mentioned before Proposition 3.1.1.

Example 3.1.7. By Theorem 2.2.1, an integral 2-Cay$(\mathbb{Z}_n, S)$ is a gcd-graph. However, an integral t-Cay$(\mathbb{Z}_n, S)$ may not be a gcd-hypergraph when $t \geq 3$. For example, if H $=$ 5-Cay$(\mathbb{Z}_5, \{\pm 1\})$ which is not a gcd-hypergraph of $\mathbb{Z}_5$, then $E(\mathrm{H}) = \{\{0, 1, 2, 3, 4\}\}$. Hence, $C = \mathbb{Z}_5 \setminus \{0\} = G_5(1)$ and $a_{0,k} = 1$ for any $k \in C$, but H is integral by Theorem 3.1.2.

Finally, we study L-integral and D-integral t-Cayley hypergraphs. We start with a simple result on L-integral t-Cayley hypergraphs obtained by Proposition 2.1.4, Theorems 3.1.2 and 3.1.5. Let H $= t$-Cay$(\mathbb{Z}_n, S)$. By Proposition 2.1.4, H is regular, so there exists $d \in \mathbb{N}$ such that $\deg k = d$ for any $0 \leq k \leq n-1$.

It follows that

$$L(\mathrm{H}) = \mathcal{D}(\mathrm{H}) - A(\mathrm{H}) = dI_n - A(\mathrm{H}).$$

Hence,

$$\mathrm{Lspec}(\mathrm{H}) = \{d - \lambda : \lambda \in \mathrm{Spec}(\mathrm{H})\}.$$

By Theorems 3.1.2 and 3.1.5, we easily get

Corollary 3.1.8. *Let* $\mathrm{H} = t\text{-}\mathrm{Cay}(\mathbb{Z}_n, S)$*. Then* H *is* L*-integral if and only if* H *is integral. In particular, a* gcd*-hypergraph of* $\mathbb{Z}_n$ *is* L*-integral.*

Now, we consider D-integral t-Cayley hypergraphs. For $t = 2$, Ilić [13] showed that a gcd-graph of $\mathbb{Z}_n$ is D-integral. Assume that $\mathrm{H} = t\text{-}\mathrm{Cay}(\mathbb{Z}_n, S)$ is connected. That is, $\langle S \rangle = G$ by Proposition 2.1.3 (1). By the natural labeling in $D(\mathrm{H})$, it is clear that $D(\mathrm{H})$ is circulant. Thus, it suffices to consider the first row of $D(\mathrm{H})$. Since H is connected, the set $\{k : d(0, k) \neq 0\} = \{1, 2, \ldots, n-1\}$. Hence, we get a characterization of D-integral t-Cayley hypergraphs similar to Theorem 3.1.2.

Theorem 3.1.9. *Assume that* $\mathrm{H} = t\text{-}\mathrm{Cay}(\mathbb{Z}_n, S)$ *is connected. Then* H *is* D*-integral if and only if for each* $d \mid n$*, there is* $c_d \in \{1, 2, \ldots, \mathrm{diam}(\mathrm{H})\}$ *such that* $d(0, k) = c_d$ *for all* $k \in G_n(d)$*.*

Let $\mathrm{H} = t\text{-}\mathrm{Cay}(\mathbb{Z}_n, S)$. We observe that $d(0, k)$ is the distance between 0 and k in $2\text{-}\mathrm{Cay}(\mathbb{Z}_n, C)$ where $C = S \cup 2S \cup \cdots \cup (t-1)S \smallsetminus \{0\}$. Hence, the distance matrix $D(\mathrm{H}) = D(2\text{-}\mathrm{Cay}(\mathbb{Z}_n, C))$. If H is a gcd-hypergraph, then $2\text{-}\mathrm{Cay}(\mathbb{Z}_n, C)$ is also a gcd-graph. This implies that $2\text{-}\mathrm{Cay}(\mathbb{Z}_n, C)$ is D-integral [13]. Consequently, H is D-integral and we obtain the following theorem.

Theorem 3.1.10. *A* gcd*-hypergraph of* $\mathbb{Z}_n$ *is* D*-integral.*

Remark 3.1.11. Let $S = S_1 \times S_2$ be a subset of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \smallsetminus \{(0,0)\}$ such that $S = -S$ and $\mathrm{H} = t\text{-}\mathrm{Cay}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S)$. Suppose that $S_1 \neq \{0\}$ and $S_2 \neq \{0\}$.

We observe that t-Cay$(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S_1 \times S_2)$ is a subgraph of t-Cay$(\mathbb{Z}_{n_1}, S_1) \otimes t$-Cay$(\mathbb{Z}_{n_2}, S_2)$. Fix two vertices $(x_1, y_1), (x_2, y_2) \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. Let $\{x + ix' : 0 \leq i \leq t-1\}$ be a hyperedge in t-Cay$(\mathbb{Z}_{n_1}, S_1)$ containing both of x_1 and x_2 and let $\{y + iy' : 0 \leq i \leq t-1\}$ a hyperedge in t-Cay$(\mathbb{Z}_{n_2}, S_2)$ containing both of y_1 and y_2. Then $\{(x, y) + i(x', y') : 0 \leq i \leq t-1\}$ is a hyperedge in t-Cay$(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S_1 \times S_2)$. But when $t \geq 3$, the problem is that it may not contain (x_1, y_1) and (x_2, y_2). This means that $A(t\text{-Cay}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S_1 \times S_2))$ may not equal to $A(t\text{-Cay}(\mathbb{Z}_{n_1}, S_1) \otimes t\text{-Cay}(\mathbb{Z}_{n_2}, S_2))$ when $t \geq 3$. Hence, a characterization of integral t-Cayley hypergraphs of finite abelian groups is still an open problem when $t \geq 3$.

3.2 Algebraic degree of spectra of t-Cayley hypergraphs of $\mathbb{Z}_n$

The main purpose of this section is Theorem 3.2.2 which shows the algebraic degree of spectra of t-Cayley hypergraphs of $\mathbb{Z}_n$. To prove this theorem, we shall recall basic properties and give a lemma which is useful to prove Theorem 3.2.2. In addition, we give formulas of algebraic degree of specific cases in Corollary 3.2.3 and 3.2.5.

Firstly, we define the algebraic degree of a hypergraph as follows.

Let H be a hypergraph on m vertices and $f(x) = \det(xI_m - A(\mathrm{H})) \in \mathbb{Z}[x]$ the characteristic polynomial of $A(\mathrm{H})$. Let E_f be the splitting field of $f(x)$ over $\mathbb{Q}$. The *algebraic degree* of H is $[E_f : \mathbb{Q}]$ and denoted by $\deg \mathrm{H}$.

By Theorem 3.1.2, we have a characterization of integral t-Cayley hypergraphs of $\mathbb{Z}_n$. They are hypergraphs of $\mathbb{Z}_n$ of algebraic degree one. We study the algebraic degree of t-Cayley hypergraphs of $\mathbb{Z}_n$ in this section.

Let $n \geq 3$ and $\mathrm{H} = t$-Cay$(\mathbb{Z}_n, S)$. Recall from the beginning of Section 2 that the eigenvalues of H are

$$\lambda_j = \sum_{k \in C} a_{0,k} (e^{2\pi j i/n})^k = \sum_{k \in C} a_{0,k} \omega^{jk}$$

where $C = \{k : a_{0,k} \neq 0\} = S \cup 2S \cup \cdots \cup (t-1)S \setminus \{0\}$, $j \in \{0, 1, \ldots, n-1\}$ and $\omega = e^{2\pi i/n}$, a primitive nth root of unity. By Theorems 1.4.4 and 1.4.5, the splitting field is $\mathbb{Q}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ and $\mathbb{Q} \subseteq \mathbb{Q}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) \subseteq \mathbb{Q}(\omega)$. By Theorems 1.4.3, 1.4.7 and 1.4.8,

$$\deg \mathrm{H} = [\mathbb{Q}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) : \mathbb{Q}] = \frac{\phi(n)}{|\mathrm{Gal}\,(\mathbb{Q}(\omega)/\mathbb{Q}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}))|}, \tag{3.1}$$

where $\mathrm{Gal}\,(\mathbb{Q}(\omega)/\mathbb{Q}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})) = \{\sigma \in \mathrm{Aut}(\mathbb{Q}(\omega)) : \sigma$ is a $\mathbb{Q}$-automorphism and $\sigma(\lambda_j) = \lambda_j$ for all $j \in \{0, 1, \ldots, n-1\}\}$. We shall determine the size of this group and obtain the algbraic degree of H.

Lemma 3.2.1. *Let $y \in \{0, 1, \ldots, n-1\}$ be such that $\gcd(y, n) = 1$ and $\sigma_y \in \mathrm{Aut}(\mathbb{Q}(\omega))$ be the $\mathbb{Q}$-automorphism defined by $\omega \mapsto \omega^y$. Then $\sigma_y(\lambda_j) = \lambda_j$ for all $j \in \{0, 1, \ldots, n-1\}$ if and only if there is $n_y \in \mathbb{N}$ with $C = C_1 \cup \cdots \cup C_{n_y}$, $yC_l \equiv C_l \mod n$ and $a_{0,k} = a_{0,yk}$ for all $k \in C_l$ and $l \in \{1, 2, \ldots, n_y\}$.*

Proof. If there is an $n_y \in \mathbb{N}$ with $C = C_1 \cup \cdots \cup C_{n_y}$, $yC_l \equiv C_l \mod n$ and $a_{0,k} = a_{0,yk}$ for all $k \in C_l$ and $l \in \{1, 2, \ldots, n_y\}$, then

$$\begin{aligned}\sigma_y(\lambda_j) = \sigma_y\left(\sum_{k \in C} a_{0,k}\omega^{jk}\right) &= \sum_{l=1}^{n_y}\sum_{k \in C_l} a_{0,k}\sigma_y\left(\omega^{jk}\right) = \sum_{l=1}^{n_y}\sum_{k \in C_l} a_{0,k}\omega^{jky} \\ &= \sum_{l=1}^{n_y}\sum_{k \in C_l} a_{0,yk}\omega^{jky} = \sum_{k \in C} a_{0,yk}\omega^{jyk} = \sum_{yk \in C} a_{0,yk}\omega^{jyk} = \lambda_j\end{aligned}$$

for all $j \in \{0, 1, \ldots, n-1\}$. On the other hand, suppose that $\sigma_y(\lambda_j) = \lambda_j$ for all $j \in \{0, 1, \ldots, n-1\}$. Then $\sum_{k \in C} a_{0,k}\left(\omega^j\right)^{yk} = \sum_{k \in C} a_{0,k}\left(\omega^j\right)^k$ for all $j \in \{0, 1, \ldots, n-1\}$. Let $p(x) = \sum_{k \in C} a_{0,k}x^{yk} - \sum_{k \in C} a_{0,k}x^k$. It is a polynomial of degree at most $n-1$. Since $1, \omega, \ldots, \omega^{n-1}$ are distinct roots of $p(x)$, we have $p(x) = 0$. Define an equivalence relation on C by $k \sim k'$ whenever $a_{0,k} = a_{0,k'}$. Let $C_1, \ldots, C_{n_y}$ be all equivalence classes of $\sim$. Then $C = C_1 \cup \cdots \cup C_{n_y}$. Since $p(x) = 0$, we have $yC_l \equiv C_l \mod n$ and so $a_{0,k} = a_{0,yk}$ for all $k \in C_l$ and $l \in \{1, 2, \ldots, n_y\}$. $\square$

Theorem 3.2.2. *Let* $\mathrm{H} = t\text{-Cay}(\mathbb{Z}_n, S)$ *and* $C = S \cup 2S \cup \cdots \cup (t-1)S \setminus \{0\}$. *Let* m *be the number of* y *in* $\{0, 1, \ldots, n-1\}$ *such that* $\gcd(y, n) = 1$ *and there is an* $n_y \in \mathbb{N}$ *with* $C = C_1 \cup \cdots \cup C_{n_y}$, $yC_l \equiv C_l \mod n$ *and* $a_{0,k} = a_{0,yk}$ *for all* $k \in C_l$ *and* $l \in \{1, 2, \ldots, n_y\}$. *Then*

$$\deg \mathrm{H} = \frac{\phi(n)}{m}.$$

Moreover, $\deg \mathrm{H} \leq \frac{\phi(n)}{2}$.

Proof. By Lemma 3.2.1, m is the size of $\mathrm{Gal}\left(\mathbb{Q}(\omega)/\mathbb{Q}\left(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\right)\right)$. It follows from Equation (3.1) that $\deg \mathrm{H} = \frac{\phi(n)}{m}$. From $S \equiv -S \mod n$, we have $C = -C \mod n$. Since $\{\pm k\} = -\{\pm k\}$ and $a_{0,k} = a_{0,-k}$ for any $k \in C$, 1 and -1 are such y. Hence, $m \geq 2$, so $\frac{\phi(n)}{m} \leq \frac{\phi(n)}{2}$. □

Consider $\mathrm{H} = 2\text{-Cay}(\mathbb{Z}_n, S)$. Then $C = S$ and $a_{0,k} = 1$ for any $k \in S$ and $a_{0,k} = 0$ otherwise. The assumption of Theorem 3.2.2 can be reduced to $yS \equiv S \mod n$. In addition, if $n = p$ is a prime number, Mönius showed in the proof of Theorem 2.5 of [18] that m in Theorem 3.2.2 is the maximum number of $M \in \{1, 2, \ldots, |S|\}$ such that M divides $\gcd(|S|, p-1)$ and

$$S = \bigcup_{l=1}^{|S|/M} S_l$$

where $|S_l| = M$ and for each $l \in \{1, 2, \ldots, |S|/M\}$, $k^M = (k')^M \mod p$ for all $k, k' \in S_l$. The next corollary gives the algebraic degree of Cayley graph of $\mathbb{Z}_n$ over S which generalizes Theorem 2.5 of [18].

Corollary 3.2.3. *Let* $\mathrm{H} = 2\text{-Cay}(\mathbb{Z}_n, S)$. *If* m *is the number of* y *in* $\{0, 1, \ldots, n-1\}$ *such that* $\gcd(y, n) = 1$ *and* $yS \equiv S \mod n$, *then*

$$\deg \mathrm{H} = \frac{\phi(n)}{m}.$$

Example 3.2.4. Consider $\mathrm{H} = 2\text{-Cay}(\mathbb{Z}_{31}, S)$ where $S = \{\pm 2, \pm 3, \pm 10, \pm 12, \pm 13, \pm 15\} = C$. Since $\pm 1, \pm 5, \pm 6$ are all elements of y such that $\gcd(y, 31) = 1$ and

$yC \equiv C \mod 31$, by Corollary 3.2.3, $\deg \mathrm{H} = \frac{\phi(31)}{6} = 5$. This coincides Example 2.10 of [18].

In the proof of Theorem 3.2.2, we have known that 1 and -1 are always such y satisfying $yC \equiv C \mod n$. If only they satisfy this congruence, we have a special case of Theorem 3.2.2 as follows.

Corollary 3.2.5. *Let* $\mathrm{H} = t\text{-Cay}(\mathbb{Z}_n, S)$ *and* $C = S \cup 2S \cup \cdots \cup (t-1)S \setminus \{0\}$. *If* $y = 1$ *and* $y = -1$ *are the only elements in* $\mathbb{Z}_n$ *such that* $\gcd(y, n) = 1$ *and* $yC \equiv C \mod n$*, then*

$$\deg \mathrm{H} = \frac{\phi(n)}{2}.$$

We provide some numerical examples using Theorem 3.2.2 and Corollary 3.2.5 as follows.

Example 3.2.6. Consider $\mathrm{H} = 3\text{-Cay}(\mathbb{Z}_{12}, \{\pm 1\})$. We have $C = \{\pm 1, \pm 2\}$. In addition, $a_{0,\pm 1} = 2$ and $a_{0,\pm 2} = 1$. The characteristic polynomial of $A(\mathrm{H})$ is

$$(x-1)^2(x+2)^3(x+3)^2(x-6)(x^2-2x-11)^2$$

and hence $\deg \mathrm{H} = 2$. Since 1 and -1 are the only elements y in $\mathbb{Z}_{12}$ such that $\gcd(y, 12) = 1$ and $yC \equiv C \mod 12$, by Corollary 3.2.5, $\deg \mathrm{H} = \frac{\phi(12)}{2} = 2$.

Example 3.2.7. Let $S = \{\pm 1\}$ be a subset of $(\mathbb{Z}_9, +)$. Them $\max\{o(x) : x \in S\} = 9$, so $2 \leq t \leq 9$. The algebraic degree of t-Cayley hypergraph of $\mathbb{Z}_9$ over S for all t are presented in the following table. The cases $t \in \{2, 3, 4\}$ are computed by Corollary 3.2.5 and the others are obtained from Theorem 3.2.2.

t	$a_{0,\pm 1}$	$a_{0,\pm 2}$	$a_{0,\pm 3}$	$a_{0,\pm 4}$	y with $yC \equiv C \mod 9$	$\deg t\text{-Cay}(\mathbb{Z}_9, S)$
2	1				± 1	3
3	2	1			± 1	3
4	3	2	1		± 1	3
5	4	3	2	1	$\pm 1, \pm 2, \pm 4$	3
6	5	4	3	3	$\pm 1, \pm 2, \pm 4$	3
7	6	5	5	5	$\pm 1, \pm 2, \pm 4$	3
8	7	7	7	7	$\pm 1, \pm 2, \pm 4$	1
9	1	1	1	1	$\pm 1, \pm 2, \pm 4$	1

Example 3.2.8. Let $S = \{\pm 2\}$ be a subset of $(\mathbb{Z}_{10}, +)$. Then $\max\{o(x) : x \in S\} = 5$, so $2 \leq t \leq 5$. The algebraic degree of t-Cayley hypergraph of $\mathbb{Z}_{10}$ over S for all t are presented in the following table. The case $t = 2$ is computed by Corollary 3.2.5 and the others are obtained from Theorem 3.2.2.

t	$a_{0,\pm 1}$	$a_{0,\pm 2}$	$a_{0,\pm 3}$	$a_{0,\pm 4}$	$a_{0,5}$	y with $yC \equiv C \mod 10$	$\deg t\text{-Cay}(\mathbb{Z}_{10}, S)$
2		1				± 1	2
3		2		1		± 1	2
4		3		3		$\pm 1, \pm 3$	1
5		1		1		$\pm 1, \pm 3$	1

Remark 3.2.9. From Examples 3.2.7 and 3.2.8, we note that when a subset S of $\mathbb{Z}_n$ is fixed, for any $2 \leq t \leq \max\{o(x) : x \in S\} - 1$, we have $\deg t\text{-Cay}(\mathbb{Z}_n, S) \geq \deg (t+1)\text{-Cay}(\mathbb{Z}_n, S)$.

Remark 3.2.10. The adjacency matrix of t-Cayley hypergraph of $\mathbb{Z}_n$ is circulant. We know the exact eigenvalues and they are in simple forms (Theorem 1.2.3). We may work on integrality or compute algebraic degree of spectra of t-Cayley hypergraphs of other finite groups in the future.

3.3 Algebraic degree of spectra of Cayley graphs of $\mathbb{Z}_n$

Corollary 3.2.3 gives algebraic degree of spectra of the Cayley graph of $\mathbb{Z}_n$ over S by using congruence properties of the set S. In this section, we discuss recent Mönius' result comparing with Corollary 3.2.3.

In 2022, Mönius [19] studied other properties of spectra of Cayley graphs of $\mathbb{Z}_n$. He provided the splitting field and algebraic degree of Cayley graphs of $\mathbb{Z}_n$ by using many results on Schur rings and Schur partitions. We shall compute algebraic degree from Mönius' result and our result when $t = 2$.

Next, we briefly introduce Schur rings and orbit Schur rings. Let $(G, \cdot)$ be a group with identity e. For a subset S of G, let $\underline{S} = \sum_{g \in S} g \in \mathbb{Q}G$.

A *Schur ring* $\mathcal{A}$ over G is a subalgebra of the group algebra $\mathbb{Q}G$ satisfying the properties below:

1. $\mathcal{A}$ has a linear basis $\underline{S_0}, \dots, \underline{S_r}$, where $S_0 = \{e\}$,

2. $\{\underline{S_0}, \dots, \underline{S_r}\}$ is a partition of G, and

3. $S_j = \{x^{-1} : x \in S_j\} \in \{\underline{S_0}, \dots, \underline{S_r}\}$ for all $j \in \{0, \dots, r\}$.

For a subgroup Γ of $\operatorname{Aut} G$, the *orbit Schur ring of* Γ *over* G, denoted by $\mathbb{Q}G^{\Gamma}$, is a Schur ring defined by

$$\mathbb{Q}G^{\Gamma} = \{\alpha \in \mathbb{Q}G : \sigma(\alpha) = \alpha \text{ for all } \sigma \in \Gamma\}.$$

We quote the first main theorem in [19].

Theorem 3.3.1 ([19]). *Let* $\mathrm{H} = 2\text{-Cay}(\mathbb{Z}_n, S)$ *and* $f(x) = \det(xI_n - A(\mathrm{H}))$. *The splitting field of* $f(x)$ *over* $\mathbb{Q}$ *is given by* $\mathbb{Q}(\omega)^{\Gamma}$, *where* $\Gamma \leq \operatorname{Aut} \mathbb{Z}_n$ *is defined by* $\langle\langle S \rangle\rangle_{\mathcal{O}} = \mathbb{Q}\mathbb{Z}_n^{\Gamma}$ *the unique least orbit Schur ring containing* $\underline{S}$. *In addition,* Γ *is the maximum subgroup of* $\operatorname{Aut} \mathbb{Z}_n$ *such that* $S = \cup_{x \in S} \{\sigma(x) : \sigma \in \Gamma\}$.

By Theorem 3.3.1, we have that

$$\phi(n) = [\mathbb{Q}(\omega) : \mathbb{Q}] = \left[\mathbb{Q}(\omega) : \mathbb{Q}(\omega)^{\Gamma}\right] \left[\mathbb{Q}(\omega)^{\Gamma} : \mathbb{Q}\right] = |\Gamma| \left[\mathbb{Q}(\omega)^{\Gamma} : \mathbb{Q}\right].$$

Since $\deg \mathrm{H} = \left[\mathbb{Q}(\omega)^\Gamma : \mathbb{Q}\right]$, we can conclude the following theorem.

Theorem 3.3.2 ([19])**.** *Let* $\mathrm{H} = 2\text{-Cay}(\mathbb{Z}_n, S)$ *and* Γ *be the maximum subgroup of* $\operatorname{Aut} \mathbb{Z}_n$ *such that* $S = \cup_{x \in S}\{\sigma(x) : \sigma \in \Gamma\}$*. Then*

$$\deg \mathrm{H} = \frac{\phi(n)}{|\Gamma|}.$$

Remark 3.3.3. Mönius [19] has a more general result of Theorem 3.3.2 by considering any subset S of $\mathbb{Z}_n$ which may not satisfy the condition $S = -S$. However, under the assumption $S = -S$, we can show that $|\Gamma| = m$, where m is defined in Corollary 3.2.3, as follows. Recall that m is the number of y in $\{0, 1, \ldots, n-1\}$ such that $\gcd(y, n) = 1$ and $yS \equiv S \mod n$. We note that $\operatorname{Aut} \mathbb{Z}_n \cong \mathbb{Z}_n^\times$ and $\operatorname{Aut} \mathbb{Z}_n = \{\sigma_y : y \in \mathbb{Z}_n^\times\}$ where $\sigma_y(1) = y$. Let $y \in \{0, 1, \ldots, n-1\}$ with $\gcd(y, n) = 1$ and $yS \equiv S \mod n$. We have that the map $\sigma_y(1) = y$ is an automorphism of $\mathbb{Z}_n$ such that $\sigma_y(s) = s\sigma_y(1) = sy$ for any $s \in S$. Since $yS \equiv S \mod n$, we can conclude that $\sigma_y \in \Gamma$. On the other hand, we can see that for any $\sigma_y \in \operatorname{Aut} \mathbb{Z}_n$, so $|\sigma_y(S)| = |S|$. If $S = \cup_{x \in S}\{\sigma_y(x) : \sigma_y \in \Gamma\} = \cup_{\sigma_y \in \Gamma}\sigma_y(S)$, then $\sigma_y(S) \subseteq S$. Hence, $S = \sigma_y(S) = yS$. This means $yS \equiv S \mod n$. Therefore, $|\Gamma| = m$.

From Mönius' result, to compute the algebraic degree of Cayley graphs of 2-Cay$(\mathbb{Z}_n, S)$ by using Schur rings, we follow the steps below.

1. Find all subgroups of $\operatorname{Aut} \mathbb{Z}_n$.
2. For each subgroup Γ of $\operatorname{Aut} \mathbb{Z}_n$, compute the orbit Schur ring $\mathbb{Q}\mathbb{Z}_n^\Gamma$.
3. Find the unique least orbit Schur ring containing $\underline{S}$, i.e. $\langle\langle S \rangle\rangle_{\mathcal{O}}$.
4. Suppose that $\langle\langle S \rangle\rangle_{\mathcal{O}} = \mathbb{Q}\mathbb{Z}_n^\Gamma$ for some subgroup Γ of $\operatorname{Aut} \mathbb{Z}_n$. By the one-to-one correspondence between the lattice of orbit Schur rings over $\mathbb{Z}_{12}$ and the lattice of subfields of $\mathbb{Q}(e^{2\pi i/n})$ and Theorem 3.3.1, the splitting field is $\mathbb{Q}(e^{2\pi i/n})^\Gamma$.

5. By Theorem 3.3.2, the algebraic degree of 2-Cay$(\mathbb{Z}_n, S) = \phi(n)/|\Gamma|$.

Example 3.3.4. Let $n = 12$ and $\omega = e^{2\pi i/12}$. The subgroups of $\operatorname{Aut}\mathbb{Z}_{12} \cong \mathbb{Z}_{12}^{\times}$ are $\{1\}, \{1,5\}, \{1,7\}, \{1,11\}$ and $\{1,5,7,11\}$. Mönius [19] computed the orbit Schur rings over $\mathbb{Z}_{12}$ as follows:

$$
\begin{aligned}
\mathbb{Q}\mathbb{Z}_{12}^{\{1\}} &= \left\langle \underline{\{0\}}, \underline{\{1\}}, \underline{\{2\}}, \underline{\{3\}}, \underline{\{4\}}, \underline{\{5\}}, \underline{\{6\}}, \underline{\{7\}}, \underline{\{8\}}, \underline{\{9\}}, \underline{\{10\}}, \underline{\{11\}} \right\rangle, \\
\mathbb{Q}\mathbb{Z}_{12}^{\{1,5\}} &= \left\langle \underline{\{0\}}, \underline{\{3\}}, \underline{\{6\}}, \underline{\{9\}}, \underline{\{1,5\}}, \underline{\{2,10\}}, \underline{\{4,8\}}, \underline{\{7,11\}} \right\rangle, \\
\mathbb{Q}\mathbb{Z}_{12}^{\{1,7\}} &= \left\langle \underline{\{0\}}, \underline{\{2\}}, \underline{\{4\}}, \underline{\{6\}}, \underline{\{8\}}, \underline{\{10\}}, \underline{\{1,7\}}, \underline{\{3,9\}}, \underline{\{5,11\}} \right\rangle, \\
\mathbb{Q}\mathbb{Z}_{12}^{\{1,11\}} &= \left\langle \underline{\{0\}}, \underline{\{6\}}, \underline{\{1,11\}}, \underline{\{2,10\}}, \underline{\{3,9\}}, \underline{\{4,8\}}, \underline{\{5,7\}} \right\rangle \quad \text{and} \\
\mathbb{Q}\mathbb{Z}_{12}^{\{1,5,7,11\}} &= \left\langle \underline{\{0\}}, \underline{\{6\}}, \underline{\{2,10\}}, \underline{\{3,9\}}, \underline{\{4,8\}}, \underline{\{1,5,7,11\}} \right\rangle.
\end{aligned}
$$

The following figure shows the one-to-one correspondence between the lattice of orbit Schur rings over $\mathbb{Z}_{12}$ and the lattice of subfields of $\mathbb{Q}(\omega)$.

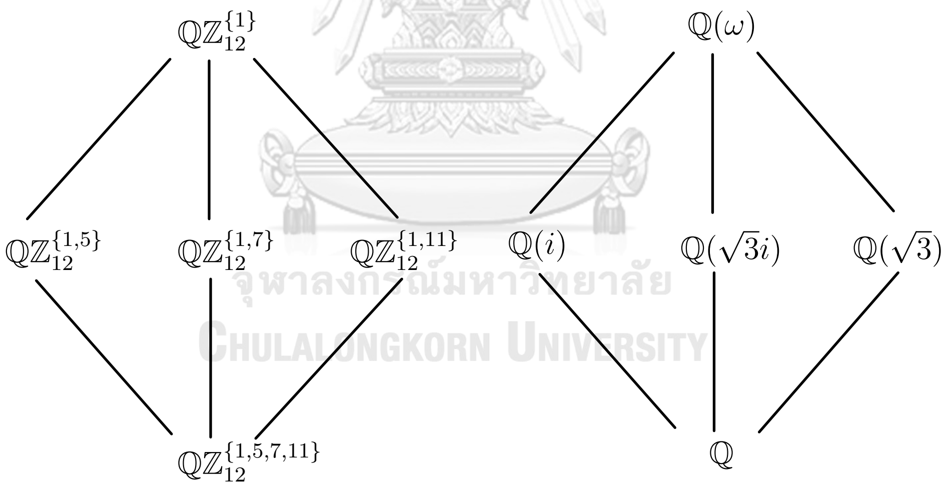

Figure 3.1: Lattice of orbit Schur rings over $\mathbb{Z}_{12}$ (left) and lattice of subfields of $\mathbb{Q}(\omega)$ (right)

The following table shows some algebraic degree of Cayley graphs of $\mathbb{Z}_{12}$ where $S = \{\pm1\}, \{\pm3\}, \{\pm1, \pm2\}$ and $\{\pm1, \pm2, \pm3\}$ by using Theorems 3.3.1 and 3.3.2.

S	$\langle\langle S\rangle\rangle_{\mathcal{O}} = \mathbb{Q}\mathbb{Z}_{12}^{\Gamma}$	Γ	$\mathbb{Q}(\omega)^{\Gamma}$	$\deg 2\text{-Cay}(\mathbb{Z}_{12}, S)$
$\{\pm 1\}$	$\mathbb{Q}\mathbb{Z}_{12}^{\{1,11\}}$	$\{1, 11\}$	$\mathbb{Q}(\sqrt{3})$	2
$\{\pm 3\}$	$\mathbb{Q}\mathbb{Z}_{12}^{\{1,5,7,11\}}$	$\{1, 5, 7, 11\}$	$\mathbb{Q}$	1
$\{\pm 1, \pm 2\}$	$\mathbb{Q}\mathbb{Z}_{12}^{\{1,11\}}$	$\{1, 11\}$	$\mathbb{Q}(\sqrt{3})$	2
$\{\pm 1, \pm 2, \pm 3\}$	$\mathbb{Q}\mathbb{Z}_{12}^{\{1,11\}}$	$\{1, 11\}$	$\mathbb{Q}(\sqrt{3})$	2

Next, we compute the algebraic degree of Cayley graphs of $\mathbb{Z}_{12}$ where $S = \{\pm 1\}, \{\pm 3\}, \{\pm 1, \pm 2\}$ and $\{\pm 1, \pm 2, \pm 3\}$ by using Corollary 3.2.3 as the following table.

S	y with $yS \equiv S \mod 12$	$\deg 2\text{-Cay}(\mathbb{Z}_{12}, S)$
$\{\pm 1\}$	$\{1, 11\}$	2
$\{\pm 3\}$	$\{1, 5, 7, 11\}$	1
$\{\pm 1, \pm 2\}$	$\{1, 11\}\}$	2
$\{\pm 1, \pm 2, \pm 3\}$	$\{1, 11\}$	2

In Example 3.3.4, we compute the algebraic degree of some Cayley graphs over $\mathbb{Z}_{12}$ in two different ways. Firstly, we use the corresponding Schur ring over $\mathbb{Z}_{12}$. It requires complicated tools and consists of many steps. By the way, an advantage of this computation is that we can find the splitting field of the characteristic polynomial of its adjacency matrix over $\mathbb{Q}$. In the other hand, we immediately compute its algebraic degree by using simple congruence property by using Corollary 3.2.3.

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