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EIGENVALUES OF UNITARY CAYLEY GRAPH OF MATRIX ALGEBRAS
AND ITS SUBCONSTITUENTS

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A Dissertation Submitted in Partial Fulfillment of the Requirements
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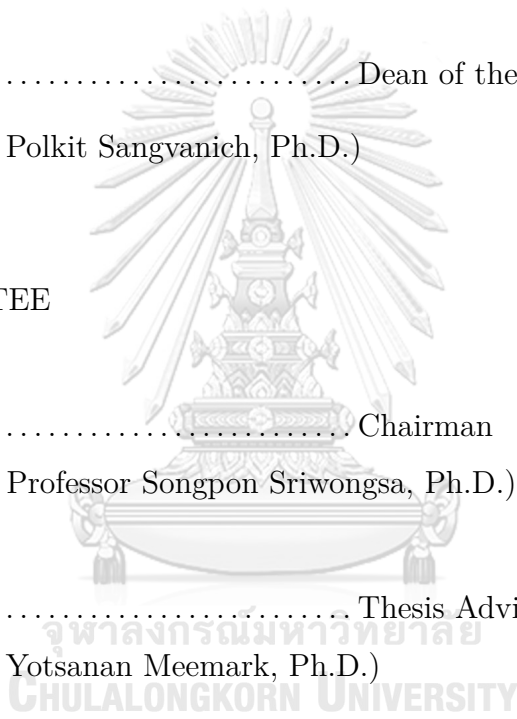
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จิตรสุพัฒน์ รัตนกั้วานวงศ์ : ค่าเฉพาะของกราฟเคย์เลย์ยูนิแทรีของพีชคณิตเมทริกซ์และส่วนประกอบย่อย (EIGENVALUES OF UNITARY CAYLEY GRAPH OF MATRIX ALGEBRAS AND ITS SUBCONSTITUENTS)

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สำหรับริงจำกัด R ที่มีเอกลักษณ์ กราฟเคย์เลย์ยูนิแทรีของริง R ซึ่งเขียนแทนด้วย C_R คือกราฟที่มีจุดยอดเป็นสมาชิกใน R และสำหรับ $x, y \in R$ จุดยอด x เชื่อมกับจุดยอด y ก็ต่อเมื่อ $x - y \in R^\times$ ให้ \mathbb{F}_q แทนฟิลด์จำกัดอันดับ q และ n เป็นจำนวนนับ ในวิทยานิพนธ์นี้ เราได้ผลลัพธ์บนกราฟเคย์เลย์ยูนิแทรีบนพีชคณิตเมทริกซ์ที่ลึกซึ่งยิ่งขึ้น เราใช้ค่าเฉพาะของกราฟ $C_{M_n(\mathbb{F}_q)}$ มาสร้างริงที่กำหนดโดยกราฟเคย์เลย์ยูนิแทรี นอกจากนี้ เราใช้ตารางคาเรคเตอร์ของ $GL_2(\mathbb{F}_q)$ มาศึกษาค่าเฉพาะของส่วนประกอบย่อยอันดับที่หนึ่งและอันดับที่สองของกราฟ $C_{M_2(\mathbb{F}_q)}$ เราได้ผลลัพธ์ของสมบัติไฮเพอร์เอเนอจีติกและสมบัติรามานูจัน และเรายังคำนวณจำนวนคลิก ริงคเลข และ จำนวนความอิสระ ของกราฟเหล่านี้ได้

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For a finite ring R with identity, the unitary Cayley graph, C_R , of R is the graph whose vertex set is R and for any $x, y \in R$, x is adjacent to y if and only if $x - y \in R^\times$. Let \mathbb{F}_q denote a finite field of order q and $n \in \mathbb{N}$. In this dissertation, we obtain deeper results on unitary Cayley graphs over matrix algebras. We use the eigenvalues of $C_{M_n(\mathbb{F}_q)}$ to construct new families of ring determined by unitary Cayley graphs. Moreover, we study the first and second subconstituents of $C_{M_n(\mathbb{F}_q)}$. We use the character table of $GL_2(\mathbb{F}_q)$ to determine the eigenvalues of subconstituents of $C_{M_2(\mathbb{F}_q)}$. We obtain their hyperenergeticity and Ramanujan property. We also compute the clique number, the chromatic number and the independence number of these subconstituents.



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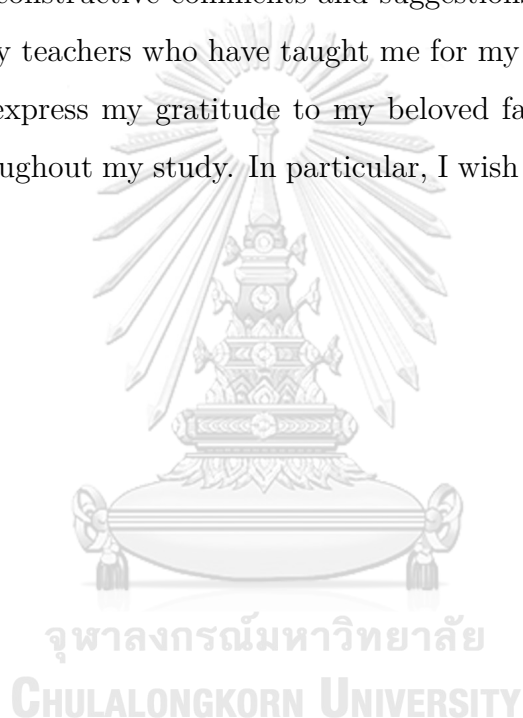
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CHAPTER I

UNITARY CAYLEY GRAPHS OF MATRIX ALGEBRAS

In this chapter, we provide background and results on the unitary Cayley graph of matrix algebras. Throughout, all rings have identity $1 \neq 0$.

1.1 Background on rings and graphs

Let R be a finite ring and R^\times denote the group of units of R . An ideal M of R is **maximal** if $M \neq R$ and for every ideal J of R , if $M \subseteq J \subseteq R$ then $J = M$ or $J = R$.

Theorem 1.1.1. ([20], Theorem 2.20) *If R is commutative and M is an ideal of R , then M is a maximal ideal of R if and only if R/M is a field.*

Next, a commutative ring R is a **local ring** if R has a unique maximal ideal. If R is a local ring with the maximal ideal M , then by Theorem 1.1.1, $\mathbb{k} := R/M$ is a field, called the **residue field**. In addition, if $u \in R^\times$ and $m \in M$, then $u + m \in R^\times$.

Example 1.1.2. 1. Every field is a local ring with the maximal ideal $\{0\}$.

2. Let p be a prime and $n \in \mathbb{N}$. Then \mathbb{Z}_{p^n} is a local ring with the maximal ideal $p\mathbb{Z}_{p^n}$ and the residue field $\mathbb{k} = \mathbb{Z}_{p^n}/p\mathbb{Z}_{p^n} \cong \mathbb{Z}_p$.

3. Since $p\mathbb{Z}$ is a maximal ideal of \mathbb{Z} for all primes p , we have \mathbb{Z} is not a local ring.

Theorem 1.1.3. [23] *Every finite commutative ring is a product of finite local rings.*

Example 1.1.4. Let $n \in \mathbb{N}$ and $n > 1$. Write $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ where p_1, \dots, p_k are distinct primes and $\alpha_1, \dots, \alpha_k \in \mathbb{N}$. Then

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \dots \times \mathbb{Z}_{p_k^{\alpha_k}}$$

where $\mathbb{Z}_{p_i^{\alpha_i}}$ is a finite local ring for all $i \in \{1, \dots, k\}$.

Let R be a finite commutative ring with identity. An R -**algebra** is a ring A such that $(A, +)$ is an R -module and $r(ab) = (ra)b = a(kb)$ for all $r \in R$ and $a, b \in A$. For $m, n \in \mathbb{N}$, let $M_n(R)$ be the algebra of $n \times n$ matrices over R . The group of all invertible matrices over R is denoted by $\text{GL}_n(R)$. Write I_n and $\mathbf{0}_{m \times n}$ for the $n \times n$ identity matrix and the $m \times n$ zero matrix, respectively. We write \mathbb{F}_q for the field of q elements, where q is a prime power.

Theorem 1.1.5. [21] $|\text{GL}_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q) \dots (q^n - q^{n-1})$.

A matrix in $M_n(\mathbb{F}_q)$ is a **linear derangement** if it is invertible and does not fix any nonzero vector. Let e_n be the number of linear derangements in $M_n(\mathbb{F}_q)$ and define $e_0 = 1$. According to [21], we obtain the recursion formula for e_n , namely, e_n satisfies the recursion

$$e_n = e_{n-1}(q^n - 1)q^{n-1} + (-1)^n q^{\frac{n(n-1)}{2}}.$$

Next, we provide some terminology and results from Graph Theory. For more details, see [2, 10]. Throughout this dissertation, our graphs are undirected and their vertex set are finite sets. Let G be a graph with n vertices and $V(G)$ denote the vertex set of G . For each $x \in V(G)$, the **degree of x** is the number of neighborhoods of x in G . The graph G is **k -regular** if every vertex has degree k . A k -regular graph G is **edge regular with parameters (n, k, λ)** if there exists a parameter λ such that for any two adjacent vertices, there are exactly λ vertices adjacent to both of them. If an edge regular graph with parameters (n, k, λ) also satisfies an additional property that for any two non-adjacent vertices, there are exactly μ vertices adjacent to both of them, then it is called a **strongly regular**

graph with parameters (n, k, λ, μ) .

Example 1.1.6. Let G be a graph shown in figure 1.1. Then G is an edge regular graph with parameters $(4, 2, 0)$ and is a strongly regular graph with parameters $(4, 2, 0, 1)$.



Figure 1.1

A **complete graph** is a graph such that any two distinct vertices are adjacent. A **clique** is a complete subgraph and the **clique number** of G is the maximum size of cliques in G , denoted by $\omega(G)$. A set I of vertices of G is called an **independent set** if no distinct vertices of I are adjacent. The **independence number** of G is the maximum size of independent sets, denoted by $\alpha(G)$. The **chromatic number** of G is the least number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. We write $\chi(G)$ for the chromatic number of G . Note that we must use at least $\omega(G)$ colors for coloring vertices of G and each color can be assigned to at most $\alpha(G)$ vertices, so we have

$$\chi(G) \geq \max \left\{ \omega(G), \frac{|V(G)|}{\alpha(G)} \right\}.$$

Example 1.1.7. Let G be a graph shown in figure 1.2. Note that G contains a 3-cycle. Then $\chi(G) \geq \omega(G) \geq 3$. Moreover, we can use precisely 3 colors to color each vertex of G , so $\chi(G) \leq 3$. It follows that $\omega(G) = \chi(G) = 3$. Next, it is easy to see that G has an independent set of size three. For each independent set I of

G, I must contain at most one vertex in the 3-cycle. Since there are two vertices outside from the 3-cycle, we have $\alpha(G) \leq 3$, so $\alpha(G) = 3$.

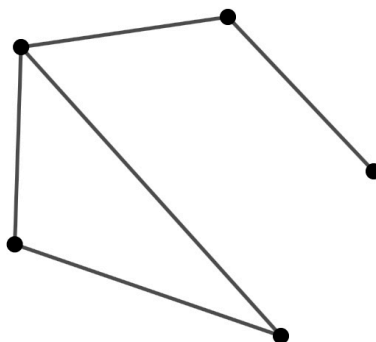


Figure 1.2

The **adjacency matrix** of G with vertex set $\{v_1, v_2, \dots, v_n\}$ is the $n \times n$ symmetric matrix A_G in which entry a_{jk} is the number of edges (0 or 1) in G with endpoints $\{v_j, v_k\}$ for all $j, k \in \{1, 2, \dots, n\}$. An **eigenvalue** of G is an eigenvalue of the adjacency matrix of G , and an **eigenvector** of G is an eigenvector of the adjacency matrix of G . The **spectrum of a matrix** is the list of its eigenvalues together with their multiplicities. The **spectrum of G** is the spectrum of its adjacency matrix. If $\lambda_1, \dots, \lambda_r$ are eigenvalues of a graph G with multiplicities m_1, \dots, m_r , respectively, we write

$$\text{Spec } G = \begin{pmatrix} \lambda_1 & \dots & \lambda_r \\ m_1 & \dots & m_r \end{pmatrix}$$

to describe the spectrum of G . Let G be a graph of n vertices and let $\lambda_1, \dots, \lambda_n$ be its eigenvalues. The **energy** of G is

$$E(G) = \sum_{i=1}^n |\lambda_i|.$$

A graph G is **hyperenergetic** if

$$E(G) > 2(n - 1).$$

We say that a graph G is **bipartite** if there is a partition V_1, V_2 of $V(G)$ such that any two vertices in V_i are not adjacent to each other for all $i \in \{1, 2\}$. It is well-known that if G is a connected k -regular graph, then k is an eigenvalue of G with the maximal modulus with multiplicity one [10]. Moreover, the eigenvalue $-k$ is an eigenvalue of G if and only if G is bipartite. A connected k -regular graph G is **Ramanujan** if

$$|\lambda| \leq 2\sqrt{k-1}$$

for any eigenvalue λ of G other than k and $-k$.

Example 1.1.8. Let G be a graph defined in Example 1.1.6. The adjacency matrix of G is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Its eigenvalues are $-2, 0, 0$ and 2 . Then

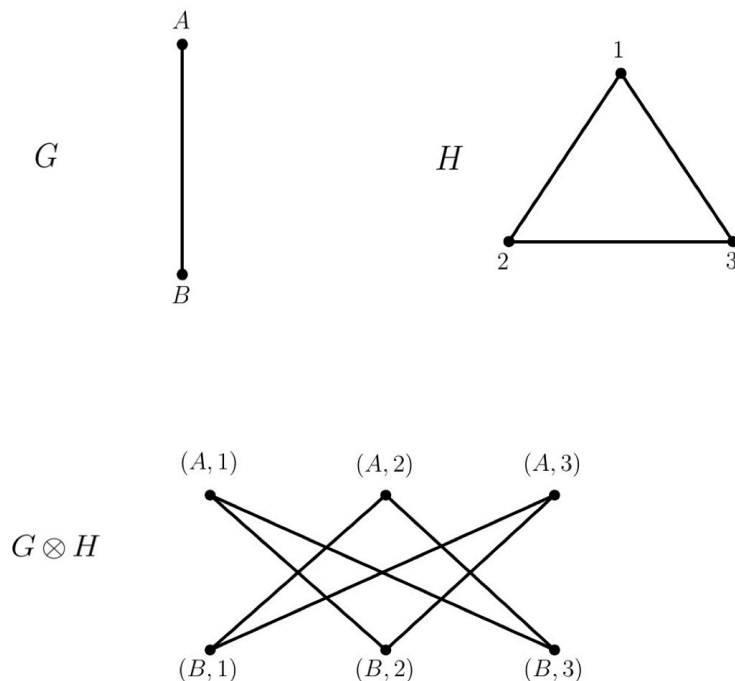
$$\text{Spec}(G) = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

Also, its energy is $E(G) = 2 + 2 = 4 < 2(4 - 1)$, so it is not hyperenergetic. Finally, G is 2-regular and 0 is the only eigenvalue of G other than ± 2 . Then G is Ramanujan.

Let G and H be undirected graphs. The **tensor product** graph $G \otimes H$ is the graph consisting of the vertex set $V(G) \times V(H)$ and the edge set

$$\{(x_1, y_1), (x_2, y_2)\} : x_1 \text{ is adjacent to } x_2 \text{ in } G \text{ and } y_1 \text{ is adjacent to } y_2 \text{ in } H\}.$$

Example 1.1.9. The following graphs show the tensor product of the graphs G and H .



Furthermore, the eigenvalues of the tensor product $G \otimes H$ can be determined from ones of G and H .

Theorem 1.1.10. [23] Let G and H be graphs on m vertices and n vertices, respectively. Assume that $\lambda_1, \dots, \lambda_m$ and μ_1, \dots, μ_n are eigenvalues of graphs G and H , respectively. Then the eigenvalues of $G \otimes H$ are $\lambda_i \mu_j$ for $i = \{1, \dots, m\}$ and $j = \{1, \dots, n\}$.

Example 1.1.11. According to Example 1.1.9, we have $\text{Spec}(G) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and

$$\text{Spec}(H) = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}. \text{ Then } \text{Spec}(G \otimes H) = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \end{pmatrix}.$$

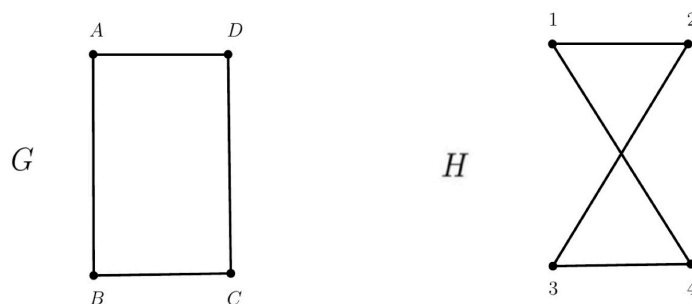
We also know that

Theorem 1.1.12. [15] For graphs G and H , $\chi(G \otimes H) \leq \min\{\chi(G), \chi(H)\}$.

Finally, we give a definition of an isomorphism of graphs. Let G and H be graphs. We say that G is **isomorphic** to H , denoted by $G \cong H$ if there is a bijection f from $V(G)$ onto $V(H)$ such that for any $x, y \in V(G)$, x is adjacent to

y in G if and only if $f(x)$ and $f(y)$ is adjacent in H . If G is isomorphic to H , then we just rename vertices of G to obtain vertices of H . Thus, two isomorphic graphs can be viewed as the same graphs but only their vertex sets are labeled differently.

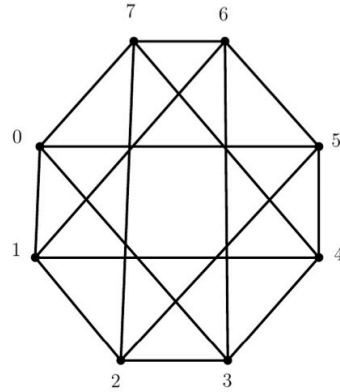
Example 1.1.13. Let G and H be graphs defined below. Define $f : \{A, B, C, D\} \rightarrow \{1, 2, 3, 4\}$ by $f(A) = 1$, $f(B) = 2$, $f(C) = 4$ and $f(D) = 3$. Then f is a graph isomorphism, so the graphs G and H are isomorphic.



1.2 Results on unitary Cayley graphs of matrix algebras

Let R be a finite ring. The **unitary Cayley graph of R** , denoted by C_R , is a graph with vertex set R and for each $x, y \in R$, x is adjacent to y if and only if $x - y \in R^\times$.

Example 1.2.1. We know that $\mathbb{Z}_8^\times = \{1, 3, 5, 7\}$, so the graph $C_{\mathbb{Z}_8}$ is regular of degree 4. Each vertex a is adjacent to $a + 1$, $a + 3$, $a + 5$ and $a + 7$. We display the graph $C_{\mathbb{Z}_8}$ below.



Note that C_R is an $|R^\times|$ -regular. If $R = R_1 \times \cdots \times R_k$ where R_1, \dots, R_k are finite rings with identity $1 \neq 0$, then $R^\times = R_1^\times \times \cdots \times R_k^\times$, so

$$C_R = C_{R_1} \otimes \cdots \otimes C_{R_k}.$$

Properties of the unitary Cayley graphs of finite rings are extensively studied in several papers (see [1, 4, 7, 12, 14, 15, 16, 17, 23]).

For commutative rings, in 2007, Klotz et al. [17] used properties of positive integers to study the graph $C_{\mathbb{Z}_n}$ where $n \in \mathbb{N}$. Klotz determined the clique number, chromatic number, the independence number, diameter and vertex connectivity of the graph. Akhtar et al. [1] generalized Klotz's results by working on C_R where R is a finite commutative ring by decomposing the ring R to a product of local rings. They also obtained an automorphism group of C_R . Next, Ilić et al. [12] computed the energy of $C_{\mathbb{Z}_n}$ where $n \in \mathbb{N}$ and characterized all positive integers n such that the graph $C_{\mathbb{Z}_n}$ is hyperenergetic. In 2011, Kiani et al. [14] showed that if R is a finite local ring with maximal ideal M of size m , then C_R is a complete multipartite graph such that each partite set is a coset of M . This gives the eigenvalues and energy of C_R , namely

$$\text{Spec}(C_R) = \left(\begin{array}{ccc} |R^\times| & -m & 0 \\ 1 & \frac{|R^\times|}{m} & \frac{|R|}{m}(m-1) \end{array} \right) \text{ and } E(C_R) = 2|R^\times|.$$

Now, let $R = R_1 \times \cdots \times R_k$ where R_i is a finite local ring with maximal ideal M_i of size m_i for all $i \in \{1, \dots, k\}$. By the above decomposition, all eigenvalues of C_R are $(-1)^{|C|} \frac{|R^\times|}{\prod_{j \in C} |R_j^\times|/m_j}$ with multiplicity $\prod_{j \in C} |R_j^\times|/m_j$ where C runs over subsets of $\{1, \dots, k\}$, 0 with multiplicity $|R| - \prod_{j=1}^k \left(1 + \frac{|R_j^\times|}{m_j}\right)$, and $E(C_R) = 2^k |R^\times|$. Moreover, assume that $|R_1|/m_1 \leq \cdots \leq |R_k|/m_k$. Kiani showed that C_R is hyperenergetic if and only if R satisfies one of the following properties:

- (a) $s = 2$, $|R_1|/m_1 \geq 3$ and $|R_2|/m_2 \geq 4$,
- (b) $s \geq 3$ with $(|R_{s-2}|/m_{s-2} \geq 3)$ or $(|R_{s-1}|/m_{s-1} \geq 3$ and $|R_s|/m_s \geq 4)$.

Hence, the unitary Cayley graphs of finite commutative rings are well-studied.

For non-commutative rings, in 2012, Kiani et al. [15] worked on the unitary Cayley graph of the ring $M_{n_1}(\mathbb{F}_{q_1}) \times \cdots \times M_{n_k}(\mathbb{F}_{q_k})$ where $n_1, \dots, n_k \in \mathbb{N}$. They computed the clique number, the chromatic number and the independence number of the graph. For $k = 1$, their main tools were a subfield K and a right ideal J of $M_n(\mathbb{F}_q)$ recalled in the next theorem.

Theorem 1.2.2. [15] *We have*

- (a) *The ring $M_n(\mathbb{F}_q)$ contains a subfield K of size q^n ,*
- (b) *Let J be the set of matrices in $M_n(\mathbb{F}_q)$ whose the entries of the first row are all zeros. Then J is a right ideal of $M_n(\mathbb{F}_q)$.*

Using the subfield K and the ideal J mentioned above, Kiani obtained the clique number, the chromatic number and the independence number of $C_{M_n(\mathbb{F}_q)}$ recorded in the next theorem. We include his proof in this theorem because we shall use similar idea to determine these parameters for the subconstituents of $C_{M_n(\mathbb{F}_q)}$ defined in Chapter 3. The subfield K and the ideal J mentioned in Theorem 1.2.2 also play an important role in our proofs. Kiani extended the result to $C_{M_{n_1}(\mathbb{F}_{q_1}) \times \cdots \times M_{n_k}(\mathbb{F}_{q_k})}$ by Theorem 1.1.12.

Theorem 1.2.3. [15] *We have*

- (a) $\omega(C_{M_n(\mathbb{F}_q)}) = \chi(C_{M_n(\mathbb{F}_q)}) = q^n$,

$$(b) \alpha(C_{M_n(\mathbb{F}_q)}) = q^{n^2-n}.$$

Proof. For (a), since the subfield K forms a clique of size q^n , we have

$$q^n \leq \omega(C_{M_n(\mathbb{F}_q)}) \leq \chi(C_{M_n(\mathbb{F}_q)}).$$

From $|M_n(\mathbb{F}_q)/J| = q^n$ and each coset of J is an independent set, it follows that $\chi(C_{M_n(\mathbb{F}_q)}) \leq q^n$

Next, we prove (b). Since each coset of J form an independent set, we have $\alpha(C_{M_n(\mathbb{F}_q)}) \geq q^{n^2-n}$. Note that K is a subgroup of $M_n(\mathbb{F}_q)$ under addition. Moreover, each coset of K is a clique in $C_{M_n(\mathbb{F}_q)}$, so

$$\alpha(C_{M_n(\mathbb{F}_q)}) \leq |M_n(\mathbb{F}_q)/K| = q^{n^2-n}.$$

This completes the proof. □

Later, in 2015, Kiani et al. [16] studied the regularity of $C_{M_n(\mathbb{F}_q)}$ where $n \geq 2$. It is clear that $C_{M_n(\mathbb{F}_q)}$ is $|\text{GL}_n(\mathbb{F}_q)|$ -regular. Furthermore, they constructed a bijection between sets of common neighborhoods of any adjacent vertices to prove that the graph $C_{M_n(\mathbb{F}_q)}$ is edge regular with parameter $(q^{n^2}, |\text{GL}_n(\mathbb{F}_q)|, e_n)$. In addition, they also proved that $C_{M_2(\mathbb{F}_q)}$ is strongly regular with parameters

$$(q^4, (q^2 - 1)(q^2 - q), q^4 - 2q^3 - q^2 + 3q, q^4 - 2q^3 + q),$$

but the graph $C_{M_3(\mathbb{F}_q)}$ is not strongly regular. Finally, they calculated the diameter of the graph $C_{M_n(\mathbb{F}_q)}$.

Theorem 1.2.4. [16] *The graph $C_{M_n(\mathbb{F}_q)}$ has diameter 2.*

In 2020, we [23] extended Kiani's results by proving that the graph $C_{M_n(\mathbb{F}_q)}$ is strongly regular if and only if $n = 2$. Their idea is to see $(M_n(\mathbb{F}_q), +)$ as $(\mathbb{F}_q^{n^2}, +)$. Since all characters of the group $(\mathbb{F}_q, +)$ is given by $\chi_a(x) = e^{\frac{2\pi i}{p} \text{Tr}(ax)}$ for all $a, x \in \mathbb{F}_q$ where Tr is the trace map and p is a characteristic of \mathbb{F}_q , the eigenvalues

of $C_{M_n}(\mathbb{F}_q)$ are of the form ([22] Theorem 2)

$$\rho_A = \sum_{S \in \text{GL}_n(\mathbb{F}_q)} \chi_A(S),$$

where $A = [a_{ij}]_{n \times n} \in M_n(\mathbb{F}_q)$ and $\chi_A = \prod_{1 \leq i, j \leq n} \chi_{a_{ij}}$. We calculated eigenvalues from the following three matrices:

$$A_1 = \mathbf{0}_{n \times n}, A_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Using some combinatorial methods on $\text{GL}_n(\mathbb{F}_q)$, it follows that

$$\begin{aligned} \rho_{A_1} &= (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}), \\ \rho_{A_2} &= -(q^n - q) \cdots (q^n - q^{n-1}), \text{ and} \\ \rho_{A_3} &= q(q^n - q^2) \cdots (q^n - q^{n-1}). \end{aligned}$$

In addition, it can be showed that if A and B are $n \times n$ matrices of the same rank, then $\rho_A = \rho_B$. This means that eigenvalues ρ_{A_1}, ρ_{A_2} and ρ_{A_3} are induced from matrices of rank 0,1 and 2, respectively. Using these eigenvalues, we can determine a strong regularity, hyperenergeticity and Ramanujan property of $C_{M_n}(\mathbb{F}_q)$.

Furthermore, we worked on the ring of matrices over finite local rings. Let R be a finite local ring with unique maximal ideal M and the residue field \mathbb{k} . Then $M_n(R)/M_n(M) \cong M_n(\mathbb{k})$. This gives a decomposition

$$C_{M_n(R)} = C_{M_n(\mathbb{k})} \otimes \mathring{M}_n(M)$$

where $\mathring{M}_n(M)$ is the complete graph on $|M_n(M)|$ vertices with a loop on any vertex. Since $\text{Spec}(\mathring{M}_n(M)) = \begin{pmatrix} m^{n^2} & 0 \\ 1 & m^{n^2} - 1 \end{pmatrix}$, eigenvalues of $C_{M_n(R)}$ can be determined from the ones of $C_{M_n(\mathbb{k})}$. Moreover, this decomposition can be used to

determine strong regularity, hyperenergeticity and Ramanujan property of $C_{M_n(R)}$.

Before we continue to discuss the next result, we introduce the Bruhat's decomposition of $GL_n(\mathbb{F}_q)$. Let S_n denote the set of permutations on $\{1, \dots, n\}$. For each $\sigma \in S_n$, define a σ -reduced matrix, denote by $W_\sigma = [W_{ij}]_{n \times n}$, is the matrix such that

- (a) $W_{j\sigma(j)} = 1$ for all $1 \leq j \leq n$,
- (b) $W_{jr} = 0$ if $r > \sigma(j)$,
- (c) $W_{r\sigma(j)} = 0$ if $r > j$.

From the definition, note that a σ -reduced matrix is not uniquely determined because there is no condition on some entries of the matrix W_σ .

Example 1.2.5. Let $\sigma \in S_3$ defined by $\sigma(1) = 2, \sigma(2) = 3$ and $\sigma(3) = 1$. Then the possible σ -reduced matrices have the form

$$W_\sigma = \begin{bmatrix} a & 1 & 0 \\ 0 & b & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

where $a, b \in \mathbb{F}_q$.

For $A \in GL_n(\mathbb{F}_q)$, we have an uppertriangular matrix L and $\sigma \in S_n$ such that $A = LW_\sigma$ ([25] p. 94). This decomposition is called the **Bruhat's decomposition** of $GL_n(\mathbb{F}_q)$.

Chen et al. [4] obtained all eigenvalues of $C_{M_n(\mathbb{F}_q)}$ using enumerative combinatorics. For each $k \in \{1, \dots, n\}$, they computed an eigenvalue ρ_A where

$$A = \begin{bmatrix} I_k & \mathbf{0}_{k \times (n-k)} \\ \mathbf{0}_{(n-k) \times k} & \mathbf{0}_{(n-k) \times (n-k)} \end{bmatrix}. \text{ Let}$$

$$G_k = \{S = [s_{ij}]_{n \times n} \in GL_n(\mathbb{F}_q) \mid s_{11} + \dots + s_{kk} \in \ker \text{Tr}\}.$$

Assume that \mathbb{F}_q has characteristic $p > 0$. It follows that

$$\rho_A = |G_k| - \frac{|\mathrm{GL}_n(\mathbb{F}_q)| - |G_k|}{p-1}.$$

To compute $|G_k|$, note that for each $S \in G_k$, $S = LW_\sigma$ for some L an upper triangular matrix and $\sigma \in S_n$ by the Bruhat's decomposition. They determined the number of possible matrices L and W_σ by counting the elements of

$$\{(x_1, \dots, x_m) \in (\mathbb{F}_q^\times)^m \mid x_1 + \dots + x_m \in \ker \mathrm{Tr}\}.$$

They found that

$$|G_k| = \frac{1}{p} \left[|\mathrm{GL}_n(\mathbb{F}_q)| + (-1)^k (p-1) q^{\frac{1}{2}k(2n-k-1)} |\mathrm{GL}_{n-k}(\mathbb{F}_q)| \right],$$

so

$$\rho_A = (-1)^k q^{\frac{1}{2}k(2n-k-1)} |\mathrm{GL}_{n-k}(\mathbb{F}_q)| = (-1)^k q^{\frac{k(k-1)}{2}} (q^n - q^{k+1}) \dots (q^n - q^{n-1}).$$

where $A = \begin{bmatrix} \mathbf{I}_k & \mathbf{0}_{k \times (n-k)} \\ \mathbf{0}_{(n-k) \times k} & \mathbf{0}_{(n-k) \times (n-k)} \end{bmatrix}$ for all $k \in \{0, \dots, n\}$. Since any two matrices of the same rank produce the same eigenvalues, They obtained all eigenvalues of $C_{M_n(\mathbb{F}_q)}$.

Theorem 1.2.6. [4] *The eigenvalues of the graph $C_{M_n(\mathbb{F}_q)}$ are*

(a) $\lambda_0(n, q) = (q^n - 1) \dots (q^n - q^{n-1})$ with multiplicity one,

(b) $\lambda_k(n, q) = (-1)^k q^{\frac{k(k-1)}{2}} (q^n - q^k) \dots (q^n - q^{n-1})$ with multiplicity $\frac{((q^n - 1) \dots (q^n - q^{k-1}))^2}{(q^k - 1) \dots (q^k - q^{k-1})}$ for all $k \in \{1, \dots, n-1\}$, and

(c) $\lambda_n(n, q) = (-1)^n q^{\frac{n(n-1)}{2}}$ with multiplicity $(q^n - 1) \dots (q^n - q^{n-1})$.

Results on the eigenvalues of $C_{M_n(\mathbb{F}_q)}$ have been extended in [19]. Huang et al. [19] defined the Cayley graph $\mathcal{G}(m, n, r)$, where $m, n \in \mathbb{N}$ and $0 \leq r \leq \min\{m, n\}$, to be the graph whose vertex set is the set of $m \times n$ matrices over the field \mathbb{F}_q and

two matrices A and B are adjacent if and only if $\text{rank}(A - B) = r$. Note that $\mathcal{G}(n, n, n)$ is indeed the unitary Cayley graph of matrix algebra. They followed the idea of Chen's paper and computed the eigenvalues of the graph $\mathcal{G}(m, n, r)$.

1.3 Objectives

In this dissertation, we examine the eigenvalues of $M_n(\mathbb{F}_q)$ and construct new families of non-commutative DU-ring, which is defined later in chapter II. Moreover, from Theorem 1.2.4, the graph $C_{M_n(\mathbb{F}_q)}$ has diameter two. This leads us to define the first and second subconstituents of $C_{M_n(\mathbb{F}_q)}$ and inspires us to study their spectral properties. We determine the eigenvalues of the subconstituents of $C_{M_2(\mathbb{F}_q)}$ and analyze their hyperenergeticity and Ramanujan property, and computing the clique numbers, the chromatic numbers and the independence numbers of the subconstituents.

The second chapter consists of two sections. We define a DU-ring and review Kiani's work in the first section. In the next section, we compare the modulus of each eigenvalue of $C_{M_n(\mathbb{F}_q)}$ obtained in [4] to show that the following rings are DU-rings.

- (a) $M_{n_1}(\mathbb{F}_{q_1}) \times \cdots \times M_{n_k}(\mathbb{F}_{q_k})$ where q_1, \dots, q_k are pairwise relatively prime.
- (b) $M_{n_1}(\mathbb{F}_{q_1}) \times M_{n_2}(\mathbb{F}_{q_2})$.
- (c) $M_{m_1}(\mathbb{F}_{p_1^{s_1}}) \times M_{n_1}(\mathbb{F}_{p_1^{t_1}}) \times \cdots \times M_{m_k}(\mathbb{F}_{p_k^{s_k}}) \times M_{n_k}(\mathbb{F}_{p_k^{t_k}})$ where p_1, \dots, p_k are distinct primes and $s_1, \dots, s_k, t_1, \dots, t_k \in \mathbb{N}$.

We divide the third chapter into five sections. We give the definition of subconstituents of a graph and describe the subconstituents $C_R^{(1)}$ and $C_R^{(2)}$ of unitary Cayley graph of a finite ring R in the first section. The second section contains terminologies of Cayley graphs of finite groups and their associated sets, and Representation Theory used in this work. In the third section, we find the associated set of the graph $C_{M_n(\mathbb{F}_q)}^{(1)}$ and prove that this set is a union of conjugacy classes. We determine all eigenvalues of the graph $C_{M_2(\mathbb{F}_q)}^{(1)}$ by using the character table of

$GL_2(\mathbb{F}_q)$. We show further that it is hyperenergetic and Ramanujan if $q \geq 3$. In the next section, we prove that the graph $C_{M_2(\mathbb{F}_q)}^{(2)}$ is the tensor product between a complete graph and a complete multipartite graph and obtain its spectrum. We apply this result to conclude that $C_{M_2(\mathbb{F}_q)}^{(2)}$ is hyperenergetic but it is not Ramanujan if $q \geq 5$. Finally, we compute the clique numbers, chromatic numbers and the independence numbers of the subconstituents of the graph $C_{M_2(\mathbb{F}_q)}$ in the final section. This chapter is a joint work with Y. Meemark. The paper has been published in *Finite Fields and Their Applications* [24]. Finally, we conclude the results we obtained in this dissertation in the fourth chapter.



CHAPTER II

RING DETERMINED BY UNITARY CAYLEY GRAPHS

2.1 Kiani's conjecture and DU-rings

Let R be a finite ring. The **Jacobson radical** of a ring R is the intersection of all maximal ideals of R . It is denoted by J_R . We say that the ring R is **semisimple** if $J_R = \{0\}$.

- Example 2.1.1.**
1. If R is a finite local ring with unique maximal ideal M . then $J_R = M$. In particular, \mathbb{F}_q is a semisimple ring. On the other hand, \mathbb{Z}_p^n is not semisimple for all primes p and $n \geq 2$.
 2. For a finite ring R , we have by [8] that $J_{M_n(R)} = M_n(J_R)$. Hence, the matrix algebra $M_n(\mathbb{F}_q)$ is semisimple for all $n \in \mathbb{N}$.

By Wedderburn-Artin theorem [8], any finite semisimple ring is of the form $M_{n_1}(\mathbb{F}_{q_1}) \times \cdots \times M_{n_k}(\mathbb{F}_{q_k})$. Moreover, it is easy to see that R/J_R is always semisimple. Hence, the ring R/J_R can be expressed to the product of matrix algebras. Kiani et al. proved the following results.

Proposition 2.1.2. [15] *Let R and S be finite rings such that $C_R \cong C_S$. Then*

- (a) $C_{R/J_R} \cong C_{S/J_S}$,
- (b) $|J_R| = |J_S|$, and
- (c) if R is semisimple, so is S .

Proposition 2.1.3. [15]

- (a) If $m, n \in \mathbb{N}$, and $C_{M_n(\mathbb{F}_{q_1})} \cong C_{M_m(\mathbb{F}_{q_2})}$, then $q_1 = q_2$ and $m = n$.

(b) If S is a finite ring and $C_S \cong C_{M_n(\mathbb{F}_q)}$, then $S \cong M_n(\mathbb{F}_q)$.

For finite rings R and S such that $C_R \cong C_S$, it is not necessary that $R \cong S$. For example, we have $C_{\mathbb{Z}_4} \cong C_{\mathbb{Z}_2[x]/(x^2)}$ but \mathbb{Z}_4 and $\mathbb{Z}_2[x]/(x^2)$ are not isomorphic because they have different characteristics. However, we can see that $\mathbb{Z}_2 \cong \mathbb{Z}_4/J_{\mathbb{Z}_4} \cong \mathbb{Z}_2[x]/(x^2)/J_{\mathbb{Z}_2[x]/(x^2)}$. Kiani also conjectured that for any rings R and S , if $C_R \cong C_S$, then $R/J_R \cong S/J_S$. They showed that the conjecture holds for the class of finite commutative rings by examining eigenvalues of these two graphs. Furthermore, by Proposition 2.1.2, it suffices to verify this conjecture for the class of finite semisimple rings. This leads to study rings determined by unitary Cayley graphs (DU-rings). A ring R is a **DU-ring** if for any ring S such that $C_R \cong C_S$, we have $R \cong S$. It follows from Proposition 2.1.3 that $M_n(\mathbb{F}_q)$ is a DU-ring. Note that Kiani's conjecture is equivalent to saying that every semisimple ring is a DU-ring. In the next section, we provide an application of eigenvalues of $M_n(\mathbb{F}_q)$ to determine new families of such rings, which makes Kiani's conjecture closer to be true.

2.2 Constructions of DU-rings

In this section, we use the eigenvalues of the unitary Cayley graph of matrix algebras to discover new families of non-commutative DU-ring.

According to Theorem 1.2.6, we note that

$$\left| \frac{\lambda_{k-1}(n, q)}{\lambda_k(n, q)} \right| = q^{n-k+1} - 1 \geq 1$$

for all $k \in \{1, \dots, n\}$, so we have

$$|\lambda_0(n, q)| > |\lambda_1(n, q)| > \dots > |\lambda_{n-1}(n, q)| \geq |\lambda_n(n, q)|. \quad (2.1)$$

Next, we prove the following lemmas.

Lemma 2.2.1. *Let $\mathcal{F} = M_{n_1}(\mathbb{F}_{q_1}) \times \dots \times M_{n_k}(\mathbb{F}_{q_k})$ and $\mathcal{E} = M_{m_1}(\mathbb{F}_{r_1}) \times \dots \times M_{m_l}(\mathbb{F}_{r_l})$. If the graphs $C_{\mathcal{F}}$ and $C_{\mathcal{E}}$ are isomorphic, then $q_1^{n_1} \dots q_k^{n_k} = r_1^{m_1} \dots r_l^{m_l}$.*

Proof. From the above discussion, the least modulus of the eigenvalues of $C_{\mathcal{F}}$ is

$$|\lambda_{n_1}(n_1, q_1)| \cdots |\lambda_{n_k}(n_k, q_k)| = q_1^{\frac{n_1(n_1-1)}{2}} \cdots q_k^{\frac{n_k(n_k-1)}{2}},$$

and the least modulus of the eigenvalues of $C_{\mathcal{E}}$ is

$$|\lambda_{m_1}(m_1, r_1)| \cdots |\lambda_{m_l}(m_l, r_l)| = r_1^{\frac{m_1(m_1-1)}{2}} \cdots r_l^{\frac{m_l(m_l-1)}{2}}.$$

Since the graphs are isomorphic, we can conclude that

$$q_1^{\frac{n_1(n_1-1)}{2}} \cdots q_k^{\frac{n_k(n_k-1)}{2}} = r_1^{\frac{m_1(m_1-1)}{2}} \cdots r_l^{\frac{m_l(m_l-1)}{2}}.$$

Also, the number of their vertices are equal, so

$$q_1^{n_1} \cdots q_k^{n_k} = r_1^{m_1} \cdots r_l^{m_l}.$$

It follows that $q_1^{n_1} \cdots q_k^{n_k} = r_1^{m_1} \cdots r_l^{m_l}$ as desired. \square

Lemma 2.2.2. *Let $\mathcal{F} = M_{n_1}(\mathbb{F}_{q_1}) \times \cdots \times M_{n_k}(\mathbb{F}_{q_k})$ and $\mathcal{E} = M_{m_1}(\mathbb{F}_{r_1}) \times \cdots \times M_{m_l}(\mathbb{F}_{r_l})$. Assume that the graphs $C_{\mathcal{F}}$ and $C_{\mathcal{E}}$ are isomorphic. The following statements hold.*

- (a) *If $q_1^{n_1} \leq \cdots \leq q_k^{n_k}$ and $r_1^{m_1} \leq \cdots \leq r_l^{m_l}$, then $k = l$ and $q_i^{n_i} = r_i^{m_i}$ for all $i \in \{1, \dots, k\}$.*
- (b) *There is a permutation σ of $\{1, \dots, k\}$ such that $q_i = r_{\sigma(i)}$ for all $i \in \{1, \dots, k\}$.*

Proof. First, we prove (a). By Lemma 2.2.1, we have

$$q_1^{n_1} \cdots q_k^{n_k} = r_1^{m_1} \cdots r_l^{m_l}. \quad (2.2)$$

Note that

$$\lambda_1(n_i, q_i) = (q_i^{n_i} - q_i) \cdots (q_i^{n_i} - q_i^{n_i-1}) = \frac{|\mathrm{GL}_{n_i}(\mathbb{F}_{q_i})|}{q_i^{n_i} - 1}$$

for all $i \in \{1, \dots, k\}$. Set

$$Q = \prod_{i=1}^k |\mathrm{GL}_{n_i}(\mathbb{F}_{q_i})| \quad \text{and} \quad R = \prod_{i=1}^l |\mathrm{GL}_{m_i}(\mathbb{F}_{r_i})|.$$

Since $C_{\mathcal{F}}$ and $C_{\mathcal{E}}$ are isomorphic and they are regular of degrees Q and R , respectively, we have $Q = R$. The k eigenvalues of $C_{\mathcal{F}}$ of the largest modulus smaller than Q are

$$\frac{1}{q_k^{n_k} - 1} Q \leq \dots \leq \frac{1}{q_1^{n_1} - 1} Q,$$

and the l eigenvalues of $C_{\mathcal{E}}$ of the largest modulus inferior to R are

$$\frac{1}{r_l^{m_l} - 1} R \leq \dots \leq \frac{1}{r_1^{m_1} - 1} R.$$

Without loss of generality, we assume that $k \leq l$. Thus,

$$\frac{1}{q_i^{n_i} - 1} Q = \frac{1}{r_i^{m_i} - 1} R$$

for all $i \in \{1, \dots, k\}$, which implies that $q^{n_i} = r^{m_i}$ for all $i \in \{1, \dots, k\}$. Applying this result to (2.2) gives $r_{k+1}^{m_{k+1}} \dots r_l^{m_l} = 1$ and it forces that $k = l$.

To prove (b), we recall that

$$|\lambda_{n_i}(n_i, q_i)| = q_i^{\frac{n_i(n_i-1)}{2}} \quad \text{and} \quad |\lambda_{n-1}(n_i, q_i)| = q_i^{\frac{(n_i-1)(n_i-2)}{2}} (q_i^{n_i} - q_i^{n_i-1}) = q_i^{\frac{(n_i)(n_i-1)}{2}} (q_i - 1)$$

for all $i \in \{1, \dots, k\}$. Set

$$M = \prod_{i=1}^k q_i^{\frac{n_i(n_i-1)}{2}} \quad \text{and} \quad N = \prod_{i=1}^k r_i^{\frac{m_i(m_i-1)}{2}}.$$

Thus, M and N come from the eigenvalue of least modulus of $C_{\mathcal{F}}$ and $C_{\mathcal{E}}$, respectively, so $M = N$. The k eigenvalues of $C_{\mathcal{F}}$ of least modulus larger than M are $(q_1 - 1)M, \dots, (q_k - 1)M$, and the k eigenvalues of $C_{\mathcal{E}}$ of least modulus larger than N are $(r_1 - 1)N, \dots, (r_k - 1)N$. This induces a permutation σ of $\{1, \dots, k\}$ such that $(q_i - 1)M = (r_{\sigma(i)} - 1)N$, so $q_i = r_{\sigma(i)}$ for all $i \in \{1, \dots, k\}$. \square

The first main result is presented in the following theorem.

Theorem 2.2.3. *Let $\mathcal{F} = M_{n_1}(\mathbb{F}_{q_1}) \times \cdots \times M_{n_k}(\mathbb{F}_{q_k})$ and $\mathcal{E} = M_{m_1}(\mathbb{F}_{r_1}) \times \cdots \times M_{m_l}(\mathbb{F}_{r_l})$. If the graphs $C_{\mathcal{F}}$ and $C_{\mathcal{E}}$ are isomorphic and q_1, \dots, q_k are pairwise relatively prime, then \mathcal{F} and \mathcal{E} are isomorphic. Consequently, if q_1, \dots, q_k are pairwise relatively prime, then $M_{n_1}(\mathbb{F}_{q_1}) \times \cdots \times M_{n_k}(\mathbb{F}_{q_k})$ is a DU-ring.*

Proof. First, we assume that $q_1^{n_1} \leq \cdots \leq q_k^{n_k}$ and $r_1^{m_1} \leq \cdots \leq r_l^{m_l}$. We can conclude from Lemma 2.2.2 (a) that $k = l$ and $q_i^{n_i} = r_i^{m_i}$ for all $i \in \{1, \dots, k\}$. For any $i \in \{1, \dots, k\}$, we write $q_i = p_i^{s_i}$ and $r_i = p_i^{t_i}$ for some $s_1, \dots, s_k, t_1, \dots, t_k \in \mathbb{N}$ and p_1, \dots, p_k are distinct prime numbers since q_1, \dots, q_k are pairwise relatively prime. It follows that

$$s_i n_i = t_i m_i \quad (2.3)$$

for all $i \in \{1, \dots, k\}$. Also, $C_{\mathcal{F}}$ and $C_{\mathcal{E}}$ have the same number of vertices, so

$$q_1^{n_1^2} \cdots q_k^{n_k^2} = r_1^{m_1^2} \cdots r_l^{m_l^2}.$$

Since q_1, \dots, q_k are pairwise relatively prime, we have $q_i^{n_i^2} = r_i^{m_i^2}$ for all $i \in \{1, \dots, k\}$. This implies that

$$s_i n_i^2 = t_i m_i^2 \quad (2.4)$$

for all $i \in \{1, \dots, k\}$. By (2.3) and (2.4), we can conclude that $s_i = t_i$ and $n_i = m_i$ for all $i \in \{1, \dots, k\}$ and hence \mathcal{F} and \mathcal{E} are isomorphic. \square

Next, we prove the second main result.

Theorem 2.2.4. *Let $\mathcal{F} = M_{n_1}(\mathbb{F}_{q_1}) \times M_{n_2}(\mathbb{F}_{q_2})$ and $\mathcal{E} = M_{m_1}(\mathbb{F}_{r_1}) \times M_{m_2}(\mathbb{F}_{r_2})$. If $C_{\mathcal{F}}$ and $C_{\mathcal{E}}$ are isomorphic, then \mathcal{F} and \mathcal{E} are isomorphic. Consequently, $M_{n_1}(\mathbb{F}_{q_1}) \times M_{n_2}(\mathbb{F}_{q_2})$ is a DU-ring.*

Proof. Assume that $q_1^{n_1} \leq q_2^{n_2}$ and $r_1^{m_1} \leq r_2^{m_2}$. By Lemma 2.2.2 (a), we get $q_1^{n_1} = r_1^{m_1}$ and $q_2^{n_2} = r_2^{m_2}$. Write $q_i = p_i^{s_i}$ and $r_i = p_i^{t_i}$ for $i \in \{1, 2\}$ where p_1, p_2 are

primes and $s_1, s_2, t_1, t_2 \in \mathbb{N}$. It follows that

$$s_1 n_1 = t_1 m_1, \quad (2.5)$$

and

$$s_2 n_2 = t_2 m_2. \quad (2.6)$$

If $p_1 \neq p_2$, then we are done by Theorem 2.2.3. Suppose further that $p_1 = p_2$. Since $C_{\mathcal{F}}$ and $C_{\mathcal{E}}$ have the same number of vertices, we have

$$s_1 n_1^2 + s_2 n_2^2 = t_1 m_1^2 + t_2 m_2^2. \quad (2.7)$$

By Lemma 2.2.2 (b), we have $\{q_1, q_2\} = \{r_1, r_2\}$. If $q_1 = r_1$ and $q_2 = r_2$, then $s_1 = t_1$ and $s_2 = t_2$, and so $n_1 = m_1$ and $n_2 = m_2$. Now, we assume that $q_1 = r_2$ and $q_2 = r_1$. Thus, $s_1 = t_2$ and $s_2 = t_1$. By (2.5) and (2.6),

$$n_1 n_2 = m_1 m_2.$$

Moreover, plugging $s_2 m_1 = s_1 n_1$ and $s_2 n_2 = s_1 m_2$ into (2.7) and dividing both sides by s_1 give

$$n_1^2 + m_2 n_2 = m_1 n_1 + m_2^2. \quad (2.8)$$

Similarly, we have

$$n_2^2 + m_1 n_1 = m_2 n_2 + m_1^2. \quad (2.9)$$

We can conclude from (2.8) and (2.9) that $n_1^2 + n_2^2 = m_1^2 + m_2^2$. Since $n_1 n_2 = m_1 m_2$, it follows that $n_1 + n_2 = m_1 + m_2$. Hence, the sets $\{n_1, n_2\}$ and $\{m_1, m_2\}$ are the set of solutions of the equation $x^2 - (m_1 + m_2)x + m_1 m_2 = 0$. Then $\{n_1, n_2\} = \{m_1, m_2\}$.

We distinguish two cases.

Case 1. $n_1 = m_1$ and $n_2 = m_2$. By (2.5) and (2.6), we have $s_1 = t_1$ and $s_2 = t_2$, so \mathcal{F} and \mathcal{E} are isomorphic.

Case 2. $n_1 = m_2$ and $n_2 = m_1$. We know that $s_1 = t_2$ and $s_2 = t_1$. It follows that \mathcal{F} and \mathcal{E} are isomorphic. \square

Remark 2.2.5. Unfortunately, for $k \geq 3$, the above arguments do not follow easily. For example, when $k = 3$ under the same set-up, we work on $\mathcal{F} = M_{n_1}(\mathbb{F}_{p_1^{s_1}}) \times M_{n_2}(\mathbb{F}_{p_2^{s_2}}) \times M_{n_3}(\mathbb{F}_{p_3^{s_3}})$ and $\mathcal{E} = M_{n_1}(\mathbb{F}_{p_1^{t_1}}) \times M_{n_2}(\mathbb{F}_{p_2^{t_2}}) \times M_{n_3}(\mathbb{F}_{p_3^{t_3}})$ with $p_1^{s_1 n_1} \leq p_2^{s_2 n_2} \leq p_3^{s_3 n_3}$ and $p_1^{t_1 m_1} \leq p_2^{t_2 m_2} \leq p_3^{t_3 m_3}$. By Lemma 2.2.2 (a), we have $p_i^{s_i n_i} = p_i^{m_i t_i}$ for all $i \in \{1, 2, 3\}$. Then

$$s_1 n_1 = t_1 m_1, s_2 n_2 = t_2 m_2, \text{ and } s_3 n_3 = t_3 m_3. \quad (2.10)$$

In addition, by Lemma 2.2.2 (b), we have $\{p_1^{s_1}, p_2^{s_2}, p_3^{s_3}\} = \{p_1^{t_1}, p_2^{t_2}, p_3^{t_3}\}$. If $p_1 = p_2 = p_3$, then $\{s_1, s_2, s_3\} = \{t_1, t_2, t_3\}$. Assume further that $s_1 = t_2$, $s_2 = t_3$ and $s_3 = t_1$. The system (2.10) becomes

$$s_1 n_1 = s_3 m_1, s_2 n_2 = s_1 m_2, \text{ and } s_3 n_3 = s_2 m_3. \quad (2.11)$$

Again, since $C_{\mathcal{F}}$ and $C_{\mathcal{E}}$ have the same number of vertices, we get

$$s_1 n_1^2 + s_2 n_2^2 + s_3 n_3^2 = s_3 m_1^2 + s_1 m_2^2 + s_2 m_3^2. \quad (2.12)$$

However, it seems difficult to derive from only (2.11) and (2.12) to reach the isomorphism of \mathcal{F} and \mathcal{E} as we have done in the proof of Theorem 2.2.4.

Finally, we let $\mathcal{F} = M_{m_1}(\mathbb{F}_{p_1^{s_1}}) \times M_{n_1}(\mathbb{F}_{p_1^{t_1}}) \times \cdots \times M_{m_k}(\mathbb{F}_{p_k^{s_k}}) \times M_{n_k}(\mathbb{F}_{p_k^{t_k}})$ and $\mathcal{E} = M_{u_1}(\mathbb{F}_{q_1}) \times M_{v_1}(\mathbb{F}_{r_1}) \times \cdots \times M_{u_k}(\mathbb{F}_{q_k}) \times M_{v_k}(\mathbb{F}_{r_k})$, where p_1, \dots, p_k are distinct primes and s_i 's, t_i 's are positive integers. We may assume that $p_i^{s_i m_i} \leq p_i^{t_i n_i}$ for all $i \in \{1, \dots, k\}$. Suppose that the graphs $C_{\mathcal{F}}$ and $C_{\mathcal{E}}$ are isomorphic. Let $i \in \{1, \dots, k\}$. According to the Lemma 2.2.2 (a), we may write $q_i = p_i^{a_i}$ and $r_i = p_i^{b_i}$, and $p_i^{s_i m_i} = p_i^{a_i u_i}$ and $p_i^{t_i n_i} = p_i^{b_i v_i}$, so we have $s_i m_i = a_i u_i$ and $t_i n_i = b_i v_i$. Since p_1, \dots, p_k are distinct primes, comparing the number of vertices of the graphs gives $s_i m_i^2 + t_i n_i^2 = a_i u_i^2 + b_i v_i^2$. In addition, Lemma 2.2.2 (b) implies $\{s_i, t_i\} = \{u_i, v_i\}$. Thus, we have the same system of equations as in the proof of Theorem 2.2.4. It follows that $M_{m_i}(\mathbb{F}_{p_i^{s_i}}) \times M_{n_i}(\mathbb{F}_{p_i^{t_i}})$ and $M_{u_i}(\mathbb{F}_{q_i}) \times M_{v_i}(\mathbb{F}_{r_i})$ are isomorphic for all $i \in \{1, \dots, k\}$. Hence, \mathcal{F} and \mathcal{E} are isomorphic. Therefore, we have proved our

final result.

Theorem 2.2.6. *Let $\mathcal{F} = M_{m_1}(\mathbb{F}_{p_1^{s_1}}) \times M_{n_1}(\mathbb{F}_{p_1^{t_1}}) \times \cdots \times M_{m_k}(\mathbb{F}_{p_k^{s_k}}) \times M_{n_k}(\mathbb{F}_{p_k^{t_k}})$ where p_1, \dots, p_k are distinct primes and s_i 's and t_i 's are positive integers. Then \mathcal{F} is a DU-ring.*



CHAPTER III
SUBCONSTITUENTS OF UNITARY CAYLEY
GRAPHS OF MATRICES OVER FINITE FIELDS

3.1 Subconstituents of graphs

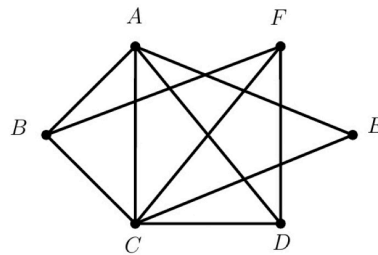
Let $n \in \mathbb{N}$. First, we provide a definition of subconstituents of a graph. Let G be a graph. A subgraph X of G is an **induced subgraph** if $V(X) \subseteq V(G)$ and for any $x, y \in V(X)$, x is adjacent to y in X if and only if they are adjacent in G . Note that the induced subgraph X is obtained by removing some vertices of G together with edges containing a removed vertex.

Now, let G be a graph with diameter 2 and $x \in V(G)$. Let $N(x)$ denote the set of neighbors of x in G . We have

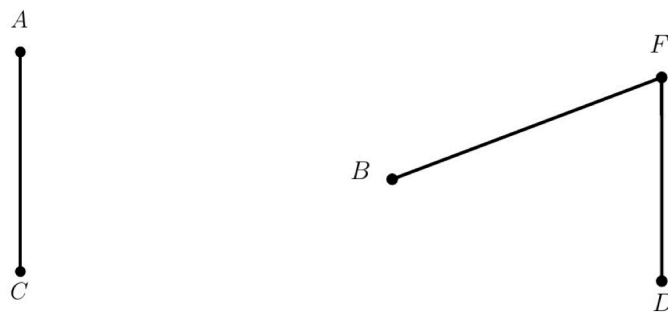
$$V(G) = \{x\} \cup N(x) \cup (V(G) \setminus (N(x) \cup \{x\})).$$

Since G has diameter 2, the set $V(G) \setminus (N(x) \cup \{x\})$ is the set of non-adjacent vertices to x except x itself. This leads us to define the first and the second subconstituents of G . The **first subconstituent of G at x** is the subgraph of G induced by the set $N(x)$ and the **second subconstituent of G at x** is the subgraph of G induced by the set $V(G) \setminus (N(x) \cup \{x\})$.

Example 3.1.1. Let G be the following graph.



Consider the vertex E , we have the set of neighborhoods of E is $\{A, C\}$, so we obtain subconstituents of G at $x = E$. From the graphs below, the left-hand side displays the first subconstituents of G at $x = E$ and the right-hand side displays the second subconstituent of G at $x = E$.



Subconstituents of strongly regular graphs are studied in many graphs and have many interesting properties. The second subconstituent of the Hoffman-Singleton graph is determined by its spectrum in [6]. Moreover, the discovery of which graph has strongly regular subconstituents interests mathematicians. For example, Cameron et al. [3] used the Bose-Mesner algebra of a strongly regular graph to classify strongly regular graphs whose subconstituents are strongly regular, and Kasikova [13] used the same tools to classify distance-regular graph which has strongly regular subconstituents. In addition, we can use eigenvalues of subconstituents to prove the uniqueness of strongly regular of some parameter, e.g., Clebsch graph is a unique strongly regular graph with parameters $(16, 5, 0, 2)$ (see [10] p.230).

According to Theorem 1.2.4, the graph $C_{M_n(\mathbb{F}_q)}$ has diameter 2, it motivates us to study subconstituents of unitary Cayley graph of matrix algebras. Let R be a finite ring. For $x \in R$, the maps $f : N(0) \rightarrow N(x)$ and $g : R \setminus (N(0) \cup \{0\}) \rightarrow R \setminus (N(x) \cup \{x\})$ which both send y to $x - y$ are graph isomorphisms. Hence, we may only study the subconstituents at $x = 0$ and we write $C_R^{(1)}$ and $C_R^{(2)}$ for the first subconstituent and the second subconstituent of C_R at $x = 0 \in R$, respectively.

3.2 Eigenvalues of normal Cayley graphs and a character table of the group $GL_2(\mathbb{F}_q)$

Let G be a finite group and V a finite-dimensional complex vector space. A **representation** of G on V is a homomorphism $\rho : G \rightarrow GL(V)$ where $GL(V)$ denotes the group of automorphisms on V . Let ρ be a representation of G on V . Then for each $g \in G$, $\rho(g)$ is a linear transformation on V . A subspace W of V is **ρ -invariant** under G if $\rho(g)(W) \subseteq W$ for all $g \in G$. If ρ has no proper invariant subspace of V , then we say that ρ is an **irreducible representation**.

Example 3.2.1. 1. Let G be a group and V a vector space. Define $\rho : G \rightarrow GL(V)$ by $\rho(g) = 1_V$ where 1_V is the identity map on V . Then ρ is a representation of G on V .

2. Let $z \in \mathbb{C}$. We know that $(\mathbb{R}, +)$ is an additive group. Define $\rho : \mathbb{R} \rightarrow GL(\mathbb{C})$ by

$$(\rho(x))(w) = e^{zx}w$$

for all $x \in \mathbb{R}$ and $w \in \mathbb{C}$. Then ρ is a representation of $(\mathbb{R}, +)$.

Example 3.2.2. 1. Every representation $\rho : G \rightarrow GL(V)$ with $\dim V = 1$ is always irreducible.

2. Define $\rho : \mathbb{R} \rightarrow GL(\mathbb{R}^2)$ by rotations of \mathbb{R}^2 , that is,

$$\rho(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

for all $x \in \mathbb{R}$. Since there is no one-dimensional subspace is mapped to itself by all rotations, it follows that ρ is irreducible.

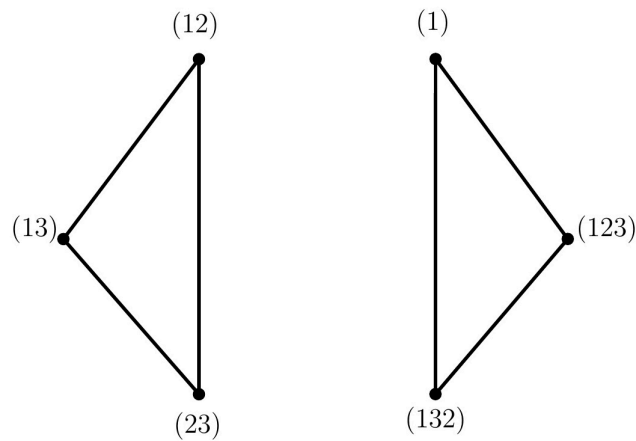
Next, we define a character of a representation. A **character** χ corresponding to ρ is the complex-valued function on G defined by $\chi(g) = \text{tr}(\rho(g))$ for all $g \in G$ where $\text{tr}(\rho(g))$ is the trace of the matrix representation of $\rho(g)$ on V . A character is said to be **irreducible** if they are induced from an irreducible representation. The **dimension** of a character is the dimension of vector space V . It is easy to see that $\chi(1) = \dim V$ where 1 is the identity of the group G , and $\chi(ghg^{-1}) = \text{tr}(\rho(g)\rho(h)\rho(g^{-1})) = \text{tr}(\rho(g)\rho(g^{-1})\rho(h)) = \chi(h)$ for all $g, h \in G$. Thus, a character is a constant on a conjugacy class of G .

Now, we focus on characters of the group \mathbb{F}_q^\times . Readers can see [18] for more details. If $\mathbb{F}_q^\times = \langle a \rangle$ for some $a \in \mathbb{F}_q^\times$, then the irreducible characters of the group $(\mathbb{F}_q^\times, \cdot)$ are $\chi_k(x) = e^{\frac{2\pi imk}{q-1}}$ for all $x = a^m \in \mathbb{F}_q^\times$ and $k \in \{0, 1, 2, \dots, q-2\}$. Write $\mathbb{F}_q^\times = \langle a \rangle$ where $a \in \mathbb{F}_q^\times$. We have for $k \in \{0, 1, \dots, q-2\}$,

$$\sum_{x \in \mathbb{F}_q^\times} \chi_k(x) = \begin{cases} q-1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let G be a finite group and S be a subset of G not containing the identity and $S = S^{-1}$ where $S^{-1} = \{s^{-1} : s \in S\}$. The **Cayley graph of G associated to S** is the undirected graph $\text{Cay}(G, S)$ whose vertex set is G and for each $g, h \in G$, g is adjacent to h if and only if $g = hs$ for some $s \in S$. We say that a Cayley graph is **normal** if S is a union of conjugacy classes of G .

Example 3.2.3. Let $G = S_3$ and $S = \{(123), (132)\}$. Since S is a conjugacy class of S_3 , the Cayley graph $\text{Cay}(G, S)$ is a normal Cayley graph. The graph $\text{Cay}(G, S)$ is shown below.



Eigenvalues of a normal Cayley graph can be determined by using the next theorem.

Theorem 3.2.4. [26]

Let G be a finite group and S be a subset of G not containing the identity and $S = S^{-1}$ where $S^{-1} = \{s^{-1} : s \in S\}$. If S is a union of conjugacy classes of G and χ_1, \dots, χ_r are irreducible characters of G , then the eigenvalues of $\text{Cay}(G, S)$ are

$$\lambda_j = \frac{1}{\chi_j(1)} \sum_{s \in S} \chi_j(s)$$

with multiplicity $m_j = \sum_{\substack{k=1 \\ \lambda_k = \lambda_j}}^r \chi_k(1)^2$ for all $j \in \{1, \dots, r\}$.

Now, we focus on the group $\text{GL}_2(\mathbb{F}_q)$. The conjugacy classes of $\text{GL}_2(\mathbb{F}_q)$ are given in the following table. The readers can see [9] for more details.

Representatives	Number of elements	Number of classes
$a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, x \neq 0$	1	$q - 1$
$b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, x \neq 0$	$q^2 - 1$	$q - 1$
$c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, x \neq y \text{ and } x, y \neq 0$	$q^2 + q$	$\frac{(q-1)(q-2)}{2}$
$d_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}, y \neq 0 \text{ (} q \text{ is odd)}$	$q^2 - q$	$\frac{q(q-1)}{2}$
$d_z = \begin{pmatrix} 0 & z^{q+1} \\ 1 & z + z^q \end{pmatrix}, z \in E \setminus \mathbb{F}_q \text{ (} q \text{ is even)}$		

Here, $c_{x,y}$ and $c_{y,x}$ are conjugate, $d_{x,y}$ and $d_{x,-y}$ are conjugate, and d_z and d_{z^q} are conjugate. Moreover, let $E = \mathbb{F}_q[\sqrt{\varepsilon}]$ be an extension of \mathbb{F}_q of degree two. We can identify the matrices $d_{x,y}$ as $\zeta = x + y\sqrt{\varepsilon}$ and the matrices d_z as z in $E \setminus \mathbb{F}_q$. Now, let α, β be distinct irreducible character of \mathbb{F}_q^\times and φ an irreducible characters of E^\times such that $\varphi^q \neq \varphi$ and φ is not an irreducible character of \mathbb{F}_q^\times . The next table presents all irreducible characters of $\text{GL}_2(\mathbb{F}_q)$. As mentioned earlier, it suffices to specify their values on each conjugacy class of $\text{GL}_2(\mathbb{F}_q)$.

Theorem 3.2.5. [9] *The character table of $\text{GL}_2(\mathbb{F}_q)$ is presented by the following table.*

	$a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	$c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	$d_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} = \zeta$ <small>q is odd</small>	$d_z = \begin{pmatrix} 0 & z^{q+1} \\ 1 & z + z^q \end{pmatrix} = z$ <small>q is even</small>
U_α	$\alpha(x^2)$	$\alpha(x^2)$	$\alpha(xy)$	$\alpha(\zeta^q)$	$\alpha(z^q)$
V_α	$q\alpha(x^2)$	0	$\alpha(xy)$	$-\alpha(\zeta^q)$	$-\alpha(z^q)$
$W_{\alpha,\beta}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y) + \alpha(y)\beta(x)$	0	0
X_φ	$(q-1)\varphi(x)$	$-\varphi(x)$	0	$-(\varphi(\zeta) + \varphi(\zeta^q))$	$-(\varphi(z) + \varphi(z^q))$

Moreover, $U_\alpha, V_\alpha, W_{\alpha,\beta}$ and X_φ are of dimension 1, $q, q+1$ and $q-1$, respectively.

3.3 Spectral properties of $C_{M_2(\mathbb{F}_q)}^{(1)}$

In this section, we study spectral properties of $C_{M_2(\mathbb{F}_q)}^{(1)}$. We start by showing that $C_{M_n(\mathbb{F}_q)}^{(1)}$ is $\text{Cay}(\text{GL}_n(\mathbb{F}_q), (\text{I}_n + \text{GL}_n(\mathbb{F}_q)) \cap \text{GL}_n(\mathbb{F}_q))$. To see this, let $A, B \in \text{GL}_n(\mathbb{F}_q)$. Then $AB^{-1} \in \text{GL}_n(\mathbb{F}_q)$ and

$$\begin{aligned} A - B \in \text{GL}_n(\mathbb{F}_q) &\iff (AB^{-1} - \text{I}_n)B \in \text{GL}_n(\mathbb{F}_q) \\ &\iff (AB^{-1} - \text{I}_n) \in \text{GL}_n(\mathbb{F}_q) \\ &\iff AB^{-1} \in (\text{I}_n + \text{GL}_n(\mathbb{F}_q)) \cap \text{GL}_n(\mathbb{F}_q). \end{aligned}$$

It also follows that the graph $C_{M_n(\mathbb{F}_q)}^{(1)}$ is regular of degree $|(\text{I}_n + \text{GL}_n(\mathbb{F}_q)) \cap \text{GL}_n(\mathbb{F}_q)| = e_n$, defined in the Chapter 1. Moreover, for $A, B \in \text{GL}_n(\mathbb{F}_q)$, we have

$$\begin{aligned} ABA^{-1} \in (\text{I}_n + \text{GL}_n(\mathbb{F}_q)) \cap \text{GL}_n(\mathbb{F}_q) &\iff ABA^{-1} - \text{I}_n \in \text{GL}_n(\mathbb{F}_q) \\ &\iff A(B - \text{I}_n)A^{-1} \in \text{GL}_n(\mathbb{F}_q) \\ &\iff (B - \text{I}_n) \in \text{GL}_n(\mathbb{F}_q) \\ &\iff B \in (\text{I}_n + \text{GL}_n(\mathbb{F}_q)) \cap \text{GL}_n(\mathbb{F}_q). \end{aligned}$$

Thus, $(\text{I}_n + \text{GL}_n(\mathbb{F}_q)) \cap \text{GL}_n(\mathbb{F}_q)$ is a union of conjugacy classes, so $C_{M_n(\mathbb{F}_q)}^{(1)}$ is a normal Cayley graph. We record this result in

Theorem 3.3.1. *The graph $C_{M_n(\mathbb{F}_q)}^{(1)}$ is the normal Cayley graph of $\text{GL}_n(\mathbb{F}_q)$ associated with $(\text{I}_n + \text{GL}_n(\mathbb{F}_q)) \cap \text{GL}_n(\mathbb{F}_q)$ and it is regular of degree e_n .*

Next, we determine all eigenvalues of $C_{M_2(\mathbb{F}_q)}^{(1)}$. Let $k \in \{0, 1, \dots, q-2\}$ and consider χ_k an irreducible character of \mathbb{F}_q^\times . We first handle the case q is odd by showing some lemmas on sums of characters of \mathbb{F}_q^\times .

Lemma 3.3.2. *If q is odd, then for $k \in \{0, 1, \dots, q-2\}$,*

$$\sum_{x \in \mathbb{F}_q^\times} \chi_k(x^2) = \begin{cases} q-1 & \text{if } k \in \left\{0, \frac{q-1}{2}\right\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We know that

$$\sum_{x \in \mathbb{F}_q^\times} \chi_k(x^2) = \sum_{m=0}^{q-2} \chi_k(a^{2m}) = \sum_{m=0}^{q-2} e^{\frac{4\pi i m k}{q-1}} = \sum_{m=0}^{q-2} \left(e^{\frac{4\pi i k}{q-1}} \right)^m.$$

Note that $e^{\frac{4\pi i k}{q-1}} = 1$ if and only if $k = 0$ or $k = \frac{q-1}{2}$. If $k \in \left\{ 0, \frac{q-1}{2} \right\}$, then

$\sum_{x \in \mathbb{F}_q^\times} \chi_k(x^2) = q-1$. Finally, if $k \notin \left\{ 0, \frac{q-1}{2} \right\}$, then

$$\sum_{x \in \mathbb{F}_q^\times} \chi_k(x^2) = \frac{1 - \left(e^{\frac{4\pi i k}{q-1}} \right)^{q-1}}{1 - \left(e^{\frac{4\pi i k}{q-1}} \right)} = 0,$$

and the proof is completed. \square

Lemma 3.3.3. *If q is odd, then for $k \in \{0, 1, \dots, q-2\}$ and $\varepsilon \in \mathbb{F}_q \setminus \mathbb{F}_q^2$, we have*

$$(a) \quad \sum_{\substack{x, y \in \mathbb{F}_q^\times \setminus \{1\} \\ \text{and } x \neq y}} \chi_k(xy) = \begin{cases} q^2 - 5q + 6 & \text{if } k = 0, \\ -q + 3 & \text{if } k = \frac{q-1}{2}, \text{ and} \\ 2 & \text{otherwise,} \end{cases}$$

$$(b) \quad \sum_{(x, y) \in \mathbb{F}_q \times \mathbb{F}_q^\times} \chi_k(x^2 - \varepsilon y^2) = \begin{cases} q^2 - q & \text{if } k = 0, \\ -q + 1 & \text{if } k = \frac{q-1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We note that

$$\begin{aligned} \sum_{\substack{x, y \in \mathbb{F}_q^\times \setminus \{1\} \\ \text{and } x \neq y}} \chi_k(xy) &= \left(\sum_{x \in \mathbb{F}_q^\times} \chi_k(x) \right) \left(\sum_{y \in \mathbb{F}_q^\times} \chi_k(y) \right) - \sum_{x \in \mathbb{F}_q^\times} \chi_k(x^2) \\ &\quad - \sum_{x \in \mathbb{F}_q^\times \setminus \{1\}} \chi_k(x) - \sum_{y \in \mathbb{F}_q^\times \setminus \{1\}} \chi_k(y) \\ &= \left(\sum_{x \in \mathbb{F}_q^\times} \chi_k(x) \right)^2 - \left(\sum_{x \in \mathbb{F}_q^\times} \chi_k(x^2) \right) - 2 \left(\sum_{x \in \mathbb{F}_q^\times} \chi_k(x) \right) + 2. \end{aligned}$$

If $k = 0$, then applying Lemma 3.3.2 gives the right-hand side equals $q^2 - 5q + 6$. If $k = \frac{q-1}{2}$, then the right-hand side is $-q + 3$. Finally, if $k \notin \left\{0, \frac{q-1}{2}\right\}$, then the summands on the right-hand side are all gone and we get 2 left. This proves (a).

For (b), since ε is not a square in \mathbb{F}_q , $E = \mathbb{F}_q[\sqrt{\varepsilon}]$ is an extension of degree two of \mathbb{F}_q . Thus, $E = \{x + y\sqrt{\varepsilon} : x, y \in \mathbb{F}_q\}$. Moreover, let N_{E/\mathbb{F}_q} be the norm map. Recall that for $x, y \in \mathbb{F}_q$, $N_{E/\mathbb{F}_q}(x + y\sqrt{\varepsilon}) = x^2 - \varepsilon y^2$ and by Hilbert's Theorem 90, N_{E/\mathbb{F}_q} is surjective with kernel of size $q + 1$. Consider the sum

$$\begin{aligned}
\sum_{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q^\times} \chi_k(x^2 - \varepsilon y^2) &= \sum_{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q \setminus \{(0,0)\}} \chi_k(x^2 - \varepsilon y^2) - \sum_{x \in \mathbb{F}_q^\times} \chi_k(x^2) \\
&= \sum_{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q \setminus \{(0,0)\}} \chi_k(N_{E/\mathbb{F}_q}(x + y\sqrt{\varepsilon})) - \sum_{x \in \mathbb{F}_q^\times} \chi_k(x^2) \\
&= |\ker N_{E/\mathbb{F}_q}| \sum_{x \in \mathbb{F}_q^\times} \chi_k(x) - \sum_{x \in \mathbb{F}_q^\times} \chi_k(x^2) \\
&= (q+1) \sum_{x \in \mathbb{F}_q^\times} \chi_k(x) - \sum_{x \in \mathbb{F}_q^\times} \chi_k(x^2).
\end{aligned}$$

If $k = 0$, then the right-hand side becomes $q^2 - q$, and if $k = \frac{q-1}{2}$, then the right-hand side is $-(q-1)$ by Lemma 3.3.2. Finally, for $k \notin \left\{0, \frac{q-1}{2}\right\}$, it also follows that each summand on the right-hand side is 0. \square

Lemma 3.3.4. For $k, l \in \{0, 1, \dots, q-2\}$ such that $k \neq l$, we have

$$\begin{aligned}
(a) \quad \sum_{x \in \mathbb{F}_q^\times} \chi_k(x) \chi_l(x) &= \begin{cases} q-1 & \text{if } k+l = q-1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \\
(b) \quad \sum_{\substack{x, y \in \mathbb{F}_q^\times \setminus \{1\} \\ \text{and } x \neq y}} [\chi_k(x) \chi_l(y) + \chi_k(y) \chi_l(x)] &= \begin{cases} 4 & \text{if } 0 < k+l < q-1, k, l \neq 0, \\ 2(3-q) & \text{otherwise.} \end{cases}
\end{aligned}$$

Proof. Let χ_k and χ_l be distinct irreducible characters of \mathbb{F}_q^\times . Then we have

$$\sum_{x \in \mathbb{F}_q^\times} \chi_k(x) \chi_l(x) = \sum_{m=0}^{q-2} e^{\frac{2\pi i(k+l)m}{q-1}}.$$

Similar to the proof of Lemma 3.3.2, we can conclude that

$$\sum_{x \in \mathbb{F}_q^\times} \chi_k(x) \chi_l(x) = \begin{cases} q-1 & \text{if } k+l \in \{0, q-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Since k, l are distinct, $k+l \neq 0$, and so we have (a).

For (b), we consider the sum

$$\begin{aligned} & \sum_{\substack{x, y \in \mathbb{F}_q^\times \setminus \{1\} \\ \text{and } x \neq y}} [\chi_k(x) \chi_l(y) + \chi_k(y) \chi_l(x)] = 2 \sum_{\substack{x, y \in \mathbb{F}_q^\times \setminus \{1\} \\ \text{and } x \neq y}} \chi_k(x) \chi_l(y) \\ & = 2 \left[\left(\sum_{x \in \mathbb{F}_q^\times} \chi_k(x) \right) \left(\sum_{y \in \mathbb{F}_q^\times} \chi_l(y) \right) - \sum_{x \in \mathbb{F}_q^\times} \chi_k(x) \chi_l(x) - \sum_{x \in \mathbb{F}_q^\times \setminus \{1\}} \chi_k(x) - \sum_{y \in \mathbb{F}_q^\times \setminus \{1\}} \chi_l(y) \right]. \end{aligned}$$

Since $k \neq l, k+l \neq 0$. If $k+l = q-1$, then $k, l \neq 0$ because $0 \leq k, l \leq q-2$. Part (a) gives

$$\sum_{\substack{x, y \in \mathbb{F}_q^\times \setminus \{1\} \\ \text{and } x \neq y}} [\chi_k(x) \chi_l(y) + \chi_k(y) \chi_l(x)] = 2(-(q-1) + 2) = 2(3-q).$$

Assume that $k+l \neq q-1$. We distinguish two cases.

Case 1. $k=0$ or $l=0$, say $k=0$. Then $l \neq 0$ and so

$$\sum_{\substack{x, y \in \mathbb{F}_q^\times \setminus \{1\} \\ \text{and } x \neq y}} [\chi_k(x) \chi_l(y) + \chi_k(y) \chi_l(x)] = 2(-(q-1) + 2) = 2(3-q).$$

Case 2. $k, l \neq 0$. Then we conclude that

$$\sum_{\substack{x, y \in \mathbb{F}_q^\times \setminus \{1\} \\ \text{and } x \neq y}} [\chi_k(x) \chi_l(y) + \chi_k(y) \chi_l(x)] = 2.$$

This completes the proof. \square

Remark 3.3.5. Assume that q is odd. Before computing the eigenvalues of $C_{M_2(\mathbb{F}_q)}^{(1)}$, we note that for each $x, y \in \mathbb{F}_q$,

- (a) $a_x \in \text{GL}_2(\mathbb{F}_q) \cap (\text{I}_2 + \text{GL}_2(\mathbb{F}_q))$ if and only if $x \neq 1$
- (b) $b_x \in \text{GL}_2(\mathbb{F}_q) \cap (\text{I}_2 + \text{GL}_2(\mathbb{F}_q))$ if and only if $x \neq 1$
- (c) $c_{x,y} \in \text{GL}_2(\mathbb{F}_q) \cap (\text{I}_2 + \text{GL}_2(\mathbb{F}_q))$ if and only if $x, y \neq 1$
- (d) $d_{x,y} \in \text{GL}_2(\mathbb{F}_q) \cap (\text{I}_2 + \text{GL}_2(\mathbb{F}_q))$ for all $x \in \mathbb{F}_q$ and $y \neq 0$.

To verify (d), we suppose that there exist $x \in \mathbb{F}_q$ and $y \in \mathbb{F}_q^\times$ such that

$$\det \begin{pmatrix} x-1 & \varepsilon y \\ y & x-1 \end{pmatrix} = 0,$$

so $(x-1)^2 - \varepsilon y^2 = 0$ in \mathbb{F}_q . Thus, $x + y\sqrt{\varepsilon} = 1$ in E . Since $\{1, \sqrt{\varepsilon}\}$ is an \mathbb{F}_q -basis of E , we have $y = 0$ which is absurd.

From the character table of $\text{GL}_2(\mathbb{F}_q)$ mentioned at the second section of this chapter, let λ_χ denote an eigenvalue induced from an irreducible character χ . Since the character U_{χ_k} has dimension one, the above remark gives

$$\begin{aligned} \lambda_{U_{\chi_k}} = & \sum_{x \in \mathbb{F}_q^\times \setminus \{1\}} \chi_k(x^2) + (q^2 - 1) \sum_{x \in \mathbb{F}_q^\times \setminus \{1\}} \chi_k(x^2) \\ & + \frac{q^2 + q}{2} \sum_{\substack{x, y \in \mathbb{F}_q^\times \setminus \{1\} \\ \text{and } x \neq y}} \chi_k(xy) + \frac{q^2 - q}{2} \sum_{(x, y) \in \mathbb{F}_q \times \mathbb{F}_q^\times} \chi_k(x^2 - \varepsilon y^2). \end{aligned}$$

According to Lemmas 3.3.2 and 3.3.3, we have $\lambda_{U_{\chi_0}} = q^4 - 2q^3 - q^2 + 3q$, $\lambda_{U_{\chi_{\frac{q-1}{2}}}} = q$ and

$$\lambda_{U_{\chi_k}} = (-1) + (q^2 - 1)(-1) + \frac{q^2 + q}{2}(1 + 1) = q$$

if $k \notin \left\{0, \frac{q-1}{2}\right\}$. It follows that the eigenvalues of $C_{M_2(\mathbb{F}_q)}^{(1)}$ obtained from U_{χ_k} are $q^4 - 2q^3 - q^2 + 3q$ and q with multiplicities 1 and $q - 2$, respectively.

Now, we work on V_{χ_k} . Since V_{χ_k} has dimension q , we have

$$\lambda_{V_{\chi_k}} = \frac{1}{q} \left(q \sum_{x \in \mathbb{F}_q^\times \setminus \{1\}} \chi_k(x^2) + \frac{q^2 + q}{2} \sum_{\substack{x, y \in \mathbb{F}_q^\times \setminus \{1\} \\ \text{and } x \neq y}} \chi_k(xy) - \frac{q^2 - q}{2} \sum_{(x, y) \in \mathbb{F}_q \times \mathbb{F}_q^\times} \chi_k(x^2 - \varepsilon y^2) \right).$$

Again, applying Lemmas 3.3.2 and 3.3.3 gives $\lambda_{V_{\chi_0}} = -q^2 + q + 1$, $\lambda_{V_{\chi_{\frac{q-1}{2}}}} = q$ and

$$\lambda_{V_{\chi_k}} = \frac{1}{q} \left(q(-1) + \frac{q^2 + q}{2}(1 + 1) \right) = q$$

if $k \notin \left\{ 0, \frac{q-1}{2} \right\}$. Thus, the eigenvalues of $C_{M_2(\mathbb{F}_q)}^{(1)}$ obtained from V_{χ_k} are $-q^2 + q + 1$ and q with multiplicities q^2 and $q^2 + q^2(q-3) = q^3 - 2q^2$, respectively.

Next, we consider the eigenvalues induced from the character W_{χ_k, χ_l} with $k \neq l$. Since W_{χ_k, χ_l} has dimension $q + 1$, we have

$$\lambda_{W_{\chi_k, \chi_l}} = \frac{1}{q+1} \left((q+1) \sum_{x \in \mathbb{F}_q^\times \setminus \{1\}} \chi_k(x) \chi_l(x) + (q^2 - 1) \sum_{x \in \mathbb{F}_q^\times \setminus \{1\}} \chi_k(x) \chi_l(x) + \frac{q^2 + q}{2} \sum_{\substack{x, y \in \mathbb{F}_q^\times \setminus \{1\} \\ \text{and } x \neq y}} (\chi_k(x) \chi_l(y) + \chi_k(y) \chi_l(x)) \right).$$

First, we assume that $k + l = q - 1$. Thus, $k, l \neq 0$. Note that there are $\frac{q-3}{2}$ choices of such k, l . It follows from Lemma 3.3.4 that

$$\lambda_{W_{\chi_k, \chi_l}} = \frac{1}{q+1} \left((q+1)(q-2) + (q^2-1)(q-2) + 2 \left(\frac{q^2+q}{2} \right) (3-q) \right) = q.$$

If $0 < k + l < q - 1$, then we have two cases to consider. If $k = 0$ or $l = 0$, then

there are $q - 2$ choices of k and l , and

$$\lambda_{W_{\chi_k, \chi_l}} = \frac{1}{q+1} \left((q+1)(-1) + (q^2-1)(-1) + 2 \left(\frac{q^2+q}{2} \right) (3-q) \right) = -q(q-2).$$

If $k, l \neq 0$, then there are $\frac{(q-3)^2}{2}$ choices of k and l , and

$$\lambda_{W_{\chi_k, \chi_l}} = \frac{1}{q+1} \left((q+1)(-1) + (q^2-1)(-1) + \left(\frac{q^2+q}{2} \right) (4) \right) = q.$$

Thus, the eigenvalues of $C_{M_2(\mathbb{F}_q)}^{(1)}$ obtained from W_{χ_k, χ_k} are $-q(q-2)$ and q with multiplicities $(q+1)^2(q-2)$ and $\frac{(q+1)^2(q-2)(q-3)}{2}$, respectively.

Finally, let φ be an irreducible character of E^\times such that $\varphi^q \neq \varphi$. Hence, φ is a non-trivial character and there are $\frac{q^2-q}{2}$ choices of φ . Since X_φ has dimension $q-1$, we have

$$\begin{aligned} \lambda_{X_\varphi} &= \frac{1}{q-1} \left((q-1) \sum_{x \in \mathbb{F}_q^\times \setminus \{1\}} \varphi(x) - (q^2-1) \sum_{x \in \mathbb{F}_q^\times \setminus \{1\}} \varphi(x) \right. \\ &\quad \left. - \frac{q^2-q}{2} \sum_{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q^\times} (\varphi(x+y\sqrt{\varepsilon}) + \varphi(x-y\sqrt{\varepsilon})) \right) \\ &= \frac{1}{q-1} \left(-(q^2-q) \sum_{x \in \mathbb{F}_q^\times} \varphi(x) + (q^2-q) - (q^2-q) \sum_{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q^\times} \varphi(x+y\sqrt{\varepsilon}) \right) \\ &= \frac{1}{q-1} \left(-(q^2-q) \sum_{x \in E^\times} \varphi(x) + (q^2-q) \right) = q. \end{aligned}$$

Hence, the eigenvalue from this case is q with multiplicity $\frac{(q-1)^2(q^2-q)}{2}$.

Summing all multiplicities of the eigenvalue q from each character gives its total multiplicity $q^4 - 2q^3 - 2q^2 + 4q + 1$. Therefore, we obtain the spectrum of $C_{M_2(\mathbb{F}_q)}^{(1)}$ in the case that q is odd. For q even and $q \geq 4$, we can find all eigenvalues corresponding to each U_χ, V_χ and X_φ in the similar manner without the case $k = \frac{q-1}{2}$. Note that the eigenvalue obtained from the case $k = \frac{q-1}{2}$ when q is odd is always

q . Hence, the eigenvalues corresponding to those characters of the case q is even and $q \geq 4$ are equal to the eigenvalues in the case q is odd. As for eigenvalues corresponding to W_{χ_k, χ_l} , we have multiplicities of q become $\frac{(q+1)^2(q-2)}{2}$ and $\frac{(q+1)^2(q-4)(q-2)}{2}$ whose sum is again $\frac{(q+1)^2(q-2)(q-3)}{2}$, so the multiplicities of q when q is even remains the same.

Finally, if $q = 2$, then the graph $C_{M_2(\mathbb{F}_q)}^{(1)}$ has $(2^2 - 1)(2^2 - 2) = 6$ vertices and is two copies of K_3 , so its spectra are 2 of multiplicity 2 and -1 of multiplicity 4. Thus, we completely determine the spectrum for the graph $C_{M_2(\mathbb{F}_q)}^{(1)}$.

Theorem 3.3.6. (a) If $q = 2$, then $\text{Spec } C_{M_2(\mathbb{F}_q)}^{(1)} = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$.

(b) If $q \geq 3$, then $\text{Spec } C_{M_2(\mathbb{F}_q)}^{(1)} =$

$$\begin{pmatrix} q^4 - 2q^3 - q^2 + 3q & q & -q^2 + q + 1 & q^3 - 2q^2 \\ 1 & q^4 - 2q^3 - 2q^2 + 4q + 1 & q^2 & (q+1)^2(q-2) \end{pmatrix}.$$

Moreover, $E(C_{M_2(\mathbb{F}_q)}^{(1)}) = 2q^5 - 2q^4 - 8q^3 + 6q^2 + 8q$ for all $q \geq 2$.

Furthermore, for all $q \geq 3$, we have

$$\begin{aligned} E(C_{M_2(\mathbb{F}_q)}^{(1)}) - 2((q^2 - 1)(q^2 - q) - 1) &= 2q^5 - 2q^4 - 8q^3 + 6q^2 + 8q - 2((q^2 - 1)(q^2 - q) - 1) \\ &= 2q^5 - 4q^4 - 6q^3 + 8q^2 + 6q + 2 > 2q^5 - 4q^4 - 6q^3 \\ &= 2q^3(q - 3)(q + 1) \geq 0. \end{aligned}$$

This proves hyperenergeticity of the graph $C_{M_2(\mathbb{F}_q)}^{(1)}$ when $q \geq 3$, while $C_{M_2(\mathbb{Z}_2)}^{(1)}$ is not hyperenergetic because its energy is $8 < 2(6 - 1)$.

Since $C_{M_2(\mathbb{Z}_2)}^{(1)}$ is disconnected, it is not Ramanujan. We show that the graph $C_{M_2(\mathbb{F}_q)}^{(1)}$ is Ramanujan for $q \geq 3$. Since $|-q^2 + q + 1| > |-q(q-2)| > q$, it suffices to show that $2\sqrt{(q^4 - 2q^3 - q^2 + 3q - 1)} \geq q^2 - q - 1$ which is equivalent

to $4(q^4 - 2q^3 - q^2 + 3q - 1) \geq (q^2 - q - 1)^2$, and we have

$$\begin{aligned} 4(q^4 - 2q^3 - q^2 + 3q - 1) - (q^2 - q - 1)^2 &= 3q^4 - 6q^3 - 3q^2 + 10q - 5 \\ &\geq 3q^4 - 6q^3 - 3q^2 = 3q^2((q - 1)^2 - 2) \geq 0. \end{aligned}$$

We record this work in

Theorem 3.3.7. *The graph $C_{M_2(\mathbb{F}_q)}^{(1)}$ is hyperenergetic and Ramanujan. Moreover, $C_{M_2(\mathbb{Z}_2)}^{(1)}$ is neither hyperenergetic nor Ramanujan.*

3.4 Spectral properties of $C_{M_2(\mathbb{F}_q)}^{(2)}$

We study the second subconstituent of $C_{M_2(\mathbb{F}_q)}$ in this section. We first show that the graph is a tensor product of a complete graph and a complete multi-partite graph and so we can calculate its eigenvalues. Let $\mathbb{F}_q^{2 \times 1}$ denote the set of column vectors of size 2×1 over \mathbb{F}_q . Since a 2×2 matrix is non-invertible if and only if its column vectors are parallel, we can conclude that

$$\begin{aligned} M_2(\mathbb{F}_q) \setminus (\text{GL}_2(\mathbb{F}_q) \cup \{\mathbf{0}_{2 \times 2}\}) \\ = \left(\bigcup_{\vec{v} \in \mathbb{F}_q^{2 \times 1} \setminus \{\vec{0}\}} \left\{ \begin{pmatrix} a\vec{v} & \vec{v} \end{pmatrix} : a \in \mathbb{F}_q \right\} \right) \cup \left\{ \begin{pmatrix} \vec{v} & \vec{0} \end{pmatrix} : \vec{v} \in \mathbb{F}_q^{2 \times 1} \setminus \{\vec{0}\} \right\} \end{aligned}$$

where $\vec{0}$ denotes the zero vector of $\mathbb{F}_q^{2 \times 1}$. Before giving a structure of the graph $C_{M_2(\mathbb{F}_q)}^{(2)}$, we need the next lemma.

Lemma 3.4.1. *Let A, B be non-invertible matrices in $M_2(\mathbb{F}_q)$, $a, b \in \mathbb{F}_q$ and $\vec{v}, \vec{w} \in \mathbb{F}_q^{2 \times 1} \setminus \{\vec{0}\}$.*

- (a) *If $A = \begin{pmatrix} a\vec{v} & \vec{v} \end{pmatrix}$ and $B = \begin{pmatrix} b\vec{w} & \vec{w} \end{pmatrix}$, then $A - B$ is non-invertible if and only if $a = b$ or \vec{v}, \vec{w} are linearly dependent, or equivalently, $A - B$ is invertible if and only if $a \neq b$ and \vec{v}, \vec{w} are linearly independent.*

(b) If $A = \begin{pmatrix} a\vec{v} & \vec{v} \end{pmatrix}$ and $B = \begin{pmatrix} \vec{w} & \vec{0} \end{pmatrix}$, then $A - B$ is non-invertible if and only if \vec{v} and \vec{w} are linearly dependent.

Proof. Observe that

$$A - B \text{ is non-invertible} \iff (a\vec{v} - b\vec{w}) = c(\vec{v} - \vec{w}) \text{ for some } c \in \mathbb{F}_q.$$

Assume that $A - B$ is non-invertible and \vec{v}, \vec{w} are linearly independent. Then $a = c$ and $b = c$, so $a = b$. Conversely, the case $a = b$ is clear. If $\vec{w} = c\vec{v}$ for some $c \in \mathbb{F}_q$, then $A - B = \begin{pmatrix} (a - bc)\vec{v} & (1 - c)\vec{v} \end{pmatrix}$ is non-invertible. This proves (a). For (b), we have

$$\begin{aligned} A - B \text{ is non-invertible} &\iff a\vec{v} - \vec{w} = c\vec{v} \text{ for some } c \in \mathbb{F}_q \\ &\iff (a - c)\vec{v} = \vec{w} \text{ for some } c \in \mathbb{F}_q, \end{aligned}$$

which is equivalent to \vec{v} and \vec{w} being linearly dependent. \square

In the next step, we define two graphs G and H as follows: G is the complete graph on $q + 1$ vertices parametrized by the set of projective lines $\mathbb{P}^1(\mathbb{F}_q) = \{[a, 1] : a \in \mathbb{F}_q\} \cup \{[1, 0]\}$ and the vertex set of H is $\mathbb{F}_q^{2 \times 1} \setminus \{\vec{0}\}$ and for any $\vec{v}, \vec{w} \in \mathbb{F}_q^{2 \times 1} \setminus \{\vec{0}\}$, \vec{v} and \vec{w} are adjacent if and only if \vec{v} and \vec{w} are not parallel. Note that H is the complete $(q + 1)$ -partite graph such that each partite has $q - 1$ vertices.

Let $f : C_{M_2(\mathbb{F}_q)}^{(2)} \rightarrow G \otimes H$ defined by $\begin{pmatrix} a\vec{v} & \vec{v} \end{pmatrix} \mapsto ([a, 1], \vec{v})$ and $\begin{pmatrix} \vec{v} & \vec{0} \end{pmatrix} \mapsto ([1, 0], \vec{v})$ for any $a \in \mathbb{F}_q$ and $\vec{v} \in \mathbb{F}_q^{2 \times 1} \setminus \{\vec{0}\}$. Thus, f is bijective. Now, let A, B be nonzero non-invertible matrices in $M_2(\mathbb{F}_q)$, $a, b \in \mathbb{F}_q$ and $\vec{v}, \vec{w} \in \mathbb{F}_q^{2 \times 1}$, $A = \begin{pmatrix} a\vec{v} & \vec{v} \end{pmatrix}$ and $B = \begin{pmatrix} b\vec{w} & \vec{w} \end{pmatrix}$. Lemma 3.4.1 (a) implies

$$\begin{aligned} A - B \in \text{GL}_2(\mathbb{F}_q) &\iff a \neq b \text{ and } \vec{v}, \vec{w} \text{ are linearly independent} \\ &\iff ([a, 1], \vec{v}) \text{ is adjacent to } ([b, 1], \vec{w}). \end{aligned}$$

Next, we assume that $A = \begin{pmatrix} a\vec{v} & \vec{v} \end{pmatrix}$ and $B = \begin{pmatrix} \vec{w} & \vec{0} \end{pmatrix}$. From Lemma 3.4.1 (b), we

have

$$\begin{aligned} A - B \in \text{GL}_2(\mathbb{F}_q) &\iff \vec{v} \text{ and } \vec{w} \text{ are linearly independent} \\ &\iff ([a, 1], \vec{v}) \text{ is adjacent to } ([1, 0], \vec{w}). \end{aligned}$$

Hence, f is a graph isomorphism, so we have the structure of the graph $C_{M_2(F)}^{(2)}$.

Theorem 3.4.2. *The graph $C_{M_2(\mathbb{F}_q)}^{(2)}$ is the tensor product of the complete graph on $q + 1$ vertices and the complete $(q + 1)$ -partite graph such that each part has $q - 1$ vertices, and it is a $(q^3 - q^2)$ -regular graph.*

Recall that eigenvalues of the tensor product can be determined by Theorem 1.1.10. Since the eigenvalues of G are q with multiplicity 1 and -1 with multiplicity q and the eigenvalues of H are $q^2 - q$, $-q + 1$ and 0 of multiplicities 1, q and $q^2 - q - 2$, respectively, we obtain the spectrum and energy of the graph $C_{M_2(\mathbb{F}_q)}^{(2)}$.

Theorem 3.4.3. *We have*

$$\text{Spec } C_{M_2(\mathbb{F}_q)}^{(2)} = \begin{pmatrix} q^3 - q^2 & -q^2 + q & q - 1 & 0 \\ 1 & 2q & q^2 & q^3 - 3q - 2 \end{pmatrix}.$$

Moreover, $E(C_{M_2(\mathbb{F}_q)}^{(2)}) = 4q^3 - 4q^2$.

Since the number of vertices of $C_{M_2(\mathbb{F}_q)}^{(2)}$ is $|M_2(\mathbb{F}_q) \setminus (\text{GL}_2(\mathbb{F}_q) \cup \{\mathbf{0}_{2 \times 2}\})| = q^3 + q^2 - q - 1$ and

$$4q^3 - 4q^2 - 2(q^3 + q^2 - q - 2) = 2q^3 - 6q^2 + 2q + 4 = 2(q - 2)(q^2 - q - 1) \geq 0.$$

Thus, $C_{M_2(\mathbb{F}_q)}^{(2)}$ is hyperenergetic unless $q = 2$. Finally, we show that the graph $C_{M_2(\mathbb{F}_q)}^{(2)}$ is not Ramanujan. Since $q^2 - q$ is an eigenvalue of $C_{M_2(\mathbb{F}_q)}^{(2)}$, we claim that $(q^2 - q)^2 > 4(q^3 - q^2 - 1)$, which is equivalent to the inequality $q^4 - 6q^3 + 5q^2 + 4 > 0$. This holds for $q \geq 5$ because $q^4 - 6q^3 + 5q^2 + 4 = q^2(q - 1)(q - 5) + 4 > 0$. For $q \in \{2, 3, 4\}$, it is easily seen that $C_{M_2(\mathbb{F}_q)}^{(2)}$ is Ramanujan in all cases. We record both results in

Theorem 3.4.4. *The graph $C_{M_2(\mathbb{F}_q)}^{(2)}$ is hyperenergetic if and only if $q \geq 3$, and it is Ramanujan if and only if $q \leq 4$.*

3.5 Clique numbers, chromatic numbers and independence numbers

In this section, we compute the clique number, the chromatic number and the independence number of subconstituents of $C_{M_2(\mathbb{F}_q)}$. Recall that we have a subfield K and the right ideal J of the ring $M_n(\mathbb{F}_q)$ as mentioned in Theorem 1.2.2. We start with the first subconstituent. Note that $\mathbf{0}_{n \times n} \in K$ and so $K \setminus \{\mathbf{0}_{n \times n}\}$ forms a complete subgraph in $C_{M_n(\mathbb{F}_q)}^{(1)}$. Hence, $\omega(C_{M_n(\mathbb{F}_q)}^{(1)}) \geq q^n - 1$. Next, write $M_n(\mathbb{F}_q) = \bigcup_{i=1}^{q^n} (B_i + J)$ as a union of cosets of J where the coset $B_1 + J = J$. Note that each coset forms an independent set and $\mathbf{0}_{n \times n} \in J$. It follows that $GL_n(\mathbb{F}_q)$ is a subset of $\bigcup_{i=2}^{q^n} (B_i + J)$ and hence $\chi(C_{M_n(\mathbb{F}_q)}^{(1)}) \leq q^n - 1$. Since $\omega(C_{M_n(\mathbb{F}_q)}^{(1)}) \leq \chi(C_{M_n(\mathbb{F}_q)}^{(1)})$, we have the following theorem.

Theorem 3.5.1. $\omega(C_{M_n(\mathbb{F}_q)}^{(1)}) = \chi(C_{M_n(\mathbb{F}_q)}^{(1)}) = q^n - 1$.

Recall that if G is a graph, then $\alpha(G) \geq \frac{|V(G)|}{\chi(G)}$. Theorem 3.5.1 gives

$$\alpha(C_{M_n(\mathbb{F}_q)}^{(1)}) \geq \frac{|GL_n(\mathbb{F}_q)|}{\chi(C_{M_n(\mathbb{F}_q)}^{(1)})} = (q^n - q) \dots (q^n - q^{n-1}).$$

Consider the group K^\times as a multiplicative subgroup of $GL_n(\mathbb{F}_q)$. Let $X = AM$ and $Y = AN$ where $M, N \in K^\times$ such that $M \neq N$ and $A \in GL_n(\mathbb{F}_q)$. Then $X - Y = A(M - N)$ is invertible because $M, N \in K^\times$. It follows that each coset forms a complete graph. This implies that $\alpha(C_{M_n(\mathbb{F}_q)}^{(1)}) \leq (q^n - q) \dots (q^n - q^{n-1})$. Hence, we have shown

Theorem 3.5.2. $\alpha(C_{M_n(\mathbb{F}_q)}^{(1)}) = (q^n - q) \dots (q^n - q^{n-1})$.

By Theorem 3.4.2, we have the second subconstituent of $C_{M_2(\mathbb{F}_q)}$ is the tensor product of the complete graph on $q + 1$ vertices G and the complete $q + 1$ -partite graph H such that each partite has $q - 1$ vertices. Since $\chi(G) = \chi(H) = q + 1$, we can conclude that $\chi(C_{M_2(\mathbb{F}_q)}^{(2)}) \leq q + 1$. Moreover, let $V(G) = \{a_1, \dots, a_{q+1}\}$ and V_1, \dots, V_{q+1} be the partites of H . Choose $v_i \in V_i$ for all $i \in \{1, \dots, q + 1\}$. We can see that the subgraph of $G \otimes H$ induced by $\{(a_1, v_1), \dots, (a_{q+1}, v_{q+1})\}$ is a complete graph, so $\omega(G \otimes H) \geq q + 1$. Thus, we obtain the clique number and the chromatic number of the graph $C_{M_2(\mathbb{F}_q)}^{(2)}$.

Theorem 3.5.3. $\omega(C_{M_2(\mathbb{F}_q)}^{(2)}) = \chi(C_{M_2(\mathbb{F}_q)}^{(2)}) = q + 1$.

Our final theorem gives the independence number of $C_{M_2(\mathbb{F}_q)}^{(2)}$.

Theorem 3.5.4. $\alpha(C_{M_2(\mathbb{F}_q)}^{(2)}) = q^2 - 1$.

Proof. Similar to the proof of Theorem 3.5.2, we know from Theorem 3.5.3 that

$$\alpha(C_{M_2(\mathbb{F}_q)}^{(2)}) \geq \frac{|M_2(\mathbb{F}_q) \setminus (GL_2(\mathbb{F}_q) \cup \{\mathbf{0}_{2 \times 2}\})|}{\chi(C_{M_n(\mathbb{F}_q)}^{(2)})} = \frac{q^3 + q^2 - q - 1}{q + 1} = q^2 - 1.$$

Write $M_2(\mathbb{F}_q) = \bigcup_{i=1}^{q^2} (A_i + K)$ as a union of cosets of K . Then an independent set of

$C_{M_2(\mathbb{F}_q)}^{(2)}$ is contained in $\bigcup_{i=2}^{q^2} (A_i + K)$. Since each coset forms a complete subgraph, we have $\alpha(C_{M_2(\mathbb{F}_q)}^{(2)}) \leq q^2 - 1$ and the result follows. \square

CHAPTER IV

CONCLUSIONS

In the second chapter, we use eigenvalues of $C_{M_n(\mathbb{F}_q)}$ to show that the following rings are DU-rings.

- (a) $M_{n_1}(\mathbb{F}_{q_1}) \times \cdots \times M_{n_k}(\mathbb{F}_{q_k})$ where q_1, \dots, q_k are pairwise relatively prime.
- (b) $M_{n_1}(\mathbb{F}_{q_1}) \times M_{n_2}(\mathbb{F}_{q_2})$.
- (c) $M_{m_1}(\mathbb{F}_{p_1^{s_1}}) \times M_{n_1}(\mathbb{F}_{p_1^{t_1}}) \times \cdots \times M_{m_k}(\mathbb{F}_{p_k^{s_k}}) \times M_{n_k}(\mathbb{F}_{p_k^{t_k}})$ where p_1, \dots, p_k are distinct primes and $s_1, \dots, s_k, t_1, \dots, t_k \in \mathbb{N}$.

In the third chapter, we obtain spectral properties of substituieren of $C_{M_n(\mathbb{F}_q)}$ and compute their clique numbers, chromatic numbers and independence numbers. First, we list the results on the graph $C_{M_2(\mathbb{F}_q)}^{(1)}$.

1. $C_{M_n(\mathbb{F}_q)}^{(1)}$ is the normal Cayley graph of $GL_n(\mathbb{F}_q)$ associated with $(I_n + GL_n(\mathbb{F}_q)) \cap GL_n(\mathbb{F}_q)$ and it is regular of degree e_n .
2. If $q = 2$, then $\text{Spec } C_{M_2(\mathbb{F}_q)}^{(1)} = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$.
3. If $q \geq 3$, then $\text{Spec } C_{M_2(\mathbb{F}_q)}^{(1)} =$

$$\begin{pmatrix} q^4 - 2q^3 - q^2 + 3q & q & -q^2 + q + 1 & q^3 - 2q^2 \\ 1 & q^4 - 2q^3 - 2q^2 + 4q + 1 & q^2 & (q+1)^2(q-2) \end{pmatrix}.$$

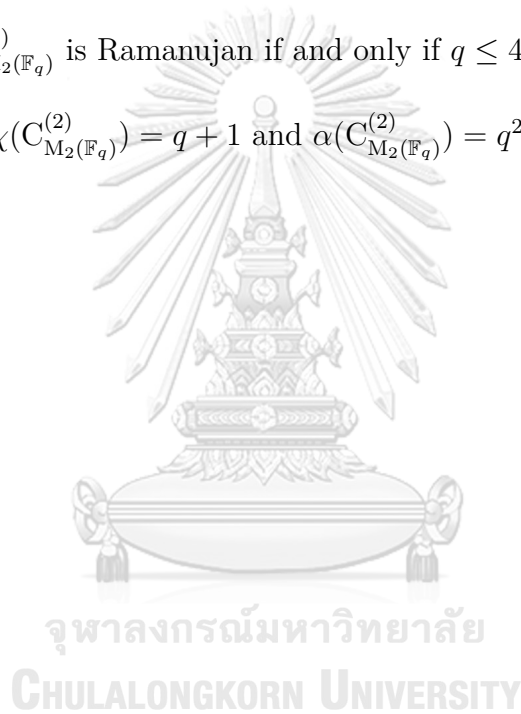
4. For $q \geq 3$, the graph $C_{M_2(\mathbb{F}_q)}^{(1)}$ is hyperenergetic and Ramanujan.

Moreover, $C_{M_2(\mathbb{Z}_2)}^{(1)}$ is neither hyperenergetic nor Ramanujan.

5. $\omega(C_{M_n(\mathbb{F}_q)}^{(1)}) = \chi(C_{M_n(\mathbb{F}_q)}^{(1)}) = q^n - 1$ and $\alpha(C_{M_n(\mathbb{F}_q)}^{(1)}) = (q^n - q) \cdots (q^n - q^{n-1})$.

Finally, we list the results on the graph $C_{M_2(\mathbb{F}_q)}^{(2)}$.

6. The graph $C_{M_2(\mathbb{F}_q)}^{(2)}$ is the tensor product of the complete graph on $q + 1$ vertices and the complete $(q + 1)$ -partite graph such that each partite has $q - 1$ vertices, and it is a $(q^3 - q^2)$ -regular graph.
7. $\text{Spec } C_{M_2(\mathbb{F}_q)}^{(2)} = \begin{pmatrix} q^3 - q^2 & -q^2 + q & q - 1 & 0 \\ 1 & 2q & q^2 & q^3 - 3q - 2 \end{pmatrix}$.
8. The graph $C_{M_2(\mathbb{F}_q)}^{(2)}$ is hyperenergetic if and only if $q \geq 3$.
9. The graph $C_{M_2(\mathbb{F}_q)}^{(2)}$ is Ramanujan if and only if $q \leq 4$.
10. $\omega(C_{M_2(\mathbb{F}_q)}^{(2)}) = \chi(C_{M_2(\mathbb{F}_q)}^{(2)}) = q + 1$ and $\alpha(C_{M_2(\mathbb{F}_q)}^{(2)}) = q^2 - 1$.



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