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A LINEAR MATRIX INEQUALITY APPROACH TO DESIGN STATE FEEDBACK CONTROL FOR NON-POLYNOMIAL NONLINEAR SYSTEMS

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พิง ลิม: แนวทางอสมการเมทริกซ์เชิงเส้นเพื่อออกแบบการควบคุมป้อนกลับสถานะสำหรับระบบไม่เป็นเชิงเส้นแบบไม่เป็นพหุนาม. (A LINEAR MATRIX INEQUALITY APPROACH TO DESIGN STATE FEEDBACK CONTROL FOR NON-POLYNOMIAL NONLINEAR SYSTEMS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ศ. ดร. เดวิด บรรเจิดพงศ์ชัย, 0 หน้า.

วิทยานิพนธ์นี้มีจุดมุ่งหมายเพื่อออกแบบตัวควบคุมการป้อนกลับสถานะสำหรับระบบไม่เป็นเชิงเส้นแบบไม่เป็นพหุนามภายใต้สัญญาณขาเข้ามีขอบเขต การกำหนดปัญหาเริ่มต้นด้วยการแปลงระบบไม่เป็นพหุนามให้เป็นระบบพหุนาม ขั้นตอนนี้ดำเนินการโดยนิยามพจน์ที่ไม่เป็นพหุนามให้เป็นตัวแปรสถานะใหม่ร่วมกับเงื่อนไขบังคับพีชคณิตที่สอดคล้องกับสมบัติของพหุนามไม่เชิงเส้น วิธีแปลงนี้หลีกเลี่ยงการประมาณของระบบพหุนาม หลังจากนั้น เราออกแบบตัวควบคุมการป้อนกลับสถานะบนพื้นฐานทฤษฎีบทเสถียรภาพเลียปูนอฟแบบขยายผลและอาศัยเกณฑ์สมรรถนะกำลังสอง เราเสนอการออกแบบการควบคุมป้อนกลับสถานะ 2 แบบ การออกแบบแรกใช้ฟังก์ชันเลียปูนอฟแบบสถิต (เมทริกซ์เลียปูนอฟเป็นเมทริกซ์คงตัว) ขณะที่การออกแบบที่สองใช้ฟังก์ชันเลียปูนอฟแบบพหุนาม (เมทริกซ์เลียปูนอฟเป็นฟังก์ชันพหุนามของตัวแปรสถานะ) สำหรับตัวควบคุมการป้อนกลับแบบแรก เงื่อนไขการออกแบบจัดอยู่ในรูปแบบอสมการเทริกซ์เชิงเส้น ค่าขอบเขตบนของฟังก์ชันต้นทุนกำลังสองแบบเหมาะที่สุดหามาได้โดยการแก้ปัญหาอสมการเมทริกซ์เชิงเส้น สำหรับตัวควบคุมการป้อนกลับแบบที่สอง เงื่อนไขการออกแบบจัดอยู่ในรูปแบบอสมการเมริกซ์เชิงเส้นคู่ เมื่อกำหนดล่วงหน้าอันดับของฟังก์ชันเลียปูนอฟ เงื่อนไขบังคับกลายเป็นอสมการเมริกซ์ที่ขึ้นกับพารามิเตอร์ และแก้หาคำตอบด้วยเทคนิคผลบวกของกำลังสอง เรานำเสนอตัวอย่างเชิงตัวเลขเพื่อสาธิตประสิทธิผลของการออกแบบการควบคุมการป้อนกลับสถานะ อีกทั้งเราประยุกต์การออกแบบกับแขนเพนดูลัมผกผันบนรถ และเปรียบเทียบผลตอบสนองของสถานะและสัญญาณควบคุมโดยแปรเปลี่ยนอันดับของฟังก์ชันเลียปูนอฟ

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PHING LIM : A LINEAR MATRIX INEQUALITY APPROACH TO DESIGN STATE FEEDBACK CONTROL FOR NON-POLYNOMIAL NONLINEAR SYSTEMS. ADVISOR : PROF. DAVID BANJERDPONGCHAI, Ph.D., 0 pp.

This thesis aims to design state feedback controller for non-polynomial nonlinear systems subject to bounded control inputs. The problem formulation begins by transforming the non-polynomial systems into polynomial systems. This can be done by defining non-polynomial terms as new state variables with algebraic constraints satisfying the non-polynomial properties. This method avoids the approximation of the recast polynomial systems. Then we design the state feedback controller based on the extended Lyapunov stability theorem and the quadratic performance criterion. Two state feedback control designs are proposed. The first design employs static Lyapunov function (Lyapunov matrix is constant matrix) whereas the second design for polynomial Lyapunov function (Lyapunov matrix is polynomial function of state variable). For the first state feedback controller, the design conditions are derived in terms of linear matrix inequality. An upper bound on the optimal quadratic cost function can be readily obtained using available LMI solvers. For the second state feedback controller, the design conditions are derived in terms of bilinear matrix inequality. With a prior fixed degree of Lyapunov function, these constraints become parameter-dependent linear matrix inequality, which can be solved using the sum of squares technique. Numerical examples are provided to demonstrate the effectiveness of the proposed control designs. Moreover,

we apply the proposed design to the inverted pendulum on cart and compare the state response and control input by varying the degree of Lyapunov function.

Department:	Electrical Engineering	Student's Signature	....
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Chapter I

INTRODUCTION

1.1 Introduction

The interest in controller design for nonlinear systems has become a popular topic in control system theory in the last decade. One of the nonlinear models used in some of the existing works (**?**), (**?**), (**?**) for state feedback controller synthesis is the polynomial system. The author proposed the sum of squares approach to the state feedback controller design problem for polynomial systems with saturation input in **?**. In this work, a polytope represents the saturation function, providing sufficient tractable stability conditions. Similarly, the sum of square based for state feedback design was proposed in **?** to guarantee the control design cost of a polynomial system with bounded input. Sufficient design conditions are obtained with the goal of obtaining a system with a minimized input amplitude constraint that is asymptotically stable. After that, S-procedure derived these sufficient conditions in terms of SOS decomposition. For **?**, the convex optimization approach was given to designate state feedback for polynomial systems. The design criteria include the quadratic cost function, bound control input magnitudes, and bound semi-algebraic set of the closed-loop system operation. In addition, the design conditions were known as state-dependent linear matrix inequality, which was easily solved by using sum of squares technique.

The design conditions for the state feedback controller above can be mainly applied to all nonlinear systems. However, the approximation still happens when the original plan is non-polynomial. Any system containing non-polynomial terms such as triangle, exponential, and logarithmic functions was referred to as a non-polynomial system. This type of system was also seen in some previous works (**?**), (**?**), which proposed the study of domains of attraction. In **?**, the author proposed the approach of a descriptor system to handle the DA analysis condition with the Lyapunov stability theorem, and two stability conditions were obtained and covert to the LMI problem. In **?**, the same technique as **?** is proposed, but the difference is that a polynomial Lyapunov function is considered. So, the non-polynomial system is approached as a descriptor or polynomial system by using Taylor expansion, which removes the non-polynomial terms before being involved with stability analysis using the Lyapunov stability theorem. More than these, another method of recasting the non-polynomial system also see in some previous work without using the approximation approach. This technique replaces all of the non-polynomial terms in the original system with additional state variables. By the way, the increased state variables and existing algebraic constraints become the problem when using this technique. Meanwhile, the extended Lyapunov stability theorem is proposed by authors in **?**. The authors proposed the recasting algorithm process for the general non-polynomial system with this work. They then considered the stability analysis with extended Lyapunov analysis involved with arising algebraic constraints.

This thesis aims to design state feedback for a non-polynomial system and avoid the approximation using Taylor expansion. The extended Lyapunov stability theorem in **?** is considered in the controller design with the quadratic criterion. The best upper bounded value of the quadratic cost function is defined using convex optimization. The LMI technique is used to formulate the problem, and the conditions are recast to be an LMI problem. More than these, inverted pendulum on the cart is considered, and the proposed design approach applies. We know that a benchmark nonlinear control design system is the inverted pendulum on a cart. In the previous work, there were controller designs such as LQR (**?**) and nonlinear predictive control (NMPC) (**?**). The dynamic of an inverted pendulum may be simply deduced from physical law. However, the non-polynomial system in the model is brought about by the trigonometric function. In this study, the polynomial model of the inverted pendulum system is obtained using the LMI technique. We create a state feedback controller using the polynomial model based on the quadratic Lyapunov function. The YALMIP is used as a solving tool in MATLAB.

1.2 Objectives

1. Establish the prerequisites for developing state feedback controls for non-polynomial systems. The quadratic cost function has a definition that is based on convex optimization. By using the LMI approach and recasting the problem as an LMI problem, the problem condition is created.
2. The accuracy of the suggested design will be demonstrated by performing a numerical example of an inverted pendulum on a cart.

1.3 Scope of Thesis

1. Consider the nonlinear system, which contains non-polynomial functions such as trigonometric, exponential, and logarithmic functions.
2. Consider the quadratic form of Lyapunov function with constant and polynomial matrix variables.
3. State feedback's gain depends on Lyapunov matrix and state variables.

1.4 Methodology

This thesis considers the recasting technique for transforming a non-polynomial system into a polynomial one using replacing technique in **Algorithm 2.1**. The purpose is to avoid error values in the model using the Taylor expansion. Next, we design a state feedback law for the recasted polynomial system based on the extended Lyapunov stability theorem with quadratic performance. The extended Lyapunov is provided in detail in **Proposition 3.1**. The interest in this extended stability criterion is the involving term algebraic constraints of recasted polynomial system. The value of the upper bound of quadratic cost can then compute easily using the LMI solver. Finally, two examples of non-polynomial systems: A two-dimensional non-polynomial system and an Inverted pendulum on cart system, are provided as numerical examples to check the effectiveness of the proposed design.

1.5 Main Outcomes

There are three main outcomes in this thesis work. Firstly, we obtain a state feedback control law for a non-polynomial system. Secondly, we improve the performance of state response with bound control inputs. Finally, the proposed design law can handle the large domain of attraction compared to the previous design.

1.6 Thesis Outline

This thesis is organized as follows. **Chapter II** introduces the preliminary, which brings in the non-polynomial system form and the algorithm used to transform this system into a rational approach. In **Chapter III**, we illustrate the problem statements. We prove the system as a state-dependent linear-like, quadratic cost function form, study domain of the system, control input domain, and the Lyapunov level set. In **Chapter IV**, we start to propose the main results of this work with two sections. The first section is to derive a theorem for state feedback design in the case of a static Lyapunov matrix. Meanwhile, the second section provides another theorem for state feedback in the case of a polynomial Lyapunov matrix. Lastly, Based on the proposed design, we apply the control law for the example of a two-dimensional system and inverted pendulum on cart as a numerical example in the last **Chapter V**.

Chapter II

PRELIMINARY

The model of a dynamic system is generally can be written as a couple of finite numbers of first-order ordinary differential equations

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)\\
\dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)\\
\vdots\ &\ \ \vdots\\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)
\end{aligned}$$

where, $x_1, \ldots, x_n$ are the state variables and $\dot{x}_i$ is time derivative of state x_i. $u_1, \ldots, u_m$ are the control inputs.

The mathematical model of a dynamic system is naturally nonlinear. A nonlinear system with all the differential equations is a polynomial known as a polynomial system. Nevertheless, if any differential equation in the model contains any non-polynomial function, such as a trigonometric, exponential, or logarithmic function, this nonlinear system is called a non-polynomial system. This chapter considers the relationship between these two classes of nonlinear systems.

We denote the vector notation for nonlinear system as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n,\ u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \in \mathbb{R}^m,\ f(x,u) = \begin{bmatrix} f_1(x,u) \\ f_2(x,u) \\ \vdots \\ f_n(x,u) \end{bmatrix}$$

Consider the general form of a non-polynomial system given by

$$\dot{x} = f_0(x,u) + \sum_{i=1}^{r} f_i(x,u) g_i(x_{\tau_r}) \tag{2.1}$$

where, $x \in \mathbb{R}^n$ is state variables, $u \in \mathbb{R}^m$ is control input, $f_0(x, u), f_1(x, u), \ldots, f_r(x, u)$ are column vector of polynomial function, $g_1(z), \ldots, g_r(z)$ are non-polynomial function, and $\tau_1, \ldots, \tau_r \in \{1, \ldots, n\}$ are indexes.

Since the main idea of this work is to derive a state feedback control law for a non-polynomial system (**??**); however, designing state feedback for this kind of system is quite challenging. Therefore, many authors have proposed the conversion technique to change a non-polynomial system to a polynomial system and then apply the Lyapunov stability theorem to derive the stability criterion. Most authors used the Taylor expansion to derive the approximation of the non-polynomial terms before replacing it with the original system. This is an uncomplicated way to solve this complex system; however, the approximation value exists in the design. The following algorithm returns the polynomial form of the non-polynomial system by replacing all of the non-polynomial terms with new state variables.

Algorithm 2.1. (**?**) Suppose that x_j is the j^{th} element of the state variables x. The recasting process follows the steps below:

1. Let the additional state variables $x_j = g_i(x_{\tau_r})$, $j = n+1, \ldots, n+r$, and $i = 1, \ldots, r$.
2. Define the time derivative of all additional state variables. The new system will represent (**??**) where state variables $x \in \mathbb{R}^{n+r}$.
3. If there still exist any non-polynomial function $g_i(x_{\tau_r})$, we again repeat step 1 and 2, or else the process is ended.

After the end of process, system (**??**) then converted to be a polynomial system with a general given as

$$\dot{x} = f(x, u), \tag{2.2}$$

Moreover, a polynomial system can be rewritten as state-dependent linear-like as

$$\dot{x} = A(x)Z(x) + B(x)u \tag{2.3}$$

where :

- $x \in \mathbb{R}^{n+p}$ is the new state variables of recasted system,
- $u \in \mathbb{R}^m$ is the control input.
- $Z(x)$ is column vector of polynomial dimension N which satisfy

$$Z(x) = 0 \iff x = x_{\text{eq}}$$

- $A(x)$ and $B(x)$ are polynomial matrices with suitable dimensions.

In fact, we known that $x_{n+1} = F_1(x), \ldots, x_{n+p} = F_p(x)$, thus we can say that $x_{eq} = \begin{bmatrix} O & F_1(O) & \ldots & F_p(O) \end{bmatrix}^T$

More than these, the algorithm above raises the algebraic constraint from the additional state variables. We know that there are two types of algebraic constraints: equality and inequality constraints. In this work, we denote two of the column vectors below for these algebraic constraints:

$$G_1(x) = 0, \tag{2.4}$$

$$G_2(x) \geq 0. \tag{2.5}$$

where :

- $G_1(x)$ is n_{G_1} dimensional vector of equality algebraic constraints.
- $G_2(x)$ is n_{G_2} dimensional vector of inequality algebraic constraints.

To better understand the algorithm above, we come up with the following examples.

Example 2.1. (**?**) A single-state variable system is given by

$$\dot{x} = \sin(e^x - 1) + 4\ln(x^2 + 1)$$

To begin the transforming process in **Algorithm 2.1**, we first check if the system contains non-polynomial terms. We start by compare the given system to (**??**), then we have:

$$f_0(x) = 0, f_1(x) = 1, f_2(x) = 4,$$

$$g_1(x) = \sin(e^x - 1),$$

$$g_2(x) = \ln(x^2 + 1)$$

We observe that $g_1(x)$, and $g_2(x)$ are two non-polynomial functions that we will consider in **Algorthm 2.1** as following:

Step 1, we introduce two additional state variables to represent the non-polynomial terms $g_1(x)$, and $g_2(x)$:

$$x_1 = x,$$

$$x_2 = g_1(x),$$

$$x_3 = g_2(x)$$

Step 2, Define the time derivative of additional variables x_1, x_2, x_3:

$$\dot{x}_1 = x_2 + 4x_3,$$

$$\dot{x}_2 = (x_2 + 4x_3)\cos(e^{x_1} - 1)e^{x_1},$$

$$\dot{x}_3 = \frac{2x_1(x_2 + 4x_3)}{x_1^2 + 1}$$

Now, we can rewrite the system as form (**??**):

$$\dot{x} = f_0(x) + f_1(x)g_1(x_1)$$

where

$$f_0(x) = \begin{bmatrix} x_2 + 4x_3 \\ 0 \\ \frac{2x_1(x_2+4x_3)}{x_1^2+1} \end{bmatrix}, \; f_1(x) = \begin{bmatrix} 0 \\ x_2 + 4x_3 \\ 0 \end{bmatrix}, \; g_1(x) = \cos(e^{x_1} - 1)e^{x_1}$$

We observe that there still exists the non-polynomial function $g_1(x)$ in the obtained system. So, we need to repeat the process in ***Step 1***, and ***Step 2***.

However, function $g_1(x)$ is the multiply of two non-polynomial terms $\cos(e^{x_1} - 1)$, and e^{x_1}. Therefore, it is better to represent two additional state $x_4 = \cos(e^{x_1} - 1)$, and $x_5 = e^{x_1}$. By defining the time derivative of these new state variables x_4, and x_5, observe that there is no more non-polynomial term in each state variable's time derivatives. Thus, the algorithm ends, and the non-polynomial system above becomes a polynomial system with state variables $x \in \mathbb{R}^5$

$$\dot{x} = \begin{bmatrix} x_2 + 4x_3 \\ x_4 x_5 (x_2 + 4x_3) \\ \frac{2x_1(x_2+4x_3)}{x_1^2+1} \\ -x_2 x_5 (x_2 + 4x_3) \\ x_5 (x_2 + 4x_3) \end{bmatrix}$$

with algebraic constraints

$$G_1(x) = x_2^2 + x_4^2 - 1 = 0$$

$$G_2(x) = x_5 \geq 0$$

Example 2.2. Consider a two-dimensional non-polynomial system from (**?**):

$$\dot{x}_1 = -x_1 + x_2 - x_1^2 - 5x_2^3 - \sin(x_1) + u_1, \tag{2.6}$$

$$\dot{x}_2 = 1 - 2x_2 - 4x_1^2 - e^{x_2} + u_2 \tag{2.7}$$

To transform this non-polynomial system to a polynomial using **Algorithm 1**, we start by comparing system (**??**) – (**??**) to form (**??**), we get:

$$f_0(x,u) = \begin{bmatrix} -x_1 + x_2 - x_1^2 - 5x_2^3 + u_1 \\ 1 - 2x_2 - 4x_1^2 + u_2 \end{bmatrix}, \ f_1(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, f_2(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

$$g_1(x) = \sin(x_1), \ g_2(x) = e^{x_2}$$

Non-polynomial system $g_1(x)$, and $g_2(x)$ are observed as trigonometric, and exponential, respectively. From Example 1, we got the idea that if any trigonometric function exists, such as cosine, or sine, we should be considered to represent two additional state variables for sine and cosine functions, respectively. So, in short, here we introduce three additional state variables $x_3 = \sin(x_1)$, $x_4 = \cos(x_1)$, and $x_5 = e^{x_2}$.

From the time derivative of this new states variables then reform the system as (**??**), system (**??**) – (**??**) can be rewritten as:

$$\dot{x} = \begin{bmatrix} -x_1 + x_2 - x_1^2 - 5x_2^3 - x_3 + u_1 \\ 1 - 2x_2 - 4x_1^2 - x_5 + u_2 \\ (-x_1 + x_2 - x_1^2 - 5x_2^3 - x_3 + u_1)x_4, \\ -(-x_1 + x_2 - x_1^2 - 5x_2^3 - x_3 + u_1)x_3, \\ (1 - 2x_2 - 4x_1^2 - x_5 + u_2)x_5 \end{bmatrix} \tag{2.8}$$

We see no more non-polynomial terms in the system (**??**). Thus, we successfully converted a non-polynomial system (**??**) – (**??**) to a polynomial system (**??**) with the algebraic constraints arise for the additional variables as

$$G_1(x) = x_3^2 + x_4^2 - 1 = 0, \tag{2.9}$$

$$G_2(x) = x_5 \geq 0 \tag{2.10}$$

To obtain the state-dependent linear-like form of recasted system (**??**), we choose

$$Z(x) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 - 1 & x_5 - 1 \end{bmatrix}^T$$

then system (**??**) can be rewritten as (**??**) where:

$$A(x) = \begin{bmatrix} -1-x_1 & 1-5x_2^2 & -1 & 0 & 0 \\ -4x_1 & -2 & 0 & 0 & -1 \\ -x_4-x_1x_4 & x_4-5x_2^2x_4 & -x_4 & 0 & 0 \\ x_3+x_1x_3 & -x_3+5x_2^2x_3 & x_3 & 0 & 0 \\ -4x_1x_5 & -2x_5 & 0 & 0 & -x_5 \end{bmatrix},$$

$$B(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ x_4 & 0 \\ -x_3 & 0 \\ 0 & x_5 \end{bmatrix}$$

Example 2.3. Consider an inverted pendulum on cart system below:

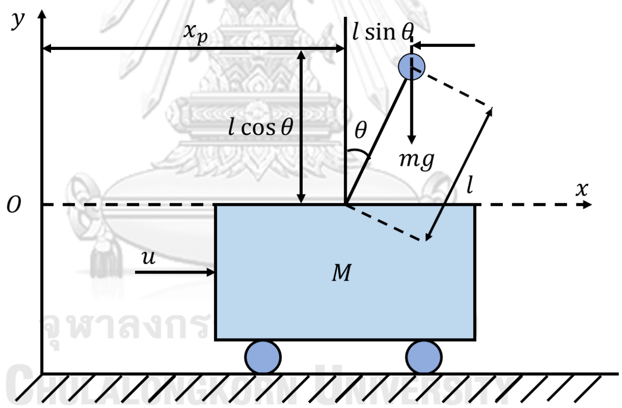


Figure 2.1: Inverted Pendulum on Cart System.

State equations of an inverted pendulum on cart are given by:

$$\dot{x}_1 = x_2, \tag{2.11}$$

$$\dot{x}_2 = \frac{u\cos(x_1) - (M+m)g\sin(x_1) + mlx_2^2\cos(x_1)\sin(x_1)}{ml\cos^2(x_1) - (M+m)l}, \tag{2.12}$$

$$\dot{x}_3 = x_4, \tag{2.13}$$

$$\dot{x}_4 = \frac{u + mx_2^2\sin(x_1) - mg\sin(x_1)\cos(x_1)}{M+m-m\cos^2(x_1)} \tag{2.14}$$

where :

- x_1 is the angle of pendulum θ
- x_2 is the angle's velocity of the pendulum $\dot{\theta}$
- x_3 is the position of the cart x_p
- x_4 is the speed of the cart $\dot{x}_p$.

We observe that in equations (**??**) – (**??**) contains of two non-polynomial terms sin(x_1), and cos(x_1). Thus, we can easily transform this non-polynomial system by replacing these terms with two new state variables.

$$x_5 = \sin(x_1),$$
$$x_6 = \cos(x_1)$$

Hence, the time derivative of x_5, and x_6 are

$$\dot{x}_5 = \dot{x}_1 \cos(x_1) = x_2 x_6,$$
$$\dot{x}_6 = -\dot{x}_1 \sin(x_1) = -x_2 x_5$$

Now, we have a recast system as

$$\dot{x}_1 = x_2, \tag{2.15}$$
$$\dot{x}_2 = \frac{ux_6 - (M+m)gx_5 + mlx_2^2 x_5 x_6}{mlx_6^2 - (M+m)l}, \tag{2.16}$$
$$\dot{x}_3 = x_4, \tag{2.17}$$
$$\dot{x}_4 = \frac{u + mx_2^2 x_5 - mgx_5 x_6}{M + m - mx_6^2}, \tag{2.18}$$
$$\dot{x}_5 = x_2 x_6, \tag{2.19}$$
$$\dot{x}_6 = -x_2 x_5 \tag{2.20}$$

To derive the system (**??**) – (**??**) to state-dependent linear-like form, we represent the column vector as

$$Z(x) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 - 1 \end{bmatrix}^T$$

Hence, the state-dependent linear-form (**??**) of inverted pendulum on cart is obtained with the matrix $A(x)$, and $B(x)$ as follows:

$$A(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{25}(x) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{45}(x) & 0 \\ 0 & x_6 & 0 & 0 & 0 & 0 \\ 0 & -x_5 & 0 & 0 & 0 & 0 \end{bmatrix}, B(x) = \begin{bmatrix} 0 \\ b_{21}(x) \\ 0 \\ b_{41}(x) \\ 0 \\ 0 \end{bmatrix}$$

where:

$$a_{25}(x) = \frac{(M+m)g - mlx_6x_2^2}{l\left(M+m-mx_6^2\right)}, \quad a_{45}(x) = \frac{mx_2^2 - mgx_6}{M+m-mx_6^2},$$
$$b_{21}(x) = -\frac{x_6}{l\left(M+m-mx_6^2\right)}, \quad b_{41}(x) = \frac{1}{M+m-mx_6^2}.$$

Finally, we consider that only triangular functions appear in this system. Therefore, only one equality algebraic constraint arises for x_5, and x_6.

$$G_1(x) = x_5^2 + x_6^2 - 1 = 0 \tag{2.21}$$

In conclusion, we can use **Algorithm 2.1** to transform non-polynomial systems into polynomial systems without using approximation methods. This technique is suitable for avoiding similar errors in the recasted system. Despite that, the increase of additional state variables happened, and the stability analysis is further associated with algebraic constraints arising from representing additional state variables. In passing, the extended Lyapunov stability theorem would be considered to solve this problem. **Examples 2.2** and **2.3** will be used as numerical examples to illustrate the effectiveness of the proposed design in **Chapter V**.

Chapter III

PROBLEM STATEMENT

This chapter introduces the problem statement of this thesis work. This problem considers the state-dependent linear-like form of recasted polynomial (**??**), and its algebraic constraints (**??**) – (**??**) which are detailed in **Chapter II**.

The state feedback controller $u = u(x)$ is aimed to remain in the set $\mathcal{U}$ which is assumed to be a hypercube form as

$$\mathcal{U} = \{u \in \mathbb{R}^m | \; \|u_j(x)\| \leq \mu_j, j = 1, 2, \ldots, m\},$$

μ_j is the specified bound of the control input.

Moreover, we also assume that the closed-loop system is operating in a hypercube set $\mathcal{D}$ such that

$$\mathcal{D} = \left\{x \in \mathbb{R}^{n+p} \mid Z(x)^T B_{\mathcal{D}} Z(x) \leq 1\right\}, B_{\mathcal{D}} \succ O$$

In this assumption, we can derive a function $G_{\mathcal{D}}(x) = 1 - Z(x)^T B_{\mathcal{D}} Z(x) \geq 0$ that use later in the process of design the state feedback law with extended Lyapunov theorem.

The quadratic cost function is considered as form

$$J(x_0, u) = \int_0^\infty \left(Z(x)^T Q Z(x) + u^T R u\right) dt \tag{3.1}$$

where, Q, and R are symmetric positive definite matrices which dimension $\mathbb{R}^{N\times N}$ and $\mathbb{R}^{m\times m}$, respectively. We denote $x_0 = x(0)$ as initial condition. Let us indicate the optimal cost function by $J^*(x_0)$.

The Lyapunov level set is defined to be $\Omega = \{x \in \mathbb{R}^{n+p} | \; V(x) \leq 1\}$. This level set must be inside the input set $u(x)$ in $\mathcal{U}$ for any x in Ω.

$$\Omega(x) \subset \mathcal{X}_{\mathcal{U}} \tag{3.2}$$

where $\mathcal{X}_{\mathcal{U}} = \{x \in \mathbb{R}^{n+p} \mid \|u_j(x)\| \leq \mu_j, j = 1, 2, \ldots, m\}$.

Convex optimization can be used to tackle the problem of computing the optimal cost function and related state feedback controller. The Lyapunov stability theorem should thus be taken into account.

On the other hand, some of the recasted system (**??**) is a rational system that contains the denominator term. This term is needed to simplify when applying the Lyapunov stability theorem for stability analysis. So, we better introduce the collective denominator for the design process, characterized by the symbol $G_c(x)$. A positive collective denominator is recommended to prevent the system from not functioning properly. Moreover, we set two matrices, $A_c(x) = G_c(x)A(x)$ and $B_c(x) = G_c(x)B(x)$, to simplify the condition written in the coming part. In case the system has no denominator, $A_c(x) = A(x)$, $B_c(x) = B(x)$.

The proposition below is the extended Lyapunov stability theorem from (**?**), which handles the recasted system's stability from **Algorithm 2.1**.

Proposition 3.1. (**?**): The system (**??**) with algebraic constraints (**??**) and (**??**) is asymptotically stable at the equilibrium point x_{eq} if there exist a function $V(x)$, a function $\hat{u}$, column vector polynomial function $\lambda_1(x)$, $\lambda_2(x)$, and column vector sum of square polynomial function $\sigma_1(x)$, $\sigma_2(x)$, and a sum of square polynomial function $\sigma_3(x)$, such that

$$V(x_{eq}) = 0, \tag{3.3}$$

$$V(x) - \lambda_1(x)^T G_1(x) - \sigma_1(x)^T G_2(x) - \phi(x) \geq 0 \tag{3.4}$$

$$\begin{aligned} G_c(x)\frac{\partial V}{\partial x}(x)\left(A(x)Z(x) + B(x)\hat{u}\right) + \lambda_2^T(x)G_1(x) + \sigma_2^T(x)G_2(x)\ldots \\ +\sigma_3^T(x)G_{\mathcal{D}}(x) + Z(x)^T Q Z(x) + \hat{u}^T R\hat{u} \leq 0, \end{aligned} \tag{3.5}$$

where $\phi(x)$ is a positive scalar polynomial function. Moreover, any initial point x_0 in Ω must be satisfy $J^*(x_0) \leq J(x_0, \hat{u}) \leq V(x_0)$ if it holds for condition (**??**) and $\Omega(x) \subset \mathcal{D}$.

Next chapter, the conditions (**??**) – (**??**) are considered to design the state feedback control law for system (**??**) – (**??**) with all assumptions above.

Chapter IV

MAIN RESULTS

This chapter demonstrates the detail of two main results of this thesis work. The first result provides a theorem for designing state feedback law with the constant Lyapunov's matrix variable. Although, the second result discusses the control law in the case of Lyapunov's matrix variable is a polynomial matrix.

In easy to derive sufficient conditions for both theorem, we consider the quadratic form of a vector of polynomial functions $\lambda_1(x)$, $\lambda_2(x)$, vector of sum of square polynomial functions $\sigma_1(x)$, $\sigma_2(x)$, and sum of square polynomial function $\sigma_3(x)$ and $\phi(x)$ in (**??**) – (**??**) as

$$\lambda_1(x) = Z(x)^T \begin{bmatrix} L_{11} \\ \vdots \\ L_{1n_{G_1}} \end{bmatrix} Z(x), \tag{4.1}$$

$$\lambda_2(x) = Z(x)^T \begin{bmatrix} L_{21} \\ \vdots \\ L_{2n_{G_1}} \end{bmatrix} Z(x), \tag{4.2}$$

$$\sigma_1(x) = Z(x)^T \begin{bmatrix} S_{11} \\ \vdots \\ S_{1n_{G_1}} \end{bmatrix} Z(x), \tag{4.3}$$

$$\sigma_2(x) = Z(x)^T \begin{bmatrix} S_{21} \\ \vdots \\ S_{2n_{G_1}} \end{bmatrix} Z(x), \tag{4.4}$$

$$\sigma_3(x) = Z(x)^T S_3 Z(x) \tag{4.5}$$

$$\phi(x) = Z(x)^T \Phi Z(x) \tag{4.6}$$

where :

- $L_{1i}, L_{2i}, (i = 1, \ldots, n_{G_1})$ are constant symmetric matrices $N \times N$.
- $S_{1i}, S_{2i}, (i = 1, \ldots, n_{G_2}), S_3$ and Φ are positive definite matrices $N \times N$.

Hence, we can be archived as the following:

$$\lambda_1^T(x)G_1(x) = Z(x)^T \left(\sum_{i=1}^{n_{G_1}} L_{1i}G_{1i}\right) Z(x), \tag{4.7}$$

$$\lambda_2^T(x)G_1(x) = Z(x)^T \left(\sum_{i=1}^{n_{G_1}} L_{2i}G_{1i}\right) Z(x), \tag{4.8}$$

$$\sigma_1^T(x)G_2(x) = Z(x)^T \left(\sum_{i=1}^{n_{G_2}} S_{1i}G_{2i}\right) Z(x), \tag{4.9}$$

$$\sigma_2^T(x)G_2(x) = Z(x)^T \left(\sum_{i=1}^{n_{G_2}} S_{2i}G_{2i}\right) Z(x), \tag{4.10}$$

Following equations (**??**) – (**??**) above, we will accomplish the state feedback control law for system (**??**) based on **Proposition 3.1**.

4.1 Static Lyapunov Function

In this section, we examine the quadratic Lyapunov function with a constant matrix $P \succ O$:

$$V(x) = Z(x)^T P^{-1} Z(x), \tag{4.11}$$

where, P is a symmetric matrix $N \times N$.

Time derivative of (**??**) is obtained as

$$\dot{V}(x) = 2Z(x)^T P^{-1} \mathcal{M}(x)\dot{x}, \tag{4.12}$$

where

$$\mathcal{M}(x) = \frac{\partial Z(x)}{\partial x}$$

Proof: From (**??**), we can say the time derivative is

$$\begin{aligned}\frac{dV(x)}{dt} &= \frac{d(Z(x)^T P^{-1} Z(x))}{dt} \\ &= \frac{d(Z(x)^T P^{-1})}{dt} Z(x) + Z(x)^T P^{-1} \frac{dZ(x)}{dt} \\ &= \frac{dZ(x)^T}{dt} P^{-1} Z(x) + Z(x)^T P^{-1} \frac{dZ(x)}{dt} \\ \Longrightarrow \frac{dV(x)}{dt} &= 2Z(x)^T P^{-1} \frac{dZ(x)}{dt} \dot{x}\end{aligned}$$

which is satisfied (**??**) by expressing $\mathcal{M}(x) = \frac{dZ(x)}{dt}$.

According to this Lyapunov candidate and the assumption above, we have attained a state feedback control law for a non-polynomial system (**??**) in **Theorem 1** below:

Theorem 1. Given the set $\mathcal{D}$ and bounded of control inputs $\mu_j, j = 1, 2, \ldots, m$, the system (**??**) with column vector of equality algebraic constraints $G_1(x)$, and inequality algebraic constraints $G_2(x)$ is stable in a region $\Omega(P)$ using state feedback controller

$$u = -R^{-1} B_c(x)^T \mathcal{M}(x)^T P^{-1} Z(x)$$

if there exists a matrix $P \succ O, L_{1i}, i = 1, 2, \ldots, n_{G_1}, S_{1i} \succ O, i = 1, 2, \ldots, n_{G_2}, Q_{pv}$, and $\Phi \succ O$, such that

$$\begin{bmatrix} \sum_{i=1}^{n_{G_1}} L_{1i} G_{1i}(x) + \sum_{i=1}^{n_{G_2}} S_{1i} G_{2i}(x) + \Phi & I_N \\ I_N & P \end{bmatrix} \prec O, \ \forall x \in \mathcal{D} \tag{4.13}$$

$$\begin{bmatrix} F(x) & Q_v^T(x) Q_{pv} \\ * & I_N \end{bmatrix} \prec O, \ \forall x \in \mathcal{D} \tag{4.14}$$

$$\begin{bmatrix} P & -\mathcal{M}(x) B_c(x) R^{-1} e_j \\ * & \mu_j^2 \end{bmatrix} \succ O, \ \forall x \in \mathcal{D}, j = 1, 2, \ldots, m \tag{4.15}$$

$$P \prec B_D^{-1} \tag{4.16}$$

where:

$$F(x) = \mathcal{M}(x) A_c(x) P + P A_c(x)^T \mathcal{M}(x)^T - \mathcal{M}(x) B_c(x) R^{-1} B_c^T \mathcal{M}(x)^T$$

- e_j is the j^{th} column of identity matrix size $m \times m$.
- I_N is an identity matrix $N \times N$.
- Q_v is a matrix of monomials of x.
- Q_{pv} is invertible matrix which satisfying that $Q_{pv}^T Q_{pv} \succ O$

Proof: Following the chosen Lyapunov candidate (**??**), condition (**??**) is automatically satisfied.

Now, we regard the condition (**??**) of **Proposition 3.1** to derive the constraint (**??**). Substitute Lyapunov candidate in (**??**), equation (**??**), (**??**), and (**??**) to condition (**??**), then simplify the vector $Z(x)$. An LMI came by follows:

$$P^{-1} - \sum_{i=1}^{n_{G_1}} L_{1i} G_{1i} - \sum_{i=1}^{n_{G_2}} S_{1i} G_{2i} - \Phi \succ O, \ \forall x \in \mathcal{D}$$

Furthermore, the constraint (**??**) is then defined using Schur's complement.

Next, we consider the proof for condition (**??**). We start by substituting time-derivative (**??**), equation (**??**), and (**??**) to the constraint (**??**) of **Proposition 3.1**. Additionally, we choose the best controller $\hat{u} = -R^{-1} B_c(x)^T \mathcal{M}(x)^T P^{-1} Z(x)$ in purpose the simplify terms $B_c(x)\hat{u}$ and $\hat{u}^T R \hat{u}$. After that, we simplify the vector $Z(x)$. So, we have:

$$\begin{aligned} P^{-1}\mathcal{M}(x)A_c(x) + A_c(x)^T \mathcal{M}(x)^T P^{-1} - P^{-1}\mathcal{M}(x)B_c(x)R^{-1}B_c(x)^T \mathcal{M}(x)^T P^{-1} \ldots \\ + \sum_{i=1}^{n_{G_1}} L_{2i} G_{1i} + \sum_{i=1}^{n_{G_2}} S_{2i} G_{2i} + S_3 G_{\mathcal{D}}(x) + Q \prec O, \end{aligned}$$

We further pre- and post-multiplying P to the LMI above, then we get:

$$\begin{aligned} \mathcal{M}(x)A_c(x) + A_c(x)^T \mathcal{M}(x)^T - \mathcal{M}(x)B_c(x)R^{-1}B_c(x)^T \mathcal{M}(x)^T \ldots \\ + P\left(\sum_{i=1}^{n_{G_1}} L_{2i} G_{1i}\right) P + P\left(\sum_{i=1}^{n_{G_2}} S_{2i} G_{2i}\right) P + P S_3 G_{\mathcal{D}}(x) P + PQP \prec O, \end{aligned}$$

Let

$$Q_v^T(x) Q_p Q_p Q_v(x) = \left(\sum_{i=1}^{n_{G_1}} L_{2i} G_{1i}\right) + \left(\sum_{i=1}^{n_{G_2}} S_{2i} G_{2i}\right) + S_3 G_{\mathcal{D}}(x) + Q,$$

where, Q_v is matrix of monomial, and Q_p is invertible matrix.
After that, we have:

$$\mathcal{M}(x)A_c(x) + A_c(x)^T\mathcal{M}(x)^T - \mathcal{M}(x)B_c(x)R^{-1}B_c(x)^T\mathcal{M}(x)^T \dots$$
$$+Q_v^T(x)Q_pPPQ_pQ_v(x) \prec O,$$

Let, $Q_{pv} = Q_pP$. Then, we can rewrite the inequality above to (**??**) by using Schur's complement.

Condition (**??**) is to make sure that the Lyapunov level set $\Omega(P)$ remains inside the control input set $\mathcal{X}_{\mathcal{U}}$ as depicted in (**??**). Thus, we can say

$$P^{-1} - \frac{1}{\mu_j^2}P^{-1}\mathcal{M}(x)B_c(x)R^{-1}e_je_j^TR^{-1}B_c(x)^T\mathcal{M}(x)^TP^{-1} \succ O$$

By applying the Schur's complement, the LMI in (**??**) is achieved after we simplify P^{-1} by Pre- and Post-multiply by $\mathrm{diag}(P, 1)$.

Lastly, the constraint (**??**) is used to guarantee the level set $\Omega \in \mathcal{D}$.

Furthermore, the main problem is obtaining an optimal cost function's upper bound. We introduce a constant matrix $T \succ O$ that satisfies

$$J^*(x_0) \leq Z(x_0)^TP^{-1}Z(x_0) \leq Z(x_0)^TTZ(x_0).$$

We called the term $Z(x_0)^TTZ(x_0)$ as the best upper bound of cost function, which is represented by $\bar{J}^*(x_0)$.

It means $T \succ P^{-1}$. Thus, it is simple to show an equivalent LMI constraint

$$\begin{bmatrix} T & I \\ I & P \end{bmatrix} \succeq O \qquad (4.17)$$

We finally attained the upper bound of the quadratic cost function and optimal controller by minimizing $Z(x_0)^TTZ(x_0)$ subject to (**??**) – (**??**).

In this proposed design, the decision variables are $P, L_{1i}, i = 1, \dots, n_{G_1}, S_{1i}, i = 1, \dots, n_{G_2}, Q_{pv}$, and Φ. Besides, they can be solved efficiently using LMI feasible problems. MATLAB optimization toolbox, YALMIP, can settle this problem using PENLAB solver. The flowchart in Figure **??** illustrates the algorithm of computing the upper bound of the cost function based on the suggested design in **Theorem 1** which describes in detail by **Algorithm 4.1**.

Algorithm 4.1. With the state-dependent linear-like form of recasted polynomial system (**??**), we compute the best upper bound cost function using **Theorem 1** by following steps:

1. Start by input the system parameters $A(x), B(x), Z(x)$, collective denominator $G_c(x)$, initial condition x_0, bound input μ, maximum iteration $itermax$, and tolerance tol.

2. Compute the best upper bound by minimizing $Z(x_0)^T T Z(x_0)$ subject to constraints (**??**) – (**??**).

3. Return the best upper bound of the cost function $\bar{J}^*(x_0)$, and all decision variables.

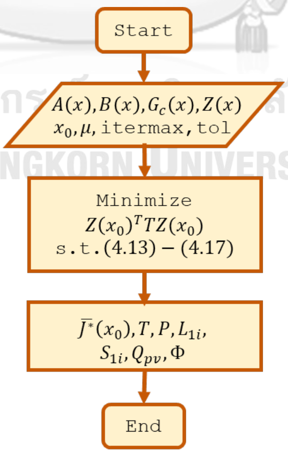


Figure 4.1: Flowchart of state feedback design with **Theorem 1**.

4.2 Polynomial Lyapunov Function

In this section, we begin a state feedback design law for system (**??**) by choosing a Lyapunov candidate as the following form:

$$V(x) = Z(x)^T P^{-1}(x) Z(x), \tag{4.18}$$

where, $P(x) \succ O$ is a polynomial matrix.

Consider the time derivative of the quadratic Lyapunov function above, which is given as

$$\dot{V}(x) = 2Z(x)^T P^{-1}(x)\mathcal{M}(x)\dot{x} + Z(x)^T \sum_{i=1}^{n} (\dot{x}_i \frac{\partial P^{-1}(x)}{\partial x_i}) Z(x), \tag{4.19}$$

where:

$$\mathcal{M}(x) = \frac{\partial Z(x)}{\partial x}$$

Proof: From (**??**), we can say the time derivative is

$$\begin{aligned}
\frac{dV(x)}{dt} &= \frac{d\left(Z(x)^T P^{-1}(x) Z(x)\right)}{dt} \\
&= Z(x)^T P^{-1}(x) \frac{dZ(x)}{dt} + \frac{d\left(Z(x)^T P^{-1}(x)\right)}{dt} Z(x) \\
&= Z(x)^T P^{-1}(x) \frac{dZ(x)}{dt} + \frac{dZ(x)^T}{dt} P^{-1}(x) Z(x) + Z(x)^T \frac{dP^{-1}(x)}{dt} Z(x) \\
\implies \frac{dV(x)}{dt} &= 2Z(x)^T P^{-1}(x) \frac{\partial Z(x)}{\partial x} \dot{x} + Z(x)^T \frac{\partial P^{-1}(x)}{\partial x} \dot{x} Z(x)
\end{aligned}$$

Suppose that $P_i^{-1}(x), i = 1, \ldots, N$ the i^{th} row of matrix $P^{-1}(x)$. Thus, we can say:

$$\frac{\partial P^{-1}(x)}{\partial x} = \begin{bmatrix} \frac{\partial P_1^{-1}(x)}{\partial x_1} & \frac{\partial P_1^{-1}(x)}{\partial x_2} & \cdots & \frac{\partial P_1^{-1}(x)}{\partial x_n} \\ \frac{\partial P_2^{-1}(x)}{\partial x_1} & \frac{\partial P_2^{-1}(x)}{\partial x_2} & \cdots & \frac{\partial P_2^{-1}(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial P_N^{-1}(x)}{\partial x_1} & \frac{\partial P_N^{-1}(x)}{\partial x_2} & \cdots & \frac{\partial P_N^{-1}(x)}{\partial x_n} \end{bmatrix}$$

Multiply $\frac{\partial P^{-1}(x)}{\partial x}$ by $\dot{x}$, we get:

$$\begin{aligned}\frac{\partial P^{-1}(x)}{\partial x}\dot{x} &= \begin{bmatrix}\sum_{i=1}^{n}(\dot{x}_i\frac{\partial P_1^{-1}(x)}{\partial x_i}) & \sum_{i=1}^{n}(\dot{x}_i\frac{\partial P_2^{-1}(x)}{\partial x_i}) & \ldots & \sum_{i=1}^{n}(\dot{x}_i\frac{\partial P_N^{-1}(x)}{\partial x_i})\end{bmatrix}^T \\ &= \begin{bmatrix}\sum_{i=1}^{n}(\dot{x}_i\begin{bmatrix}\frac{\partial P_{11}^{-1}(x)}{\partial x_i} & \frac{\partial P_{12}^{-1}(x)}{\partial x_i} & \ldots & \frac{\partial P_{1N}^{-1}(x)}{\partial x_i}\end{bmatrix}) \\ \sum_{i=1}^{n}(\dot{x}_i\begin{bmatrix}\frac{\partial P_{21}^{-1}(x)}{\partial x_i} & \frac{\partial P_{22}^{-1}(x)}{\partial x_i} & \ldots & \frac{\partial P_{2N}^{-1}(x)}{\partial x_i}\end{bmatrix}) \\ \vdots \\ \sum_{i=1}^{n}(\dot{x}_i\begin{bmatrix}\frac{\partial P_{N1}^{-1}(x)}{\partial x_i} & \frac{\partial P_{N2}^{-1}(x)}{\partial x_i} & \ldots & \frac{\partial P_{NN}^{-1}(x)}{\partial x_i}\end{bmatrix})\end{bmatrix} \\ &= \sum_{i=1}^{n}(\dot{x}_i\frac{\partial P^{-1}(x)}{\partial x_i})\end{aligned}$$

Substitute $\frac{\partial P^{-1}(x)}{\partial x}$ to $\frac{dV(x)}{dt}$, we obtain the solution as (**??**).

The following **Theorem 2** is a state feedback law for system (**??**) based on the extended Lyapunov theorem in **Proposition 3.1**:

Theorem 2. Given the set $\mathcal{D}$ and bounded of control inputs $\mu_j, j = 1, 2, \ldots, m$, the system (**??**) with column vector of equality algebraic constraints $G_1(x)$, and inequality algebraic constraints $G_2(x)$ is stable in a region $\Omega(P)$ using state feedback controller

$$u = -R^{-1}B_c(x)^T\mathcal{M}(x)^TP^{-1}(x)Z(x)$$

if there exists a matrix $P(x) \succ O, L_{1i}, i = 1, 2, \ldots, n_{G_1}, S_{1i} \succ O, i = 1, 2, \ldots, n_{G_2}$, and $\Phi \succ O$, such that

$$\begin{bmatrix}\sum_{i=1}^{n_{G_1}} L_{1i}G_{1i}(x) + \sum_{i=1}^{n_{G_2}} S_{1i}G_{2i}(x) + \Phi & I_N \\ I_N & P(x)\end{bmatrix} \prec O,\ \forall x \in \mathcal{D} \tag{4.20}$$

$$\begin{bmatrix}F(x) - \sum_{j=1}^{m} \mp \left(\mu_j \sum_{i=1}^{n} \frac{\partial P(x)}{\partial x_i}B_{cij}(x)\right) & P(x)Q_v^T(x)Q_p \\ * & I_N\end{bmatrix} \prec O,\ \forall x \in \mathcal{D} \tag{4.21}$$

$$\begin{bmatrix}P(x) & -\mathcal{M}(x)B_c(x)R^{-1}e_j \\ * & \mu_j^2\end{bmatrix} \succ O,\ \forall x \in \mathcal{D}, j = 1, 2, \ldots, m \tag{4.22}$$

$$P(x) \prec B_D^{-1} \tag{4.23}$$

where:

$$F(x) = \mathcal{M}(x)A_c(x)P(x) + P(x)A_c(x)^T\mathcal{M}(x)^T - \mathcal{M}(x)B_c(x)R^{-1}B_c^T\mathcal{M}(x)^T \dots$$
$$-\sum_{i=1}^{n}\frac{\partial P(x)}{\partial x_i}A_{ci}(x)Z(x)$$

- e_j is the j^{th} column of identity matrix size $m \times m$.
- I_N is an identity matrix $N \times N$.
- $A_{ci}(x)$ is the i^{th} row of matrix $A_c(x)$.
- B_{cij} is the element $(i,j)^{th}$ of matrix $B_c(x)$.
- Q_v is a matrix of monomials of x.
- Q_p is invertible matrix which satisfying that $Q_p^TQ_p \succ O$

Remark 4.1: The symbol $\sum_{i=1}^{m} \mp(.)$ in (**??**) shows that the inequality holds for all combination of $+(.)$ and $-(.)$ which means the constraint (**??**) contains 2^m constraints.

Proof: Theorem 2 is the extension of Theorem 1 in case the Lyapunov matrix is not constant. Following the proof of (**??**), (**??**), and (**??**), we easily obtain the constraints (**??**), (**??**), and (**??**), respectively. So, we look at here the proof of (**??**) only.

Similarly, by considering the constraint (**??**) in **Proposition 3.1**, then substitute the time-derivative of Lyapunov candidate (**??**), equations (**??**), and (**??**).

$$2Z(x)^TP^{-1}(x)\mathcal{M}(x)A_c(x)Z(x) + 2Z(x)^TP^{-1}(x)\mathcal{M}(x)B_c(x)\hat{u}\dots$$
$$+Z(x)^TG_c(x)\left(\sum_{i=1}^{n}(\dot{x}_i\frac{\partial P^{-1}(x)}{\partial x_i})\right)Z(x) + Z(x)^T\left(\sum_{i=1}^{n_{G_1}}L_{2i}G_{1i}\right)Z(x)\dots$$
$$+Z(x)^T\left(\sum_{i=1}^{n_{G_2}}S_{2i}G_{2i}\right)Z(x) + Z(x)^TS_3G_D(x)Z(x) + \hat{u}^TR\hat{u} + Z(x)^TQZ(x) \le 0$$

With the best controller $\hat{u} = -R^{-1}B_c(x)^T\mathcal{M}(x)^T P^{-1}Z(x)$, the inequality below is attained after simplify $Z(x)$:

$$P^{-1}\mathcal{M}(x)A_c(x) + A_c(x)^T\mathcal{M}(x)^T P^{-1} - P^{-1}\mathcal{M}(x)B_c(x)R^{-1}B_c(x)^T\mathcal{M}(x)^T P^{-1}\dots$$
$$+G_c(x)\left(\sum_{i=1}^{n}(\dot{x}_i\frac{\partial P^{-1}(x)}{\partial x_i})\right) + \sum_{i=1}^{n_{G_1}} L_{2i}G_{1i} + \sum_{i=1}^{n_{G_2}} S_{2i}G_{2i} + S_3 G_{\mathcal{D}}(x) + Q \prec O,$$

Regard term $\frac{\partial P^{-1}(x)}{\partial x_i}$:

Assume that $p_{jk}(x)$ and $p^{kl}(x)$ are component functions of $P(x)$ and $P^{-1}(x)$, respectively. Then, we have

$$\sum_{k=1}^{N} p_{jk}(x)p^{kl}(x) = \delta_k^l$$

where δ_k^l is the **Kronecker** delta symbol. Hence, for each $x_i, i = 1,\dots,n$ we get

$$\sum_{k=1}^{N}\left(\frac{\partial p_{jk}(x)}{\partial x_i}p^{kl}(x) + p_{jk}(x)\frac{\partial p^{kl}(x)}{\partial x_i}\right) = 0$$

that is,

$$\frac{\partial P(x)}{\partial x_i}P^{-1}(x) = -P(x)\frac{\partial P^{-1}(x)}{\partial x_i}$$

Therefore,

$$\frac{\partial P^{-1}(x)}{\partial x_i} = -P^{-1}(x)\frac{\partial P(x)}{\partial x_i}P^{-1}(x)$$

Notice that this proof is true if $P(x)$ is a square matrix only. We substitute the derivative and $\dot{x}_i = A_i(x)Z(x) + B_i(x)u$ where $A_i(x)$, and $B_i(x)$ are i^{th} row of matrices $A(x)$, and $B(x)$, respectively.

$$P^{-1}(x)\mathcal{M}(x)A_c(x) + A_c(x)^T\mathcal{M}(x)^T P^{-1}(x) + \left(\sum_{i=1}^{n_{G_1}} L_{2i}G_{1i}\right)\dots$$
$$+\left(\sum_{i=1}^{n_{G_2}} S_{2i}G_{2i}\right) - P^{-1}(x)\mathcal{M}(x)B_c(x)R^{-1}B_c(x)^T L(x)^T P^{-1}(x) + Q\dots$$
$$+S_3G_D(x) - G_c(x)P^{-1}(x)\left(\sum_{i=1}^{n}((A_i(x)Z(x) + B_i(x)u)\frac{\partial P(x)}{\partial x_i})\right)P^{-1}(x) \prec 0$$

We are further pre- and post-multiplying $P(x)$ to the LMI above, then we acquire:

$$\begin{aligned}\mathcal{M}(x)A_c(x) + A_c(x)^T\mathcal{M}(x)^T - \mathcal{M}(x)B_c(x)R^{-1}B_c(x)^T\mathcal{M}(x)^T \ldots\\ -\sum_{i=1}^{n}\frac{\partial P(x)}{\partial x_i}A_{ci}(x)Z(x) - \sum_{j=1}^{m}\mp\left(\mu_j\sum_{i=1}^{n}\frac{\partial P(x)}{\partial x_i}B_{cij}(x)\right)\ldots\\ +P(x)\left(\sum_{i=1}^{n_{G_1}}L_{2i}G_{1i}\right)P(x) + P(x)\left(\sum_{i=1}^{n_{G_2}}S_{2i}G_{2i}\right)P(x)\ldots\\ +P(x)S_3G_{\mathcal{D}}(x)P(x) + P(x)QP(x) \prec O,\end{aligned}$$

Let,

$$Q_v^T(x)Q_pQ_pQ_v(x) = \left(\sum_{i=1}^{n_{G_1}}L_{2i}G_{1i}\right) + \left(\sum_{i=1}^{n_{G_2}}S_{2i}G_{2i}\right) + S_3G_{\mathcal{D}}(x) + Q,$$

where, Q_v is matrix of monomial, and Q_p is invertible matrix. Thus, we hold the following:

$$\begin{aligned}\mathcal{M}(x)A_c(x) + A_c(x)^T\mathcal{M}(x)^T - \mathcal{M}(x)B_c(x)R^{-1}B_c(x)^T\mathcal{M}(x)^T \ldots\\ -\sum_{i=1}^{n}\frac{\partial P(x)}{\partial x_i}A_{ci}(x)Z(x) - \sum_{j=1}^{m}\mp\left(\mu_j\sum_{i=1}^{n}\frac{\partial P(x)}{\partial x_i}B_{cij}(x)\right)\ldots\\ +P(x)Q_v^T(x)Q_pQ_pQ_v(x)P(x) \prec O,\end{aligned}$$

Again, we can rewrite the inequality above to (**??**) by using Schur's complement.

The same as **Theorem 1**, the best upper bound of the optimal cost function is considered. The constant matrix $T \succ O$ that satisfies

$$J^*(x_0) \leq Z(x_0)^TP^{-1}(x_0)Z(x_0) \leq Z(x_0)^TTZ(x_0 = \bar{J}^*(x_0)$$

An LMI constraint which equivalent to $T \succ P^{-1}(x_0)$ is:

$$\begin{bmatrix} T & I \\ I & P(x_0) \end{bmatrix} \succ O \tag{4.24}$$

We finally obtain the upper bound of the quadratic cost function and optimal controller by minimizing $Z(x_0)^TTZ(x_0)$ subject to (**??**) – (**??**).

For **Theorem 2** above, we observe that the decisions variables are $L_{1i}, i = 1,\ldots,n_{G_1}, S_{1i}, i = 1,\ldots,n_{G_2}, Q_p$, and coefficients of $P(x)$. The constraints then become parameter-independent linear matrix inequalities for a fixed degree of $P(x)$, which can be solved using the sum of square technique. In MATLAB, we use SeDuMi as a solver for this problem. The iterative method is suggested to compute the best upper bounded of the optimal cost function. We provide **Algorithm 4.2** with details about the algorithm of computing the best upper bound cost function using **Theorem 2**. Figure **??** shows the flowchart of **Algorithm 4.2**.

Algorithm 4.2: With the state-dependent linear-like form of recasted polynomial system (**??**), we compute the best upper bound of cost function $\bar{J}^*(x_0)$ using **Theorem 2** as following

1. Input the system parameters $A(x), B(x), Z(x)$, collective denominator $G_c(x)$, initial condition x_0, bound input μ, maximum iteration $itermax$, and tolerance tol.
2. Let initial condition Q_p, then set iteration number $iter = 0$.
3. Compute best upper bound cost function $\bar{J}^*(x_0)$ by minimizing $Z(x_0)^T T Z(x_0)$ subject to constraints (**??**) – (**??**).
4. $iter = iter + 1$.
5. If $iter < itermax$, continue to step 6, or else the algorithm is ended.
6. If the decrease of best upper bound cost function $\Delta\bar{J}^*(x_0) \leq tol$, return $\bar{J}^*(x_0)$, and all decision variables then the algorithm is ended, or else go to the step 7.
7. If $iter$ is odd, go back to step 3 with $P(x)$ obtained. If $iter$ is even, go back to step 3 with Q_p obtained.

Remark 4.2: For degree of $P(x) = 0$, the constraints (**??**) – (**??**) in **Theorem 2** are reduced to the design condition in **Theorem 1**.

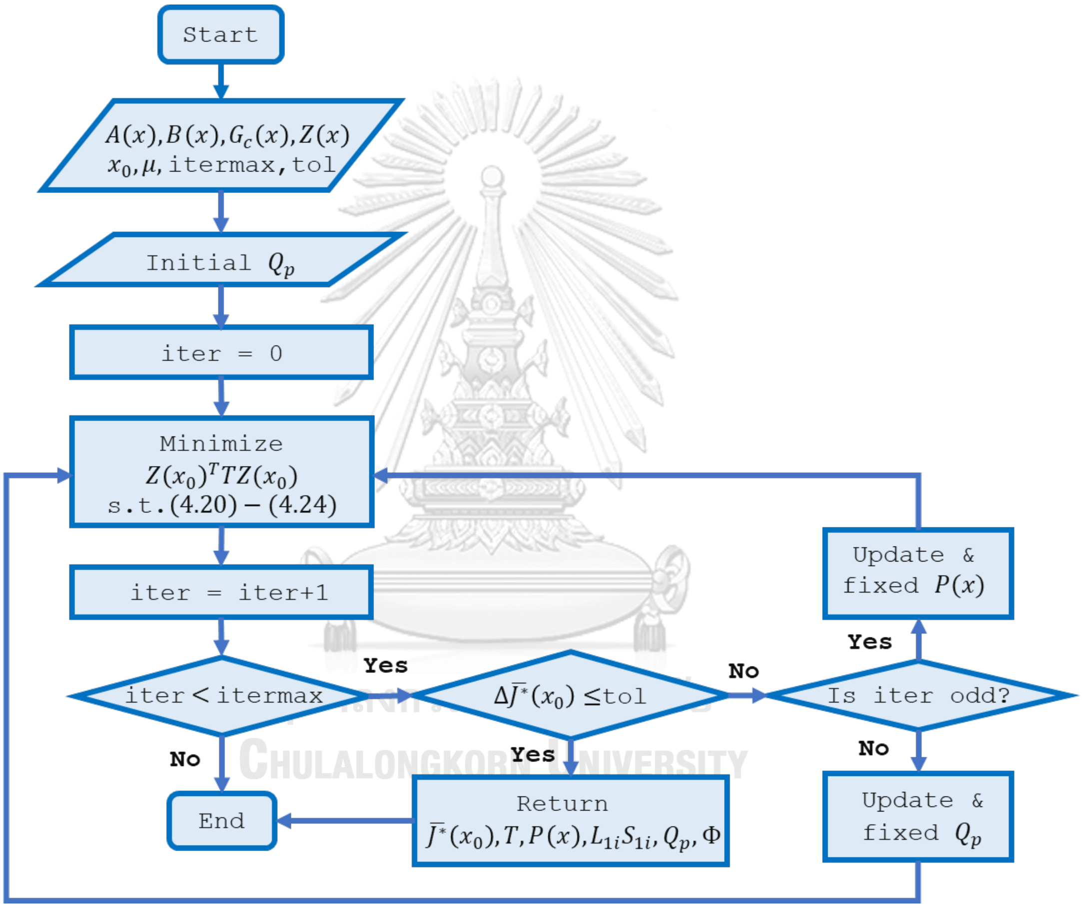


Figure 4.2: Flowchart of state feedback design with **Theorem 2**.

Chapter V

NUMERICAL EXAMPLES

We offer numerical examples in this chapter to demonstrate the accuracy and efficiency of the suggested design in the study results. The first numerical example is Han's two-dimensional non-polynomial system (**?**). We compare the proposed approach in **Theorem 1** to existing work results to determine which is better at handling the most extraordinarily constrained level set and minor control input. The detail of the previous work's proposed design can see in **?**. As a second example, we prefer the inverted pendulum on cart system, which is regarded as a standard system for the stability analysis of this control system theory.

5.1 Two-dimension non-polynomial system

In this section, we consider the state feedback controller design for a two-dimension non-polynomial system (**??**) – (**??**) (**?**), which is detailed in **Chapter II**, **Example 2.2**. Moreover, to indicate the worth of obtained control law in the previous chapter, we examine the results from **?** compared with the proposed design in **Theorem 1**.

Previous work's Result

In **?**, the author provided a controller design law to enlarge the domain of attraction $\hat{\gamma}$. With Lyapunov level set:

$$\Omega = \{x \in \mathbb{R}^2 \mid V(x) = x_1^2 + x_2^2 \leq \hat{\gamma}\}$$

the result of the estimated domain of attraction vs. degree of state feedback d_u is shown in Table **??**:

The interest of the results is that the estimation of domain of attraction $\hat{\gamma}$ increase when the degree of state feedback controller d_u increases. Along with this

Table 5.1: Estimated of Domain of Attraction with varying degree of state feedback controller (**?**)

d_u	1	2	3	4
$\hat{\gamma}$	1.1818	3.3012	133.44	352.73

result, we will select the bigger value of $\hat{\gamma}$ in each case of d_u as the domain of attraction when applying **Theorem 1**. It will show how better that the proposed design can handle the large domain of attraction compared to **?**.

Additionally, we are also interested in the state feedback controller given by **?** in case $d_u = 1$, and $d_u = 2$ as follows.

$$d_u = 1 : u = \begin{bmatrix} -x_1 + 0.2369x_2 \\ 0.2369x_1 - x_2 \end{bmatrix} \tag{5.1}$$

$$d_u = 2 : u = \begin{bmatrix} -x_1 + x_2 + 0.5003x_1^2 + 0.9508x_1x_2 - 0.9151x_2^2 \\ x_1 - x2 + 0.9508x_1^2 - 0.9151x_1x_2 + 0.1778x_2^2 \end{bmatrix} \tag{5.2}$$

By giving the same initial condition, we will show the smallest bound control input in which the design in **?** can be handled and compare it to the proposed design.

Results of Proposed design

Following **Example 2.2** in **Chapter II**, the state-dependent linear-like matrix $A(x), B(x)$, and $Z(x)$ of system (**??**) – (**??**) are obtained with the algebraic constraints (**??**), and (**??**). We notice that the recasted system has one equality constraint ($n_{G_1} = 1$) and one inequality constraint ($n_{G_2} = 1$). Hence, the decision variables from **Theorem 1** are $L_{11}S_{11}, Q_{pv}, \Phi$, and P. **Theorem 2**'s decision variables are $L_{11}, S_{11}, Q_p, \Phi$, and coefficients of polynomial matrix $P(x)$.

From $Z(x)$, we have:

$$\mathcal{M}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ x_4 & 0 & 1 & 0 & 0 \\ -x_3 & 0 & 0 & 1 & 0 \\ 0 & x_5 & 0 & 0 & 1 \end{bmatrix}$$

Given the quadratic matrix of the cost function and initial condition:

$$Q = I_5,\ R = I_2,\ x_0 = \begin{bmatrix} \frac{\pi}{10} & -1 & \sin\frac{\pi}{10} & \cos\frac{\pi}{10} & e^{-1} \end{bmatrix}^T$$

We aim to compare the domain of attraction in this work and the previous ones. We introduce the Lyapunov level set as the variables of $\hat{\gamma}$:

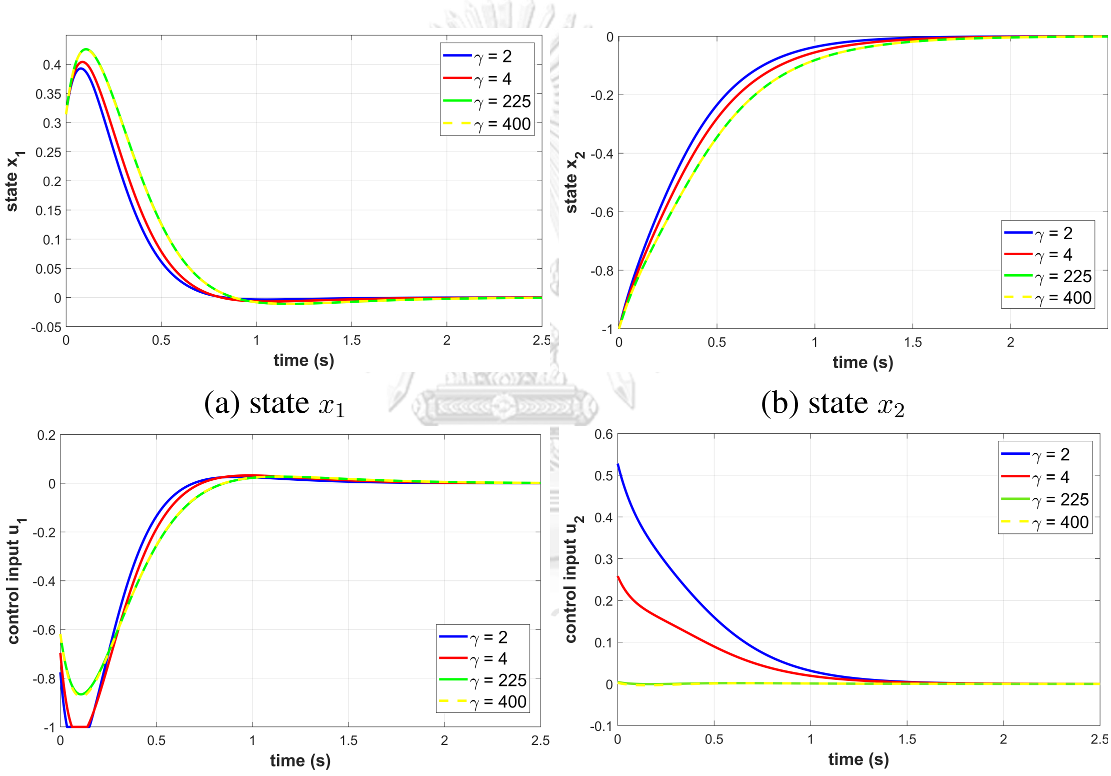
$$\Omega = \left\{ x \in \mathbb{R}^5 \middle| \frac{x_1^2}{\hat{\gamma}} + \frac{x_2^2}{\hat{\gamma}} + x_3^2 + x_4^2 + \frac{x_5^2}{e^{\sqrt{\hat{\gamma}}}} \leq 1 \right\}$$

As mentioned above, the bound of level set $\hat{\gamma}$ in this study is selected as larger than each case of $\hat{\gamma}$ obtained in **?**. Based on the level set chosen, the upper bound of the cost function is shown in Table **??**.

Table 5.2: Upper bound of the optimal cost of two-dimensional non-polynomial system with varying bound input using **Theorem 1**

μ_1	μ_2	$\hat{\gamma}$			
		2	4	225	400
0.54	0.25	infeasible	infeasible	infeasible	infeasible
1	1	0.670852279	0.379879699	0.102771113	0.100634698
2	2	0.670852262	0.379879484	0.102770637	0.100633725
3	3	0.670852262	0.379879484	0.102770634	0.100633725
4	4	0.670852262	0.379879484	0.102770351	0.100633725
5	5	0.670852262	0.379879484	0.102770071	0.100633725

We observe that the upper bound of the optimal cost function in Table **??** decreases when we increase the bounded of x_1 and x_2. More than these, we know that the maximum bounded of DA in **?** $\hat{\gamma} = 352.73$ in case the degree of state feedback $d_u = 4$ (shown in Table **??**). And the maximum of bounded DA chosen in the proposed design $\hat{\gamma} = 400$. This result reveals that the proposed design can be handled with the extensive domain of attraction and is better than the previous result from **?**. On top of that, we also see that the upper bounded is decreasing while the bounded input μ is increasing, similar to the discovery in Jennawasin (**?**), which illustrates that the proposed design in **Theorem 1** can hold the tight input constraints. More than these, the results reveal that the LMI problem is infeasible when $\mu = \begin{bmatrix} 0.54 & 0.25 \end{bmatrix}^T$. The LMI problem is feasible when the control input constraint is at least $\mu = \begin{bmatrix} 0.55 & 0.26 \end{bmatrix}^T$.

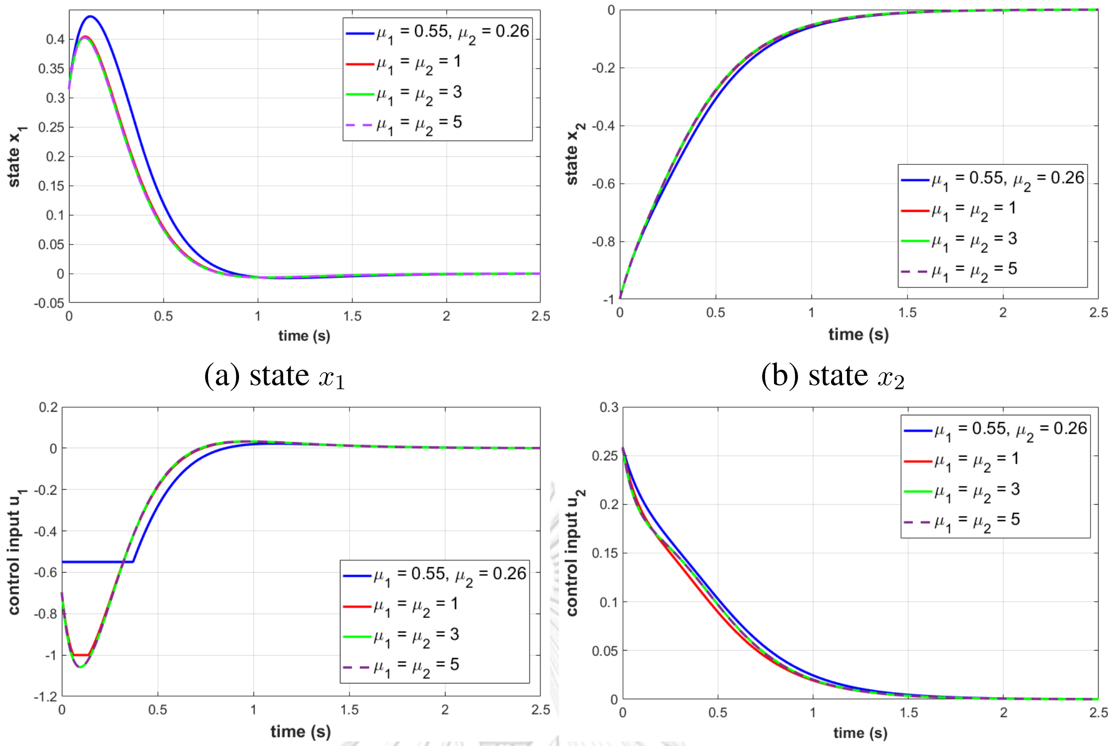


(a) state x_1 (b) state x_2

(c) control input u_1 (d) control input u_2

Figure 5.1: Comparison results of **Theorem 1** by varying $\hat{\gamma}$ with bound input $\mu = 1$.

Figure **??** shows the states response and control input of two-dimensional non-polynomial system (**??**) – (**??**) in all cases of $\hat{\gamma}$ in Table **??** above when bound input $\mu = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. With the graph, we focus on the control input u_1. Control input u_1 has affected by bound input μ when $\hat{\gamma} = 2$, and $\hat{\gamma} = 4$ and the control input remains inside the bound value in case $\hat{\gamma} = 255$, and $\hat{\gamma} = 400$. So, we can say that the narrow value of bound level set $\hat{\gamma}$ can achieve the proposed control law.



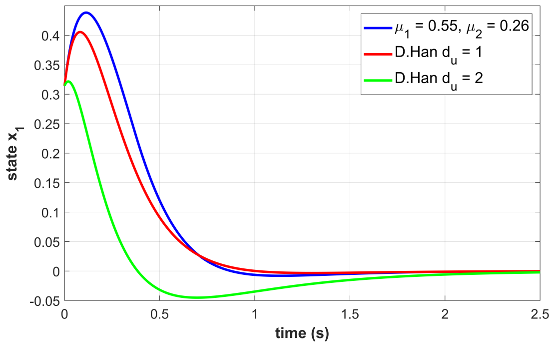
(a) state x_1 (b) state x_2

(c) control input u_1 (d) control input u_2

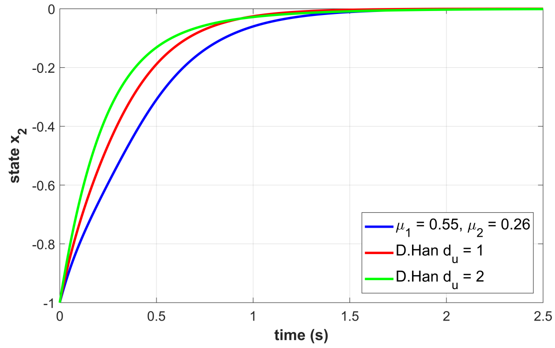
Figure 5.2: State variables and control inputs with varying input bound using **Theorem 1** and $\hat{\gamma} = 4$

Figure **??** shows the performance of state response and control inputs from **Theorem 1** in case $\hat{\gamma} = 4$. By varying bound input μ_1 and μ_2, the performance of both state variables tends to be stable, and control input u_2 tends to be zero without any affected by the bound input value. By the way, we are interested that the performance of control input u_1 got the effect from bound input value in case $\mu_1 = 1$, and the smallest one. This reveals that the small bound value of the control input accomplishes the state feedback law.

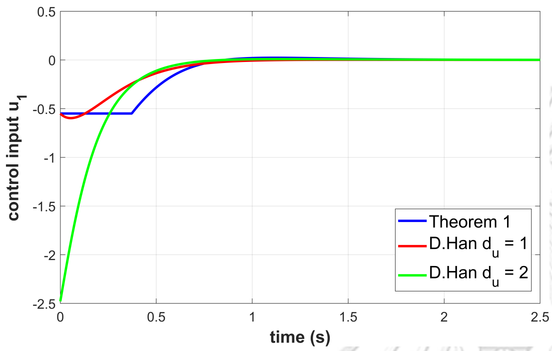
The comparison between state feedback controller (**??**) and (**??**) versus the controller acquired by **Theorem 1** case the smallest μ, $\hat{\gamma} = 4$ is illustrated in Figure **??**. In Figure **??**, the smallest control input found by state feedback controller from **?** is $\mu = [0.60, 1.07]^T$ (case $d_u = 1, \hat{\gamma} = 1.1818$) bigger than smallest control input from **Theorem 1** which is $\mu = [0.55, 0.26]^T$. Thus, we can say that the proposed design **Theorem 1** can handle minor control input compared with the previous result (**?**).



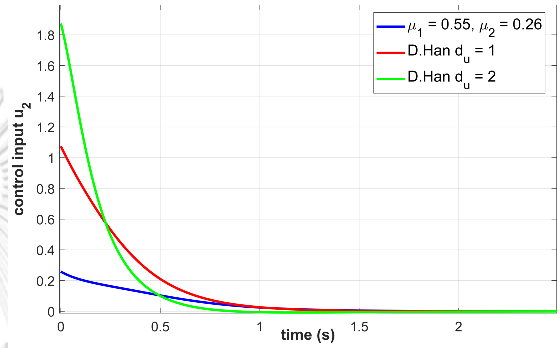
(a) state x_1



(b) state x_2



(c) control input u_1



(d) control input u_2

Figure 5.3: Comparison of **Theorem 1** $\hat{\gamma} = 4$, $\mu_1 = 0.55, \mu_2 = 0.26$ vs **?** $d_u = 1, \hat{\gamma} = 1.1818, d_u = 2, \hat{\gamma} = 3.3012$.

Now, we consider the results from **Theorem 2** of proposed state feedback law for two-dimensional non-polynomial system above. We start by assuming that the polynomial matrix $P(x)$ depends only on two main state variables x_1 and x_2. Table **??** shows form of matrix $P(x)$ from degree 0 to 2, which will consider in this study. The iterative search for the best upper bound of cost function follows process in the flowchart (Figure **??**) where the tolerance is 10^{-8}, and the maximum iteration is 100.

Table 5.3: Polynomial matrix $P(x)$ for two-dimensional system.

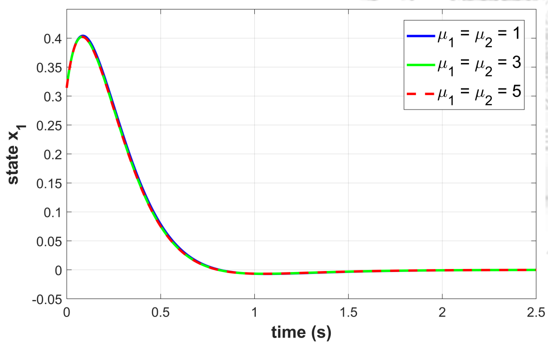
Degree	$P(x)$
0	P_{00}
1	$P_{00} + P_{10}x_1 + P_{01}x_2$
2	$P_{00} + P_{10}x_1 + P_{01}x_2 + P_{11}x_1x_2 + P_{20}x_1^2 + P_{02}x_2^2$

The best upper bound of cost function and the number of iterations is demonstrated in Table **??** with varying bound input μ for each case of degree $P(x)$.

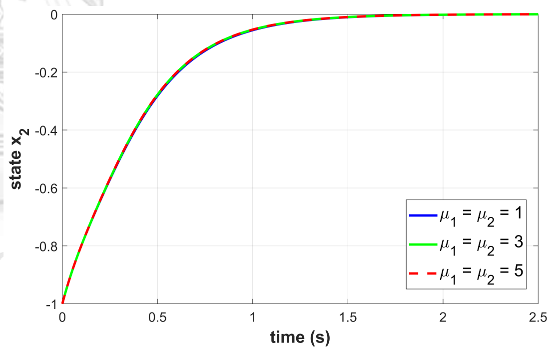
Table 5.4: Upper bound of the optimal cost of two-dimensional non-polynomial system with varying bound input using **Theorem 2**

μ_1	μ_2	degree $P(x)$=0		degree $P(x)$=1		degree $P(x) = 2$	
		J^*	iter	J^*	iter	J^*	iter
1	1	0.379879699	1	0.379799324	4	0.379799324	5
2	2	0.379879484	1	0.379674222	4	0.379674222	5
3	3	0.379879484	1	0.379685564	5	0.379685564	5
4	4	0.379879484	1	0.379647215	6	0.379637215	6
5	5	0.379879484	1	0.379413731	5	0.379413731	6

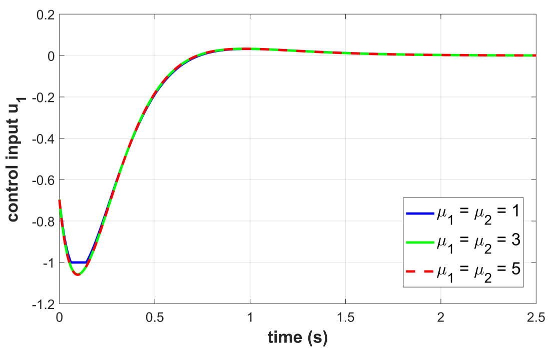
From Table **??**, we spot that the upper bound optimal cost is better when the degree of $P(x)$ increases. Furthermore, the bound value is smaller with large μ. We also notice that the upper bound value is still the same for the degree $P(x)$ equal to 1 and 2. In other words, for this example, the best upper bounds are determined with $P(x)$ of degree 1. Reducing the input bound provides worse upper bounds when the degree of $P(x)$ is fixed.



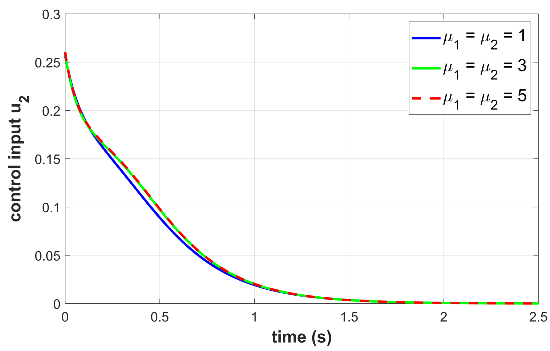
(a) state x_1



(b) state x_2

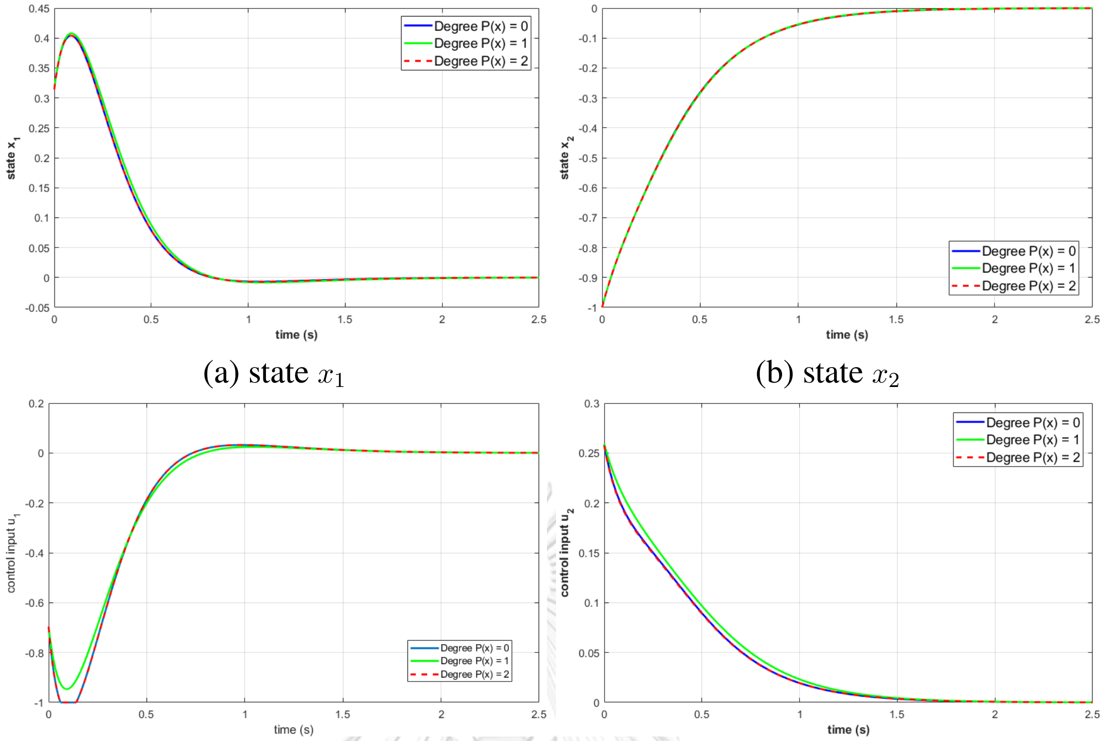


(c) control input u_1



(d) control input u_2

Figure 5.4: State variables and control input with varying input bound using **Theorem 2** and degree $P(x)$ = 2.



(a) state x_1 (b) state x_2

(c) control input u_1 (d) control input u_2

Figure 5.5: State variables and control input with varying degree $P(x)$ using **Theorem 2** and $\mu = \begin{bmatrix}1 & 1\end{bmatrix}^T$.

Figure **??** reveals the performance of state variables and control inputs from **Theorem 2** case degree of $P(x) = 2$. We fascinate that the performance of control input u_1 carry out by the small control input. It seems similar to the result from **Theorem 1** or degree of $P(x) = 0$. This is clearly seen in Figure **??**, which show the performance of state variables and control inputs by varying degree of Lyapunov matrix in case $\mu = \begin{bmatrix}1 & 1\end{bmatrix}^T$. From this point, we can conclude that small bound input affected the control input u_1 for all degrees of $P(x)$.

Finally, we plot the best upper bound cost function versus the number of iterations as shown in Figure **??** to indicate the convergence of iterative search in **Theorem 2**. The graph manifests that each iteration's upper bound cost function decreases until the stopping criteria when the decrease of the upper bound cost function is slighter than tolerance.

Remark 5.1: The results above are computed using MATLAB R2018b with a laptop Dell Processor Intel(R) Core(TM) i5-4210U CPU @1.70GHz; Ram installed 8.00GB. The solver SeDuMi and PENLAB are used to compute the best upper bound of the cost function using Theorem 1 and 2, respectively. The average running time is 133min.

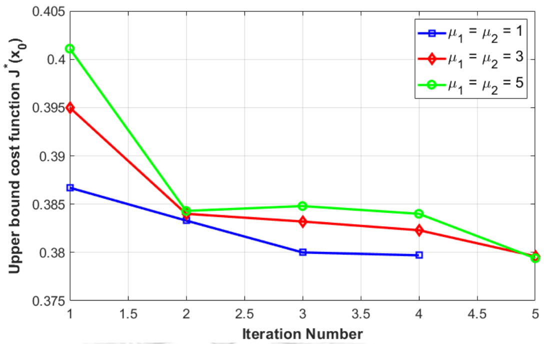


Figure 5.6: Upper bound cost function vs Iteration Number with varying input bound case degree $P(x) = 1$ using iterative search in **Theorem 2**.

5.2 Inverted Pendulum on Cart System

We already discussed the process of transforming the dynamic model of inverted pendulum on cart to polynomial state-dependent linear-like form in **Chapter II**, **Example 2.3**. In this section, we think about the state feedback controller design based on the proposed design in the main results in **Chapter IV**. In this part, The parameters of this system are given in Table **??**.

Table 5.5: Parameters of Inverted Pendulum on Cart

Symbols	Definition	units	Value
M	mass of cart	kg	2.40
m	mass of pendulum	kg	0.23
l	length of pendulum	m	0.36
g	gravity	m/s^2	9.81

We are interested that the recasted system contains denominator terms that need to be simplified for design process. So, we better choose the collective denominator as

$$G_c(x) = M + m - mx_6^2 \geq 0, \ \forall x \in \mathcal{D}$$

Moreover, the algebraic constraint in this transforming process is only the equality constraint (**??**). Consequently, the decision variables from **Theorem 1** are L_{11}, Q_{pv}, Φ, and P. And, **Theorem 2**'s decision variables are L_{11}, Φ, Q_p, and coefficients of matrix $P(x)$.

From $Z(x)$ chosen, we attain matrix $\mathcal{M}(x)$ as

$$\mathcal{M}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ x_6 & 0 & 0 & 0 & 1 & 0 \\ -x_5 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This study, the domain of variables x, quadratic cost function matrices Q, R, and initial condition x_0 are given as

$$\mathcal{D} = \left\{ x \in \mathbb{R}^6 \mid \frac{x_1^2}{10^2} + \frac{x_2^2}{20^2} + \frac{x_3^2}{5^2} + \frac{x_4^2}{20^2} + x_5^2 + x_6^2 \leq 1 \right\}$$

$$Q = I_6,\ R = 1,\ x_0 = \begin{bmatrix} 0.1 & 0 & 0 & 0 & \sin(0.1) & \cos(0.1) \end{bmatrix}^T$$

Additionally, we provide the Table **??** to detail the form of polynomial matrix $P(x)$ from degree 0 to 2 in variables of original states x_1, x_2, x_3, and x_4 where the coefficients are constant symmetric matrix 6×6.

Table 5.6: Polynomial matrix $P(x)$ for inverted pendulum on cart system.

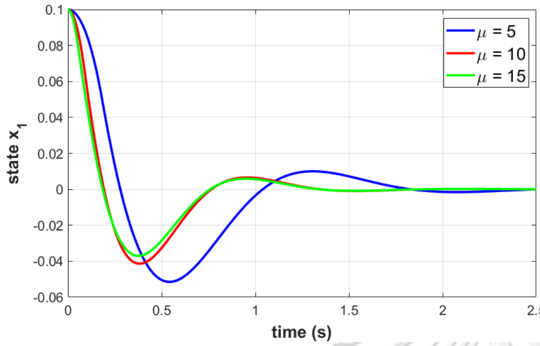
Degree	$P(x)$
0	P_{0000}
1	$P_{0000} + P_{1000}x_1 + P_{0100}x_2 + P_{0010}x_3 + P_{0001}x_4$
2	$P_{0000} + P_{1000}x_1 + P_{0100}x_2 + P_{0010}x_3 + P_{0001}x_4$
	$+P_{1100}x_1x_2 + P_{1010}x_1x_3 + P_{1001}x_1x_4 + P_{0110}x_2x_3 + P_{0101}x_2x_4$
	$+P_{0011}x_3x_4 + P_{2000}x_1^2 + P_{0200}x_2^2 + P_{0020}x_3^2 + P_{0002}x_4^2$

For **Theorem 2**, the iterative search follows the flowchart in Figure **??** with the tolerance 10^{-7}, and the maximum iteration 100. Table **??** reveal the best upper bound of the cost function with varying of bound control input for each degree of $P(x)$. **Theorem 1**'s results are the same as degree $P(x) = 0$.

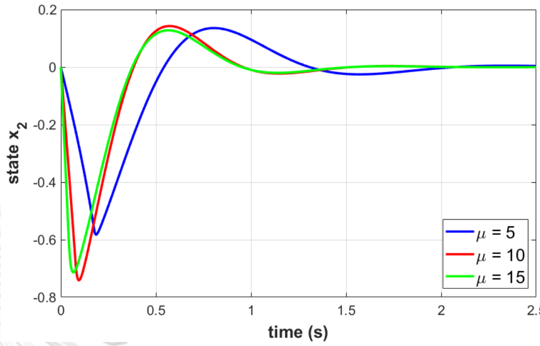
Table 5.7: Upper bound cost value of Inverted Pendulum on Cart with varying control input bound

μ	degree $P(x) = 0$		degree $P(x) = 1$		degree $P(x) = 2$	
	J^*	iter	J^*	iter	J^*	iter
2.5	infeasible	1	infeasible	1	infeasible	1
5	0.1234	1	0.1205	5	0.1189	6
10	0.1226	1	0.1187	5	0.1163	7
15	0.1222	1	0.1181	5	0.1152	7

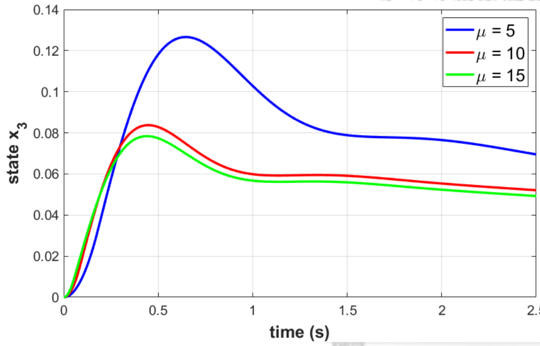
The results in Table **??** demonstrate that as the bound of control input increases, so does the lower bound of the cost function. For **Theorem 2**, the better upper bound cost function is apparent when we increase the degree of $P(x)$. In addition, the designed condition is applicable for severe input limitations, which is similar to the conclusion in **?** both for **Theorem 1** and **2**. We are also concerned that the issue is impractical in the case of $\mu = 2.5$. The feasible solution requires at least $\mu = 2.51$.



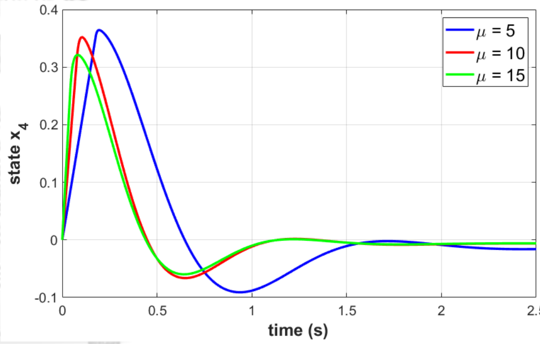
(a) state x_1



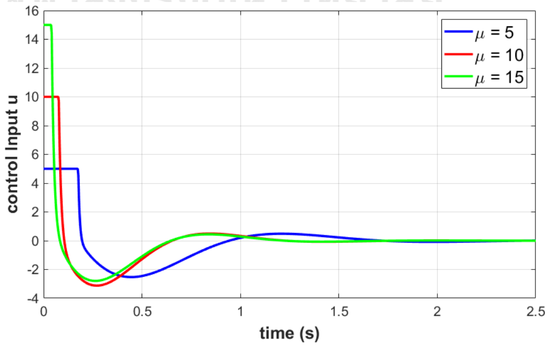
(b) state x_2



(c) state x_3



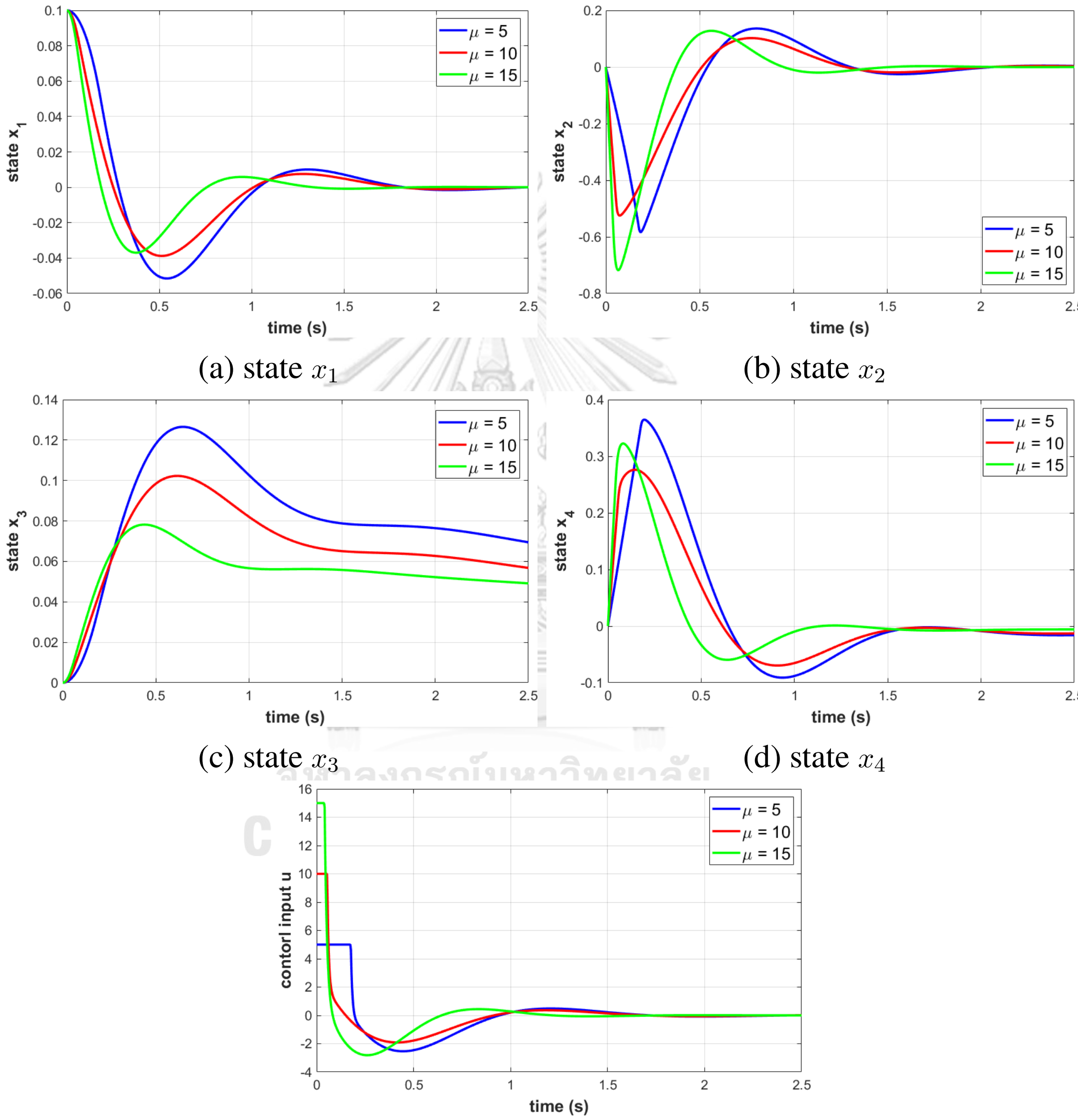
(d) state x_4



(e) control input u

Figure 5.7: State variables and control input with varying input bound using **Theorem 1**.

With varying bound control input, we plot the time response of state variables, and control input from **Theorem 2** degree $P(x) = 0$ (**Theorem 1**), degree $P(x) = 1$, and degree $P(x) = 2$ as shown in Figure **??**, **??**, and **??**, respectively. We discern that the effect from bound input to control input value is more apparent in case of small control input. Moreover, the figures indicate that the state response in case large bound control input is faster than the response in case small bound control input.



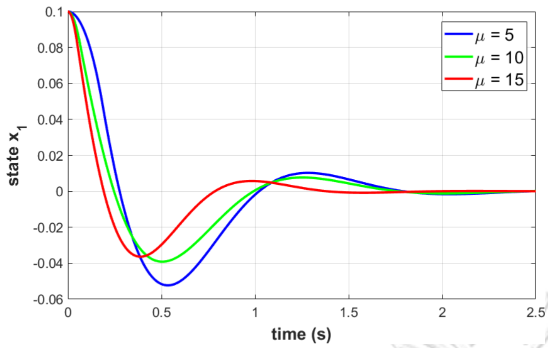
(a) state x_1 (b) state x_2

(c) state x_3 (d) state x_4

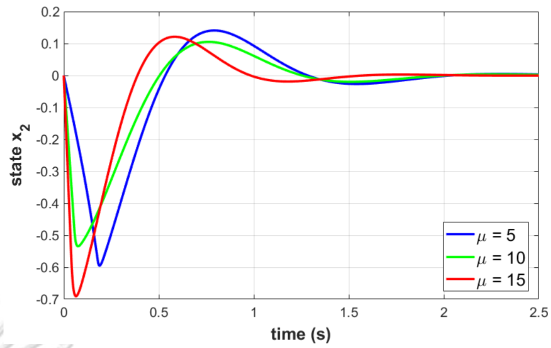
(e) control input u

Figure 5.8: State variables and control input with varying input bound using **Theorem 2** for degree $P(x) = 1$.

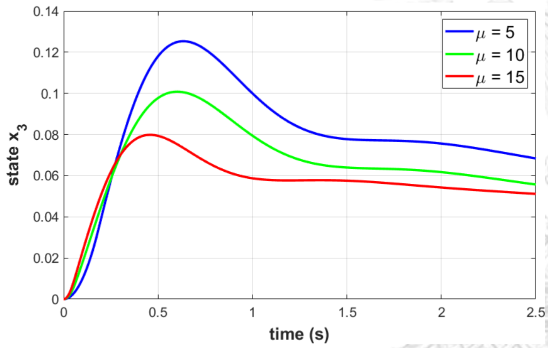
We also consider the plot of state variables and control input by varying degrees of $P(x)$ in case $\mu = 5$ and $\mu = 10$. The Figures are shown in Figure **??** and **??**, respectively. From these plots, we are interested that the case of degree $P(x) = 1$ and 2 are quite similar. We can say that at the high degree $P(x)$, the responses' performance is close to each other.



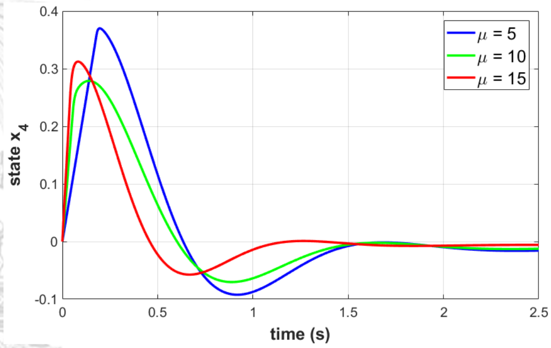
(a) state x_1



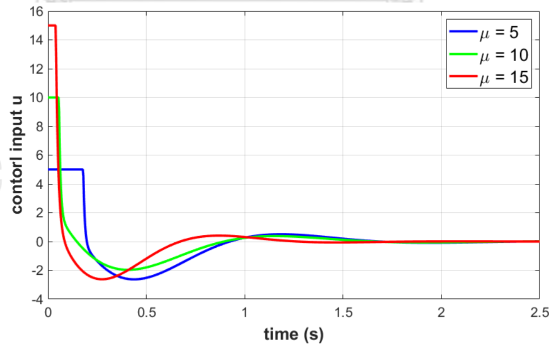
(b) state x_2



(c) state x_3



(d) state x_4

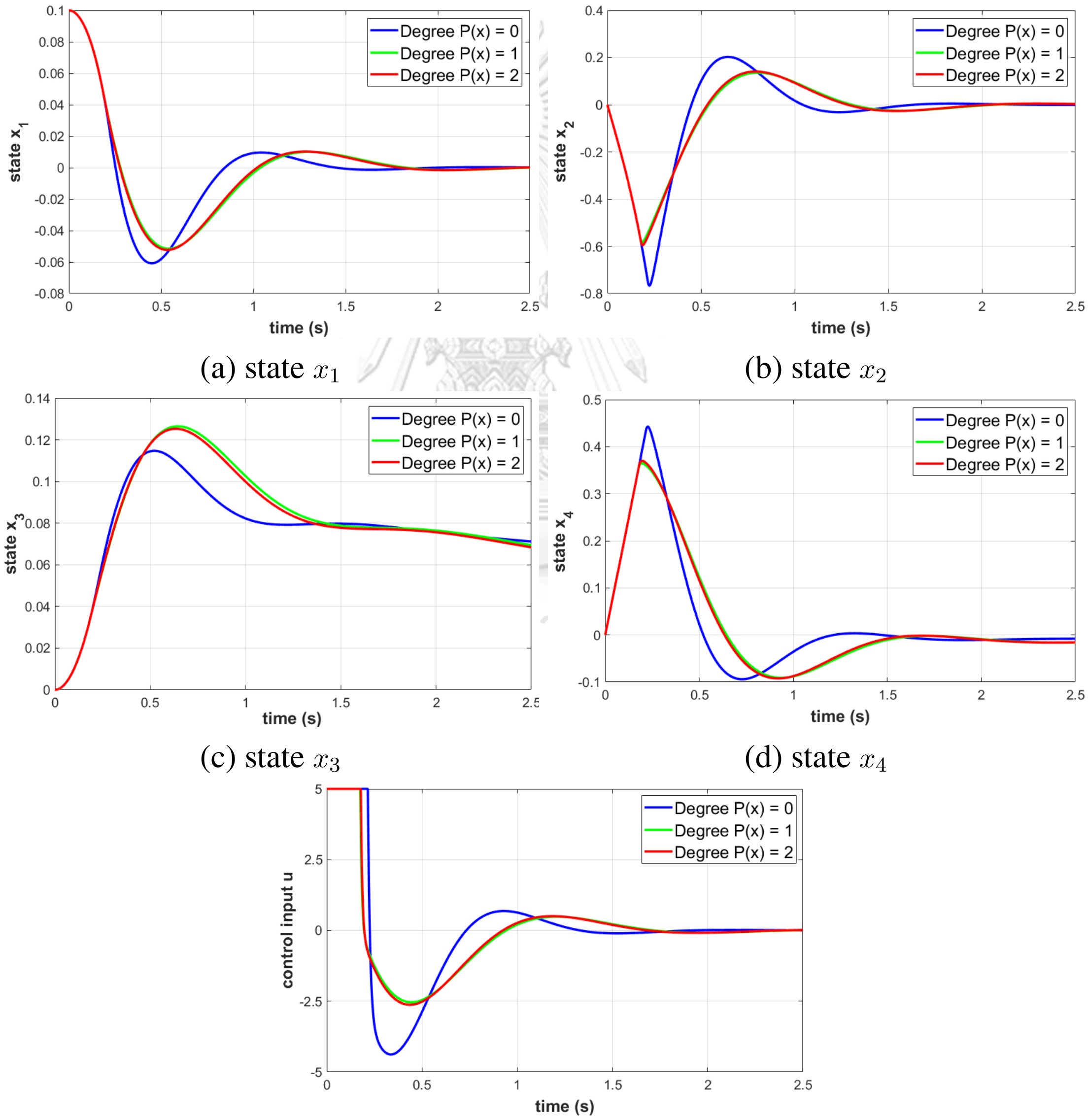


(e) control input u

Figure 5.9: State variables and control input with varying input bound using **Theorem 2** for degree $P(x) = 2$.

At the end of this example, we also provide the plot between the best upper bound cost function in each iteration to indicate the convergence of iterative search in **Theorem 2**. The value of the upper bound cost function decreases every iteration until the stopping criteria that the gape of decreasing value is smaller than tolerance as shown in Figure **??**.

Remark 5.2: we are merely presenting the main findings of the inverted pendulum on cart using **Theorem 1** and **2**, YALMIP toolbox and MATLAB R2018b are utilized to solve this case study. SeDuMi and PENLAB are used as a solver for both Theorem, respectively. Like the first study case, we compute all of the results in this example with a laptop Dell Processor Intel(R) Core(TM) i5-4210U CPU @1.70GHz; Ram installed 8.00GB. Average time consumption is about 140min.

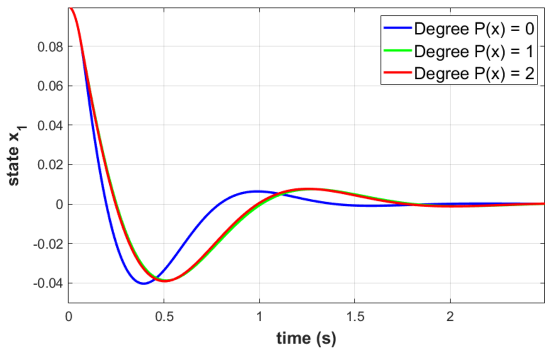


(a) state x_1 (b) state x_2

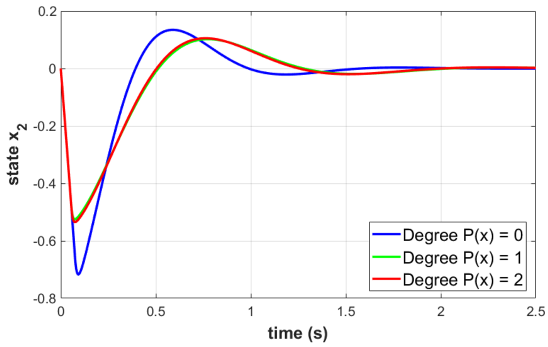
(c) state x_3 (d) state x_4

(e) control input u

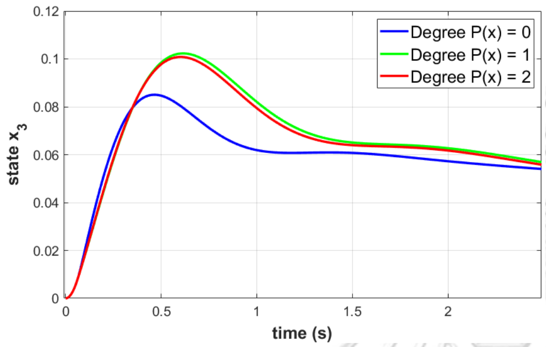
Figure 5.10: State variables and control input with varying degree $P(x)$ using **Theorem 2** and $\mu = 5$.



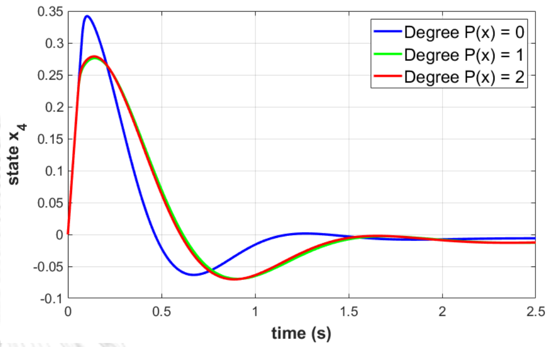
(a) state x_1



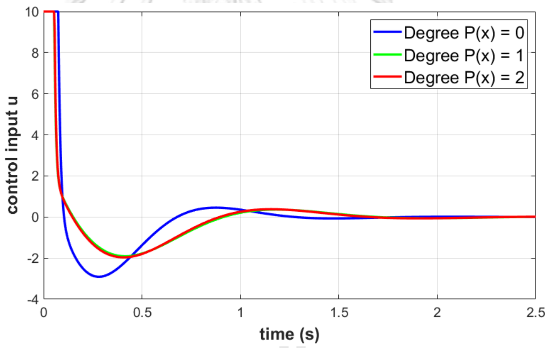
(b) state x_2



(c) state x_3



(d) state x_4



(e) control input u

Figure 5.11: State variables and control input with varying degree $P(x)$ using **Theorem 2** and $\mu = 10$.

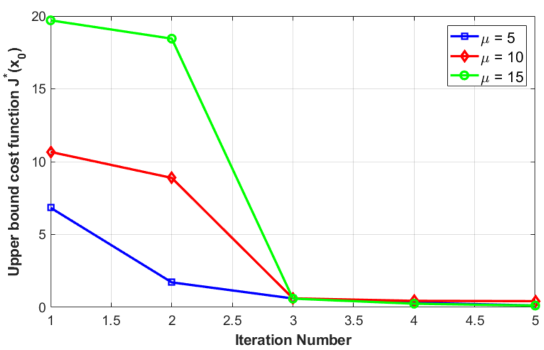


Figure 5.12: Upper bound cost function vs Iteration Number with varying input bound case degree $P(x) = 1$ using iterative search in **Theorem 2**.

Chapter VI

CONCLUSIONS

6.1 Summary of Main Results

This thesis proposed two primary state feedback design laws for non-polynomial systems with control input constraints. We start the problem formulation by converting non-polynomial systems into polynomial systems. To prevent the existence of approximation error in model design, from Taylor expansion, we replace non-polynomial terms with new state variables in the transforming algorithm (**Algorithm 2.1**). Then, the extended Lyapunov stability theorem with quadratic performance criterion is considered for the state feedback design. We introduce the candidate of Lyapunov in quadratic form with constant matrix variable in the first state feedback design law. Moreover, the design constraints are developed in LMI form, and the best upper bound of the quadratic cost function can quickly compute using the LMI technique. Meanwhile, we investigated the Lyapunov candidate with polynomial matrix variables for the second design. The design constraints are derived in terms of BMIs, which are then known as PDLMIs for any fixed prior degree of polynomial matrix variables degree. Furthermore, the best upper bound can be easily expressed using SOS approach. A two-dimensional non-polynomial system and an inverted polynomial system are provided as numerical examples to indicate the worth of the proposed designs.

6.2 Future Works

The interest in domain attraction estimation for uncertain non-polynomial systems based on Lyapunov stability theorem is given in some previous work (**?**), (**?**). In **?** for the most significant estimate of the robust domain of attraction based on rational Lyapunov function and squared matrix representation (SMR). The authors suggest converting the uncertain non-polynomial system to an uncertain polynomial system using Taylor expansion. The authors offer a similar process in **?**. The system is recast to be polynomial by approximation technique. Then, they computed the most significant guaranteed subset of the DOA via a fixed Lyapunov function. Therefore, it would be interesting to design a state feedback control law for an uncertain non-polynomial system in future work.

Biography

Phing Lim was born in Battambang, Cambodia, in March 1998. He graduated from Phreah Monivong high school and then went to Institute of Technology of Cambodia where he received B.Eng in electrical engineering. His research interest includes control systems, robotics, embedded systems, and power electronics. He received Graduate Scholarship for ASEAN and Non-ASEAN Countries from Chulalongkorn University to study in a master's program at the department of Electrical Engineering, Faculty of Engineering, Chulalongkorn University. He has a publication:

Lim, P., and Banjerdpongchai, D. 2023. An LMI Approach to State Feedback Control of Non-polynomial Systems. In 2023 3rd International Symposium on Instrumentation, Control, Artificial Intelligence, and Robotics (ICA-SYMP), pp. 24–27.