ผลเฉลยแบบยอมได้เฉพาะที่และผลเฉลยแบบควบคุมเฉพาะที่ของระบบสมการเชิงเส้นแบบช่วง

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TOLERANCE–LOCALIZED AND CONTROL–LOCALIZED SOLUTIONS TO SYSTEM OF INTERVAL LINEAR EQUATIONS

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วิทยานิพนธ์เรื่องนี้มีความสนใจศึกษาระบบสมการเชิงเส้นแบบช่วง $\mathbf{A}x = \mathbf{b}$ ซึ่งเป็นระบบสมการที่มีเมทริกซ์สัมประสิทธิ์ $\mathbf{A}$ และเมทริกซ์ด้านขวามือ $\mathbf{b}$ เปลี่ยนแปลงได้ในบางช่วงของจำนวนจริง เราได้ทำการศึกษาสองผลเฉลยของระบบสมการเชิงเส้นแบบช่วงที่มีชื่อว่าผลเฉลยแบบยอมได้เฉพาะที่และผลเฉลยแบบควบคุมเฉพาะที่ และได้ทำการนำเสนอลักษณะเฉพาะของแต่ละผลเฉลยข้างต้นแบ่งออกเป็นสองทฤษฎีบทหลักด้วยกัน โดยทฤษฎีบทแรกนั้นนำเสนอลักษณะเฉพาะของผลเฉลยในรูปแบบของเมทริกซ์กึ่งกลางและเมทริกซ์รัศมี ซึ่งสามารถพิสูจน์ได้โดยตรงจากการศึกษาคำนิยามของผลเฉลยนั้นๆ ในส่วนของทฤษฎีบทที่สองได้แสดงลักษณะเฉพาะของผลเฉลย โดยใช้แนวคิดเกี่ยวกับระยะทางเข้ามาช่วย และจากทฤษฎีบทที่สองนี้ เราจะได้รูปแบบปิดของเซตของผลเฉลยทั้งหมดของแต่ละผลเฉลยด้วย ยิ่งไปกว่านั้นเราได้นำแนวคิดเกี่ยวกับผลเฉลยแบบยอมได้เฉพาะที่ ไปปรับใช้กับโจทย์ปัญหาการจัดตารางสอน จากวัตถุประสงค์เพื่อที่จะหาค่าที่เหมาะสมที่สุดของความชอบและงานส่วนที่ไม่เป็นไปตามแผน เราได้สร้างแบบจำลองกำหนดการเชิงเส้นจำนวนเต็มเพื่อใช้ในการแก้ปัญหานี้ หลังจากทำการแก้ปัญหาพบว่า ผลที่ได้จากแบบจำลองข้างต้นไม่ได้มีความแตกต่างอย่างมีนัยสำคัญเมื่อเทียบกับผลจากแบบจำลองตั้งต้น ยิ่งไปกว่านั้นเมื่อนำผลที่ได้ไปเปรียบเทียบกับการมอบหมายงานที่เกิดขึ้นจริงในปีการศึกษาที่ 2/2561 พบว่าค่าความชอบโดยรวมต่องานที่ได้รับมอบหมายของอาจารย์นั้นมีค่าที่สูงกว่า

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KANOKWAN BURIMAS : TOLERANCE–LOCALIZED AND CONTROL–LOCALIZED SOLUTIONS TO SYSTEM OF INTERVAL LINEAR EQUATIONS. ADVISOR : ASSOC. PROF. PHANTIPA THIPWIWATPOTJANA, Ph.D., THESIS COADVISOR : ASST. PROF. WORRAWATE LEELA-APIRADEE, Ph.D., 57 pp.

In this thesis, we are interested in a system of interval linear equations $\mathbf{A}x = \mathbf{b}$ whose coefficient $\mathbf{A}$ and right hand side $\mathbf{b}$ vary in some real intervals. We study two types of solutions called tolerance–localized and control–localized solutions of interval linear equations system. The characterizations of each solution are proposed in two main theorems. First, the proposed theorem is stated in terms of center and radius matrices which is directly proved by following their definitions. The other theorem is presented as magnitude sense with new notation. Based on the second theorem, the closed form of all solution sets is released. In addition, we apply the idea of tolerance–localized solution to deal with the course assignment problem. To optimize the preference and over/under workload of the instructors, we formulate the modified integer linear programming model to solve the problem. The obtained result is not significantly different from the original model. Moreover, we found that the obtained result gives higher overall preference of the instructors than actual course assignment in the 2$^{\text{nd}}$ semester of 2018.

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CHAPTER I

INTRODUCTION

This thesis is concerned about a system of interval linear equations $\mathbf{A}x = \mathbf{b}$ whose coefficients $\mathbf{A}$ and right hand side $\mathbf{b}$ are defined as interval terms. By reviewing the literatures, there are many researches in this field. The uncertainty is one that can be solved by applying interval linear equations. For instance, Fan *et al.* [1] developed a robust interval linear programming (RILP) method over the conventional interval linear programming method for dealing with environmental decision making under uncertainties in 2012. Simic *et al.* [2] formulated a model for optimal long-term planning of vehicle recycling in the Republic of Serbia under uncertainty by using interval linear programming approach. In addition, finding the exact solution of interval linear equations has gotten much attention from researchers such as Gay [3] and Hansen [4]. Moreover, Lodwick *et al.* [5] proposed extended version of interval linear systems where coefficients are fuzzy intervals. The interval extended zero method solving upon interval linear equations was proposed by Sevastjanov *et al.* [6] in 2007.

The literature [7, 8, 9] said that there are many types of solutions to the system proposed such as weak, strong, tolerance, control, L–localized and R–localized solutions. Oettli and Prager [10] gave conditions that result in the exact solution of linear system whose coefficients and right hand side are interval. Next, Hladı *et al.* [11] extended Oettli and Prager work in more general. In the same fashion, the general strong solvability was proposed by Hladı *et al.* as well. Shary [12] established a solvability theory for the linear tolerance problem which is evolved from tolerance solution. Moreover, the sufficient criteria for the existence controllable solution set was proposed by Shary [13] in 1997 as well. Next, Li *et al.* [8] introduced L–localized and R–localized solutions and gave the basic numerical examples to find L and R–localized solutions. By the way, in real world problems, the solution might not be specified by only one characteristic but it could merge with several characteristics. For example, Tian *et al.* [14] proposed new type of solution that merges between tolerance and control solutions called tolerance-control solution. Li *et al.* [8] proposed a localized solution by mixing the concept of L-localized and R–localized solutions together. In addition, the new characterization of weak solution set to the system considered in magnitude sense was proposed by Shary [15] in 2015. In the similar idea, tolerance–control and localized solution set can be found by Leela–apiradee [16].

In this thesis, we study two types of solutions called tolerance–localized and control–localized solutions. Tolerance–localized solutions are solutions that merge together between tolerance, L–localized and R–localized concepts. Similarly, control–localized solutions can be defined in the same fashion. By following the idea of Leela–apiradee [16], we propose the characterization of these two solutions. By reviewing the research studies of Thipwiwatpotjana *et al.* [17, 18], we found that the concept of interval linear equations can applied to course assignment problem with uncertainty. Therefore, we apply the proposed results, specified Theorem 3.2, to deal with a course assignment problem as an application example. Basic knowledge about a system of interval linear equations is provided in Chapter 2. Any important definitions that are used throughout this thesis are also stated in this chapter. In Chapters 3 and 4, the proposed theorems of tolerance–localized and control–localized solutions are presented, respectively. The applied application is stated in Chapter 5. The conclusion of this thesis is given in the last chapter.

CHAPTER II

BACKGROUND KNOWLEDGE

In this chapter, we introduce the basic knowledge and some useful theorems that use throughout this thesis. We separate the details into two sections. Starting with introduction to interval linear equations, we describe the basic understanding and give new notations that use to find the solutions in Chapters 3 and 4. The important theorem which is the main equipment to solve our proposed theorems is stated in this section also. The definitions and their properties of the solutions to the system are provided in the last section.

2.1 System of interval linear equations

Firstly, let us introduce the general system of interval linear equations which is the main topic of this thesis. Let m and n be positive integers. The set of all $m \times n$ (interval) matrices over $\mathbb{R}$ and the set of all column (interval) vectors of size n over $\mathbb{R}$ are denoted by $\mathbb{R}^{m\times n}$ ($\mathbb{IR}^{m\times n}$) and $\mathbb{R}^n$ ($\mathbb{IR}^n$), respectively. Let $\mathbf{A} \in \mathbb{IR}^{m\times n}$ and $\mathbf{b} \in \mathbb{IR}^m$ such that

$$\mathbf{A} = [\underline{A}, \overline{A}] = \begin{bmatrix} [\underline{a}_{11}, \overline{a}_{11}] & [\underline{a}_{12}, \overline{a}_{12}] & \cdots & [\underline{a}_{1n}, \overline{a}_{1n}] \\ [\underline{a}_{21}, \overline{a}_{21}] & [\underline{a}_{22}, \overline{a}_{22}] & \cdots & [\underline{a}_{2n}, \overline{a}_{2n}] \\ \vdots & \vdots & \ddots & \vdots \\ [\underline{a}_{m1}, \overline{a}_{m1}] & [\underline{a}_{m2}, \overline{a}_{m2}] & \cdots & [\underline{a}_{mn}, \overline{a}_{mn}] \end{bmatrix}$$

and

$$\mathbf{b} = [\underline{b}, \overline{b}] = ([\underline{b}_1, \overline{b}_1], [\underline{b}_2, \overline{b}_2], \dots, [\underline{b}_m, \overline{b}_m])^T = (\mathbf{b}_1, \dots, \mathbf{b}_m)^T.$$

Remark 2.1. The matrices of left and right boundaries of each interval component of $\mathbf{A}$ denoted by $\underline{A}$ and $\overline{A}$, respectively, are written as real matrices

$$\underline{A} = \begin{bmatrix} \underline{a}_{11} & \underline{a}_{12} & \cdots & \underline{a}_{1n} \\ \underline{a}_{21} & \underline{a}_{22} & \cdots & \underline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{a}_{m1} & \underline{a}_{m2} & \cdots & \underline{a}_{mn} \end{bmatrix} \quad \text{and} \quad \overline{A} = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\ \overline{a}_{21} & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{m1} & \overline{a}_{m2} & \cdots & \overline{a}_{mn} \end{bmatrix}.$$

The boundaries $\underline{b}$ and $\overline{b}$ can be explained in similar fashion as above.

Remark 2.2. An interval matrix $\mathbf{A}$ and an interval vector $\mathbf{b}$ can also be represented as sets

$$\mathbf{A} = \{A = [a_{ij}] \in \mathbb{R}^{m\times n} : \underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij},\ i = 1, \ldots, m,\ j = 1, \ldots, n\}$$

and

$$\mathbf{b} = \{b = [b_i] \in \mathbb{R}^m : \underline{b}_i \leq b_i \leq \overline{b}_i,\ i = 1, \ldots, m\}.$$

The interval system of m linear equations with vector of variables $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ is written in the following form:

$$\begin{aligned} [\underline{a}_{11}, \overline{a}_{11}]x_1 + [\underline{a}_{12}, \overline{a}_{12}]x_2 + \cdots + [\underline{a}_{1n}, \overline{a}_{1n}]x_n &= [\underline{b}_1, \overline{b}_1] &\iff (\mathbf{A}x)_1 = \mathbf{b}_1 \\ [\underline{a}_{21}, \overline{a}_{21}]x_1 + [\underline{a}_{22}, \overline{a}_{22}]x_2 + \cdots + [\underline{a}_{2n}, \overline{a}_{2n}]x_n &= [\underline{b}_2, \overline{b}_2] &\iff (\mathbf{A}x)_2 = \mathbf{b}_2 \\ &\vdots & \vdots \\ [\underline{a}_{m1}, \overline{a}_{m1}]x_1 + [\underline{a}_{m2}, \overline{a}_{m2}]x_2 + \cdots + [\underline{a}_{mn}, \overline{a}_{mn}]x_n &= [\underline{b}_m, \overline{b}_m] &\iff (\mathbf{A}x)_m = \mathbf{b}_m \end{aligned}$$

or, briefly,

$$\mathbf{A}x = \mathbf{b}.$$

Also, $\mathbf{A}$ and $\mathbf{b}$ can be written in terms of their center and radius matrix and vector as

$$\mathbf{A} = [A_c - \Delta, A_c + \Delta] \text{ and } \mathbf{b} = [b_c - \delta, b_c + \delta],$$

where the centers are $A_c = \frac{1}{2}(\bar{A} + \underline{A})$ and $b_c = \frac{1}{2}(\bar{b} + \underline{b})$ and the radii are $\Delta = \frac{1}{2}(\bar{A} - \underline{A})$ and $\delta = \frac{1}{2}(\bar{b} - \underline{b})$. Normally, the set $\mathbf{A}x$ cannot be written as $[\underline{A}x, \overline{A}x]$, but it can be represented by an explicit interval as shown in the theorem below.

Theorem 2.3 (See [7]). Let $\mathbf{A} = [\underline{A}, \overline{A}] \in \mathbb{IR}^{m\times n}$ and $x \in \mathbb{R}^n$. Then

$$\mathbf{A}x = [A_c x - \Delta|x|, A_c x + \Delta|x|].$$

Next, we introduce the notations $\lfloor * \rfloor$, $\lceil * \rceil$ and $\langle * \rangle$, which will show up in the main results in Chapters 3 and 4. Let us define $\lfloor \mathbf{b} \rfloor$, $\lceil \mathbf{b} \rceil$ and $\langle \mathbf{b} \rangle$ for a given interval vector $\mathbf{b}$ as follows:

$$\lceil \mathbf{b} \rceil = \max\{|\underline{b}|, |\bar{b}|\} = \begin{pmatrix} \max\{|\underline{b}_1|, |\bar{b}_1|\} \\ \max\{|\underline{b}_2|, |\bar{b}_2|\} \\ \vdots \\ \max\{|\underline{b}_m|, |\bar{b}_m|\} \end{pmatrix},$$

$$\lfloor \mathbf{b} \rfloor = \min\{|\underline{b}|, |\bar{b}|\} = \begin{pmatrix} \min\{|\underline{b}_1|, |\bar{b}_1|\} \\ \min\{|\underline{b}_2|, |\bar{b}_2|\} \\ \vdots \\ \min\{|\underline{b}_m|, |\bar{b}_m|\} \end{pmatrix}$$

and

$$\langle \mathbf{b} \rangle = \begin{pmatrix} \langle b_1 \rangle \\ \vdots \\ \langle b_m \rangle \end{pmatrix} \text{ where } \langle b_i \rangle = \begin{cases} \min\{|\underline{b}_i|, |\bar{b}_i|\}, & \text{if } 0 \notin [\underline{b}_i, \bar{b}_i]; \\ 0, & \text{otherwise.} \end{cases}$$

The example of finding $\lfloor \mathbf{b} \rfloor$, $\lceil \mathbf{b} \rceil$ and $\langle \mathbf{b} \rangle$ of $\mathbf{b}$ in $\mathbb{IR}$ are shown in Figure 2.1.

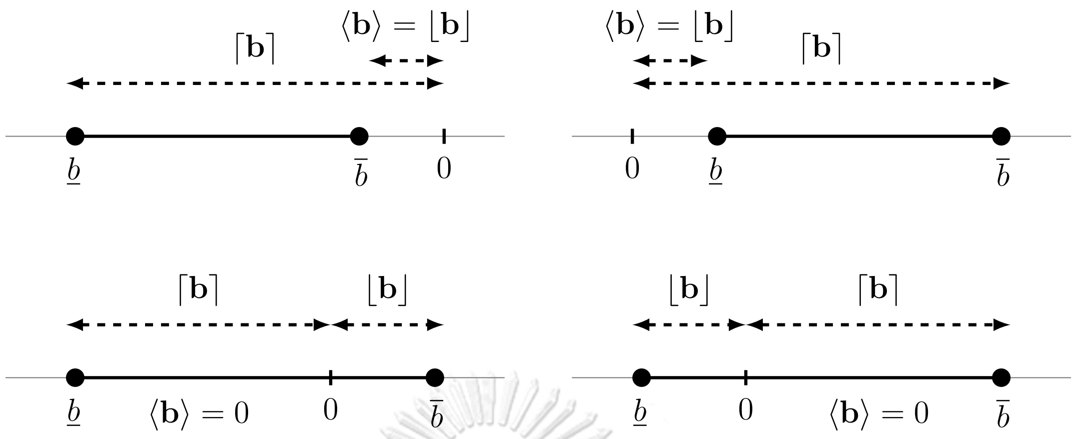

Figure 2.1: The $\lfloor \mathbf{b} \rfloor$, $\lceil \mathbf{b} \rceil$ and $\langle \mathbf{b} \rangle$ of $\mathbf{b}$ in $\mathbb{IR}$.

2.2 Basic properties of solutions to system of interval linear equations

The definitions and properties of each solution to system of interval linear equations are presented in this section. To recognize each solution clearly, we illustrate them as figures as well. The concepts of new solutions are defined in this section also.

Definition 2.4. (See [7]) A vector $x \in \mathbb{R}^n$ is called

(i) a *weak solution* of $\mathbf{A}x = \mathbf{b}$ if it satisfies $Ax = b$ for some $A \in \mathbf{A}, b \in \mathbf{b}$

(ii) a *strong solution* of $\mathbf{A}x = \mathbf{b}$ if it satisfies $Ax = b$ for each $A \in \mathbf{A}, b \in \mathbf{b}$.

In 1964, Oettli and Prager [10] focused on the interval linear system $\mathbf{A}x = \mathbf{b}$ and gave a characterization of weak solution.

Theorem 2.5. (See [10]) A vector $x \in \mathbb{R}^n$ is a weak solution of $\mathbf{A}x = \mathbf{b}$ if and only if x satisfies

$$|A_c x - b_c| \leq \Delta |x| + \delta.$$

A characterization of strong solution was represented by Rohn [7] in the same way with weak solution shown in Theorem 2.6.

Theorem 2.6. (See [7]) A vector $x \in \mathbb{R}^n$ is a strong solution of $\mathbf{A}x = \mathbf{b}$ if and only if x satisfies

$$A_c x = b_c,$$
$$\Delta|x| = \delta = 0.$$

A tolerance solution of the interval linear system $\mathbf{A}x = \mathbf{b}$ was motivated by the studies of a crane construction problem in [19] and the problem of input–output planning with inexact data in [20]. Its characterization was first provided by Rohn [21] in 1986.

Definition 2.7. (See [7]) A vector $x \in \mathbb{R}^n$ is called a *tolerance solution* of $\mathbf{A}x = \mathbf{b}$ if for each $A \in \mathbf{A}$ there exists $b \in \mathbf{b}$ such that $Ax = b$.

Lemma 2.8. (See [22]) The set of all tolerance solutions of $\mathbf{A}x = \mathbf{b}$, denoted by $\sum_{\forall\exists}(\mathbf{A}, \mathbf{b})$ can be written as follows:

$$\sum_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n : \mathbf{A}x \subseteq \mathbf{b}\}.$$

After the new version to the characterization of a tolerance solution described by using the center and radius matrices.

Theorem 2.9. (See [7]) A vector $x \in \mathbb{R}^n$ is called a tolerance solution of $\mathbf{A}x = \mathbf{b}$ if and only if x satisfies

$$|A_c x - b_c| \leq -\Delta|x| + \delta.$$

In 1992, a control solution was proposed by Shary in [23]. The characterization of control solution can be seen in [7] in the forms of center and radius matrices.

Definition 2.10. (See [7]) A vector $x \in \mathbb{R}^n$ is called a *control solution* of $\mathbf{A}x = \mathbf{b}$ if for each $b \in \mathbf{b}$ there exists $A \in \mathbf{A}$ such that $Ax = b$.

Lemma 2.11. (See [13]) The set of all control solutions of $\mathbf{A}x = \mathbf{b}$, denoted by $\sum_{\exists\forall}(\mathbf{A}, \mathbf{b})$ can be written as follows:

$$\sum_{\exists\forall}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n : \mathbf{A}x \supseteq \mathbf{b}\}.$$

Theorem 2.12. (See [7]) A vector $x \in \mathbb{R}^n$ is called a control solution of $\mathbf{A}x = \mathbf{b}$ if and only if x satisfies

$$|A_c x - b_c| \leq \Delta|x| - \delta.$$

By Lemmas 2.8 and 2.11, we can depict the set of tolerance and control solutions of $\mathbf{A}x = \mathbf{b}$ where $x \in \mathbb{R}$ in Figures 2.2 and 2.3, respectively.

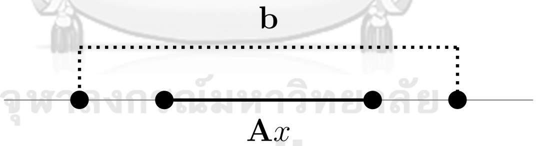

Figure 2.2: $\mathbf{A}x \subseteq \mathbf{b}$, where $x \in \mathbb{R}$.

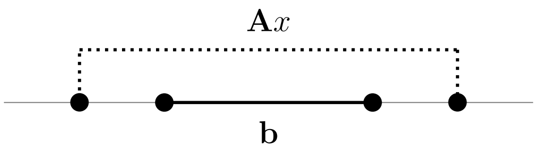

Figure 2.3: $\mathbf{A}x \supseteq \mathbf{b}$, where $x \in \mathbb{R}$.

Based on the definitions of tolerance and control solutions, a new type solution called tolerance-control solution was presented by Tian *et al.* [14] described in Definition 2.13.

Definition 2.13. (See [14]) A vector $x \in \mathbb{R}^n$ is called a *tolerance-control solution* of $\mathbf{A}x = \mathbf{b}$ if $(\mathbf{A}x)_i = \mathbf{b}_i$ is either tolerance or control, for each $i = 1, ..., m$.

Theorem 2.14. (See [14]) For $\mathbf{A}x = \mathbf{b}$, let $T = \{i|(\mathbf{A}x_i) \subseteq \mathbf{b}_i\}$, $C = \{i|(\mathbf{A}x_i) \supseteq \mathbf{b}_i\}$ and $U = \{1, ..., m\}$. A vector $x \in \mathbb{R}^n$ is a tolerance-control solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies

$$|(A_cx)_i - (b_c)_i| \leq -(\Delta|x|)_i + \delta_i,\ i \in T$$
$$|(A_cx)_i - (b_c)_i| \leq (\Delta|x|)_i - \delta_i,\ i \in C,$$

where $T \cup C = U$.

Alternatively, Leela-apiradee [16] proposed the new characterization of a tolerance-control solution to the system considered in magnitude sense.

Theorem 2.15. (See [16]) A vector $x \in \mathbb{R}^n$ is a tolerance-control solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies

$$\lceil (\mathbf{A}x)_i - (b_c)_i \rceil \leq \delta_i \ \text{ or } \ \lfloor (\mathbf{A}x)_i - (b_c)_i \rfloor \geq \delta_i,$$

for each $i \in M$ where M is the sets of indices $\{1, 2, ..., m\}$.

Moreover, Li *et al.* [8] introduced L–localized and R–localized solutions and proposed the characterization of them as shown in Theorems 2.20 and 2.21, respectively. Next, let us introduce the important relations $\leq_{st}$ and $\leq_s$ between two interval vectors in Definition 2.16 which will be used to define L–localized and R–localized solutions of $\mathbf{A}x = \mathbf{b}$ in Definitions 2.17 and 2.18, respectively.

Definition 2.16. (See [8]) Let $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$ be any two interval vectors.

(i) If $\underline{x} \leq \underline{y} \leq \overline{x} \leq \overline{y}$, then $\mathbf{x}$ is *strictly less than or equal* to $\mathbf{y}$, denoted by $\mathbf{x} \leq_{st} \mathbf{y}$.

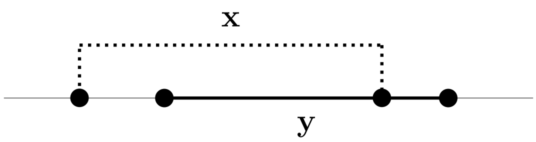

Figure 2.4: $\mathbf{x} \leq_{st} \mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in \mathbb{IR}$.

(ii) If $\overline{x} \leq \underline{y}$, then $\mathbf{x}$ is *strongly less than or equal* to $\mathbf{y}$, denoted by $\mathbf{x} \leq_s \mathbf{y}$.

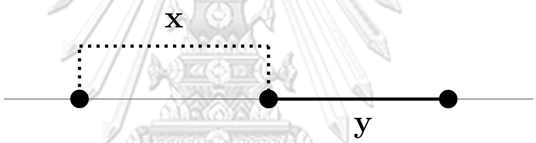

Figure 2.5: $\mathbf{x} \leq_s \mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in \mathbb{IR}$.

Definition 2.17. (See [8]) A vector $x \in \mathbb{R}^n$ is called an *L–localized solution* of $\mathbf{A}x = \mathbf{b}$ if there is at least one $A \in \mathbf{A}$ such that $Ax \in \mathbf{b}$. For the other $A \in \mathbf{A}$, $Ax \leq_s \mathbf{b}$.

Definition 2.18. (See [8]) A vector $x \in \mathbb{R}^n$ is called an *R–localized solution* of $\mathbf{A}x = \mathbf{b}$ if there is at least one $A \in \mathbf{A}$ such that $Ax \in \mathbf{b}$. For the other $A \in \mathbf{A}$, $-Ax \leq_s -\mathbf{b}$.

Lemma 2.19. (See [8]) The set of all L–localized and R–localized solution of $\mathbf{A}x = \mathbf{b}$, denoted by $\sum_{LS}(\mathbf{A}, \mathbf{b})$ and $\sum_{RS}(\mathbf{A}, \mathbf{b})$ respectively, are written as

$$\sum_{LS}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n : \mathbf{A}x \leq_{st} \mathbf{b}\} \text{ and } \sum_{RS}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n : -\mathbf{A}x \leq_{st} -\mathbf{b}\}.$$

Theorem 2.20. (See [8]) A vector $x \in \mathbb{R}^n$ is an L–localized solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies

$$-\Delta|x| - \delta \leq A_c x - b_c \leq -|\Delta|x| - \delta|.$$

Theorem 2.21. (See [8]) A vector $x \in \mathbb{R}^n$ is an R–localized solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies

$$|\Delta|x| - \delta| \leq A_c x - b_c \leq \Delta|x| + \delta.$$

According to Lemma 2.19, the set of L–localized and R–localized solutions of $\mathbf{A}x = \mathbf{b}$ where $x \in \mathbb{R}$ are illustrated as Figures 2.6 and 2.7.

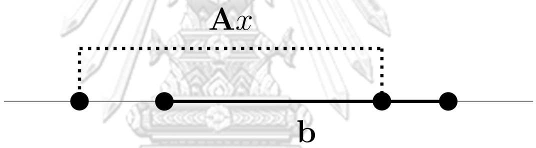

Figure 2.6: $\mathbf{A}x \leq_{st} \mathbf{b}$, where $x \in \mathbb{R}$.

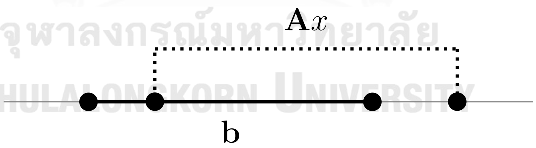

Figure 2.7: $-\mathbf{A}x \leq_{st} -\mathbf{b}$, where $x \in \mathbb{R}$.

By mixing the concepts of L-localized and R–localized solutions together, Li *et al.* [8] proposed a new solution called a localized solution that is defined in Definition 2.22.

Definition 2.22. (See [8]) A vector $x \in \mathbb{R}^n$ is called a *localized solution* of $\mathbf{A}x = \mathbf{b}$ if $(\mathbf{A}x)_i = \mathbf{b}_i$ is either L–localized or R–localized, for each $i = 1, ..., m$.

Theorem 2.23. (See [8]) Let e_j be a m-dimensional column vector which has 1 at row j and 0 elsewhere and E_{jj} be a $m \times m$ matrix which has 1 at the position (j,j) and 0 elsewhere. A vector $x \in \mathbb{R}^n$ is a localized solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies

$$-\Delta|x| - \delta \leq \hat{A}_c x - \hat{b}_c \leq -|\delta - \Delta|x||,$$

where $\hat{A}_c = (I_{m\times m} - 2\sum_{j\in M} E_{jj})A_c$, $\hat{b}_c = (I_{m\times m} - 2\sum_{j\in M} E_{jj})b_c$ and $M = \{j|\mathbf{b}_j \leq_{st} (\mathbf{A}x)_j, 1 \leq j \leq m\}$.

In Theorem 2.24, Leela-apiradee [16] offered the new characterization of localized solution to the system instead of Theorem 2.23.

Theorem 2.24. (See [16]) A vector $x \in \mathbb{R}^n$ is a localized solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies

$$\lfloor \mathbf{A}x - b_c \rfloor \leq \delta \leq \lceil \mathbf{A}x - b_c \rceil.$$

Based on the definitions of tolerance, control, L–localized and R–localized solutions, we develop the concepts of a tolerance–localized and a control–localized solution as shown in the following definition.

Definition 2.25. A vector $x \in \mathbb{R}^n$ is called a *tolerance–localized solution* of $\mathbf{A}x = \mathbf{b}$ if x in $(\mathbf{A}x)_i = \mathbf{b}_i$ is either one of tolerance, L–localized or R–localized solution, for each $i = 1, \ldots, m$.

Remark 2.26. The set of all tolerance–localized solution of $\mathbf{A}x = \mathbf{b}$, denoted by $\sum_{TL}(\mathbf{A}, \mathbf{b})$, can be written as follows:

$$\sum_{TL}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n : (\mathbf{A}x)_i \subseteq (\mathbf{b})_i,\ \forall i \in S,\ \ (\mathbf{A}x)_j \leq_{st} \mathbf{b}_j,\ \forall j \in Q$$
$$\text{and } \ -(\mathbf{A}x)_k \leq_{st} -\mathbf{b}_k,\ \forall k \in R\},$$

where S, Q and R be disjoint index subsets of $\{1, \ldots, m\}$ such that $S \cup Q \cup R = \{1, \ldots, m\}$. That is, the rows $i \in S, j \in Q$ and $k \in R$ of the system are tolerance, L–localized and R–localized, respectively.

Definition 2.27. A vector $x \in \mathbb{R}^n$ is called a *control–localized solution* of $\mathbf{A}x = \mathbf{b}$ if x in $(\mathbf{A}x)_i = \mathbf{b}_i$ is either one of control, L–localized or R–localized solution, for each $i = 1, \ldots, m$.

Remark 2.28. The set of all control–localized solution of $\mathbf{A}x = \mathbf{b}$, defined by $\sum_{CL}(\mathbf{A}, \mathbf{b})$, is written as

$$\sum_{CL}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n : (\mathbf{A}x)_i \supseteq (\mathbf{b})_i,\ \forall i \in P,\ \ (\mathbf{A}x)_j \leq_{st} \mathbf{b}_j,\ \forall j \in Q$$
$$\text{and}\ \ -(\mathbf{A}x)_k \leq_{st} -\mathbf{b}_k,\ \forall k \in R\},$$

where P, Q and R be disjoint index subsets of $\{1, \ldots, m\}$ such that $P \cup Q \cup R = \{1, \ldots, m\}$. That is, the rows $i \in P, j \in Q$ and $k \in R$ of the system are control, L–localized and R–localized, respectively.

By these definitions of solutions to system of interval linear equations, we provide a diagram of their relationship in Figure 2.8.

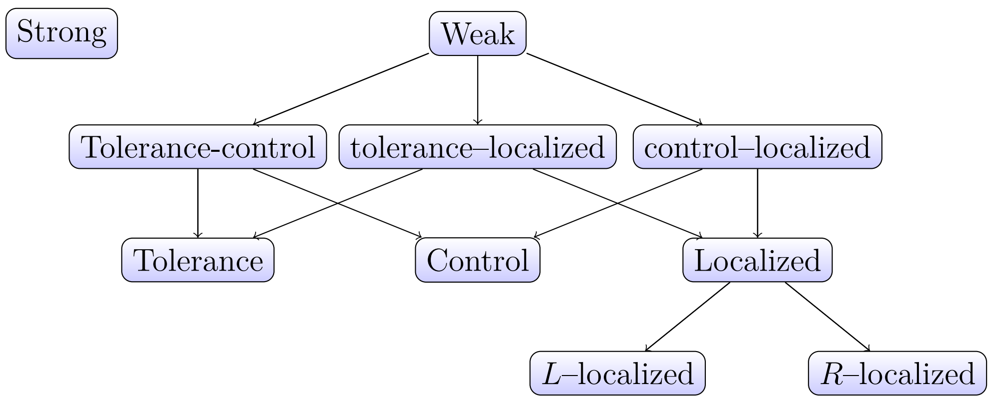

Figure 2.8: The relationship to solutions of an interval linear equations system, where A $\longrightarrow$ B means A is a superset of B.

To show the characterization of each solution type clearly, we give a simple example about a daily energy requirement of human. The daily energy requirement of each person might be different depend on gender, ages, weight or physical structure. In addition, type of activities can determine the daily energy requirement as well. Thus, we found that the daily energy requirement can present as interval form. For example, a 25-year-old woman with 50 kilograms weight and 160 centimeters tall requires 1457 kcal/day for light activity lifestyle and 2307 kcal/day for vigorously active lifestyle [24], then her daily energy requirement can be rewritten as $\mathbf{b} = [1457, 2307]$ kcal/day. In order to maintain weight and harness remaining energy, her actual daily energy receiving $\mathbf{A}x$ should be between 1457 and 2307 where $\mathbf{A}$ refers to the amount of kcal/1 gram of the nutrients and x is the number of grams of receiving nutrients. Thus, $\mathbf{A}x \subseteq \mathbf{b}$ for this case which refers that the relation of her actual daily energy receiving and daily energy requirement is tolerance. However, not everyone can reach their daily energy requirement by some reasons such as their increasing work, personal errand or the after work party. So, their actual daily energy receiving might be less or more than usual that is $\mathbf{A}x \supseteq \mathbf{b}$. This situation leads control relation of the actual daily energy receiving and daily energy requirement. In addition, some people who want to gain weight or pregnant women have to receive energy not less than daily energy requirement and it would be better if he/she is able to receive energy more than usual. By this case, the relation of the actual daily energy receiving and daily energy requirement is described as R–localized or $\mathbf{b} \leq_{st} \mathbf{A}x$. For people who want to lose weight, we can describe in the same way as the previous case. So, in this case, the actual daily energy receiving and daily energy requirement refer to L–localized relation or $\mathbf{A}x \leq_{st} \mathbf{b}$.

Now, we have enough information to understand the system of interval linear equations. Particularly, the definitions of the two new solutions, tolerance–

localized and control–localized solutions are clearly described. In the two next chapter, we proposed a set of theorems that related to these two new solutions. The proofs are also completely shown.

CHAPTER III

TOLERANCE–LOCALIZED SOLUTION

This chapter provides a series of theorems related to the set of all tolerance–localized solutions of the system $\mathbf{A}x = \mathbf{b}$. Throughout Chapters 3 and 4, we denote M for convenience as the index set $\{1, \dots, m\}$. Let $I_{m\times m}$ be an identity matrix of size m and E_{ii} be a $m \times m$ matrix which has 1 at the position (i, i) and 0 elsewhere.

Theorem 3.1. A vector x is a tolerance–localized solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies

$$-\hat{\Delta}|x| - \delta \leq \hat{A}_c x - \hat{b}_c \leq \hat{e}^T|\delta - \Delta|x||.$$

The terms $\hat{A}_c = (I_{m\times m} - 2\sum_{i\in R} E_{ii})A_c$, $\hat{b}_c = (I_{m\times m} - 2\sum_{i\in R} E_{ii})b_c$, $\hat{\Delta} = (I_{m\times m} - 2\sum_{i\in S} E_{ii})\Delta$ where $S = \{i \in M; (\mathbf{A}x)_i \subseteq \mathbf{b}_i\}$ and $R = \{k \in M; -(\mathbf{A}x)_k \leq_{st} -\mathbf{b}_k\}$. The m-dimensional vector $\hat{e}$ has 1 at row $i \in S$ and -1 elsewhere.

Proof. Assume that x is a tolerance–localized solution of $\mathbf{A}x = \mathbf{b}$. Let S, Q and R be disjoint subsets of M such that rows $i \in S, j \in Q$ and $k \in R$ of the system are tolerance, L–localized and R–localized, respectively. This means

$$(\mathbf{A}x)_i \subseteq \mathbf{b}_i,\ \forall i \in S,\ (\mathbf{A}x)_j \leq_{st} \mathbf{b}_j,\ \forall j \in Q \text{ and } -(\mathbf{A}x)_k \leq_{st} -\mathbf{b}_k,\ \forall k \in R. \quad (3.1)$$

Let us focus on the first relation of (3.1) for each $i \in S$. From Theorem 2.3,

$$[(A_c x)_i - (\Delta|x|)_i, (A_c x)_i + (\Delta|x|)_i] = (\mathbf{A}x)_i \subseteq \mathbf{b}_i = [(b_c)_i - \delta_i, (b_c)_i + \delta_i],$$

which implies

$$(b_c)_i - \delta_i \le (A_c x)_i - (\Delta|x|)_i \le (A_c x)_i + (\Delta|x|)_i \le (b_c)_i + \delta_i.$$

That is, $(\Delta|x|)_i - \delta_i \le (A_c x)_i - (b_c)_i$ and $(A_c x)_i - (b_c)_i \le -(\Delta|x|)_i + \delta_i$. Thus

$$(\Delta|x|)_i - \delta_i \le (A_c x)_i - (b_c)_i \le -(\Delta|x|)_i + \delta_i.$$

Since $(\mathbf{A}x)_i \subseteq \mathbf{b}_i$, then $\delta_i \ge (\Delta|x|)_i$, i.e., $0 \le \delta_i - (\Delta|x|)_i = |\delta_i - (\Delta|x|)_i|$. Therefore

$$(\Delta|x|)_i - \delta_i \le (A_c x)_i - (b_c)_i \le |\delta_i - (\Delta|x|_i)| \text{ for each } i \in S. \tag{3.2}$$

Let $j \in Q$. The second relation of (3.1) becomes

$$[(A_c x)_j - (\Delta|x|)_j, (A_c x)_j + (\Delta|x|)_j] = (\mathbf{A}x)_j \le_{st} \mathbf{b}_j = [(b_c)_j - \delta_j, (b_c)_j + \delta_j],$$

which implies

$$(A_c x)_j - (\Delta|x|)_j \le (b_c)_j - \delta_j \le (A_c x)_j + (\Delta|x|)_j \le (b_c)_j + \delta_j.$$

that is,

$$\begin{aligned} -(\Delta|x|)_j - \delta_j \le (A_c x)_j - (b_c)_j \le (\Delta|x|)_j - \delta_j \text{ and} \\ (A_c x)_j - (b_c)_j \le -(\Delta|x|)_j + \delta_j. \end{aligned} \tag{3.3}$$

Therefore

$$-(\Delta|x|)_j - \delta_j \le (A_c x)_j - (b_c)_j \le -|\delta_j - (\Delta|x|)_j| \text{ for each } j \in Q. \tag{3.4}$$

Let $k \in R$. We first derive the terms $-(\mathbf{A}x)_k$ and $-(\mathbf{b})_k$ as follows:

$$\begin{aligned} -(\mathbf{A}x)_k &= -[(A_cx)_k - (\Delta|x|)_k, (A_cx)_k + (\Delta|x|)_k] \\ &= [-(A_cx)_k - (\Delta|x|)_k, -(A_cx)_k + (\Delta|x|)_k] \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} -\mathbf{b}_k &= -[(b_c)_k - \delta_k, (b_c)_k + \delta_k] \\ &= [-(b_c)_k - \delta_k, -(b_c)_k + \delta_k] \end{aligned} \tag{3.6}$$

By using (3.5) and (3.6), the third relation of (3.1) turns into

$$-[(A_cx)_k - (\Delta|x|)_k, (A_cx)_k + (\Delta|x|)_k] = -(\mathbf{A}x)_k \leq_{st} -\mathbf{b}_k = -[(b_c)_k - \delta_k, (b_c)_k + \delta_k],$$

which means

$$-(A_cx)_k - (\Delta|x|)_k \leq -(b_c)_k - \delta_k \leq -(A_cx)_k + (\Delta|x|)_k \leq -(b_c)_k + \delta_k,$$

i.e.,

$$\begin{aligned} -(\Delta|x|)_k + \delta_k \leq (A_cx)_k - (b_c)_k \leq (\Delta|x|)_k + \delta_k \text{ and} \\ (\Delta|x|)_k - \delta_k \leq (A_cx)_k - (b_c)_k. \end{aligned} \tag{3.7}$$

Then

$$|-(\Delta|x|)_k + \delta_k| \leq (A_cx)_k - (b_c)_k \leq (\Delta|x|)_k + \delta_k.$$

Therefore

$$-(\Delta|x|)_k - \delta_k \leq -(A_cx)_k + (b_c)_k \leq -|\delta_k - (\Delta|x|)_k| \text{ for each } k \in R. \tag{3.8}$$

Let e_i be a m-dimensional column vector which has 1 at row i and 0 elsewhere. By putting (3.2), (3.4) and (3.8) together, the left, the middle and the right terms are

$$-\Delta|x| + 2\sum_{i\in S}(\Delta|x|)_i e_i - \delta \tag{3.9}$$

$$A_c x - b_c - 2\sum_{i\in R}(A_c x - b_c)_i e_i = A_c x - b_c - 2\sum_{i\in R}(A_c x)_i e_i + 2\sum_{i\in R}(b_c)_i e_i \tag{3.10}$$

and

$$\hat{e}^T|\delta - \Delta|x|| = \sum_{i\in M}\hat{e}_i(|\delta - \Delta|x||)_i \;\text{ where }\; \hat{e}_i = \begin{cases} 1, & \text{if } i \in S; \\ -1, & \text{if } i \in Q \cup R = M \setminus S, \end{cases}$$

respectively. As a result of

$$\sum_{i\in S}(\Delta|x|)_i e_i = \left(\sum_{i\in S} E_{ii}\right)\Delta|x|,$$

$$\sum_{i\in R}(A_c x)_i e_i = \left(\sum_{i\in R} E_{ii}\right)A_c x, \text{ and}$$

$$\sum_{i\in R}(b_c)_i e_i = \left(\sum_{i\in R} E_{ii}\right)b_c,$$

the terms (3.9) and (3.10) are

$$-(I_{m\times m} - 2\sum_{i\in S} E_{ii})\Delta|x| - \delta = -\hat{\Delta}|x| - \delta$$

and

$$(I_{m\times m} - 2\sum_{i\in R} E_{ii})A_c x - (I_{m\times m} - 2\sum_{i\in R} E_{ii})b_c = \hat{A}_c x - \hat{b}_c,$$

respectively. Hence $-\hat{\Delta}|x| - \delta \le \hat{A}_c x - \hat{b}_c \le \hat{e}^T|\delta - \Delta|x||$. Reverse this proof to complete the proof of the theorem. □

In the next theorem, we propose an alternative characterization of tolerance–localized solution generated in magnitude sense. By rewriting the result of Theorem 3.2 as TL function, we obtain the solution set $\sum_{TL}(\mathbf{A},\mathbf{b})$ in a level set form shown in Theorem 3.3.

Theorem 3.2. x is a tolerance–localized solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies

$$\lfloor \mathbf{A}x - b_c \rfloor \leq \delta. \tag{3.11}$$

Proof. Assume that x is a tolerance–localized solution of $\mathbf{A}x = \mathbf{b}$. Let S, Q and R be disjoint subsets of M such that rows $i \in S, j \in Q$ and $k \in R$ of the system are tolerance, L–localized and R–localized, respectively. Then

$$(\mathbf{A}x)_i \subseteq \mathbf{b}_i,\ \forall i \in S,\ (\mathbf{A}x)_j \leq_{st} \mathbf{b}_j,\ \forall j \in Q \text{ and } -(\mathbf{A}x)_k \leq_{st} -\mathbf{b}_k,\ \forall k \in R. \tag{3.12}$$

Let tolerance row $i \in S$. By the proof of Theorem 3.1, we find that the first relation of (3.1) and (3.12) are identical. Then we have

$$(b_c)_i - \delta_i \leq (A_c x)_i - (\Delta|x|)_i \leq (b_c)_i + \delta_i \text{ and } (b_c)_i - \delta_i \leq (A_c x)_i + (\Delta|x|)_i \leq (b_c)_i + \delta_i,$$

i.e.,

$$|(A_c x)_i - (\Delta|x|)_i - (b_c)_i| \leq \delta_i \text{ and } |(A_c x)_i + (\Delta|x|)_i - (b_c)_i| \leq \delta_i,$$

which implies

$$\delta_i \geq \max\{|(A_c x)_i - (\Delta|x|)_i - (b_c)_i|, |(A_c x)_i + (\Delta|x|)_i - (b_c)_i|\}.$$

As the fact that the maximum value between $|(A_c x)_i - (\Delta|x|)_i - (b_c)_i|$ and $|(A_c x)_i + (\Delta|x|)_i - (b_c)_i|$ is greater than or equal to the minimum value between them, we obtain

$$\begin{aligned}\delta_i &\geq \min\{|(A_c x)_i - (\Delta|x|)_i - (b_c)_i|, |(A_c x)_i + (\Delta|x|)_i - (b_c)_i|\} \\ &= \lfloor[(A_c x)_i - (\Delta|x|)_i - (b_c)_i, (A_c x)_i + (\Delta|x|)_i - (b_c)_i]\rfloor \\ &= \lfloor[(A_c x)_i - (\Delta|x|)_i, (A_c x)_i + (\Delta|x|)_i] - (b_c)_i\rfloor \\ &= \lfloor(\mathbf{A}x)_i - (b_c)_i\rfloor. \end{aligned} \tag{3.13}$$

Therefore, (3.11) holds for each $i \in S$.

For each L–localized row $j \in Q$. By the derivation of the second relation of (3.1), the second relation of (3.12) turns into

$$-\delta_j \leq (A_c x)_j + (\Delta|x|)_j - (b_c)_j \leq \delta_j,$$

that is,

$$|(A_c x)_j + (\Delta|x|)_j - (b_c)_j| \leq \delta_j. \tag{3.14}$$

By the rule of disjunctive amplification, (3.14) implies to the following statement:

$$|(A_c x)_j + (\Delta|x|)_j - (b_c)_j| \leq \delta_j \text{ or } |(A_c x)_j - (\Delta|x|)_j - (b_c)_j| \leq \delta_j.$$

According to the same derivation of (3.13), it implies that (3.11) also holds for each $j \in Q$.

Next, let us consider each R–localized row $k \in R$. In the same manner as (3.1), the third relation of (3.12) becomes

$$-\delta_k \leq -(A_c x)_k + (\Delta|x|)_k + (b_c)_k \leq \delta_k,$$

that can be rewritten as

$$\delta_k \geq |-(A_cx)_k + (\Delta|x|)_k + (b_c)_k| = |(A_cx)_k - (\Delta|x|)_k - (b_c)_k|.$$

Therefore, we now have

$$|(A_cx)_k - (\Delta|x|)_k - (b_c)_k| \leq \delta_k. \tag{3.15}$$

By the same fashion as executing to (3.14), finally we obtain (3.11) for each $k \in R$. Hence, if x is a tolerance–localized solution of $\mathbf{A}x = \mathbf{b}$, then (3.11) holds as desired.

Conversely, assume that a given x satisfies the condition (3.11). Then $\lfloor \mathbf{A}x - b_c \rfloor \leq \delta$. Let $i \in M$. By reversing the proof of (3.13), we have

$$\delta_i \geq \min\{|(A_cx)_i - (\Delta|x|)_i - (b_c)_i|, |(A_cx)_i + (\Delta|x|)_i - (b_c)_i|\},$$

that is, $\delta_i \geq |(A_cx)_i - (\Delta|x|)_i - (b_c)_i|$ or $\delta_i \geq |(A_cx)_i + (\Delta|x|)_i - (b_c)_i|$. To prove the theorem, we have to separate the proof into two parts:

1. The first part assumes $\delta_i \geq |(A_cx)_i - (\Delta|x|)_i - (b_c)_i|$
2. The second part assumes $\delta_i \geq |(A_cx)_i + (\Delta|x|)_i - (b_c)_i|$.

The first part assumption can be distinguished into the following two cases below.

Case 1.1: $\delta_i \geq |(A_cx)_i - (\Delta|x|)_i - (b_c)_i|$ and $\delta_i \leq |(A_cx)_i + (\Delta|x|)_i - (b_c)_i|$, which implies

$$\begin{aligned}\delta_i \geq |(A_cx)_i - (\Delta|x|)_i - (b_c)_i| &\Leftrightarrow -\delta_i \leq (A_cx)_i - (\Delta|x|)_i - (b_c)_i \leq \delta_i \\ &\Leftrightarrow -(b_c)_i + \delta_i \geq -(A_cx)_i + (\Delta|x|)_i \\ &\quad \text{and } -(A_cx)_i + (\Delta|x|)_i \geq -(b_c)_i - \delta_i \end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\delta_i \leq |(A_cx)_i + (\Delta|x|)_i - (b_c)_i| &\Leftrightarrow -(A_cx)_i - (\Delta|x|)_i + (b_c)_i \geq \delta_i \\ &\quad \text{or } (A_cx)_i + (\Delta|x|)_i - (b_c)_i \geq \delta_i \\ &\Leftrightarrow -(b_c)_i + \delta_i \leq -(A_cx)_i - (\Delta|x|)_i \quad (3.17) \\ &\quad \text{or } (b_c)_i + \delta_i \leq (A_cx)_i + (\Delta|x|)_i. \quad (3.18)\end{aligned}$$

By putting (3.16) and (3.17) together, we get

$$-(b_c)_i - \delta_i \leq -(A_cx)_i + (\Delta|x|)_i \leq -(b_c)_i + \delta_i \leq -(A_cx)_i - (\Delta|x|)_i,$$

which contradicts Theorem 2.3. We next combine (3.16) and (3.18), then

$$-(A_cx)_i - (\Delta|x|)_i \leq -(b_c)_i - \delta_i \leq -(A_cx)_i + (\Delta|x|)_i \leq -(b_c)_i + \delta_i,$$

which means $-(\mathbf{A}x)_i \leq_{st} -\mathbf{b}_i$. Hence the row i of the system satisfying Case 1.1 is R–localized.

Case 1.2: $\delta_i \geq |(A_cx)_i - (\Delta|x|)_i - (b_c)_i|$ and $\delta_i \geq |(A_cx)_i + (\Delta|x|)_i - (b_c)_i|$, where

$$\delta_i \geq |(A_cx)_i - (\Delta|x|)_i - (b_c)_i| \Leftrightarrow (b_c)_i - \delta_i \leq (A_cx)_i - (\Delta|x|)_i \leq (b_c)_i + \delta_i \tag{3.19}$$

and

$$\begin{aligned}\delta_i \geq |(A_c x)_i + (\Delta|x|)_i - (b_c)_i| \ &\Leftrightarrow -\delta_i \leq (A_c x)_i + (\Delta|x|)_i - (b_c)_i \leq \delta_i \\ &\Leftrightarrow (b_c)_i - \delta_i \leq (A_c x)_i + (\Delta|x|)_i \\ &\quad \text{and } (A_c x)_i + (\Delta|x|)_i \leq (b_c)_i + \delta_i. \end{aligned} \tag{3.20}$$

By getting (3.19) and (3.20) together, we have

$$(b_c)_i - \delta_i \leq (A_c x)_i - (\Delta|x|)_i \leq (A_c x)_i + (\Delta|x|)_i \leq (b_c)_i + \delta_i,$$

which implies $(\mathbf{A}x)_i \subseteq \mathbf{b}_i$. Therefore, the row i of the system satisfying Case 1.2 is tolerance. Now, the first part is finished. Let us continue to prove the second part. Similar to the first part, the second part can be separated into two cases below.

Case 2.1: $\delta_i \geq |(A_c x)_i + (\Delta|x|)_i - (b_c)_i|$ and $\delta_i \leq |(A_c x)_i - (\Delta|x|)_i - (b_c)_i|$, which can be rewritten as

$$\begin{aligned}\delta_i \geq |(A_c x)_i + (\Delta|x|)_i - (b_c)_i| \ &\Leftrightarrow (b_c)_i - \delta_i \leq (A_c x)_i + (\Delta|x|)_i \\ &\quad \text{and } (A_c x)_i + (\Delta|x|)_i \leq (b_c)_i + \delta_i \end{aligned} \tag{3.21}$$

and

$$\begin{aligned}\delta_i \leq |(A_c x)_i - (\Delta|x|)_i - (b_c)_i| \ &\Leftrightarrow \delta_i \leq -(A_c x)_i + (\Delta|x|)_i + (b_c)_i \\ &\quad \text{or } \delta_i \leq (A_c x)_i - (\Delta|x|)_i - (b_c)_i \\ &\Leftrightarrow (A_c x)_i - (\Delta|x|)_i \leq (b_c)_i - \delta_i \qquad (3.22) \\ &\quad \text{or } (b_c)_i + \delta_i \leq (A_c x)_i - (\Delta|x|)_i. \qquad (3.23) \end{aligned}$$

By putting (3.21) and (3.22) together, we get

$$(A_c x)_i - (\Delta|x|)_i \leq (b_c)_i - \delta_i \leq (A_c x)_i + (\Delta|x|)_i \leq (b_c)_i + \delta_i,$$

which implies $(\mathbf{A}x)_i \leq_{st} \mathbf{b}_i$. On the other hand, putting (3.21) and (3.23) together returns

$$(b_c)_i - \delta_i \leq (A_c x)_i + (\Delta|x|)_i \leq (b_c)_i + \delta_i \leq (A_c x)_i - (\Delta|x|)_i,$$

which contradicts Theorem 2.3. Then we can summarize that the row i of the system satisfying Case 2.1 is L–localized.

Case 2.2: $\delta_i \geq |(A_c x)_i + (\Delta|x|)_i - (b_c)_i|$ and $\delta_i \geq |(A_c x)_i - (\Delta|x|)_i - (b_c)_i|$, which is the same as Case 1.2 of the first part assumption. Therefore, the row i of the system satisfying Case 2.2 is tolerance. By the two parts of the proof, x that satisfies (3.11) is a tolerance–localized solution of $\mathbf{A}x = \mathbf{b}$. Hence, we complete the proof of Theorem 3.2. □

Theorem 3.3. Let TL be a function from $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$TL(x \mid \mathbf{A}, \mathbf{b}) = \delta_i - \left| \sum_{j=1}^{n} \mathbf{a}_{ij} x_j - (b_c)_i \right|, \ \forall i \in M,$$

where $\mathbf{A} = [\mathbf{a}_{ij}] \in \mathbb{IR}^{m \times n}$ and $\mathbf{b} = [b_c - \delta, b_c + \delta] \in \mathbb{IR}^m$ are given. A vector $x \in \mathbb{R}^n$ is a tolerance–localized solution of $\mathbf{A}x = \mathbf{b}$ if it satisfies

$$TL(x \mid \mathbf{A}, \mathbf{b}) \geq 0.$$

That is $\sum_{TL}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n : TL(x \mid \mathbf{A}, \mathbf{b}) \geq 0\}$.

Proof. Recall from Theorem 3.2 that x is a tolerance–localized solution of $\mathbf{A}x = \mathbf{b}$

if it satisfies

$$\begin{aligned}\lfloor \mathbf{A}x - b_c \rfloor \leq \delta &\Leftrightarrow \delta - \lfloor \mathbf{A}x - b_c \rfloor \geq 0\\ &\Leftrightarrow \delta_i - \left\lfloor \sum_{j=1}^{n} \mathbf{a}_{ij} x_j - (b_c)_i \right\rfloor \geq 0,\ \forall i \in M\\ &\Leftrightarrow TL(x \mid \mathbf{A}, \mathbf{b}) \geq 0.\end{aligned}$$

By reversing this proof, the proof of the theorem is completed. □

The three theorems related to tolerance–localized solution are released in this section. First two theorems refer to the characterizations that one presents in center and radius matrices and the other one presents as geometric sense. In the last theorem of this section, closed form of the set of all tolerance–localized solution is represented. In the next section, the theorems related to control–localized solution are mentioned. The concepts of them are similar to the proposed theorems of tolerance–localized solution.

CHAPTER IV

CONTROL–LOCALIZED SOLUTION

In this chapter, the provided theorems are involved with the set of all control–localized solutions of the system $\mathbf{A}x = \mathbf{b}$. In the same fashion as the previous chapter, we obtain two theorems that are related to a characterization of control–localized solution and provide a theorem described the control–localized solution set in terms of a level set as well.

Theorem 4.1. Let $I_{m\times m}$ be an identity matrix of size m and E_{ii} be a $m\times m$ matrix which has 1 at the position (i,i) and 0 elsewhere. A vector x is a control–localized solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies

$$-\Delta|x| - \hat{\delta} \leq \hat{A}_c x - \hat{b}_c \leq \hat{e}^T|\delta - \Delta|x||.$$

The terms $\hat{A}_c = (I_{m\times m} - 2\sum_{i\in R} E_{ii})A_c$, $\hat{b}_c = (I_{m\times m} - 2\sum_{i\in R} E_{ii})b_c$, $\hat{\delta} = (I_{m\times m} - 2\sum_{i\in P} E_{ii})\delta$ where $P = \{i \in M; \mathbf{b}_i \subseteq (\mathbf{A}x)_i\}$ and $R = \{k \in M; -(\mathbf{A}x)_k \leq_{st} -\mathbf{b}_k\}$. The m-dimensional vector $\hat{e}$ has 1 at row $i \in P$ and -1 elsewhere.

Proof. Assume that x is a control–localized solution of $\mathbf{A}x = \mathbf{b}$. Let P, Q and R be disjoint subsets of M such that rows $i \in P, j \in Q$ and $k \in R$ of the system are control, L–localized and R–localized, respectively. This means

$$\mathbf{b}_i \subseteq (\mathbf{A}x)_i,\ \forall i \in P,\ (\mathbf{A}x)_j \leq_{st} \mathbf{b}_j,\ \forall j \in Q \text{ and } -(\mathbf{A}x)_k \leq_{st} -\mathbf{b}_k,\ \forall k \in R. \quad (4.1)$$

We begin the proof by considering the first relation of (4.1) for each $i \in P$. From Theorem 2.3,

$$[(b_c)_i - \delta_i, (b_c)_i + \delta_i] = \mathbf{b}_i \subseteq (\mathbf{A}x)_i = [(A_c x)_i - (\Delta|x|)_i, (A_c x)_i + (\Delta|x|)_i],$$

which implies

$$(A_c x)_i - (\Delta|x|)_i \leq (b_c)_i - \delta_i \leq (b_c)_i + \delta_i \leq (A_c x)_i + (\Delta|x|)_i.$$

That is, $(A_c x)_i - (b_c)_i \leq (\Delta|x|)_i - \delta_i$ and $-(\Delta|x|)_i + \delta_i \leq (A_c x)_i - (b_c)_i$. Thus

$$-(\Delta|x|)_i + \delta_i \leq (A_c x)_i - (b_c)_i \leq (\Delta|x|)_i - \delta_i.$$

Since $\mathbf{b}_i \subseteq (\mathbf{A}x)_i$, then $(\Delta|x|)_i \geq \delta_i$, i.e., $0 \leq (\Delta|x|)_i - \delta_i = |\delta_i - (\Delta|x|)_i|$. Therefore,

$$-(\Delta|x|)_i + \delta_i \leq (A_c x)_i - (b_c)_i \leq |\delta_i - (\Delta|x|_i)| \text{ for each } i \in P. \tag{4.2}$$

Next, let us focus on $j \in Q$ and $k \in R$. By the same derivation as (3.1), the second and the last relations of (4.1) becomes

$$-(\Delta|x|)_j - \delta_j \leq (A_c x)_j - (b_c)_j \leq -|\delta_j - (\Delta|x|_j)| \text{ for each } j \in Q \tag{4.3}$$

and

$$-(\Delta|x|)_k - \delta_k \leq -(A_c x)_k + (b_c)_k \leq -|\delta_k - (\Delta|x|_k)| \text{ for each } k \in R, \tag{4.4}$$

respectively. Let e_i be a m-dimensional column vector which has 1 at row i and 0 elsewhere. By putting (4.2), (4.3) and (4.4) together, the left, the middle and the right terms are

$$-\Delta|x| - \delta + 2\sum_{i \in P} \delta_i e_i \tag{4.5}$$

$$A_c x - b_c - 2\sum_{i \in R}(A_c x - b_c)_i e_i = A_c x - b_c - 2\sum_{i \in R}(A_c x)_i e_i + 2\sum_{i \in R}(b_c)_i e_i \tag{4.6}$$

and

$$\hat{e}^T|\delta - \Delta|x|| = \sum_{i\in M} \hat{e}_i(|\delta - \Delta|x||)_i \text{ where } \hat{e}_i = \begin{cases} 1, & \text{if } i \in P; \\ -1, & \text{if } i \in Q \cup R = M \setminus P, \end{cases}$$

respectively. As a result of

$$\sum_{i\in R}(A_c x)_i e_i = \left(\sum_{i\in R} E_{ii}\right) A_c x, \;\; \sum_{i\in R}(b_c)_i e_i = \left(\sum_{i\in R} E_{ii}\right) b_c$$

the terms (4.5) and (4.6) are

$$-\Delta|x| - (I_{m\times m} - 2\sum_{i\in P} E_{ii})\delta = -\Delta|x| - \hat{\delta}$$

and

$$(I_{m\times m} - 2\sum_{i\in R} E_{ii})A_c x - (I_{m\times m} - 2\sum_{i\in R} E_{ii})b_c = \hat{A}_c x - \hat{b}_c,$$

respectively. Hence $-\Delta|x| - \hat{\delta} \le \hat{A}_c x - \hat{b}_c \le \hat{e}^T|\delta - \Delta|x||$. Reverse this proof to complete the proof of the theorem. □

Theorem 4.2. A vector x is a control–localized solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies

$$\langle \mathbf{A}x - b_c \rangle \le \delta \le \lceil \mathbf{A}x - b_c \rceil. \tag{4.7}$$

Proof. Assume that x is a control–localized solution of $\mathbf{A}x = \mathbf{b}$. Let P, Q and R be disjoint subsets of M such that rows $i \in P, j \in Q$ and $k \in R$ of the system are control, L–localized and R–localized, respectively. Then

$$\mathbf{b}_i \subseteq (\mathbf{A}x)_i, \; \forall i \in P, \; (\mathbf{A}x)_j \le_{st} \mathbf{b}_j, \; \forall j \in Q \text{ and } -(\mathbf{A}x)_k \le_{st} -\mathbf{b}_k, \; \forall k \in R. \tag{4.8}$$

Let $i \in P$. In accordance with the proof of Theorem 4.1, the first relation of (4.8) is equivalent to (4.2), which means

$$(A_c x)_i - (\Delta|x|)_i - (b_c)_i \le -\delta_i \ \text{ and } \ (A_c x)_i + (\Delta|x|)_i - (b_c)_i \ge \delta_i. \tag{4.9}$$

By using the fact that $\delta_i \ge 0$, (4.9) can be written as

$$0 \le \delta_i \le -(A_c x)_i + (\Delta|x|)_i + (b_c)_i = |(A_c x)_i - (\Delta|x|)_i - (b_c)_i| \tag{4.10}$$

and

$$0 \le \delta_i \le (A_c x)_i + (\Delta|x|)_i - (b_c)_i = |(A_c x)_i + (\Delta|x|)_i - (b_c)_i|. \tag{4.11}$$

From (4.10) and (4.11), it implies that

$$\delta_i \le \min\{|(A_c x)_i - (\Delta|x|)_i - (b_c)_i|, |(A_c x)_i + (\Delta|x|)_i - (b_c)_i|\}.$$

As the fact that the minimum value between $|(A_c x)_i - (\Delta|x|)_i - (b_c)_i|$ and $|(A_c x)_i + (\Delta|x|)_i - (b_c)_i|$ is less than or equal to the maximum value between them, we get

$$\begin{aligned}\delta_i &\le \max\{|(A_c x)_i - (\Delta|x|)_i - (b_c)_i|, |(A_c x)_i + (\Delta|x|)_i - (b_c)_i|\} \\ &= \lceil[(A_c x)_i - (\Delta|x|)_i - (b_c)_i, (A_c x)_i + (\Delta|x|)_i - (b_c)_i]\rceil \\ &= \lceil[(A_c x)_i - (\Delta|x|)_i, (A_c x)_i + (\Delta|x|)_i] - (b_c)_i\rceil \\ &= \lceil(\mathbf{A}x)_i - (b_c)_i\rceil. \end{aligned} \tag{4.12}$$

Moreover, since $\mathbf{b}_i \subseteq (\mathbf{A}x)_i$, $0 = (b_c)_i - (b_c)_i \in \mathbf{b}_i - (b_c)_i \subseteq (\mathbf{A}x)_i - (b_c)_i$ for each $i \in P$. By the definition of $\langle * \rangle$ and $0 \in (\mathbf{A}x)_i - (b_c)_i$, it provides $\langle (\mathbf{A}x)_i - (b_c)_i \rangle = 0 \le \delta_i$ and therefore (4.7) satisfies for each $i \in P$.

Next, let us consider L–localized row j belongs to Q. The second relation of

(4.8) is directly derived from (3.1) as

$$(A_cx)_j - (\Delta|x|)_j - (b_c)_j \leq -\delta_j \text{ and } -\delta_j \leq (A_cx)_j + (\Delta|x|)_j - (b_c)_j \leq \delta_j. \quad (4.13)$$

As a result of $\delta_j \geq 0$, (4.13) becomes

$$0 \leq \delta_j \leq -(A_cx)_j + (\Delta|x|)_j + (b_c)_j = |(A_cx)_j - (\Delta|x|)_j - (b_c)_j|$$

and

$$|(A_cx)_j + (\Delta|x|)_j - (b_c)_j| \leq \delta_j,$$

that is,

$$|(A_cx)_j - (\Delta|x|)_j - (b_c)_j| \geq \delta_j \text{ and } |(A_cx)_j + (\Delta|x|)_j - (b_c)_j| \leq \delta_j,$$

which imply

$$\begin{aligned}\delta_j &\geq \min\{|(A_cx)_j - (\Delta|x|)_j - (b_c)_j|, |(A_cx)_j + (\Delta|x|)_j - (b_c)_j|\} \\ &\geq \langle[(A_cx)_j - (\Delta|x|)_j - (b_c)_j, (A_cx)_j + (\Delta|x|)_j - (b_c)_j]\rangle \\ &= \langle[(A_cx)_j - (\Delta|x|)_j, (A_cx)_j + (\Delta|x|)_j] - (b_c)_j\rangle \\ &= \langle(\mathbf{A}x)_j - (b_c)_j\rangle. \end{aligned} \quad (4.14)$$

In the same simplification as (4.12), it gives $\delta_j \leq \lceil(\mathbf{A}x)_j - (b_c)_j\rceil$ and hence (4.7) holds for each $j \in Q$.

For each R–localized row k belongs to R, (3.1) implies that the last relation of (4.8) is simplified as

$$-(A_cx)_k - (\Delta|x|)_k + (b_c)_k \leq -\delta_k \text{ and } -\delta_k \leq -(A_cx)_k + (\Delta|x|)_k + (b_c)_k \leq \delta_k. \quad (4.15)$$

Since $\delta_k \geq 0$, (4.15) turns into

$$0 \leq \delta_k \leq -(-(A_cx)_k - (\Delta|x|)_k + (b_c)_k) = |(A_cx)_k + (\Delta|x|)_k - (b_c)_k|$$

and

$$\delta_k \geq |-(A_cx)_k + (\Delta|x|)_k + (b_c)_k| = |(A_cx)_k - (\Delta|x|)_k - (b_c)_k|$$

that is,

$$|(A_cx)_k + (\Delta|x|)_k - (b_c)_k| \geq \delta_k \text{ and } |(A_cx)_k - (\Delta|x|)_k - (b_c)_k| \leq \delta_k. \quad (4.16)$$

In similar derivations to (4.12) and (4.14), we finally obtain (4.7) for each $k \in R$.

Conversely, assume that a given x satisfies the condition (4.7). This means $\langle \mathbf{A}x - b_c \rangle \leq \delta$ and $\delta \leq \lceil \mathbf{A}x - b_c \rceil$. Let $i \in M$. By Reversing the proof of (4.12), we have

$$\delta_i \leq \max\{|(A_cx)_i - (\Delta|x|)_i - (b_c)_i|, |(A_cx)_i + (\Delta|x|)_i - (b_c)_i|\},$$

that is, $\delta_i \leq |(A_cx)_i - (\Delta|x|)_i - (b_c)_i|$ or $\delta_i \leq |(A_cx)_i + (\Delta|x|)_i - (b_c)_i|$. Let us consider the condition $\langle \mathbf{A}x - b_c \rangle \leq \delta$. For the case $0 \notin (\mathbf{A}x)_i - (b_c)_i$, we reverse the proof of (4.14) to complete the inequality

$$\delta_i \geq \min\{|(A_cx)_i - (\Delta|x|)_i - (b_c)_i|, |(A_cx)_i + (\Delta|x|)_i - (b_c)_i|\},$$

that is, $\delta_i \geq |(A_cx)_i - (\Delta|x|)_i - (b_c)_i|$ or $\delta_i \geq |(A_cx)_i + (\Delta|x|)_i - (b_c)_i|$. If $0 \in (\mathbf{A}x)_i - (b_c)_i$, then $\delta_i \geq \langle (\mathbf{A}x)_i - (b_c)_i \rangle = 0$. To prove the theorem, we need to distinguish the assumption into two parts:

1. Assume that $\delta \leq \lceil \mathbf{A}x - b_c \rceil$ and $\langle \mathbf{A}x - b_c \rangle \leq \delta$ where $0 \notin (\mathbf{A}x)_i - (b_c)_i$

2. Assume that $\delta \leq \lceil \mathbf{A}x - b_c \rceil$ and $\langle \mathbf{A}x - b_c \rangle \leq \delta$ where $0 \in (\mathbf{A}x)_i - (b_c)_i$.

Let us begin with the first part assumption that can be separated into the following three cases below.

Case 1.1: $\delta_i \leq |(A_c x)_i - (\Delta|x|)_i - (b_c)_i|$ and $\delta_i \geq |(A_c x)_i + (\Delta|x|)_i - (b_c)_i|$ are the same as Case 2.1 in Theorem 3.2. Then we can conclude that the row i of the system satisfying Case 1.1 is L–localized.

Case 1.2: $\delta_i \geq |(A_c x)_i - (\Delta|x|)_i - (b_c)_i|$ and $\delta_i \leq |(A_c x)_i + (\Delta|x|)_i - (b_c)_i|$ are identical to Case 1.1 in Theorem 3.2. This supports that the row i of the system satisfying Case 1.2 is R–localized.

Case 1.3: $\delta_i = |(A_c x)_i - (\Delta|x|)_i - (b_c)_i|$ and $\delta_i = |(A_c x)_i + (\Delta|x|)_i - (b_c)_i|$ where

$$\begin{aligned} \delta_i = |(A_c x)_i - (\Delta|x|)_i - (b_c)_i| &\Leftrightarrow \delta_i = (A_c x)_i - (\Delta|x|)_i - (b_c)_i \\ &\quad \text{or } \delta_i = -(A_c x)_i + (\Delta|x|)_i + (b_c)_i \\ &\Leftrightarrow (A_c x)_i - (\Delta|x|)_i = (b_c)_i + \delta_i \quad (4.17) \\ &\quad \text{or } (A_c x)_i - (\Delta|x|)_i = (b_c)_i - \delta_i \quad (4.18) \end{aligned}$$

and

$$\begin{aligned} \delta_i = |(A_c x)_i + (\Delta|x|)_i - (b_c)_i| &\Leftrightarrow (A_c x)_i + (\Delta|x|)_i = (b_c)_i + \delta_i \quad (4.19) \\ &\quad \text{or } (A_c x)_i + (\Delta|x|)_i = (b_c)_i - \delta_i. \quad (4.20) \end{aligned}$$

By putting (4.17) and (4.19) together, we get

$$(A_c x)_i - (\Delta|x|)_i = (b_c)_i + \delta_i = (A_c x)_i + (\Delta|x|)_i,$$

it gives $-(\mathbf{A}x)_i \leq_{st} -\mathbf{b}_i$. However, putting (4.17) and (4.20) together contradict with Theorem 2.3. For another case, a combine of (4.18) and (4.19) recurs

$$(b_c)_i - \delta_i = (A_c x)_i - (\Delta|x|)_i \leq (A_c x)_i + (\Delta|x|)_i = (b_c)_i + \delta_i,$$

it provides $(\mathbf{A}x)_i = \mathbf{b}_i$. For the rest one, including (4.18) and (4.20) each other is turned

$$(A_c x)_i - (\Delta|x|)_i = (b_c)_i - \delta_i = (A_c x)_i + (\Delta|x|)_i,$$

it means $(\mathbf{A}x)_i \leq_{st} \mathbf{b}_i$. Therefore, we can summarize that the row i of the system satisfying Case 1.3 is control–localized.

Now, the first part is done. We begin the second part by assuming that $\delta \leq \lceil \mathbf{A}x - b_c \rceil$ and $\langle \mathbf{A}x - b_c \rangle \leq \delta$ where $0 \in (\mathbf{A}x)_i - (b_c)_i$. These assumptions are separated into the following three cases below.

Case 2.1: $\delta_i \leq |(A_c x)_i - (\Delta|x|)_i - (b_c)_i|$ and $\delta_i \geq |(A_c x)_i + (\Delta|x|)_i - (b_c)_i|$ are the same as the assumption of Case 2.1 in Theorem 3.2. It can be concluded that the row i of the system satisfying Case 2.1 is L–localized.

Case 2.2: $\delta_i \geq |(A_c x)_i - (\Delta|x|)_i - (b_c)_i|$ and $\delta_i \leq |(A_c x)_i + (\Delta|x|)_i - (b_c)_i|$ are the same as the assumption of Case 1.1 in Theorem 3.2. This is verified that the row i of the system satisfying Case 2.2 is R–localized.

Case 2.3: $\delta_i \leq |(A_c x)_i - (\Delta|x|)_i - (b_c)_i|$ and $\delta_i \leq |(A_c x)_i + (\Delta|x|)_i - (b_c)_i|$ where

$$\delta_i \leq |(A_c x)_i - (\Delta|x|)_i - (b_c)_i| \;\Leftrightarrow (A_c x)_i - (\Delta|x|)_i \geq (b_c)_i + \delta_i \tag{4.21}$$

$$\text{or } (A_c x)_i - (\Delta|x|)_i \leq (b_c)_i - \delta_i \tag{4.22}$$

and

$$\delta_i \leq |(A_cx)_i + (\Delta|x|)_i - (b_c)_i| \Leftrightarrow (A_cx)_i + (\Delta|x|)_i \geq (b_c)_i + \delta_i \tag{4.23}$$

$$\text{or } (A_cx)_i + (\Delta|x|)_i \leq (b_c)_i - \delta_i. \tag{4.24}$$

There is a contradiction with Theorem 2.3 for the case when combine (4.21) and (4.24) together. Another case is combined (4.21) and (4.23). This gives

$$(b_c)_i + \delta_i \leq (A_cx)_i - (\Delta|x|)_i \leq (A_cx)_i + (\Delta|x|)_i,$$

which refers to $-(\mathbf{A}x)_i \leq_{st} -\mathbf{b}_i$. By putting next (4.22) and (4.23) together, we obtain

$$(A_cx)_i - (\Delta|x|)_i \leq (b_c)_i - \delta_i \leq (b_c)_i + \delta_i \leq (A_cx)_i + (\Delta|x|)_i,$$

it verifies that $(\mathbf{b}_i \subseteq \mathbf{A}x)_i$. For the last case, putting (4.22) and (4.24) together return

$$(A_cx)_i - (\Delta|x|)_i \leq (A_cx)_i + (\Delta|x|)_i \leq (b_c)_i - \delta_i,$$

which implies $(\mathbf{A}x)_i \leq_{st} \mathbf{b}_i$. Therefore, the row i of the system satisfying Case 2.3 is control–localized and this complete the proof of the theorem. □

Theorem 4.3. Let CL be a function from $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$CL(x \mid \mathbf{A}, \mathbf{b}) = \min_{i \in M} \left\{ \delta_i - \left\langle \sum_{j=1}^{n} \mathbf{a}_{ij} x_j - (b_c)_i \right\rangle, \left\lceil \sum_{j=1}^{n} \mathbf{a}_{ij} x_j - (b_c)_i \right\rceil - \delta_i \right\},$$

where $\mathbf{A} = [\mathbf{a}_{ij}] \in \mathbb{IR}^{m \times n}$ and $\mathbf{b} = [b_c - \delta, b_c + \delta] \in \mathbb{IR}^m$ are given. A vector $x \in \mathbb{R}^n$ is a control–localized solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies

$$CL(x \mid \mathbf{A}, \mathbf{b}) \geq 0.$$

That is, $\sum_{CL}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n : CL(x \mid \mathbf{A}, \mathbf{b}) \geq 0\}$.

Proof. Recall from Theorem 4.2 that x is a control–localized solution of $\mathbf{A}x = \mathbf{b}$ if and only if it satisfies $\langle \mathbf{A}x - b_c \rangle \leq \delta \leq \lceil \mathbf{A}x - b_c \rceil$. That is,

$$\delta - \langle \mathbf{A}x - b_c \rangle \geq 0 \text{ and } \lceil \mathbf{A}x - b_c \rceil - \delta \geq 0,$$

which implies

$$\min\{\delta - \langle \mathbf{A}x - b_c \rangle, \lceil \mathbf{A}x - b_c \rceil - \delta\} \geq 0. \tag{4.25}$$

For each $i = \{1, \ldots, m\}$ and $j = \{1, \ldots, n\}$, (4.25) can be simplified as

$$\min_{i \in M} \left\{ \delta_i - \left\langle \sum_{j=1}^{n} \mathbf{a}_{ij} x_j - (b_c)_i \right\rangle, \left\lceil \sum_{j=1}^{n} \mathbf{a}_{ij} x_j - (b_c)_i \right\rceil - \delta_i \right\} \geq 0,$$

which refers to

$$CL(x \mid \mathbf{A}, \mathbf{b}) \geq 0.$$

By reversing this proof, the proof of the theorem is completed. □

This section, two proposed theorems mention the characterizations of control–localized solution of interval linear equation system. One theorem is shown in terms of center and radius matrices. The other is described as magnitude sense. Based on the magnitude sense characterization, the closed form of the set of all control–localized solution of interval linear equation system is proposed in the third theorem. Now, we show all of proposed theorems related to tolerance–localized and control–localized solutions. To show that the proposed theorems are practical, we present an application that applies concept of tolerance–localized solution to deal with the problem in the next section.

CHAPTER V

COURSE ASSIGNMENT PROBLEM WITH UNCERTAINTY

This chapter presents a problem that can be handled by applying the proposed theorems. By researching, we specifically take an interest to deal with course assignment problem because we found some conditions of course assignment problem that can apply the result of Theorem 3.2. The details of the problem and the results are described in the next sections.

5.1 Introduction to course assignment problem

Course assignment problem is the problem of matching subjects and instructors under some constraints. Creating a course assignment by hand may achieve bias result. To solve this problem more efficiently, there are many approaches to deal with such as integer programming, linear network optimization, graph theory coloring and other metaheuristic methods. We focus on using a linear integer programming model to deal with this problem. By studying the information in the second semesters of 2018 academic year of the Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, we would like to assign subjects to instructors that lead to the overall maximum assignment preferences and minimum over/under workload of the instructors. However, the instructors might want to change their requested workload for some reasons such as their own or family illness, or any other accidents. By these situations, the instructors might have uncertain requested workload. In addition, the variation of a number of students can affect the workload of subjects. By these reasons, course assignment problem with uncertainty is generated. Thus, in this thesis, we can use the concept of interval linear equations system to deal with the problem.

The concept of applying tolerance–localized characterization or Theorem 3.2 to course assignment problem with interval requested workload and interval workload of subjects is revealed in the next section.

5.2 Tolerance–localized to course assignment problem with uncertainty

In this section, the reason that tolerance–localized solution can exist to the course assignment problem is described. Normally, the department might have an observed data mentioned to the requested workload of each instructor. However, some instructors maybe unsure with the process of giving the requested workload because of their uncertain assigned workload in the past. For example, instructor A requested 21 workload for this semester but his/her real assigned workload maybe less than or greater than 21 workload. There are many reasons for the uncertain workload such as annual leave, research leave, increment and retirement of instructors in that semester. The enrollment termination of some subjects because of lacking enrolled students might be one of the reasons. Then, instructor A may determine his/her requested workload as the interval $[21, 24]$ and does not expect or depress, even though his/her assigned workload is not as planned. According to this description, the total assigned workload of instructors might be in L–localized, R–localized or tolerance solution terms. Thus, the concept of tolerance–localized to course assignment problem is already declared. The data, descriptions and notation of the problem is stated in the next section.

5.3 Descriptions and notation of course assignment problem

In this section, we give the information used for performing the problem. The basic restrictions and declared variables are also shown in this section also.

The mentioned information in this section and the model study in the next section have cited from [18, 17]. Firstly, we divide the teaching preference value into 6 ranks with different fuzzy values of ranks as presented in [18]. The provided ranks of instructors represent the expertise or interest in each subject. In this context, we assume that the decisions of all instructors to clarify their rank of subjects are the same way.

Rank Description	Rank	Value of rank
The most preferable subject to teach	1	1
A preferable subject to teach	2	0.95
A subject is able to teach	3	0.9
A non-preferable subject but is able to teach	4	0.85
A non-preferable subject and needless to teach	5	0.8
A subject is unable to teach	6	0

Table 5.1: Teaching preference rank descriptions.

By considering the data collected from the Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, there are 61 instructors with 84 subjects. Subjects in this context include of subjects with only one section and subjects with multiple sections, then a total number of sections of 84 subjects is 121 sections. The workload of each subject depends on class level, the number of students, the number of credits of subject, hour and a type of class (lecture or lab). Next, we will present the standard restrictions of the course assignment problem, more details can be seen in [18].

1. No more than 3 sections that each instructor can be assigned.
2. One section is required for only one instructor.
3. Multiple teaching sections of the same subject do not allow to assign to one instructor. To avoid the unfair teaching preparation, this restriction is generated.

4. Each instructor should reach the amount of his/her own requested workload.

Let us next define variables that used to formulate the model in Section 5.4. For $n = 61$ instructors and $m = 121$ sections, we denote $I = \{1, 2, \ldots, n\}$ and $J = \{1, 2, \ldots, m\}$ as sets of indices of the i^{th} instructor and the j^{th} section, respectively. Let $x = (x_{ij}) \in \mathbb{R}^{n \times m}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

- x_{ij} : Decision variable

$$x_{ij} = \begin{cases} 1, & \text{the } i^{th} \text{ instructor teaches the } j^{th} \text{ section;} \\ 0, & \text{otherwise.} \end{cases}$$

- c_{ij} : a value of rank of the j^{th} section that related to the i^{th} instructor.
- a_j : the workload of the j^{th} section.
- b_i : the requested workload of the i^{th} instructor.
- d_i : the known workload of the i^{th} instructor such as seminar, project and thesis duties. Then $b_i - d_i$ refers to the number of teaching workload of the i^{th} instructor.
- α_i : the extra workload (overload) of the i^{th} instructor and $\alpha_i \geq 0$.
- β_i : the residual workload (underload) of the i^{th} instructor and $\beta_i \geq 0$.

The tools for generating the model are completely defined. Thus, in the next section, the formulation of course assignment model is presented.

5.4 Formulation the course assignment model

By the variables defined in the previous section, we can formulate a mathematical model in this section. Let us begin with the formulation of the standard restrictions.

1. No more than 3 subjects that each instructor can be assigned.

$$\sum_{j=1}^{m} x_{ij} \leq 3, \forall i \in I.$$

2. One subject is required for only one instructor.

$$\sum_{i=1}^{n} x_{ij} = 1, \forall j \in J.$$

3. Multiple teaching sections of the same subject do not allow to assign to one instructor.

$$\sum_{j \in J_k} x_{ij} \leq 1, \forall i \in I,$$

where J_k is the index set of multiple sections of the k^{th} subject for all $k = 1, 2, \ldots, 84$.

4. Each instructor should reach the amount of his/her own requested workload.

$$\sum_{j=1}^{m} a_j x_{ij} - \alpha_i + \beta_i = b_i - d_i, \forall i \in I. \tag{5.1}$$

Next, we formulate the objective function of the course assignment problem model related to the objective of this work that is maximizing the assignment preferences and minimizing over/under workload of the instructors. Let M_1 and M_2 be large positive penalty constants of extra and residual workload, respectively. The objective function is presented in the following.

$$\max \quad \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij} - M_1 \sum_{i=1}^{n} \alpha_i - M_2 \sum_{i=1}^{n} \beta_i.$$

Normally, the value of M_2 should be greater than M_1 because having positive residual workload may refer that an instructor cannot keep his or her promise to complete work. By literature [18], we choose $(M_1, M_2) = (0.0015, 10)$ which was experimented for a reasonable result.

However, we found that the workload of the subjects can be changed by the number of enrolled students. Therefore, we can consider the workload of the j^{th} section as interval term. Similarly, if the instructors have faced some problems such as the illness or other accidents, the requested workload of the instructors is changeable as well. Thus, the requested workload of the i^{th} instructor also states in the interval. Then we can rewrite (5.1) as interval term as follows.

$$\sum_{j=1}^{m}[\underline{a_j}, \overline{a_j}]x_{ij} - \alpha_i + \beta_i = [\underline{b_i}, \overline{b_i}] - d_i, \forall i \in I. \tag{5.2}$$

By the description in Section 5.2, we can modify Constraint (5.2) as tolerance–localized constraint or (3.11). By directly deriving (3.11), we get the new constraint as follows. Let $\mathbf{A} = ([\underline{a}_j, \overline{a}_j]), \forall j = 1, 2, \ldots, m$, where $0 \leq \underline{a}_j \leq \overline{a}_j$ be an interval vector of the workload of the j^{th} section. Let $\mathbf{b} = ([\underline{b}_i, \overline{b}_i]$, where $0 \leq \underline{b}_i \leq \overline{b}_i$ and $d = (d_i), \forall i = 1, 2, \ldots, n$ be an interval vector of the requested workload and the workload of known duties of the i^{th} instructor, respectively. For all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, the solution set $x = (x_{ij})$ that satisfies (3.11) can be rewrite as follows.

$$\begin{aligned}
&\{x \mid \lfloor \mathbf{A}x - (b_c - d) \rfloor \leq \delta\} \\
&= \{(x_{ij}) \mid \min\{|\underline{a_j x_{ij}} - ((b_c)_i - d_i)|, |\overline{a_j x_{ij}} - ((b_c)_i - d_i)|\} \leq \delta_i, \forall i \in I, j \in J\} \\
&= \{(x_{ij}) \mid |\underline{a_j x_{ij}} - ((b_c)_i - d_i)| \leq \delta_i \\
&\quad \text{or } |\overline{a_j x_{ij}} - ((b_c)_i - d_i)| \leq \delta_i, \forall i \in I, j \in J\}.
\end{aligned} \tag{5.3}$$

Since $0 \leq \underline{a}_j \leq \overline{a}_j$ and $x_{ij} = \{0, 1\}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, it implies $\underline{a_j x_{ij}} = \underline{a_j} x_{ij}$ and $\overline{a_j x_{ij}} = \overline{a_j} x_{ij}$. Therefore, (5.3) is simplified as

$$
\begin{aligned}
&\{x \mid \lfloor \mathbf{A}x - (b_c - d) \rfloor \leq \delta\} \\
&= \{(x_{ij}) \mid |\underline{a_j} x_{ij} - ((b_c)_i - d_i)| \leq \delta_i \text{ or } |\overline{a_j} x_{ij} - ((b_c)_i - d_i)| \leq \delta_i, \forall i \in I, j \in J\} \\
&= \{(x_{ij}) \mid -\delta_i \leq \underline{a_j} x_{ij} - ((b_c)_i - d_i) \leq \delta_i \\
&\quad \text{or } -\delta_i \leq \overline{a_j} x_{ij} - ((b_c)_i - d_i) \leq \delta_i, \forall i \in I, j \in J\}.
\end{aligned} \tag{5.4}
$$

However, (5.4) is stated as a logical constraint which cannot apply to linear integer programming model. We have to reformulate this constraint into integer programming formulation by using Big M method. Let binary variable (0 or 1) $y = (y_i), \forall i \in I$. For a large positive penalty constant M, (5.4) becomes

$$
\begin{aligned}
\{x \mid \lfloor \mathbf{A}x - (b_c - d) \rfloor \leq \delta\} = \{(x_{ij}) \mid\ & \underline{a_j} x_{ij} - ((b_c)_i - d_i) \leq \delta_i + M y_i, \\
& -\underline{a_j} x_{ij} + ((b_c)_i - d_i) \leq \delta_i + M y_i, \\
& \overline{a_j} x_{ij} - ((b_c)_i - d_i) \leq \delta_i + M(1 - y_i), \\
& -\overline{a_j} x_{ij} + ((b_c)_i - d_i) \leq \delta_i + M(1 - y_i), \forall i \in I, j \in J\}.
\end{aligned}
$$

Therefore, Constraint (5.2) can be rewritten as tolerance–localized constraint as follows.

$$
\begin{aligned}
\textstyle\sum_{j=1}^{m} \underline{a_j} x_{ij} - ((b_c)_i - d_i) \leq\ & \delta_i + M y_i,\ \forall i \in I \\
\textstyle\sum_{j=1}^{m} -\underline{a_j} x_{ij} + ((b_c)_i - d_i) \leq\ & \delta_i + M y_i,\ \forall i \in I \\
\textstyle\sum_{j=1}^{m} \overline{a_j} x_{ij} - ((b_c)_i - d_i) \leq\ & \delta_i + M(1 - y_i),\ \forall i \in I \\
\textstyle\sum_{j=1}^{m} -\overline{a_j} x_{ij} + ((b_c)_i - d_i) \leq\ & \delta_i + M(1 - y_i),\ \forall i \in I \\
x_{ij} \in\ & \{0, 1\},\ \forall i \in I \text{ and } \forall j \in J \\
y_i \in\ & \{0, 1\},\ \forall i \in I.
\end{aligned} \tag{5.5}
$$

By the above derivation, if $y_i = 0$ for some $i \in I$,

$$\begin{aligned}\{x \mid \lfloor \mathbf{A}x - (b_c - d) \rfloor \le \delta\} = &\{(x_{ij}) \mid \underline{a_j} x_{ij} - ((b_c)_i - d_i) \le \delta_i,\\ &- \underline{a_j} x_{ij} + ((b_c)_i - d_i) \le \delta_i,\\ &\overline{a_j} x_{ij} - ((b_c)_i - d_i) \le \delta_i + M,\\ &- \overline{a_j} x_{ij} + ((b_c)_i - d_i) \le \delta_i + M\}.\end{aligned}$$

Since M is a large positive constant,

$$\begin{aligned}\{x \mid \lfloor \mathbf{A}x - (b_c - d) \rfloor \le \delta\} = &\{(x_{ij}) \mid \underline{a_j} x_{ij} - ((b_c)_i - d_i) \le \delta_i, -\underline{a_j} x_{ij} + ((b_c)_i - d_i) \le \delta_i\}\\ = &\{(x_{ij}) \mid -\delta_i \le \underline{a_j} x_{ij} - ((b_c)_i - d_i) \le \delta_i\}.\end{aligned}$$

On the other hand, if $y_i = 1$ for some $i \in I$,

$$\{x \mid \lfloor \mathbf{A}x - (b_c - d) \rfloor \le \delta\} = \{(x_{ij}) \mid -\delta_i \le \overline{a_j} x_{ij} - ((b_c)_i - d_i) \le \delta_i\}.$$

Before combining all constraints together to formulate the model, we have added a new assumption since the data of all instructors probably not set as interval form. For formulating the extended model, assume that there are two groups of instructors:

1. the instructors who provide their requested workload as interval and do not concern about the number of enrolled students that lead to the changeable workload of each subject and
2. the instructors who give a precise requested workload and also know the number of enrolled students by some reasons such as the enrolled students are usually same number in that subject or the specific subjects that design for some group of students, then the workload of the subjects is exact.

This means there are some instructors that can apply (5.5) as a constraint. Thus, the new generated model is performed by mixing the concept of the original model (See Appendix C) and Constraint (5.5). To create the new model, let us define new two index sets. Let I_{notinv} and I_{inv} be the index sets of instructors who give a precise and an interval data, respectively. Although some instructors give interval requested workload, the actual assigned workload might not come as expected. The difference terms that occur by this reason are defined as υ_i and ω_i, $\forall i \in I_{inv}$ shown in the figures below.

Figure 5.1: Difference terms υ and ω related to interval teaching workload.

The constraints for the instructors who provide their interval requested workload are added to complete the modified model. The added constraints state as follows.

$$\sum_{j=1}^{m} \overline{a_j} x_{ij} - (\overline{b_i} - d_i) \leq \upsilon_i, \ \forall i \in I_{inv}$$

$$(\underline{b_i} - d_i) - \sum_{j=1}^{m} \underline{a_j} x_{ij} \leq \omega_i, \ \forall i \in I_{inv}$$

$$\upsilon_i, \omega_i \geq 0, \ \forall i \in I_{inv}.$$

Now, we have fully useful constraints for generating the new model. The extended model is presented as the following.

$$\max \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij} - M_1\left(\sum_{i\in I_{notinv}} \alpha_i + \sum_{i\in I_{inv}} \upsilon_i\right) - M_2\left(\sum_{i\in I_{notinv}} \beta_i + \sum_{i\in I_{inv}} \omega_i\right)$$

subject to

$$\begin{aligned}
\textstyle\sum_{j=1}^{m} x_{ij} &\leq 3,\ \forall i \in I \\
\textstyle\sum_{i=1}^{n} x_{ij} &= 1,\ \forall j \in J \\
\textstyle\sum_{j\in J_j} x_{ij} &\leq 1, \forall i \in I \\
\textstyle\sum_{j=1}^{m} a_j x_{ij} - \alpha_i + \beta_i &= b_i - d_i,\ \forall i \in I_{notinv} \\
\textstyle\sum_{j=1}^{m} \underline{a_j} x_{ij} - ((b_c)_i - d_i) &\leq \delta_i + My_i,\ \forall i \in I_{inv} \\
\textstyle\sum_{j=1}^{m} -\underline{a_j} x_{ij} + ((b_c)_i - d_i) &\leq \delta_i + My_i,\ \forall i \in I_{inv} \\
\textstyle\sum_{j=1}^{m} \overline{a_j} x_{ij} - ((b_c)_i - d_i) &\leq \delta_i + M(1-y_i),\ \forall i \in I_{inv} \\
\textstyle\sum_{j=1}^{m} -\overline{a_j} x_{ij} + ((b_c)_i - d_i) &\leq \delta_i + M(1-y_i),\ \forall i \in I_{inv} \\
\textstyle\sum_{j=1}^{m} \overline{a_j} x_{ij} - (\overline{b_i} - d_i) &\leq \upsilon_i,\ \forall i \in I_{inv} \\
(\underline{b_i} - d_i) - \textstyle\sum_{j=1}^{m} \underline{a_j} x_{ij} &\leq \omega_i,\ \forall i \in I_{inv} \\
\alpha_i, \beta_i &\geq 0,\ \forall i \in I_{notinv} \\
\upsilon_i, \omega_i &\geq 0,\ \forall i \in I_{inv} \\
x_{ij} &\in \{0,1\},\ \forall i \in I \text{ and } \forall j \in J \\
y_i &\in \{0,1\},\ \forall i \in I.
\end{aligned} \tag{5.6}$$

We achieve to formulate our new extended model for the course assignment problem with uncertainty. By running the model (5.6) with CPLEX version 12.10, we can match the instructors and the subjects under considered constraints. In the next section, we will test the teaching preference rank of the obtained course assignment by using Python 3.9.1.

5.5 Results and discussion

In this section, the teaching preference rank result of Problem (5.6) or modified problem is stated. The obtained result is shown in Table 5.2. By the obtained

Rank	Number of subjects
1	84
2	18
3	2
4	0
5	3
6	10

Table 5.2: Teaching preference rank of modified problem.

result, most number of subjects lie on rank 1 and 2 because of the objective function of the model. Moreover, we found that the number of subjects in rank 6 is greater than the number in rank 3,4 and 5 since most of instructors just score preferable and the most non preferable subjects.

In order to recheck the efficiency of the modified model, we have compared the obtained result and result of the original model as shown in Table 5.3. The original model for course assignment problem in this thesis referred to the model without interval data is stated as follows.

$$
\begin{aligned}
\max \quad & \textstyle\sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij} - && M_1\textstyle\sum_{i=1}^{n}\alpha_i - M_2\sum_{i=1}^{n}\beta_i \\
\text{s.t.} \quad & \textstyle\sum_{j=1}^{m} x_{ij} \leq && 3,\ \forall i \in I \\
& \textstyle\sum_{i=1}^{n} x_{ij} = && 1,\ \forall j \in J \\
& \textstyle\sum_{j\in J_j} x_{ij} \leq && 1, \forall i \in I \\
& \textstyle\sum_{j=1}^{m} a_j x_{ij} - \alpha_i + \beta_i = && b_i - d_i,\ \forall i \in I \\
& \alpha_i, \beta_i \geq && 0,\ \forall i \in I \\
& x_{ij} \in && \{0,1\},\ \forall i \in I \text{ and } \forall j \in J.
\end{aligned}
$$

Rank	Number of subjects
1	85
2	17
3	2
4	0
5	3
6	10

Table 5.3: Teaching preference rank of original problem.

We found that the results of them are slightly different at the number of subjects in rank 1 and 2. Although the number of subjects in rank 1 of modified problem is less than the number of the original problem result, the number of subjects in rank 3, 4, 5 and 6 remain the same. By studying the data of the Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, we can conclude that the results of them are not significantly different and the modified model can guarantee the maximized preference and minimized over/under workload of the instructors.

In addition, we have compared the result from modified problem and actual course assignment in the 2^{nd} semester of 2018 as shown in Table 5.4. The actual course assignment has less number of subjects in rank 1 and more number of subjects in rank 6 comparing with the modified result. Then, we can confirm that the modified model is effectiveness. To maximize the overall preferences and

Rank	Number of subjects
1	77
2	17
3	1
4	0
5	2
6	21

Table 5.4: Actual data of teaching preference rank in the 2^{nd} semester of 2018.

minimized over/under workload of the instructors, the department can use the modified model to assign the subjects to the instructors. This modified model may help the department to save time and have an alternative model that comprehends the precise and uncertain data.

CHAPTER VI

CONCLUSION

In this thesis, we pay attention to a system of interval linear equations where the coefficients **A** and right hand side **b** are stated as interval. By literature reviews, we found that there are many researches in this field. The details can be found in Chapter 1. In Chapter 2, the description about interval linear equations is mentioned. However, the part that we indeed focus is types of solutions to the system of interval linear equations. For more details, Chapter 2 is the place to find out also.

Since the aim of this work is to study the two new solutions called tolerance–localized and control–localized solutions, the main theorems of this thesis are divided in two parts. First part or Chapter 3 includes of three theorems related to tolerance–localized solution. Two of them refer to the characterizations that state in form of center and radius matrices and magnitude notation form. The second theorem or Theorem 3.2 is an alternative characterization represented to necessary and sufficient conditions or Theorem 3.1. In addition, based on concept of Theorem 3.2, we obtain the closed form of the all solutions set of tolerance–localized solution stated as Theorem 3.3. In Chapter 4, the three theorems related to control–localized solution are mentioned. The concepts of them can be described by following Chapter 3. The summarization of the characterizations of each solution types is shown as Table 6.1 below.

After, we formulate the course assignment problem with uncertainty which applies the concept of tolerance–localized solution to deal with the problem in

Chapter 5. To optimize the preference and over/under workload of the instructors, we formulate an integer linear programming model. The obtained results by running the modified and original model with CPLEX version 12.10 and Python 3.9.1, we found that the results of them are not significantly different. However, by comparing the results of modified model and actual assignment in the 2^{nd} semester of 2018, the results of modified model give more number of subjects in rank 1 and less number of subjects in rank 6 than actual assignment in the 2^{nd} semester of 2018. This means assigning the subjects to instructors by modified model give better teaching preference rank result by comparing with actual assigning in the 2^{nd} semester of 2018. Therefore, the department may save time and have an alternative model that comprehends the precise and uncertain data for assigning the subjects to the instructors and can give maximized preferences to the instructors.

Types of solutions	Necessary and sufficient conditions	Characterizations
Weak	$\|A_cx - b_c\| \leq \triangle\|x\| + \delta$, [7]	$\langle \mathbf{A}x - b_c \rangle \leq \delta$, [15]
Tolerance	$\|A_cx - b_c\| \leq -\triangle\|x\| + \delta$, [7]	$\lceil (\boldsymbol{A}x)_i - (b_c)_i \rceil \leq \delta_i$, [16]
Control	$\|A_cx - b_c\| \leq \triangle\|x\| - \delta$, [7]	$\lfloor (\boldsymbol{A}x)_i - (b_c)_i \rfloor \geq \delta_i$, [16]
Tolerance-control	$\|A_cx - b_c\| \leq \triangle\|x\| + \delta - 2\hat{\alpha}$, [16]	$\lceil (\boldsymbol{A}x)_i - (b_c)_i \rceil \leq \delta_i$ or $\lfloor (\boldsymbol{A}x)_i - (b_c)_i \rfloor \geq \delta_i$, [16]
L–localized	$-\triangle\|x\| - \delta \leq A_cx - b_c \leq -\|\triangle\|x\| - \delta\|$, [8]	$\lfloor \boldsymbol{A}x - b_c \rfloor \leq \delta \leq \lceil \boldsymbol{A}x - b_c \rceil$, [16]
R–localized	$\|\triangle\|x\| - \delta\| \leq A_cx - b_c \leq \triangle\|x\| + \delta$, [8]	$\lfloor \boldsymbol{A}x - b_c \rfloor \leq \delta \leq \lceil \boldsymbol{A}x - b_c \rceil$, [16]
Localized	$-\triangle\|x\| - \delta \leq \hat{A}_cx - \hat{b}_c \leq -\|\delta - \triangle\|x\|\|$, [8]	$\lfloor \boldsymbol{A}x - b_c \rfloor \leq \delta \leq \lceil \boldsymbol{A}x - b_c \rceil$, [16]
tolerance–localized	$\hat{\triangle}\|x\| - \delta \leq \hat{A}_cx - \hat{b}_c \leq -\|\delta - \triangle\|x\|\|\hat{e}$	$\lfloor \mathbf{A}x - b_c \rfloor \leq \delta$
control–localized	$-\triangle\|x\| - \hat{\delta} \leq \hat{A}_cx - \hat{b}_c \leq -\|\delta - \triangle\|x\|\|\hat{e}$	$\langle \mathbf{A}x - b_c \rangle \leq \delta \leq \lceil \mathbf{A}x - b_c \rceil$

Table 6.1: Characterizations of each type of solutions.

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APPENDIX

This appendix includes of the raw data and source codes which relate to the computational results of teaching preference rank of Problem (5.6) and Original problem.

APPENDIX A : Timetabling a course assignment source code

Figure 1: The collected data for formulating the model and the obtained results.

Figure 2: The CPLEX source code for course assignment problem with uncertainty.

APPENDIX B : Finding the preference ranks source code by Python

Figure 3: The python source code for finding the preference ranks.

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