การวางนัยทั่วไปของบางทฤษฎีบทในทฤษฎีมอดูลไปยังมอดูลเสมือน

นางสาวกนกพร ช่างทอง

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

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GENERALIZATION OF SOME THEOREMS IN MODULE THEORY TO SKEWMODULES

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กำหนดให้ R เป็นวงเสมือน เราจะเรียก M ว่า **มอดูลเสมือน**บน R ก็ต่อเมื่อ M เป็นกลุ่มภายใต้การดำเนินการการบวก และมีการกระทำทางซ้าย $R\times M\rightarrow M$ ซึ่งกำหนดโดย $(r,m)\alpha\ rm$ มีสมบัติว่า สำหรับทุกๆ $r,s\in R$ และ $m,n\in M$, (1) $(r+s)m=rm+sm$, (2) $r(m+n)=rm+rn$ และ (3) $(rs)m=r(sm)$

เราจะเรียกกลุ่มย่อย N ของมอดูลเสมือน M บน R ว่า **มอดูลเสมือนย่อย**ของ M ก็ต่อเมื่อ สำหรับทุกๆ $n\in N$ และ $r\in R$ จะได้ $rn\in N$ และจะเรียก N ว่า **มอดูลเสมือนย่อยปกติ** ก็ต่อเมื่อ N เป็นมอดูลเสมือนย่อยของ M และ สำหรับทุกๆ $m\in M$, $N+m=m+N$

เราจะเรียกมอดูลเสมือน M บน R ว่า**ซิมเปิล** ก็ต่อเมื่อ M มีมอดูลเสมือนย่อยปกติเพียงสองตัวเท่านั้น คือ $\{0\}$ และ M

กำหนดให้ M เป็นมอดูลเสมือนบน R เราจะเรียก มอดูลเสมือนย่อยปกติ M_1 และ M_2 ของ M ว่า **ซับพลีเมนเทอรี** ก็ต่อเมื่อ $M=M_1\oplus M_2$ และเราจะเรียกมอดูลเสมือนย่อยปกติ N ของ M ว่า**ไดเรคซัมมานด์** ก็ต่อเมื่อ มีมอดูลเสมือนย่อยปกติ P ของ M ซึ่ง N และ P เป็นซับพลีเมนเทอรี

ผลสำคัญของงานวิจัยมีดังนี้

การทำให้ทฤษฎีบทไอโซมอร์ฟิซึมพื้นฐาน 4 ทฤษฎีบท ทฤษฎีบทไชเออร์และทฤษฎีบท จอร์แดน-โฮลเดอ ในทฤษฎีมอดูล เป็นกรณีทั่วไปในมอดูลเสมือน นอกจากนี้จะได้ทฤษฎีบทดังต่อไปนี้

<u>ทฤษฎีบท 1</u> กำหนดให้ M เป็นมอดูลเสมือนบน R ถ้า M เป็นมอดูลเสมือนอาธิเนียนและโนธีเรียนแล้ว M จะมีอนุกรมคอมโพสิชัน

<u>ทฤษฎีบท 2</u> กำหนดให้ M เป็นมอดูลเสมือนบน R ถ้า M เป็นผลรวมของมอดูลเสมือนย่อยปกติของ M ซึ่งซิมเปิลแล้ว ทุกๆ มอดูลเสมือนย่อยปกติของ M เป็นไดเรคซัมมานด์

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ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต...

สาขาวิชา **คณิตศาสตร์** ลายมือชื่ออาจารย์ที่ปรึกษา..

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Let R be a skewring. An R **-skewmodule** M is an additive group with a left action $R\times M\rightarrow M$, defined by $(r,m)\mapsto rm$, such that (1) $(r+s)m=rm+sm$, (2) $r(m+n)=rm+rn$ and (3) $(rs)m=r(sm)$ for all $r,s\in R$ and $m,n\in M$.

A subgroup N of an R -skewmodule M is called a **subskewmodule** of M if for all $n\in N$ and $r\in R$, then $rn\in N$. Moreover, N is called a **normal subskewmodule** if N is a subskewmodule of M such that $N+m=m+N$ for all $m\in M$.

An R -skewmodule M is **simple** if $\{0\}$ and M are only normal subskewmodules of M.

Let M be an R -skewmodule. Normal subskewmodules M_1 and M_2 of M are said to be **supplementary** if $M=M_1\oplus M_2$. A normal subskewmodule N of M is called a **direct summand** if there exists a normal subskewmodule P of M such that N and P are supplementary.

The main results of this research are follows:
Generalization the notion of the four Isomorphism Theorems, the Schreier's theorem and the Jordan Hölder theorem in module theory to skewmodules. Moreover we obtain the following theorems:

<u>Theorem1</u> Let M be an R -skewmodule. If M is both artinian and noetherian, then M has a composition series.

<u>Theorem2</u> Let M be an R -skewmodule. If M is the sum of a family of its normal simple subskewmodules, then every normal subskewmodule of M is a direct summand.

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Department **Mathematics** Student's signature……………………………..
Field of Study **Mathematics** Advisor's signature…………………………….
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CHAPTER I

INTRODUCTION

A Construction of great versatility is that of a module over a ring. For this research, we are interested in a more general structure. Sureeporn has been introduced the concept of a skewring in [1]: A skewring is a ring dropping an additively commutative property. An object analogous to a module over a ring which is called a skewmodule can be defined over a skewring. Moreover, we study which theorems in Module Theory can be generalized to skewmodules. In this research,we study the theorems in [1], [2], [4] and [5].

There are four chapters in this thesis. In Chapter I, we introduce the concept of a normal subskewmodule. We find that skewmodules can be studied in much the same way as modules if we replace submodules in Module Theory by normal subskewmodules.

In Chapter II, we give definitions, examples and prove some fundamental theorems about skewmodules.

In Chapter III, we study the concept of the composition series and generalize the four basic Isomorphism Theorems and the Jordan Hölder Theorem to skewmodules.

In Chapter IV, we give definitions and theorems related artinian and noetherian skewmodules. Moreover, we prove the relation between artinian, noetherian skewmodules and the composition series.

CHAPTER II

PRELIMINARIES

In this chapter we give some definitions and theorems which are used in this thesis. Moreover, some examples are given.

Notation My general notation conventions are as follows:

$\mathbb{N}$ is the set of all natural numbers,

0_R (or 0) is the additive identity of a group $(R, +)$,

$A \subset B$ (or $B \supset A$) means that A is a proper subset of B.

Definition 2.1. A triple $(R, +, \cdot)$ is a **skewring** if

(1) $(R, +)$ is a group,

(2) $(R, \cdot)$ is a semigroup and

(3) $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$ for all $x, y, z \in R$.

Definition 2.2. Let R be a skewring. A **left** R**-skewmodule** M or a **left skewmodule** M **over** R is an additive group M with a left action $R \times M \to M$, given by $(r, m) \mapsto rm$, such that

(1) $(r + s)m = rm + sm$,

(2) $r(m + n) = rm + rn$,

(3) $(rs)m = r(sm)$

for all $r, s \in R$ and all $m, n \in M$. If R has a multiplicative identity 1, we define $1m = m$ for all $m \in M$.

A left R-skewmodule M is called a **left R-module** or a **left module over R** if M is an abelian group.

A **right R-skewmodule** or a **right skewmodule over R** and a **right R-module** or a **right module over R** are defined in the similar way by replacing a left action with a right action with corresponding properties to (1)–(3). In what follows, we make the convention that the term R-skewmodule always means a left R-skewmodule.

Remark 2.3. Let M be a skewmodule with additive identity 0_M over a skewring R with additive identity 0_R. It is easy to prove that, for all $r \in R$, $m \in M$, $r0_M = 0_M$, $0_R m = 0_M$ and $(-r)m = -(rm) = r(-m)$.

Lemma 2.4. Let M be an R-skewmodule. For $r, s \in R$ and $m, n \in M$, $rn + sm = sm + rn$.

Proof. Consider

$$(r+s)(m+n) = r(m+n) + s(m+n) = rm + rn + sm + sn \qquad (1)$$

$$(r+s)(m+n) = (r+s)m + (r+s)n = rm + sm + rn + sn \qquad (2)$$

By (1), (2) and the definition of an R-skewmodule, we obtain that $rn + sm = sm + rn$.

□

Remark 2.5. Let R be a skewring and M an R-skewmodule. The following statements hold.

(1) $RM = \{\sum_{i=1}^{n} r_i m_i \mid r_i \in R, m_i \in M, n \in \mathbb{N}\}$ is a module over R.

(2) If $RM = M$, then M is a module over R.

(3) If R has a multiplicative identity, then R is a ring, and M is an R-module.

Proof. (1) Apply Lemma 2.4 to prove the commutativity of addition.

(2) The result is obtained immediately from (1).

(3) If R has a multiplicative identity, Sureepron proved that R is a ring in [1], then by (2), we obtain that M is an R-module.

□

Lemma 2.6. Let R be a skewring and M an R-skewmodule. If M is finite and there exists an $r \in R\setminus\{0\}$ such that $rm \neq 0$ for all $m \in M\setminus\{0\}$, then M is a module over R.

Proof. Assume that M is finite and there exists an $r \in R\setminus\{0\}$ such that $rm \neq 0$ for all $m \in M\setminus\{0\}$. Define $f : M\setminus\{0\} \to M\setminus\{0\}$ by

$$f(m) = rm \quad \text{for all } m \in M\setminus\{0\}.$$

To show that f is 1-1, let $m_1, m_2 \in M\setminus\{0\}$ be such that $f(m_1) = f(m_2)$. Then $rm_1 = rm_2$. Thus $r(m_1 - m_2) = 0$. By the assumption, we have $m_1 - m_2 = 0$, i.e., $m_1 = m_2$. Hence f is 1-1. Since M is finite, f is onto. Then $RM = M$. By Remark 2.5(2), M is a module over R.

□

Definition 2.7. Let R be a skewring and I a nonempty subset of R.

(1) If I is a skewring under the operations of R, then I is a **subskewring** of R, denoted by $I \leq R$.

(2) If I is a subskewring of R and $\{yx \mid x \in I, y \in R\} \subseteq I$ $\big(\{xy \mid x \in I, y \in R\} \subseteq I\big)$, then I is a **left** (**right**) **ideal** of R.

If I is both a left and right ideal of R, then I is a **two-sided ideal** or **ideal** of R.

(3) If I is a subskewring of R and $\{r + x - r \mid r \in R, x \in I\} \subseteq I$, then I is a **normal subskewring** of R.

(4) If I is a left (right) ideal of R and I is a normal subskewring of R, then I is a **normal left (right) ideal** of R.

If I is both a normal left and right ideal of R, then I is a **normal two-sided ideal** or **normal ideal** of R.

Definition 2.8. Let R and S be skewrings and $f : R \to S$. f is called a **homomorphism** if and only if for all $x, y \in R$,

$$f(x + y) = f(x) + f(y) \text{ and } f(xy) = f(x)f(y).$$

Let R be a skewring and I a normal ideal of R. Let $R/I = \{x + I \mid x \in R\}$ and define the binary operations $+, \cdot$ on R/I as follows : for all $x+I, y+I \in R/I$,

$$(x + I) + (y + I) = x + y + I \text{ and}$$
$$(x + I)(y + I) = xy + I.$$

We, now, give some examples of skewmodule.

Example 2.9. Any a skewring R is an R-skewmodule.

Example 2.10. If S is a skewring and R a subskewring of S, then S is an R-skewmodule with $rs(r \in R, s \in S)$ being the multiplication in S.

Example 2.11. If I is a left ideal of a skewring R, then I is a left R-skewmodule with $ra(r \in R, a \in I)$ being the multiplication in R.

Example 2.12. If I is a normal left ideal of a skewring R, then R/I is an R-skewmodule with

$$r(\overline{r} + I) = r\overline{r} + I \quad \text{where } r, \overline{r} \in R.$$

Example 2.13. Let R and S be skewrings and $\varphi : R \to S$ a homomorphism. Then every S-skewmodule M can be made into an R-skewmodule by defining $rm(r \in R, m \in M)$ to be $\varphi(r)m$.

To prove this, let $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$. We obtain that $(r_1+r_2)m = \big(\varphi(r_1+r_2)\big)m = \big(\varphi(r_1)+\varphi(r_2)\big)m = \varphi(r_1)m+\varphi(r_2)m = r_1m+r_2m$, $r(m_1 + m_2) = \varphi(r)(m_1 + m_2) = \varphi(r)m_1 + \varphi(r)m_2 = rm_1 + rm_2$ and $(r_1r_2)m = \varphi(r_1r_2)m = \big(\varphi(r_1)\varphi(r_2)\big)m = \varphi(r_1)\big(\varphi(r_2)m\big) = r_1(r_2m)$. Then M is an R-skewmodule.

Sureeporn introduced the next two examples for skewring and we continue studying the same examples for skewmodules.

Example 2.14. Let $(R,+,\cdot)$ be the ring of all strictly upper triangular 3×3 matrices over $\mathbb{R}$ under the usual of addition and multiplication of matrix. Then $R^3 = \{0\}$. Define a binary operation $\oplus$ on R by $a \oplus b = a + b + ab$ for all $a, b \in R$. By [1], $(R, \oplus, \cdot)$ is a skewring which is not a ring. Then from Example 2.9, $(R, \oplus)$ is an $(R, \oplus, \cdot)$-skewmodule.

Example 2.15. Let $(G, +)$ be a nonabelian group, K an abelian subgroup of G and X a nonempty set such that $X \cap G = \emptyset$ and $|X| \geq 1$.

Let $\text{Map}\,(G, X, K) = \{f : G \cup X \to G \mid\ f|_G : G \to K$ is a homomorphism$\}$. For all $f, g \in \text{Map}\,(G, X, K)$, define

$$(f +' g)(x) = f(x) + g(x) \quad \text{and}$$

$$(f \cdot g)(x) = (f \circ g)(x)$$

for all $x \in G \cup X$. Then

(1) $\big(\mathrm{Map}\,(G,X,K),+',\cdot\big)$ is a skewring which is not always a ring,

(2) G is a $\mathrm{Map}\,(G,X,K)$-skewmodule with fa defined to be $f(a)$ for all $a \in G, f \in \mathrm{Map}\,(G,X,K)$.

The first result is already proved in [1]. Next, let $a,b \in G$ and $f,g \in \mathrm{Map}\,(G,X,K)$. We obtain that

(2.1) $(f+' g)a = (f+' g)(a) = f(a)+g(b) = fa+ga$.

(2.2) $f(a+b) = f(a)+f(b) = fa+fb$.

The second equality holds since $a,b \in G$ and $f|_G$ is a homomorphism.

(2.3) $(f \cdot g)a = (f \circ g)(a) = f\big(g(a)\big) = f(ga)$.

Therefore, G is a $\mathrm{Map}\,(G,X,K)$-skewmodule.

We now define a homomorphism from an R-skewmodule to another.

Definition 2.16. If M and N are R-skewmodules, then a mapping $\varphi : M \to N$ is called an R-**homomorphism** if

(1) $\varphi(m+n) = \varphi(m)+\varphi(n)$ and

(2) $\varphi(rm) = r\varphi(m)$

for all $r \in R$ and $m,n \in M$.

An R-homomorphism φ is called an R-**monomorphism**, R-**epimorphism**, R-**isomorphism** if it is injective, surjective, bijective, respectively. In the case φ is an R-isomorphism, M and N are said to be **isomorphic**, denoted by $M \cong N$. The **kernel** of φ is its kernel as on R-homomorphism of modules, namely $\mathrm{Ker}\,\varphi = \{m \in M \mid \varphi(m) = 0\}$. Similarly the **image** of φ is the set $\mathrm{Im}\,\varphi = \{n \in N \mid \varphi(m) = n \text{ for some } m \in M\}$.

If $\varphi : M \to N$ is an R-homomorphism, then φ is a group homomorphism of $(M,+)$ into $(N,+)$, so

(1) $\varphi(0_M) = 0_N$,

(2) $\varphi(-m) = -\varphi(m)$ for all $m \in M$.

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Example 2.17. Obviously, the zero map from M into M' and the identity map on M are R-homomorphisms.

Definition 2.18. A subgroup N of an R-skewmodule M is an R-**subskewmodule**, denoted by $N < M$, is stable under the action of R on M in the sense that if $n \in N$ and $r \in R$, then $rn \in N$.

For simplicity we use the term subskewmodule instead of R-subskewmodule.

Remark 2.19. It is easy to show that a nonempty subset N of an R-skewmodule M is a subskewmodule of M if and only if

(1) $n_1 - n_2 \in N$ for all $n_1, n_2 \in N$, and

(2) $rn \in N$ for all $r \in R, n \in N$.

Example 2.20. Any R-skewmodule M has trivial subskewmodules M and $\{0\}$.

Lemma 2.21. (1) If M and M' are R-skewmodules and $f : M \to M'$ an R-homomorphism, then $\operatorname{Ker} f < M$ and $\operatorname{Im} f < M'$.

(2) If $\{M_i \mid i \in I\}$ is a family of subskewmodules of an R-skewmodule, then $\bigcap_{i \in I} M_i < M$.

Theorem 2.22. (Modular Law) If M is an R-skewmodule and if A, B, C are subskewmodules of M with $C \subseteq A$, then $A \cap (B + C) = (A \cap B) + C$.

Proof. Let M be an R-skewmodule. Assume that A, B, C are subskewmodules of M with $C \subseteq A$. Since $C \subseteq A$, it follows that $A+C = A$. Now $(A \cap B)+C \subseteq A+C$ and $(A \cap B)+C \subseteq B+C$. Thus $(A \cap B)+C \subseteq (A+C) \cap (B+C) = A \cap (B+C)$. Next, let $a \in A \cap (B+C)$. Then $a = b+c$ for some $b \in B, c \in C$. Since $C \subseteq A$, we have $c \in A$. Then $b = a - c \in A$, that is $b \in A \cap B$. Thus $a = b + c \in (A \cap B) + C$. Therefore $A \cap (B + C) = (A \cap B) + C$.

$\square$

Definition 2.23. A subskewmodule N of an R-skewmodule M is a **normal subskewmodule**, denoted by $N \lhd M$, if $N + m = m + N$ for all $m \in M$.

Remark 2.24. Let M be an R-skewmodule. The followings are equivalent.

(1) N is a normal subskewmodule of M.

(2) $m + N - m = N$ for all $m \in M$.

(3) $m + N - m \subseteq N$ for all $m \in M$.

We can see that the skewring and skewmodules in Example 2.15 are significant and interesting. From this example, we shall give various examples of definitions given previously.

Example 2.25. It is clear that $\big\langle (1\ 2) \big\rangle$ is an abelian subgroup of S_3. Let $X = \{a\}$ be such that $a \notin S_3$. Then $S_3 \cap X = \emptyset$. It is easy to check that

$$\begin{aligned} R &= \text{Map}\Big(S_3, \{a\}, \big\langle (1\ 2) \big\rangle\Big) \\ &= \{\varphi : S_3 \cup \{a\} \to S_3 \ \big|\ \varphi|_{S_3} : S_3 \to \big\langle (1\ 2) \big\rangle \text{ is a homomorphism}\} \\ &= \big\{\varphi_i \mid i \in \{1, 2, \dots, 12\}\big\} \ \text{ where} \end{aligned}$$

$$\varphi_1(x) = (1) \ \text{ for all } x \in S_3 \cup \{a\} \qquad \varphi_2(x) = \begin{cases} (1), & \text{if } x \in S_3 \\ (1\ 2), & \text{if } x = a \end{cases}$$

$$\varphi_3(x) = \begin{cases} (1), & \text{if } x \in S_3 \\ (1\ 3), & \text{if } x = a \end{cases} \qquad \varphi_4(x) = \begin{cases} (1), & \text{if } x \in S_3 \\ (2\ 3), & \text{if } x = a \end{cases}$$

$$\varphi_5(x) = \begin{cases} (1), & \text{if } x \in S_3 \\ (1\ 2\ 3), & \text{if } x = a \end{cases} \qquad \varphi_6(x) = \begin{cases} (1), & \text{if } x \in S_3 \\ (1\ 3\ 2), & \text{if } x = a \end{cases}$$

$$\varphi_7(x) = \begin{cases} (1), & \text{if } x \text{ is even permutation and x=a} \\ (1\ 2), & \text{if } x \text{ is odd permutation} \end{cases}$$

$$\varphi_8(x) = \begin{cases} (1), & \text{if } x \text{ is even permutation} \\ (1\ 2), & \text{if } x \text{ is odd permutation and x=a} \end{cases}$$

$$\varphi_9(x) = \begin{cases} (1), & \text{if } x \text{ is even permutation} \\ (1\ 2), & \text{if } x \text{ is odd permutation} \\ (1\ 3), & \text{if } x = a \end{cases}$$

$$\varphi_{10}(x) = \begin{cases} (1), & \text{if } x \text{ is even permutation} \\ (1\ 2), & \text{if } x \text{ is odd permutation} \\ (2\ 3), & \text{if } x = a \end{cases}$$

$$\varphi_{11}(x) = \begin{cases} (1), & \text{if } x \text{ is even permutation} \\ (1\ 2), & \text{if } x \text{ is odd permutation} \\ (1\ 2\ 3), & \text{if } x = a \end{cases}$$

$$\varphi_{12}(x) = \begin{cases} (1), & \text{if } x \text{ is even permutation} \\ (1\ 2), & \text{if } x \text{ is odd permutation} \\ (1\ 3\ 2), & \text{if } x = a \end{cases}$$

Then R is a skewring which is not a ring since $\varphi_4\varphi_5 \neq \varphi_5\varphi_4$.

$R_1 = \{\varphi_1, \varphi_5, \varphi_6\}$ is a subskewring of R which is a ring. Moreover, R_1 is a left ideal of R, but it is not a right ideal because $\varphi_5 \circ \varphi_{10} = \varphi_2 \notin R_1$. $\{\varphi_1, \varphi_2, \varphi_7, \varphi_8\}$ is an ideal of R which is a ring and $R_2 = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6\}$ is a normal ideal of R which is not a ring. Moreover, R_1 is a normal ideal of R_2, but it is not normal ideal of R since $\varphi_7\varphi_5\varphi_7 = \varphi_{12} \notin R_1$.

We obtain that S_3 is an R-skewmodule which is not a module and R is an R_2-skewmodule. Moreover, A_3 is a normal subskewmodule of S_3.

Example 2.26. $N = \{(1), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ is an abelian subgroup of S_4. Let $X = \{a\}$ be such that $a \notin S_4$. Then $\text{Map}\,(S_4, \{a\}, N)$ is a skewring which is not a ring and S_4 is a $\text{Map}\,(S_4, \{a\}, N)$-skewmodule. Moreover, A_4 is a normal subskewmodule of S_4 over $\text{Map}\,(S_4, \{a\}, N)$.

$\langle(1\ 2\ 3\ 4)\rangle$ is a subskewmodule of S_4 over $\text{Map}\,(S_4, \{a\}, N)$, but it is not a normal subskewmodule since $(1\ 3\ 4\ 2)(1\ 4\ 3\ 2)(1\ 3\ 4\ 2) = (3\ 4) \notin \langle(1\ 2\ 3\ 4)\rangle$

Lemma 2.27. (1) If M and M' are R-skewmodules and $\varphi : M \to M'$ an R-homomorphism, then $\text{Ker}\,\varphi \lhd M$ and φ is a monomorphism if and only if $\text{Ker}\,\varphi = \{0\}$.

(2) If $\{M_i \mid i \in I\}$ is a family of normal subskewmodules of an R-skewmodule M, then $\bigcap\limits_{i\in I} M_i \lhd M$.

Definition 2.28. Let M be an R-skewmodule and $X \subseteq M$. The intersection of all normal subskewmodules of M containing X is called a **normal subskewmodule generated** by X. If X is finite, and X generates the skewmodule M, M is said to be **finitely generated**. If $X = \emptyset$, then X clearly generates the zero skewmodule.

If $\{M_i \mid i \in I\}$ is a family of normal subskewmodules of M, then the normal subskewmodule generated by $X = \bigcup\limits_{i\in I} M_i$ is called the **sum** of the skewmodules M_i, which is denoted by $\sum_{i\in I} M_i$. If $I = \{1, 2, \ldots, n\}$, then the sum of $M_1, M_2, \ldots, M_n$ is $M_1 + M_2 + \ldots + M_n$.

Lemma 2.29. Let M be an R-skewmodule. If P and N are subskewmodules of M such that P is normal, then the following statements hold.

(1) P is contained in N implies that P is a normal subskewmodule of N.

(2) $P \cap N$ is a normal subskewmodule of N.

(3) $N + P$ is a subskewmodule of M.

(4) N is normal implies that $N + P$ is a normal subskewmodule of M.

Proof. Let M be an R-skewmodule. Assume that P and N are subskewmodules of M such that P is normal.

(1) The proof is obvious.

(2) Clearly, $P \cap N < N$. Let $n \in N, k \in P \cap N$. Then $n + k - n \in N$ since $N < M$ and $n + k - n \in P$ since $P \lhd M$. Thus $n + k - n \in P \cap N$. Hence $P \cap N$ is a normal subskewmodule of N.

(3) Notice that $N + P \neq \emptyset$ since $0 \in N + P$. Let $n + p, n' + p' \in N + P$ be such that $n, n' \in N$ and $p, p' \in P$. Then $(n + p) - (n' + p') = n + p - p' - n' = n + (p - p') - n' \in P \subseteq N + P$ since $P \lhd M$. Next, let $r \in R$. Then $r(n + p) = rn + rp \in N + P$. Hence $N + P$ is a subskewmodule of M.

(4) By (3), it is already proved that $N + P < M$. Let $m \in M$. Then

$$\begin{aligned}(N + P) + m &= N + (P + m)\\ &= N + (m + P)\\ &= (N + m) + P\\ &= (m + N) + P\\ &= m + (N + P).\end{aligned}$$

The second and the fourth equalities hold since $P \lhd M$ and $N \lhd M$, respectively. Hence $N + P$ is a normal subskewmodule of M.

□

Theorem 2.30. Let N be a normal subskewmodule of an R-skewmodule M and $M/N = \{m + N \mid m \in M\}$ the set of all cosets of M by N. Then M/N is an R-skewmodule relative to the addition and scalar multiplication defined by

$$(x + N) + (y + N) = (x + y) + N \qquad \text{and}$$

$$r(x + N) = rx + N$$

for all $x, y \in M, r \in R$.

Proof. First, we prove that these are indeed well-defined operations. Let $m_1, m_2, m_1', m_2' \in M$ be such that $m_1 + N = m_1' + N$ and $m_2 + N = m_2' + N$. Then $m_1 = m_1' + n$ and $m_2 = m_2' + \overline{n}$ for some $n, \overline{n} \in N$. Thus $m_1 + m_2 = (m_1' + n) + (m_2' + \overline{n}) = m_1' + (n + m_2') + \overline{n} = m_1' + m_2' + \widehat{n} + \overline{n}$ for some $\widehat{n} \in N$ since $N \lhd M$. Thus $m_1 + m_2 \in (m_1' + m_2') + N$. Hence $(m_1 + m_2) + N = (m_1' + m_2') + N$. Let $r \in R$. Then $rm_1 = r(m_1' + n) = rm_1' + rn \in rm_1' + N$ since $N < M$. Hence $rm_1 + N = rm_1' + N$. Therefore these operations are well-defined. It is straightforward that M/N is an R-skewmodule.

□

Definition 2.31. Let N be a normal subskewmodule of an R-skewmodule M. The R-skewmodule M/N defined in Theorem 2.30 is called the **quotient skewmodule** of M by N.

The map $\pi : M \to M/N$, defined by $\pi(x) = x + N$ for all $x \in M$, is called the **canonical projection**. It is an epimorphism with kernel N.

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Definition 2.32. Let M be an R-skewmodule. M is **simple** if $\{0\}$ and M are only its normal subskewmodules.

Lemma 2.33. Let M be an R-skewmodule. If $M = Rx = \{rx \mid r \in R\}$ for every nonzero $x \in M$, then M is simple.

Proof. Assume that $M = Rx$ for all $x \in M \setminus \{0\}$. Let N be a nonzero normal subskewmodule of M and $n \in N \setminus \{0\}$. We obtain that $M = Rn \subseteq N$. Thus $M = N$. Hence M is simple.

□

Lemma 2.34. Let M and N be R-skewmodules and $f : M \to N$ a nonzero R-homomorphism. If M is simple, then f is a monomorphism.

Proof. Let $f : M \to N$ be a nonzero R-homomorphism. Assume that M is simple. Since f is a nonzero mapping, we obtain that $\operatorname{Ker} f \neq M$. Hence $\operatorname{Ker} f = \{0\}$ since $\operatorname{Ker} f \lhd M$ and M is simple. Therefore f is a monomorphism.

□

Lemma 2.35. Let M and M' be R-skewmodules and $\varphi : M \to M'$ an R-homomorphism. Then the following statements hold.

(1) If N is a subskewmodule of M, then $\varphi[N]$ is a subskewmodule of M'. Hence $\operatorname{Im} \varphi$ is a subskewmodule of M'.

(2) If φ is an epimorphism and N is a normal subskewmodule of M, then $\varphi[N]$ is a normal subskewmodule of M'. Hence $\varphi[N]$ is a normal subskewmodule of $\operatorname{Im} \varphi$.

(3) If N is a subskewmodule of M, then $\varphi^{-1}\big(\varphi[N]\big) = \big(\operatorname{Ker} \varphi\big) + N$. Moreover if N contains $\operatorname{Ker} \varphi$, then $\varphi^{-1}\big(\varphi[N]\big) = N$.

(4) If N' is a subskewmodule of M', then $\varphi^{-1}[N']$ is a subskewmodule of M containing $\operatorname{Ker}\varphi$.

(5) If N' is a normal subskewmodule of M', then $\varphi^{-1}[N']$ is a normal subskewmodule of M containing $\operatorname{Ker}\varphi$.

Proof. Let M and M' be R-skewmodules and $\varphi : M \to M'$ an R-homomorphism.

(1) Assume that N is a subskewmodule of M. Then $\varphi[N] \neq \emptyset$ since $\varphi(0) = 0_{M'}$. Let $x, y \in \varphi[N]$. Then $\varphi(a) = x$ and $\varphi(b) = y$ for some $a, b \in N$. Thus $x - y = \varphi(a) - \varphi(b) = \varphi(a - b) \in \varphi[N]$. Let $r \in R$. Then $rx = r\varphi(a) = \varphi(ra) \in \varphi[N]$. Hence $\varphi[N]$ is a subskewmodule of M'.

(2) Assume that φ is an epimorphism and N is a normal subskewmodule of M. By (1) we have $\varphi[N] < M'$. Let $x \in \varphi[N]$ and $m' \in M'$. Then $\varphi(a) = x$ for some $a \in N$. Since φ is onto, $\varphi(m) = m'$ for some $m \in M$. It follows that $m+a-m \in N$ since $N \lhd M$. Thus $m' + x - m' = \varphi(m) + \varphi(a) - \varphi(m) = \varphi(m + a - m) \in \varphi[N]$. Hence $\varphi[N]$ is a normal subskewmodule of M'.

(3) Assume that N is a subskewmodule of M. To show that $\varphi^{-1}\big(\varphi[N]\big) = (\operatorname{Ker}\varphi)+N$, first, let $a+b \in (\operatorname{Ker}\varphi)+N$ be such that $a \in \operatorname{Ker}\varphi$ and $b \in N$. Then $\varphi(a) = 0$, so that $\varphi(a+b) = \varphi(a)+\varphi(b) = \varphi(b) \in \varphi[N]$. Hence $a+b \in \varphi^{-1}\big(\varphi[N]\big)$. This shows that $(\operatorname{Ker}\varphi) + N \subseteq \varphi^{-1}\big(\varphi[N]\big)$. Next, let $x \in \varphi^{-1}\big(\varphi[N]\big)$. Then $\varphi(x) \in \varphi[N]$, so $\varphi(x) = \varphi(n)$ for some $n \in N$. Thus $\varphi(x - n) = 0$, i.e., $x - n \in \operatorname{Ker}\varphi$. Hence $x = (x-n)+n \in (\operatorname{Ker}\varphi)+N$. Therefore $\varphi^{-1}\big(\varphi[N]\big) \subseteq (\operatorname{Ker}\varphi)+N$, so that $\varphi^{-1}\big(\varphi[N]\big) = (\operatorname{Ker}\varphi) + N$. Then if N contains $\operatorname{Ker}\varphi$ then it is obvious that $\varphi^{-1}\big(\varphi[N]\big) = N$.

(4) Assume that N' is a subskewmodule of M'. Let $x \in \operatorname{Ker}\varphi$. Then $\varphi(x) = 0 \in N'$, so that $x \in \varphi^{-1}[N']$. Hence $\operatorname{Ker}\varphi \subseteq \varphi^{-1}[N']$. Let $x, y \in \varphi^{-1}[N']$ and $r \in R$. Then $\varphi(x), \varphi(y) \in N'$. So that $\varphi(x - y) = \varphi(x) - \varphi(y) \in N'$ since $N' < M'$. Hence $x - y \in \varphi^{-1}[N']$. Next, $\varphi(rx) = r\varphi(x) \in N'$ since $N' < M'$.

Then $rx \in \varphi^{-1}[N']$. Therefore $\varphi^{-1}[N']$ is a subskewmodule of M.

(5) Assume that N' is a normal subskewmodule of M'. By (4), we already proved $\operatorname{Ker}\varphi \subseteq \varphi^{-1}[N'] < M$. Let $x \in \varphi^{-1}[N']$ and $m \in M$. Then $\varphi(x) \in N'$. Since $N' \lhd M'$ and $\varphi(m) \in M'$, it follows that $\varphi(m) + \varphi(x) - \varphi(m) \in N'$. Hence $\varphi(m+x-m) = \varphi(m)+\varphi(x)-\varphi(m) \in N'$. Thus $m+x-m \in \varphi^{-1}[N']$. Therefore $\varphi^{-1}[N']$ is a normal subskewmodule of M.

□

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CHAPTER III

JORDAN HÖLDER THEOREM

In this chapter, we discuss the basic Isomorphism Theorems and generalize Schreier's Theorem and Jordan Hölder Theorem of modules to skewmodules.

Theorem 3.1. Let M, M', N, N' be R-skewmodules and $f : M \to N$ an R-homomorphism.

(1) If $g : M \to M'$ is an epimorphism with $\operatorname{Ker} g \subseteq \operatorname{Ker} f$, then there exists a unique R-homomorphism $h : M' \to N$ such that $f = h \circ g$. Moreover, $\operatorname{Ker} h = g[\operatorname{Ker} f]$ and $\operatorname{Im} h = \operatorname{Im} f$, so that h is a monomorphism if and only if $\operatorname{Ker} g = \operatorname{Ker} f$ and h is an epimorphism if and only if f is an epimorphism.

(2) If $g : N' \to N$ is a monomorphism with $\operatorname{Im} f \subseteq \operatorname{Im} g$, then there exists a unique R-homomorphism $h : M \to N'$ such that $f = g \circ h$. Moreover, $\operatorname{Ker} h = \operatorname{Ker} f$ and $\operatorname{Im} h = g^{-1}[\operatorname{Im} f]$, so that h is a monomorphism if and only if f is a monomorphism and h is an epimorphism if and only if $\operatorname{Im} g = \operatorname{Im} f$.

Proof. (1) Assume that $g : M \to M'$ is an epimorphism with $\operatorname{Ker} g \subseteq \operatorname{Ker} f$. For each $m' \in M'$, there exists $m \in M$ such that $g(m) = m'$ since g is onto. Then we define $h : M' \to N$ by

$$h(m') = f(m) \quad \text{for all } m' \in M'.$$

To show that h is well-defined, let $m_1, m_2 \in M$ be such that $g(m_1) = g(m_2)$. We must show that $f(m_1) = f(m_2)$. Since $g(m_1 - m_2) = g(m_1) - g(m_2) = 0$, $m_1 - m_2 \in \operatorname{Ker} g \subseteq \operatorname{Ker} f$. Hence $f(m_1 - m_2) = 0$ and then $f(m_1) = f(m_2)$.

Thus h is well-defined, and it is clear that $f = h \circ g$. Moreover, it is easy to prove that h is an R-homomorphism and it is unique.

Next, we show that $\operatorname{Ker} h = g[\operatorname{Ker} f]$. Let $x \in \operatorname{Ker} h \subseteq M'$. Then $h(x) = 0$ and, since g is onto, $g(m) = x$ for some $m \in M$. Thus $f(m) = (h \circ g)(m) = h(g(m)) = h(x) = 0$, i.e., $m \in \operatorname{Ker} f$. Hence $x = g(m) \in g[\operatorname{Ker} f]$. Now, let $y \in g[\operatorname{Ker} f]$. Then $g(x) = y$ for some $x \in \operatorname{Ker} f$. Thus $h(y) = h \circ g(x) = f(x) = 0$, so that $y \in \operatorname{Ker} h$. Hence $\operatorname{Ker} h = g[\operatorname{Ker} f]$.

It is easy to prove that $\operatorname{Im} f = \operatorname{Im} h$, so that h is an epimorphism if and only if f is an epimorphism. Hence it remains to show that h is a monomorphism if and only if $\operatorname{Ker} g = \operatorname{Ker} f$. First, assume that h is a monomorphism. Let $x \in \operatorname{Ker} f$. Then $h(g(x)) = f(x) = 0$. Since h is a monomorphism, $g(x) = 0$. It follows that $x \in \operatorname{Ker} g$. This shows that $\operatorname{Ker} f \subseteq \operatorname{Ker} g$. By the assumption, we can conclude that $\operatorname{Ker} f = \operatorname{Ker} g$.

Conversely, assume that $\operatorname{Ker} f = \operatorname{Ker} g$ and let $x \in M'$ be such that $h(x) = 0$. Since g is onto, there exists $m \in M$ such that $g(m) = x$. Thus $f(m) = h \circ g(m) = h(x) = 0$. Hence $m \in \operatorname{Ker} f = \operatorname{Ker} g$, so that $x = g(m) = 0$. Therefore h is a monomorphism.

(2) Assume that $g : N' \to N$ is a monomorphism with $\operatorname{Im} f \subseteq \operatorname{Im} g$. We claim that for each $m \in M$ there exists a unique $m' \in N'$ such that $g(m') = f(m)$. Let $m \in M$. Then $f(m) \in \operatorname{Im} f \subseteq \operatorname{Im} g$. Thus there exists $m' \in N'$ such that $g(m') = f(m)$. Let $n' \in N'$ be such that $g(n') = f(m)$. Then $g(n') = g(m')$. Since g is 1-1, it follows that $n' = m'$. Now, the claim is proved. Next, define $h : M \to N'$ by

$$h(m) = g^{-1}\big(f(m)\big) \text{ for all } m \in M.$$

By the claim, h is well-defined, and it is clear that $f = g \circ h$. It is routine to check that h is an R-homomorphism. To prove the uniqueness of h, let $k : M \to N'$

be an R-homomorphism such that $f = g \circ k$. Then $g\big(h(m)\big) = g\Big(g^{-1}\big(f(m)\big)\Big) = f(m) = g(k(m))$. Since g is 1-1, $h(m) = k(m)$. This proves that $h = k$.

To show that $\operatorname{Ker} h = \operatorname{Ker} f$, first, let $x \in \operatorname{Ker} h$. Then $h(x) = 0$. But $h(x) = g^{-1}\big(f(x)\big)$, so that $f(x) = g\big(h(x)\big) = g(0) = 0$. Thus $x \in \operatorname{Ker} f$. Next, let $x \in \operatorname{Ker} f \subseteq M$. Then $f(x) = 0$. We obtain that $h(x) = g^{-1}\big(f(x)\big) = g^{-1}(0) = 0$ since g is 1-1. Thus $x \in \operatorname{Ker} h$. This shows that $\operatorname{Ker} f = \operatorname{Ker} h$. Moreover, it is easy to prove that $\operatorname{Im} h = g^{-1}[\operatorname{Im} f]$.

To prove that h is an epimorphism if and only if $\operatorname{Im} f = \operatorname{Im} g$, first, assume that h is an epimorphism. By the assumption, we have that $\operatorname{Im} f \subseteq \operatorname{Im} g$. Next, let $n \in \operatorname{Im} g$. Then $g(n') = n$ for some $n' \in N'$. Since $h : M \to N'$ is an epimorphism, there exists $m \in M$ such that $h(m) = n'$. But $h(m) = g^{-1}\big(f(m)\big)$, so that $f(m) = g\big(h(m)\big) = g(n') = n$. Then $n \in \operatorname{Im} f$. We obtain that $\operatorname{Im} f = \operatorname{Im} g$. It is clear that if $\operatorname{Im} f = \operatorname{Im} g$, then h is an epimorphism.

□

Corollary 3.2. Let M, N be R-skewmodules and $\varphi : M \to N$ an R-homomorphism. Then $M/\operatorname{Ker} \varphi \cong \operatorname{Im} \varphi$.

Proof. Let $\pi : M \to M/\operatorname{Ker} \varphi$ be the canonical projection. Then π is an epimorphism and $\operatorname{Ker} \pi = \operatorname{Ker} \varphi$. By Theorem 3.1, there exists a unique R-homomorphism $h : M/\operatorname{Ker} \varphi \to N$ such that $\operatorname{Im} h = \operatorname{Im} \varphi$. Moreover, h is a monomorphism since $\operatorname{Ker} \pi = \operatorname{Ker} \varphi$. Then $M/\operatorname{Ker} \varphi \cong \operatorname{Im} h = \operatorname{Im} \varphi$.

□

Corollary 3.3. Let M be an R-skewmodule and P and N normal subskewmodules of M such that $P \subseteq N$. Then $M/N \cong (M/P)\big/(N/P)$.

Proof. Define $\varphi : M/P \to M/N$ by

$$\varphi(m + P) = m + N \quad \text{for all } m \in M.$$

Since $P \subseteq N$, we obtain that φ is well-defined, and it is easy to prove that φ is an epimorphism. Next, we show that $\operatorname{Ker}\varphi = N/P$. Let $m \in M$ be such that $N = \varphi(m+P) = m+N$. Then $m \in N$. Thus $m+P \in N/P$. This proves that $\operatorname{Ker}\varphi \subseteq N/P$. Next, let $n \in N$. Then $\varphi(n+P) = n+N = N$. Thus $n+P \in \operatorname{Ker}\varphi$. Hence $\operatorname{Ker}\varphi = N/P$. By Corollary 3.2, $M/N \cong (M/P)\big/(N/P)$.

□

Corollary 3.4. Let M be an R-skewmodule and P and N subskewmodules of M such that P is normal. Then $N/N \cap P \cong (N+P)/P$.

Proof. Assume that P and N are subskewmodules of M such that $P \lhd M$. By Lemma 2.29 (2) and (3), we have $N \cap P \lhd N$ and $N + P < M$, respectively. Since $P \lhd M$, we obtain that $P \lhd (N+P)$. Next, define $\varphi : N \to (N+P)/P$ by

$$\varphi(n) = n + P \quad \text{for all } n \in N.$$

Clearly, φ is an R-homomorphism. To prove that φ is onto, let $k \in N+P$. Then $k = n+p$ for some $n \in N$ and $p \in P$. Thus $k+P = (n+p)+P = n+P$, so that $\varphi(n) = n+P = k+P$. Hence φ is onto. It is easy to show that $\operatorname{Ker}\varphi = N \cap P$. By Corollary 3.2, $N/N \cap P \cong (N+P)/P$.

□

Corollary 3.5. Let M, N be R-skewmodules and L a normal subskewmodule of N. If $\varphi : M \to N$ is an epimorphism, then $M/\varphi^{-1}[L] \cong N/L$.

Proof. By Lemma 2.35 (5), $\varphi^{-1}[L]$ is a normal subskewmodule of M. Define $f : M \to N/L$ by

$$f(m) = \varphi(m) + L \quad \text{for all } \ m \in M.$$

Since φ is an epimorphism, f is also an epimorphism. To show that $\operatorname{Ker} f = \varphi^{-1}[L]$, let $m \in \varphi^{-1}[L]$. Then $\varphi(m) \in L$. Thus $f(m) = \varphi(m) + L = L$

which is the zero in N/L. Hence $m \in \operatorname{Ker} f$. Next, let $m \in M$ be such that $L = f(m) = \varphi(m) + L$. Then $\varphi(m) \in L$. Thus $m \in \varphi^{-1}[L]$. We obtain that $\operatorname{Ker}\ f = \varphi^{-1}[L]$. By Corollary 3.2, $M/\varphi^{-1}[L] \cong N/L$.

□

The following theorem is generalized from the butterfly of Zazzenhaus Theorem of modules.

Theorem 3.6. Let M be an R-skewmodule and $N, P, N'$and P' subskewmodules of M such that $N \lhd P$ and $N' \lhd P'$. Then

(1) $N + (P \cap N')$ is a normal subskewmodule of $N + (P \cap P')$;

(2) $N' + (P' \cap N)$ is a normal subskewmodule of $N' + (P \cap P')$;

(3) $[N + (P \cap P')]/[N + (P \cap N')] \cong [N' + (P \cap P')]/[N' + (P' \cap N)]$.

Proof. Assume that $N, P, N'$and P' are subskewmodules of M such that $N \lhd P$ and $N' \lhd P'$

(1) Clearly, $N + (P \cap N')$ is a subskewmodule of $N + (P \cap P')$. Let $n + k \in N + (P \cap N')$ and $n' + l \in N + (P \cap P')$ be such that $n, n' \in N$, $k \in P \cap N'$ and

$l \in P \cap P'$. Then

$$\begin{aligned}(n'+l)+(n+k)-(n'+l) &= n'+l+n+k-l-n' \\ &= n'+l+n+\overline{n}+k-l \qquad \text{for some } \overline{n} \in N \\ &= n'+n''+l+k-l \qquad \text{for some } n'' \in N\end{aligned}$$

The second equality holds because $N \lhd P$ and $k-l \in P$, and the last one holds because $N \lhd P$ and $l \in P$. Since $l, k \in P$, we have $l+k-l \in P$, and since $k \in N'$, $l \in P'$ and $N' \lhd P'$, we also have $l+k-l \in N'$. Hence $(n'+l)+(n+k)-(n'+l) = (n'+n'')+(l+k-l) \in N+(P \cap N')$. Therefore $N+(P \cap N')$ is a normal subskewmodule of $N+(P \cap P')$.

(2) The proof is similar to the proof of (1).

(3) First, we prove that

$$[N+(P \cap P')]/[N+(P \cap N')] \cong [P \cap P']/[(P' \cap N)+(P \cap N')].$$

Since $P' \cap N \subseteq P \cap P'$ and $N \lhd P$, we obtain that $P' \cap N \lhd P \cap P'$, Moreover, since $P \cap N' \subseteq P \cap P'$ and $N' \lhd P'$, we have $P \cap N' \lhd P \cap P'$. By Lemma 2.29(4), $(P' \cap N)+(P \cap N')$ is a normal subskewmodule of $P \cap P'$.

Let $K = (P' \cap N)+(P \cap N')$. Define $\varphi : N+(P \cap P') \to (P \cap P')/K$ by

$$\varphi(n+q) = q+K \quad \text{for all } n \in N \text{ and } q \in P \cap P'.$$

To show that φ is well-defined, let $n_1, n_2 \in N$ and $q_1, q_2 \in P \cap P'$ be such that $n_1+q_1 = n_2+q_2$. Then $q_1-q_2 = n_2-n_1 \in (P \cap P') \cap N \subseteq P' \cap N \subseteq (P' \cap N)+(P \cap N') = K$. Thus $q_1+K = q_2+K$. Hence φ is well-defined.

To prove that φ is an R-homomorphism, let $n_1, n_2 \in N$, $q_1, q_2 \in P \cap P'$ and $r \in R$. Then

$$
\begin{aligned}
\varphi\big((n_1+q_1)+(n_2+q_2)\big) &= \varphi(n_1+q_1+n_2+q_2)\\
&= \varphi(n_1+n_2'+q_1+q_2) \qquad \text{for some } n_2' \in N\\
&= (q_1+q_2)+K\\
&= (q_1+K)+(q_2+K)\\
&= \varphi(n_1+q_1)+\varphi(n_2+q_2).
\end{aligned}
$$

The second equality holds because $q_1 \in P$, $n_2 \in N$ and $N \lhd P$, and we also obtain that $\varphi\big(r(n_1+q_1)\big) = \varphi(rn_1+rq_1) = rq_1+K = r(q_1+K) = r\varphi(n_1+q_1)$. Hence φ is an R-homomorphism.

For each $q \in P \cap P'$, $\varphi(0+q) = q+K$ since $0 \in N$, so that φ is onto. Next, we prove that $\operatorname{Ker}\varphi = N+(P\cap N')$. Let $n \in N$ and $q \in P\cap P'$ be such that $\varphi(n+q) = K$. Then $q+K = \varphi(n+q) = K$. Thus $q \in K = (P'\cap N)+(P\cap N') \subseteq N+(P\cap N')$. Next, let $n+q \in N+(P\cap N')$ be such that $n \in N$ and $q \in P\cap N'$. Then $\varphi(n+q) = q+K = K$ since $q = 0+q \in (P'\cap N)+(P\cap N') = K$. Thus $n+q \in \operatorname{Ker}\ \varphi$. Hence $\operatorname{Ker}\varphi = N+(P\cap N')$. By Corollary 3.2, $[N+(P\cap P')]/[N+(P\cap N')] \cong [P\cap P']/[(P'\cap N)+(P\cap N')]$. Similarly, we prove that $[N'+(P\cap P')]/[N'+(P'\cap N] \cong [P\cap P']/[(P'\cap N)+(P\cap N')]$. Therefore the result is proved.

□

Remark 3.7. Let M, N be R-skewmodules and L a normal subskewmodule of M. If $f: M \to N$ is an R-isomorphism,then $N/f[L] \cong M/L$.

The proof of the following two theorems are similar to the analogous Theorems in Module Theory.

Theorem 3.8. Let M be an R-skewmodule and N a normal subskewmodule of M. Then there is an inclusion-preserving bijection from the set of subskewmodules of M/N to the set of subskewmodules of M containing N.

Theorem 3.9. Let M be an R-skewmodule and N a normal subskewmodule of M. Then there is an inclusion-preserving bijection from the set of normal subskewmodules of M/N to the set of normal subskewmodules of M containing N.

Definition 3.10. Let M be an R-skewmodule and let

$$C : M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_r \text{ and } C' : M = M'_0 \supseteq M'_1 \supseteq \ldots \supseteq M'_s$$

be two decreasing finite chains of subskewmodules of M. We say that C' is a **refinement** of C if every member of C occurs in C'; if $C \neq C'$, then C is a **proper refinement** of C.

Definition 3.11. Let M be an R-skewmodule. A finite chain of subskewmodules $M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_r$ is called a **finite subnormal series** of M if $M_i \lhd M_{i-1}$ for all $i = 1, 2, \ldots, r$.

Let $M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_r$ be a finite subnormal series of an R-skewmodule M. The quotient skewmodule M_{i-1}/M_i is called the **factor** of the series. The **length** of this series is the number of nontrivial factors M_{i-1}/M_i. A finite subnormal series such that $M_i \lhd M$ for all $i = 1, 2, \ldots r$ is said to be a **finite normal series**.

Definition 3.12. A strictly decreasing finite subnormal series $C : M = M_0 \supset M_1 \supset \ldots \supset M_n = \{0\}$ is called a **composition series** of an R-skewmodule M if C has no proper refinement.

Definition 3.13. Let M be an R-skewmodule and

$$C : M = M_0 \supset M_1 \supset \ldots \supset M_r = \{0\} \qquad \text{and}$$

$$C' : M = M'_0 \supset M'_1 \supset \ldots \supset M'_s = \{0\}$$

two strictly decreasing finite subnormal series of M. Then C and C' are called **equivalent**, denoted by $C \equiv C'$, if $r = s$ and there exists a permutation π of $\{0, 1, \ldots, r-1\}$ such that $M'_i/M'_{i+1} \cong M_{\pi(i)}/M_{\pi(i)+1}$ for all $i = 0, 1, \ldots, r-1$.

Definition 3.14. Let M be an R-skewmodule and $C : M = M_0 \supseteq M_1 \supseteq \ldots$ a chain of subskewmodules of M. Let $r_1 < r_2 < \ldots < r_n < \ldots$ be a strictly increasing sequence of natural numbers. Then the chain C' given by $M_{r_1} \supseteq M_{r_2} \supseteq \ldots \supseteq M_{r_n} \supseteq \ldots$ is called a **subchain** of C.

The following lemma is generalized from Schreier's Theorem of modules in [5].

Lemma 3.15. Any two strictly decreasing finite subnormal series of an R-skewmodule M have equivalent refinements.

Proof. Let M be an R-skewmodule and

$$C : M = M_0 \supset M_1 \supset \ldots \supset M_r = \{0\} \ \ \text{and}$$

$$C' : M = M'_0 \supset M'_1 \supset \ldots \supset M'_s = \{0\}$$

two strictly decreasing finite subnormal series of M. Define

$$M_{i,0} = M_{i-1} = M_{i-1,s} \quad ; \quad M'_{j,0} = M'_{j-1} = M'_{j-1,r} \ ,$$
$$M_{i,j} = M_i + (M_{i-1} \cap M'_j) \quad \text{and} \quad M'_{j,i} = M'_j + (M'_{j-1} \cap M_i)$$

for all $i = 1, 2, \ldots, r$, for all $j = 1, 2, \ldots, s$. Then we obtain

$$C_1 : M = M_0 = M_{1,0} \supseteq M_{1,1} \supseteq M_{1,2} \supseteq \ldots \supseteq M_{1,s} = M_1 = M_{2,0} \supseteq$$
$$M_{2,1} \supseteq \ldots \supseteq M_{r,s} = \{0\} \text{ and}$$
$$C_2 : M = M'_0 = M'_{1,0} \supseteq M'_{1,1} \supseteq M'_{1,2} \supseteq \ldots \supseteq M'_{1,r} = M'_1 = M'_{2,0} \supseteq$$
$$M'_{2,1} \supseteq \ldots \supseteq M'_{s,r} = \{0\}.$$

We claim that C_1 and C_2 are decreasing finite subnormal series of M. For each $i = 1, 2, \ldots, r$, Theorem 3.6 shows that

$$M_i + (M_{i-1} \cap M'_j) \lhd M_i + (M_{i-1} \cap M'_{j-1}) \text{ since } M'_j \lhd M'_{j-1}.$$

Thus we have the claim for C_1. Similarly, we have the claim for C_2. Note that C_1 and C_2 are refinement of C and C', respectively. By Theorem 3.6, we obtain that

$$\begin{aligned} M_{i,j}/M_{i,j+1} &= [M_i + (M_{i-1} \cap M'_j)]/[M_i + (M_{i-1} \cap M'_{j+1})] \\ &\cong [M'_{j+1} + (M'_j \cap M_{i-1})]/[M'_{j+1} + (M'_j \cap M_i)] \\ &= M'_{j+1,i-1}/M'_{j+1,i} \end{aligned}$$

for all $i = 1, 2, \ldots, r$ and $j = 0, 1, \ldots, s-1$. Hence it follows that $M_{i,j} = M_{i,j+1}$ if and only if $M'_{j+1,i-1} = M'_{j+1,i}$. Let $\overline{C}_1$ be a series obtained from C_1 by dropping every skewmodules which is equal to its predecessor and $\overline{C}_2$ a series obtained in the similar way to $\overline{C}_1$ from C_2. Hence $\overline{C}_1 \equiv \overline{C}_2$.

□

The next theorem is generalized from Jordan Hölder Theorem of modules in [5].

Theorem 3.16. If an R-skewmodule M has composition series, then

(1) any strictly decreasing subnormal series of M is finite and admits a refinement which is a composition series and

(2) any two composition series of M are equivalent.

Proof. (1) Let C_1 be a composition series of M and C a strictly decreasing subnormal series of M. We prove that C is finite. Let C_2 be a finite subchain of C. By Lemma 3.15, there exist finite chains C_1' and C_2' such that C_1' and C_2' are refinements of C_1 and C_2, respectively, and $C_1' \equiv C_2'$. Since C_1 is a composition series, $C_1 \equiv C_1'$. Hence $C_2' \equiv C_1$. These equivalences show that C_2' is a composition series and, also, it is a refinement of C. Then C is finite.

(2) By the definition of a composition series, any refinement is equivalent to itself. Thus the theorem holds by Lemma 3.15.

□

CHAPTER IV

ARTINIAN AND NOETHERIAN SKEWMODULES

In this chapter, we study artinian and noetherian modules in [2] and [4] and generalize some theorems to skewmodules. Furthermore, we prove the relation between artinian, noetherian skewmodules and the composition series.

Definition 4.1. An R-skewmodule M is said to be **artinian** if for every decreasing normal series $M_1 \supseteq M_2 \supseteq \dots$, there exists an integer n such that $M_i = M_n$ for all $i \geq n$.

An R-skewmodule M is said to be **noetherian** if for every increasing normal series $M_1 \subseteq M_2 \subseteq \dots$, there exists an integer n such that $M_i = M_n$ for all $i \geq n$.

Theorem 4.2. Let M be an R-skewmodule. Then M is artinian (noetherian) if and only if for every nonempty collection of normal subskewmodules of M has a minimal (maximal) element.

Proof. Assume that M is artinian and $\mathcal{A}$ a nonempty set of normal subskewmodules of M. Then we choose $N_1 \in \mathcal{A}$. If N_1 is not minimal, then there exists $N_2 \in \mathcal{A}$ such that $N_1 \supset N_2$. If we choose $N_i \in \mathcal{A}$ which is not minimal, then there exists an $N_{i+1} \in \mathcal{A}$ such that $N_i \supset N_{i+1}$. After a finite step, we obtain a minimal element of $\mathcal{A}$, otherwise we would have a chain of normal subskewmodules of M such that $N_1 \supset N_2 \supset N_3 \supset \dots$ which contradicts the assumption that M is artinian.

Conversely, assume that every nonempty collection of normal subskewmodules of M has a minimal element. Let $N_1 \supseteq N_2 \supseteq N_3 \supseteq \ldots$ be a decreasing normal series of M. Then the set $\{N_1, N_2, \ldots\}$ has a minimal element, say N_k. By the minimality of N_k, we have $N_k = N_{k+i}$ for all $i \in \mathbb{N}$. Thus M is artinian.

□

Theorem 4.3. Let M be an R-skewmodule. If every normal subskewmodule of M is finitely generated, then M is noetherian.

Proof. Let $M_1 \subseteq M_2 \subseteq \ldots$ be an increasing normal series of M. Clearly, $\bigcup_{i \geq 1} M_i \lhd M$. Let $P = \bigcup_{i \geq 1} M_i$. By the assumption, P is finitely generated, say by $m_1, m_2, \ldots, m_k$. Since m_j is an element of some M_k for all j, there exists an $n_0 \in \mathbb{N}$ such that $m_j \in M_{n_0}$ for all $j = 1, 2, \ldots, k$. Hence $P \subseteq M_{n_0}$. Thus, for all $l \geq n_0$, we have $M_{n_0} \subseteq M_l$ by the hypothesis and $M_l \subseteq P \subseteq M_{n_0}$. Then $M_{n_0} = M_l$ for all $l \geq n_0$. Therefore M is noetherian.

□

Theorem 4.4. Let N be a normal subskewmodule of an R-skewmodule M. If M is artinian (noetherian), then the following statements hold.

(1) For every chain $N_1 \supseteq N_2 \supseteq \ldots$ $(N_1 \subseteq N_2 \subseteq \ldots)$ of subskewmodules of N such that $N_i \lhd M$ for all $i \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $N_k = N_{k+i}$ for all $i \in \mathbb{N}$.

(2) The quotient skewmodule M/N is artinian (noetherian).

Proof. Assume that M is artinian and N is a normal subskewmodule of M.

(1) Let $C : N_1 \supseteq N_2 \supseteq \ldots$ be a chain of subskewmodules of N such that $N_i \lhd M$ for all $i \in \mathbb{N}$. Then C is a decreasing normal series of M. Since M is artinian, there exists a $k \in \mathbb{N}$ such that $N_k = N_{k+i}$ for all $i \in \mathbb{N}$.

(2) This follows immediately by Theorem 3.9. □

Theorem 4.5. Let N be a normal subskewmodule of an R-skewmodule M. If N and M/N are artinian (noetherian), then M is artinian(noetherian).

Proof. Assume that N and M/N are artinian. Let $D_1 \supseteq D_2 \supseteq \dots$ be a decreasing sequence of normal subskewmodules of M. Let $\pi : M \rightarrow M/N$ be the canonical projection. Then $D_1 \cap N \supseteq D_2 \cap N \supseteq \dots$ and $\pi(D_1) \supseteq \pi(D_2) \supseteq \dots$ are decreasing sequences of normal subskewmodules of N and M/N, respectively. By the assumption, there exists an $n_0 \in \mathbb{N}$ such that $D_n \cap N = D_{n_0} \cap N$ and $\pi(D_n) = \pi(D_{n_0})$ for all $n \geq n_0$.

We claim that $D_n = D_{n_0}$ for all $n \geq n_0$. Let $n \geq n_0$. We know from the assumption, $D_n \subseteq D_{n_0}$. It remains to show that $D_{n_0} \subseteq D_n$. Let $x \in D_{n_0}$. Since $\pi(D_n) = \pi(D_{n_0})$, there exists a $y \in D_n$ such that $\pi(x) = \pi(y)$, that is, $x - y \in \operatorname{Ker} \pi = N$. Since $y \in D_n \subseteq D_{n_0}$, it follows that $x - y \in D_{n_0} \cap N = D_n \cap N \subseteq D_n$. Thus $x \in y + D_n = D_n$. Hence $D_{n_0} \subseteq D_n$. Thus we obtain the claim. This shows that M is artinian.

The proof for the noetherian case is similar . □

Theorem 4.6. Let M be an R-skewmodule. If M is both artinian and noetherian, then M has a composition series.

Proof. Assume that M is both artinian and noetherian. Let C be the collection of all normal subskewmodules of M that have a composition series. Clearly, $\{0\} \in C$. Thus $C \neq \emptyset$. Note that C has a maximal element, say M^*, since M is noetherian. We now show that $M^* = M$. Suppose that $M^* \neq M$. Then M/M^* is not the zero skewmodule. Let $M/M^* = M_0/M^* \supseteq M_1/M^* \supseteq \dots$ be decreasing nonzero normal series of M/M^*. Since M is artinian, so is M/M^* by Theorem 4.4(2). Then there exists an integer p such that $M_p/M^* = M_{p+i}/M^*$ for all $i \in \mathbb{N}$. We can choose $M^{**} \lhd M$ such that $M^* \subset M^{**} \subseteq M$ and M^{**}/M^* is simple. Since

$M^* \in C$, it has a composition series : $M^* \supset M_1^* \supset M_2^* \supset \ldots \supset M_n^* = \{0\}$. Since M^{**}/M^* is simple, $M^{**} \supset M^* \supset M_1^* \supset M_2^* \supset \ldots \supset M_n^* = \{0\}$ is a composition series of M^{**}. Hence $M^{**} \in C$ which contradicts the maximality of M^*. Hence $M^* = M$, whence M has a composition series.

□

Theorem 4.7. Let M be an R-skewmodule. If M has a composition series which is a normal series then M is both artinian and noetherian.

Proof. Assume that M has a composition series which is a normal series and let n be its length. We prove that M is both artinian and noetherian by induction on n. Clearly, if $n = 0$ then $M = \{0\}$ and there is nothing to prove. Assume that the result is true for all R-skewmodules having composition series which is a normal series of length less than $n > 1$.

Let M be an R-skewmodule having a composition series which is a normal series of length n, say $M = M_0 \supset M_1 \supset \ldots \supset M_{n-1} \supset M_n = \{0\}$. Then we observe that

$$M/M_{n-1} = M_0/M_{n-1} \supset M_1/M_{n-1} \supset \ldots \supset M_{n-1}/M_{n-1} = \{0\} \;.....\circledast$$

By Corollary 3.3, $(M_i/M_{n-1})\Big/(M_{i+1}/M_{n-1}) \cong M_i/M_{i+1}$ for all $i = 0, 1, \ldots, n-2$. Since M_i/M_{i+1} is simple, so is $(M_i/M_{n-1})\Big/(M_{i+1}/M_{n-1})$ and we also obtain that the inclusions in the claim $\circledast$ are strict. Then $\circledast$ is a composition series which is a normal series of M/M_{n-1} with length $n-1$. By the induction hypothesis, M/M_{n-1} is both artinian and noetherian. Since $M = M_0 \supset M_1 \supset \ldots \supset M_{n-1} \supset M_n = \{0\}$ is a composition series, M_{n-1} is simple. Then M_{n-1} is trivially both artinian and noetherian. Since M/M_{n-1} and M_{n-1} are both artinion and noetherian and by Theorem 4.5, we deduce that M is both artinian and noetherian.

This then shows that the result holds for all skewmodules of length n and complete the induction.

□

Theorem 4.8. Let M be an R-skewmodule. If M can be written as $M = M_1 + M_2 + \ldots + M_n$ where each M_i is artinian (noetherian) and $M_n \lhd M$, then M is artinian (noetherian).

Proof. It is enough to consider the case $n = 2$. By Corollary 3.4,

$$M/M_2 = (M_1 + M_2)/M_2 \cong M_1/(M_1 \cap M_2).$$

Since M_1 is artinian, so is $M_1/(M_1 \cap M_2)$ by Theorem 4.4 (2). Then M/M_2 is also artinian. Since M_2 is artinian, by Theorem 4.5, we deduce that M is artinian.

□

Theorem 4.9. Let M be an R-skewmodule and $f : M \to M$ an R-homomorphism. For each $p \in \mathbb{N}$, let a positive integer. Let $I_p = \mathrm{Im}\,(f^p)$ and $N_p = \mathrm{Ker}\,(f^p)$. Then the following statements hold.

(1) $I_1 = I_2$ implies that $I_1 + N_1 = M = N_1 + I_1$ and

$N_1 = N_2$ implies that $I_1 \cap N_1 = \{0\}$.

(2) If M is artinian and $I_p \lhd M$ for all $p \in \mathbb{N}$, then

(2.1) there exists an $r \in \mathbb{N}$ such that $M = I_k + N_k$ for all $k \geq r$,

(2.2) f is a monomorphism implies that f is an epimorphism.

(3) If M is noetherian, then

(3.1) there exists an $r \in \mathbb{N}$ such that $I_k \cap N_k = \{0\}$ for all $k \geq r$,

(3.2) f is an epimorphism implies that f is a monomorphism.

Proof. Assume that $f : M \to M$ an R-homomorphism. For each $p \in \mathbb{N}$, let $I_p = \operatorname{Im}(f^p)$ and $N_p = \operatorname{Ker}(f^p)$.

(1) Assume that $I_1 = I_2$. Let $x \in M$. Then there exists a $y \in M$ such that $f(x) = f^2(y)$. So $f\big(f(y) - x\big) = f^2(y) - f(x) = 0$ implies that $f(y) - x \in \operatorname{Ker} f = N_1$. But $x = f(y) - \big(f(y) - x\big) \in I_1 + N_1$. Hence $M = I_1 + N_1$. Similarly, $M = N_1 + I_1$.

Assume that $N_1 = N_2$. Let $x \in I_1 \cap N_1$. That is, $x \in \operatorname{Im} f \cap \operatorname{Ker} f$. Then $f(x) = 0$ and $x = f(a)$ for some $a \in M$. Thus $f^2(a) = f\big(f(a)\big) = f(x) = 0$. Hence $a \in \operatorname{Ker} f^2 = N_2 = N_1 = \operatorname{Ker} f$. We obtain that $f(a) = 0$ and then $x = f(a) = 0$. This shows that $I_1 \cap N_1 \subseteq \{0\}$. Therefore $I_1 \cap N_1 = \{0\}$.

(2) Assume that M is artinian and $I_p \lhd M$ for all $p \in \mathbb{N}$.

(2.1) We observe that $I_1 \supseteq I_2 \supseteq \ldots$ is a decreasing normal series of M. Since M is artinian, there exists an $r \in \mathbb{N}$ such that $I_k = I_{2k}$ for all $k \geq r$. We apply (1) to f^k. Then we have $M = I_k + N_k$ for all $k \geq r$.

(2.2) Assume that f is a monomorphism. By the hypothesis and (2.1), there exists an $r \in \mathbb{N}$ such that $M = I_r + N_r$. Since f is a monomorphism, so is f^r. Hence $N_r = \operatorname{Ker}(f^r) = \{0\}$. Then $M = I_r$. From $M \supseteq I_1 \supseteq I_2 \supseteq \ldots \supseteq I_r = M$, it follows that $M = I_1 = \operatorname{Im} f$. Thus f is an epimorphism.

(3) Assume that M is noetherian.

(3.1) We observe that $N_1 \subseteq N_2 \subseteq \ldots$ is an increasing normal series of M. Then there exists an $r \in \mathbb{N}$ such that $N_k = N_{2k}$ for all $k \geq r$. We apply (1) to f^k. So $I_k \cap N_k = \{0\}$ for all $k \geq r$.

(3.2) Assume that f is an epimorphism. By the hypothesis and (3.1), there exists an $r \in \mathbb{N}$ such that $I_r \cap N_r = \{0\}$. Since f is an epimorphism, so is f^r. Hence $I_r = M$, then $N_r = \{0\}$. From $0 \subseteq N_1 \subseteq N_2 \subseteq \ldots \subseteq N_r = \{0\}$, it follows that $N_1 = \{0\}$. That is, $\operatorname{Ker} f = \{0\}$. Thus f is a monomorphism. $\square$

Definition 4.10. Let M be an R-skewmodule and $\{M_i \mid i \in I\}$ a family of normal subskewmodules of M. Then M is called the **direct sum** of $\{M_i \mid i \in I\}$, denoted by $M = \bigoplus\limits_{i \in I} M_i$, if

(1) for each $m \in M$, there exists an $m_{i_k} \in M_{i_k}$, where $k = 1, 2, \ldots, n$, such that $m = m_{i_1} + m_{i_2} + \ldots + m_{i_n}$ and

(2) for all $i, j \in I$, if $i \neq j$, then $M_i \cap \big(\sum\limits_{j \neq i} M_j\big) = \{0\}$.

Definition 4.11. Let M be an R-skewmodule. Then normal subskewmodules M_1 and M_2 are said to be **supplementary** if $M = M_1 \bigoplus M_2$. A normal subskewmodule N of M is called a **direct summand** if there exists a normal subskewmodule P of M such that N and P are supplementary.

Theorem 4.12. Let M be an R-skewmodule. If M is a sum of a family of its normal simple subskewmodules, then every normal subskewmodule of M is a direct summand.

Proof. Assume that $(M_i)_{i \in I}$ is a family of normal simple subskewmodules of M such that $M = \sum\limits_{i \in I} M_i$. We claim that for each normal subskewmodule N of M there exists a $J \subseteq I$ such that $M = N \oplus \big(\bigoplus\limits_{i \in J} M_i\big)$. If $N = M$, then, clearly, $J = \emptyset$. Suppose that $N \subset M$. Then there exists a $k \in I$ such that $M_k \nsubseteq N$. Since $N \cap M_k \lhd M_k$ and M_k is simple, we deduce that either $N \cap M_k = \{0\}$ or $N \cap M_k = M_k$. But $M_k \nsubseteq N$, so that $N \cap M_k = \{0\}$. That is, $N + M_k$ is a direct sum. Let

$$A = \big\{H \subseteq I \;\big|\; N + \sum_{i \in H} M_i \text{ is direct}\big\}.$$

We have just shown that $A \neq \emptyset$. Let $\subseteq$ be a partially order on A. Let $\mathcal{C}$ be a totally ordered subset of A and let $K^* = \bigcup\limits_{K \in \mathcal{C}} K$. We claim that $K^* \in A$. To see this, we observe that if $x \in \sum_{i \in K^*} M_i$, then $x = m_{i_1} + m_{i_2} + \ldots + m_{i_n}$ where each i_j belongs to some subset I_J of $\mathcal{C}$. Since $\mathcal{C}$ is totally ordered, all the set $I_1, I_2, \ldots, I_n$ are contained in one of them, say I_p. Then $N \cap \sum_{i \in I_p} M_i = \{0\}$ since $I_p \in A$. Hence $N \cap \sum_{i \in K^*} M_i \subseteq N \cap \sum_{i \in I_p} M_i = \{0\}$, so that $N + \sum_{i \in K^*} M_i$ is a direct sum. This shows that $K^* \in A$. Hence K^* is an upper bound of $\mathcal{C}$ in A. By Zorn's Lemma, A has a maximal element, say J.

Next, we show that $N \oplus \big(\bigoplus_{i \in J} M_i\big) = M$. Suppose that $N \oplus \big(\bigoplus_{i \in J} M_i\big) \subset M$. Then there exists a $j \in J$ such that $M_j \nsubseteq N \oplus \big(\bigoplus_{i \in J} M_i\big)$. Since M_j is simple, we deduce that $M_j \cap \Big(N \oplus \big(\bigoplus_{i \in J} M_i\big)\Big) = \{0\}$. Hence $M_j + \Big(N \oplus \big(\bigoplus_{i \in J} M_i\big)\Big)$ is a direct sum. Thus $J \cup \{j\}$ belongs to A which contradicts the maximality of J. Hence $M = N \oplus \big(\bigoplus_{i \in J} M_i\big)$. Therefore the result holds.

□

Corollary 4.13. Let M be an R-skewmodule. Then the followings are equivalent.

(1) M is the sum of a family of normal simple subskewmodules of M.

(2) M is the direct sum of a family of normal simple subskewmodules of M.

Proof. (1)⇒(2) This follows immediately by Theorem 4.12.

(2)⇒(1) This is obvious. □

Theorem 4.14. Let M be an R-skewmodule. If $M = M_1 \oplus M_2$, then $M/M_1 \cong M_2$.

Proof. Let $\pi : M \to M_2$ be a projection mapping. We claim that $\operatorname{Ker}\pi = M_1$. Let $x \in \operatorname{Ker}\pi \subseteq M$. Then $x = m_1 + m_2$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Thus $m_2 = \pi(x) = 0$. So $x = m_1 \in M_1$. Then $\operatorname{Ker}\pi \subseteq M_1$. Moreover, $\pi(x) = 0$ for all $x \in M_1$. Thus $x \in \operatorname{Ker}\pi$. Now, the claim is proved. By Corollary 3.2, $M/M_1 \cong M_2$. □

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