CHAPTER II

QUOTIENTS AND JORDAN-HOLDER THEOREM

We refer the reader to Chapter I (Definition 1.30) for the definition of a quotient skewring. In this chapter we shall give some theorems of quotients. Moreover, we shall generalize the five basic isomorphism theorems of group and ring theory, and the Jordan-Holder Theorem of group theory to skewrings.

Theorem 2.1. ***(First Isomorphism Theorem)***

Let R,S be skewrings and $f:R\to S$ be a homomorphism. If f is surjective, then $R/_{Ker(f)} \cong S$.

Proof. Define $\varphi: R/_{Ker(f)} \to S$ by $\varphi(x+Ker(f)) = f(x)$ for every $x+Ker(f) \in R/_{Ker(f)}$. By definition of quotient skewring and f is an epimorphism, φ is a homomorphism and an isomorphism from $(R/_{Ker(f)},+)$ to $(S,+)$. Hence $R/_{Ker(f)} \cong S$. #

Theorem 2.2. ***(Second Isomorphism Theorem)***

Let R,S be skewrings and $f:R\to S$ be an epimorphism. Let I be a normal ideal in S. Then $R/_{f^{-1}[I]} \cong S/_I$.

Proof. Define $\varphi: R \to S/_I$ by $\varphi(x) = f(x)+I$ for every $x \in R$. By definition of quotient skewring and f is an epimorphism, φ is an epimorphism. By group theory, $Ker(\varphi) = f^{-1}[I]$. By First Isomorphism Theorem, $R/_{f^{-1}[I]} \cong S/_I$. #

Proposition 2.3. *Let R be a skewring and I,J be normal ideals of R such that $I \subseteq J$. Define $\varphi: R/I \to R/J$ by $\varphi(x+I) = x+J$ for every $x+I \in R/I$. Then φ is an epimorphism with $Ker(\varphi) = J/I$.*

Proof. Let $x+I, y+I \in R/I$ such that $x+I = y+I$. Then $x-y \in I \subseteq J$ and $x+J = y+J$, so φ is well-defined. By definition of quotient skewring, φ is a homomorphism. Clearly, φ is surjective. By group theory, $Ker(\varphi) = J/I$.#

Theorem 2.4. *(Third Isomorphism Theorem)*

Let R be a skewring and I_1, I_2 be normal ideals of R such that $I_1 \subseteq I_2$. Then $\dfrac{R/I_1}{I_2/I_1} \cong R/I_2$.

Proof. Define $\varphi: R/I_1 \to R/I_2$ by $\varphi(x+I_1) = x+I_2$ for every $x+I_1 \in R/I_1$. By Proposition 2.3, φ is an epimorphism and $Ker(\varphi) = I_2/I_1$. By the First Isomorphism Theorem, $\dfrac{R/I_1}{I_2/I_1} \cong R/I_2$. #

Let R be a skewring and K be a subskewring of R.

Define $RI_R(K) = \{x \in R / kx \in K$ for every $k \in K\}$ and

$LI_R(K) = \{x \in R / xk \in K$ for every $k \in K\}$.

Since $0 \in RI_R(K) \cap LI_R(K)$, $RI_R(K)$ and $LI_R(K)$ are nonempty sets. We shall show that $RI_R(K)$ and $LI_R(K)$ are normal subskewrings of R. Let $x,y \in RI_R(K)$ and $r \in R$. Then $kx, ky \in K$ for every $k \in K$. Let $k \in K$. Then $k(x-y) = kx-ky$, $k(xy) = (kx)y \in K$, since K is a skewring. Therefore $RI_R(K)$ is a subskewring of R. By Remark 1.5(2), $k(r+x-r) = kr+kx-kr = kx \in K$. So $RI_R(K)$ is normal. Hence $RI_R(K)$ is a normal subskewring of R and is also true for $LI_R(K)$.

Let $I_R(K) = RI_R(K) \cap LI_R(K)$. Clearly, $I_R(K)$ is a normal subskewring of R.

Proposition 2.5. *Let R be a skewring and K be a subskewring of R. Then $I_R(K)$ is the largest subskewring of R having K as an ideal.*

Proof. Clearly, $K \subseteq RI_R(K) \cap LI_R(K) = I_R(K)$. Then K is a subskewring of $I_R(K)$. Let $r \in I_R(K)$ and $k \in K$. Then $r \in RI_R(K)$ and $r \in LI_R(K)$. Therefore $rk, kr \in K$ which implies that K is an ideal in $I_R(K)$.

Let J be a subskewring of R such that K is an ideal in it. Let $x \in J$. Then $kx, xk \in K$ for every $k \in K$ which implies that $x \in RI_R(K) \cap LI_R(K) = I_R(K)$. Hence this proposition holds. #

Definition 2.6. *Let R be a skewring and S be a subskewring of R. The* ***normalizer*** *of S in R ; denoted by $N_R(S) = \{x \in R / x+s-x \in S \text{ for every } s \in S\}$. Hence every normal subskewring S of R, $N_R(S) = R$.*

Remark 2.7. *$N_R(S)$ is an additive subgroup of R which contains S as an additive subgroup and is the largest additive subgroup of R containing S as a normal subgroup.*

Theorem 2.8. ***(Fourth Isomorphism Theorem)***

Let R be a skewring and H,K be subskewrings of R such that $N_R(K) \cap I_R(K)$ contains H. Then the following statements hold:

(1) H+K is a subskewring of R.

(2) K is a normal ideal in H+K.

(3) $H/(H \cap K) \cong (H+K)/K$.

Proof. (1) Let $h_1, h_2 \in H$, $k_1, k_2 \in K$. We shall show that $(h_1+k_1)-(h_2+k_2)$, $(h_1+k_1)(h_2+k_2) \in H+K$. Since $H \subseteq N_R(K)$, $h_2+k_1-k_2-h_2 \in K$. Then $(h_1+k_1)-(h_2+k_2) = .$

$h_1+k_1-k_2-h_2 = (h_1-h_2)+(h_2+k_1-k_2-h_2) \in H+K$. Since $H \subseteq I_R(K)$, $k_1h_2, h_1k_2 \in K$. Then $(h_1+k_1)(h_2+k_2) = h_1h_2+(k_1h_2+h_1k_2+k_1k_2) \in H+K$. Hence H+K is a subskewring of R.

(2) Clearly, K is a subskewring of H+K. Let $h \in H$ and $x,k \in K$. Since $H \subseteq I_R(K)$, $xh, hx \in K$ which implies that $x(h+k) = xh+xk$, $(h+k)x = hx+kx \in K$. Since $H \subseteq N_R(K)$, $(h+k)+x-(h+k) = h+(k+x-k)-h \in K$. Hence K is a normal ideal of H+K.

(3) Define $\varphi: H \to (H+K)/K$ by $\varphi(x) = x+K$ for every $x \in H$. Clearly, φ is a homomorphism. Since $(H+K)/K = \{h+k+K / h \in H, k \in K\} = \{h+K / h \in H\}$, φ is surjective. By group theory, $\text{Ker}(\varphi) = H \cap K$. By First Isomorphism Theorem, $H/(H \cap K) \cong (H+K)/K$. #

Corollary 2.9. *Let R be a skewring, H a subskewring of R and K a normal ideal of R. Then the following statements hold:*

(1) H+K = K+H is a subskewring of R.

(2) K is a normal ideal of H+K.

(3) $H \cap K$ is a normal ideal of H.

(4) H is a normal ideal of R implies that H+K is a normal ideal of R.

Proof. Since K is a normal ideal of R, $N_R(K) = R$ and $I_R(K) = R$. Therefore $H \subseteq N_R(K) \cap I_R(K)$.

(1) It is well-known that H+K = K+H. By Theorem 2.8 (1), H+K is a subskewring of R.

(2) and (3) follow from Theorem 2.8 (2) and (3) respectively.

(4) Suppose that H is a normal ideal of R. By (1), H+K is a subskewring of R. It is well-known that H+K is normal in (R,+). Let $h \in H$, $k \in K$, $r \in R$. Then $rh, hr \in H$ and $rk, kr \in K$. Therefore $r(h+k) = rh+rk$, $(h+k)r = hr+kr \in H+K$ which imply that H+K is a normal ideal of R. #

Corollary 2.10. *Let R be a ring. Let H,K be subrings of R such that $H \trianglelefteq_R (K)$. Then* (1) *$H+K$ is a subring of R,*

(2) *K is an ideal in $H+K$,*

(3) *$H \cap K$ is an ideal in H and*

(4) $H/(H \cap K) \cong (H+K)/K$.

Proposition 2.11. *Let R be a skewring. Then for all left[right, two-sided] ideals I_1, I_2, $\{\sum_{i=1}^{n} x_i / n \in \mathbb{Z}^+, x_i \in I_1 \cup I_2$ for every $i \in \{1,2,\ldots,n\}\}$.*

Proof. Let $I = \{\sum_{i=1}^{n} x_i / n \in \mathbb{Z}^+, x_i \in I_1 \cup I_2$ for every $i \in \{1,2,\ldots,n\}\}$. Then $(I,+)$ is a subgroup of R. Let $r \in R$. Since I_1 and I_2 are ideals, $rx_1,\ldots,rx_m \in I_1 \cup I_2$. Thus $rx = r(\sum_{i=1}^{m} x_i) = rx_1 + \ldots + rx_m \in I$ which implies that $rx \in I$, so $(I,\cdot)$ is a semigroup. Similarly, $xr \in I$. Then I is an ideal.

Let R be a skewring and $J(R)$ be the set of all ideals in R. For all $I_1, I_2 \in J(R)$, we define

$$I_1 \leq I_2 \text{ if and only if } I_1 \trianglelefteq I_2.$$

Then $(J(R),\leq)$ is a partially ordered set.

Let $NJ(R)$ be a set of all normal ideals in R. Similarly, we have $(NJ(R),\leq)$ is a partially ordered set.

Proposition 2.12. *Let R be a skewring. Then*

(1) *for all left[right, two-sided] ideals I_1,I_2, $lub(I_1,I_2) = \{\sum_{i=1}^{n} x_i / n \in \mathbb{Z}^+, x_i \in I_1 \cup I_2$ for every $i \in \{1,2,\ldots,n\}\}$ and $glb(I_1,I_2) = I_1 \cap I_2$ and*

(2) *for all left[right, two-sided] normal ideals I_1,I_2, $lub(I_1,I_2) = I_1+I_2$ and $glb(I_1,I_2) = I_1 \cap I_2$.*

Proof. (1) By Proposition 2.11, I and $I_1 \cap I_2$ are ideals in R. Clearly, I and $I_1 \cap I_2$ are an upper bound and a lower bound of $\{I_1, I_2\}$ respectively. Let J be an ideal of R such that $I_1, I_2 \leq J$. Then $(I_1 \cup I_2) \leq J$ which implies that $I \leq J$. Therefore $\text{lub}(I_1, I_2) = I$. Let J be an ideal of R such that $J \leq I_1$ and $J \leq I_2$. Clearly, $J \leq I_1 \cap I_2$. Therefore $\text{glb}(I_1, I_2) = I_1 \cap I_2$.

(2) Similarly, we have $\text{glb}(I_1, I_2) = I_1 \cap I_2$. Let J be a normal ideal of R such that $I_1, I_2 \leq J$. By definition of $I_1 + I_2$, $I_1 + I_2 \leq J$. Therefore $\text{lub}(I_1, I_2) = I_1 + I_2$. #

Theorem 2.13. *For any skewring R, **J(R)** and **NJ(R)** are lattices.*

Proof. It follows from Proposition 2.12. #

Theorem 2.14. *Let R be a skewring, I be a normal ideal of R.*
*Let **N** be the set of all normal ideals of R contains I and*

N′ be the set of all normal ideals of R/I.

Then there exists an order-isomorphism $\varphi: N \to N'$.

Corollary 2.15. *Let R be a skewring and I be a normal ideal of R.*
*Let **N** be the set of all normal ideals of R strictly contains I and*

N′ be the set of all normal ideals of R/I *except {I}.*

*Then there exists an order-isomorphism from **N** to **N′**.*

Corollary 2.16. *Let R be a skewring and I be a normal ideal of R.*
*Let **M** be the set of all maximal normal ideals of R containing I and*

***M′** be the set of all maximal normal ideals of* R/I.

*Then there exists an order-isomorphism from **M** to **M′**.*

Theorem 2.17. *Let R be a skewring and I be a normal ideal of R.*
*Let **P** be the set of all prime ideals of R containing I and*

P' be the set of all prime normal ideals of R/I.

Then there exists an order-isomorphism from P to P'.

Proof. Let φ be the function given in Theorem 2.14. Let $\Phi = \varphi|_P$. We shall show that Φ is a bijection from P to P'.

Suppose J is a prime normal ideal in R. Let A',B' be normal ideals in R/I such that $A'B' \subseteq \Phi(J) = J/I$. By Theorem 2.14., there exists normal ideals A,B in R which containing I such that $A' = A/I$ and $B' = B/I$. Then $AB/I = (A/I)(B/I) = A'B' \subseteq \Phi(J) = J/I$ which implies that $AB \subseteq J$. Since J is prime, $A \subseteq J$ or $B \subseteq J$ which implies that $A' \subseteq \Phi(J)$ or $B' \subseteq \Phi(J)$. Hence $\Phi(J)$ is prime.

Conversely, suppose J' is a prime normal ideal in R/I. Similarly as above, $J' = J/I$ for some normal ideal J in R which contains I. Let A,B be normal ideals in R containing I such that $AB \subseteq \Phi^{-1}(J')$. Then $(A/I)(B/I) = \Phi(AB) \subseteq J'$. Since J' is prime, $A/I \subseteq J' = J/I$ or $B/I \subseteq J' = J/I$. Since φ is an order-isomorphism $A \subseteq J$ or $B \subseteq J$, so that $A \subseteq \Phi^{-1}(J')$ or $B \subseteq \Phi^{-1}(J')$. Hence $\Phi^{-1}(J')$ is prime and this proof is finished. #

Theorem 2.18. *Let R be a skewring. Let I,I',J,J' be subskewrings of R such that I' and J' are normal ideals of I and J respectively. Then the following statements hold :*

(1) $(I \cap J')+I'$ is a normal ideal in $(I \cap J)+I'$.

(2) $(I' \cap J)+J'$ is a normal ideal in $(I \cap J)+J'$.

(3) $\dfrac{(I \cap J)+I'}{(I \cap J')+I'} \cong \dfrac{(I \cap J)+J'}{(I' \cap J)+J'}$.

Proof. (1) By group theory, $(I \cap J')+I'$ is a normal subgroup in $((I \cap J)+I',+)$.Let $x \in I \cap J$, $t,z \in I'$ and $y \in I \cap J'$. Then $x+t \in (I \cap J)+I'$, $y+z \in (I \cap J')+I'$, $x,y \in I$, $x \in J$ and $y \in J'$. Then $xy, yx \in I \cap J'$ and $yt, ty, zx, xz, zt, tz \in I'$. Then

$(y+z)(x+t) = y(x+t)+z(x+t) = yx+(yt+zx+zt) \in (I\cap J')+I'$ and $(x+t)(y+z) = (x+t)y+(x+t)z = xy+(ty+xz+tz) \in (I\cap J')+I'$. Therefore $(I\cap J')+I'$ is an ideal of $(I\cap J)+I'$.

(2) Similar proof in (1).

(3) By Corollary 2.9 (3), $((I\cap J')+I')\cap(I\cap J)$ is a normal ideal of $I\cap J$.

By Theorem 2.8 (3), $(I\cap J)/((I\cap J')+I')\cap(I\cap J) \cong ((I\cap J)+(I\cap J')+I')/(I\cap J')+I'$.

Since $((I\cap J)+(I\cap J')+I')/(I\cap J')+I' = ((I\cap J)+I')/(I\cap J')+I'$,

$(I\cap J)/((I\cap J')+I')\cap(I\cap J) \cong ((I\cap J)+I')/(I\cap J')+I'$. ....(i)

Clearly, $((I\cap J')+I')\cap(I\cap J) = ((I\cap J')+I')\cap J$. ....(ii)

Claim that $((I\cap J')+I')\cap J = (I\cap J')+(I'\cap J)$. ....(iii)

Let $y\in I\cap J'$, $z\in I'$ such that $y+z\in J$. Then $y+z\in((I\cap J')+I')\cap J$. Then $z = -y+(y+z)\in J$, so that $z\in I'\cap J$. Thus $y+z\in(I\cap J')+(I'\cap J)$, that is $((I\cap J')+I')\cap J\subseteq (I\cap J')+(I'\cap J)$. Conversely, let $x\in I\cap J'$, $y\in I'\cap J$. Then $x+y\in(I\cap J')+(I'\cap J)$, $x\in I$, $x\in J'$, $y\in I'$ and $y\in J$. Then $x+y\in((I\cap J')+I')\cap J$. Therefore $(I\cap J')+(I'\cap J)\subseteq ((I\cap J')+I')\cap J$ and hence we have the claim.

By (i),(ii) and (iii), $(I\cap J)/((I\cap J')+(I'\cap J)) \cong ((I\cap J)+I')/(I\cap J')+I'$.

Similarly, $(I\cap J)/((I\cap J')+(I'\cap J)) \cong (I\cap J)+J'/(I'\cap J)+J'$. Hence we have the theorem. #

Definition 2.19. *Let R be a skewring and ρ an eqivalence relation on R. Then ρ is called a* ***congruence*** *on R if and only if $x\rho y$ implies $(x+z)\rho(y+z)$, $(z+x)\rho(z+y)$, $(xz)\rho(yz)$ and $(zx)\rho(zy)$ for all $x,y,z\in R$.*

*Let **L(R)** be the set of all congruence on a skewring R. Define $\rho\leq\sigma$ if and only if $\rho\subseteq\sigma$ for all $\rho,\sigma\in$ **L(R)**. Then **(L(R),≤)** is a partially ordered set.*

Remark 2.20. *Let R be a skewring. Then* ***L(R)*** *is clearly a lattice where for all* $\rho,\sigma\in$ ***L(R)***, *lub*(ρ,σ) = *the intersection of all congruences containing* $\rho\cup\sigma$ *and glb*$(\rho,\sigma) = \rho\cap\sigma$.

We shall show that the least upper bound of two congruences is easily computed.

Remark 2.21. *Let R be a skewring. Let* $\rho\in$ ***L(R)*** *and* $x,x',y,y'\in R$. *Then the following statemants hold:*

(1) $x\rho y$ *and* $x'\rho y'$ *imply* $xx'\rho yy'$.

(2) $x\rho y$ *and* $x'\rho y'$ *imply* $(x+x')\rho(y+y')$.

(3) $x\rho y$ *implies* $(-x)\rho(-y)$.

Theorem 2.22. *Let R be a skewring. Then there exists an order-isomorphism* Φ *of* ***L(R)*** *to* ***NJ(R)*** *such that the congruence classes of* ρ *are the cosets of* $\Phi(\rho)$.

Proof. Let $\rho\in$ **L(R)**, define $I_\rho = \{x\in R / x\rho 0\}$. Let $I\in$ **NJ(R)**, define $x\rho_I y$ if and only if $x-y\in I$ for all $x,y\in R$.

Step1. We shall show that $I_\rho\in$ **NJ(R)** and $\rho_I\in$ **L(R)**.

Since $0\in I_\rho$, $I_\rho\neq\varnothing$. Let $x,y\in I_\rho$ and $r\in R$. Then $x\rho 0$ and $y\rho 0$. By Remark 2.21 (3) and (2), $-y\rho 0$ and $(x-y)\rho 0$ respectively, that is $x-y\in I_\rho$. Since $x\rho 0$, $(r+x-r)\rho(r+0-r)$, that is $(r+x-r)\rho 0$. Then $r+x-r\in I_\rho$. Since $x\rho 0$, $(rx)\rho(r0)$ and $(xr)\rho(x0)$, that is $(rx)\rho 0$ and $(xr)\rho 0$. Then $xr, rx\in I_\rho$ Therefore I_ρ is a normal ideal of R and hence $I_\rho\in$ **NJ(R)**.

Let $x,y,z\in R$. Since $x-x = 0\in I$, $x\rho_I x$ which implies that ρ_I is reflexive. Suppose that $x\rho_I y$. Then $x-y\in I$, that is $y-x = -(x-y)\in I$. Thus $y\rho_I x$ which implies that ρ_I is symmetric. Suppose that $x\rho_I y$ and $y\rho_I z$. Then $x-y, y-z\in I$, so $x-z = (x-y)+(y-z)\in I$. Thus $x\rho_I z$ which implies that ρ_I is transitive. Therefore ρ_I is an equivalence relation. To show that ρ_I is a congruence. Suppose that $x\rho_I y$. Then

$x-y \in I$. Since I is an ideal of R, $xz-yz = (x-y)z$, $zx-zy = z(x-y) \in I$, that is $(xz)\rho_I(yz)$ and $(zx)\rho_I(zy)$. Since I is normal in R, $(z+x)-(z+y) = z+(x-y)-z$, $(x+z)-(y+z) = x+z-z-y = x-y \in I$, that is $(z+x)\rho_I(z+y)$ and $(x+z)\rho_I(y+z)$. Therefore ρ_I is a congruence and hence $\rho_I \in \mathbf{L(R)}$.

Define $\Phi:\mathbf{L(R)} \to \mathbf{NJ(R)}$ by $\Phi(\rho) = I_\rho$ for every $\rho \in \mathbf{L(R)}$ and

$\Psi:\mathbf{NJ(R)} \to \mathbf{L(R)}$ by $\Psi(I) = \rho_I$ for every $I \in \mathbf{NJ(R)}$.

Step2. We shall show that Φ and Ψ are bijections.

Claim1. $\Psi \circ \Phi = Id_{\mathbf{L(R)}}$.

Let $\sigma \in \mathbf{L(R)}$. We shall show that $\sigma = \Psi \circ \Phi(\sigma) = \rho_{I_\sigma}$. Let $(x,y) \in \rho_{I_\sigma}$. Then $x-y \in I_\sigma$, that is $(x-y)\sigma 0$. Since σ is a congruence, $(x-y+y)\sigma(0+y)$, that is $x\sigma y$. So $\rho_{I_\sigma} \subseteq \sigma$. Conversely, let $(x,y) \in \sigma$. Since σ is a congruence, $(x-y,y-y) \in \sigma$, so $(x-y,0) \in \sigma$. Then $x-y \in I_\sigma$. By definition of ρ_{I_σ}, $x\rho_{I_\sigma}y$. Therefore $\sigma \subseteq \rho_{I_\sigma}$. Hence $\sigma = \rho_{I_\sigma}$ and we have Claim1.

Claim2. $\Phi \circ \Psi = Id_{\mathbf{NJ(R)}}$.

Let $J \in \mathbf{NJ(R)}$. We shall show that $J = \Phi \circ \Psi(J) = I_{\rho_J}$. Let $x \in J$. Since $x-0 = x \in J$, $x\rho_J 0$. By definition of I_{ρ_J}, $x \in I_{\rho_J}$, that is $J \subseteq I_{\rho_J}$. Conversely, let $x \in I_{\rho_J}$. Then $x\rho_J 0$, that is $x = x-0 \in J$. Therefore $I_{\rho_J} \subseteq J$, so that $J = I_{\rho_J}$ and we have Claim2.

By Claim1 and Claim2, Φ and Ψ are bijections and $\Psi = \Phi^{-1}$.

Step3. We shall show that Φ and Ψ are order-isomorphisms.

Let $\rho,\sigma \in \mathbf{L(R)}$ be such that $\rho \le \sigma$. Then $\rho \subseteq \sigma$. We shall show that $\Phi(\rho) \subseteq \Phi(\sigma)$. Let $x \in \Phi(\rho) = I_\rho$. Then $x\rho 0$. Since $\rho \subseteq \sigma$, $x\sigma 0$, that is $x \in I_\sigma = \Phi(\sigma)$. Therefore, $\Phi(\rho) \subseteq \Phi(\sigma)$, that is $\Phi(\rho) \le \Phi(\sigma)$.

Let $I,J \in \mathbf{NJ(R)}$ be such that $I \le J$. Then $I \subseteq J$. We shall show that $\Psi(I) \subseteq \Psi(J)$. Let $(x,y) \in \Psi(I) = \rho_I$. Then $x-y \in I \subseteq J$. Thus $x\rho_J y$, so $(x,y) \in \rho_J = \Psi(J)$. Therefore, $\Psi(I) \subseteq \Psi(J)$, that is $\Psi(I) \le \Psi(J)$. Hence Φ and Ψ are both order-isomorphisms.

Next, we shall show that the equivalence classes of ρ_I are the cosets of I. Note that $x\rho_I y$ if and only if $x-y\in I$ if and only if $x\in I+y$. Thus we see that if we know one equivalence class of a congruence on R then we know a coset. If we know one coset on R then we know the whole congruence.

To summarize, if we know one equivalence class of a congruence on a skewring then we know all equivalence classes of the congruence. #

Theorem 2.23. *For any skewring R, L(R) is commutative with respect to composition of binary relations.*

Proof. By Theorem 2.22, there exists an order-isomorphism $\Psi:\mathbf{NJ(R)}\to\mathbf{L(R)}$.

Claim that $\rho_{I_1+I_2} = \rho_{I_2}\circ\rho_{I_1}$ for all $I_1,I_2\in\mathbf{NJ(R)}$.

Let $I_1,I_2\in\mathbf{NJ(R)}$ and $(x,y)\in\rho_{I_1+I_2}$. Then $x-y\in I_1+I_2$. Thus there exist $i_1\in I_1$ and $i_2\in I_2$ such that $x-y = i_1+i_2$. Then $x-(i_2+y) = x-y-i_2 = i_1\in I_1$ and $(i_2+y)-y = i_2\in I_2$. Then $(x,i_2+y)\in\rho_{I_1}$ and $(i_2+y,y)\in\rho_{I_2}$ and hence $(x,y)\in\rho_{I_2}\circ\rho_{I_1}$. So that $\rho_{I_1+I_2}\subseteq \rho_{I_2}\circ\rho_{I_1}$.

Conversely, let $(x,y)\in\rho_{I_2}\circ\rho_{I_1}$. Then there exists a $z\in R$ such that $(x,z)\in \rho_{I_1}$ and $(z,y)\in\rho_{I_2}$. Then $x-z\in I_1$ and $z-y\in I_2$. Thus $x-y = (x-z)+(z-y)\in I_1+I_2$, that is $(x,y)\in\rho_{I_1+I_2}$ and $\rho_{I_2}\circ\rho_{I_1}\subseteq\rho_{I_1+I_2}$. Hence we have the claim.

Let $\rho_1,\rho_2\in\mathbf{L(R)}$. Since Ψ is surjective, there exist $I_1,I_2\in\mathbf{NJ(R)}$ such that $\rho_1 = \Psi(I_1) = \rho_{I_1}$ and $\rho_2 = \Psi(I_2) = \rho_{I_2}$. Then $\rho_1\circ\rho_2 = \rho_{I_1}\circ\rho_{I_2} = \rho_{I_2+I_1} = \Psi(I_2+I_1) = \Psi(I_1+I_2) = \rho_{I_1+I_2} = \rho_{I_2}\circ\rho_{I_1} = \rho_2\circ\rho_1$, by Corollary 2.9 (1). Hence $\mathbf{L(R)}$ is commutative. #

From Theorem 2.23, we see that the composition of congruences is always a congruence and given two congruences ρ_1,ρ_2, $\text{lub}(\rho_1,\rho_2) = \rho_1\circ\rho_2$.

Note. Using the facts that in group theory $HK = H \vee K$ for normal subgroups in a group G and $AB = A \vee B$ (see[3]) for a-convex subgroups of a semifield K, we get that the above theorem is true for groups and semifields.

Definition 2.24. *A finite sequence of skewring homomorphisms,*

$R_0 \xrightarrow{f_1} R_1 \longrightarrow \cdots \longrightarrow R_{n-1} \xrightarrow{f_n} R_n$, *is exact provided* $Im(f_i) = Ker(f_{i+1})$ *for every* $i \in \{1,\ldots,n-1\}$.

For every normal ideal I of a skewring R, by Proposition 1.35, $0 \longrightarrow I \xrightarrow{i} R \xrightarrow{\pi} R/I \longrightarrow 0$ is an exact sequence where i is the natural injection map and π is a canonical epimorphism.

For any skewring R, define $[R,R] = \langle\{x+y-x-y \,/\, x,y \in R\}\rangle_n$.

Consider an element in [R,R]. Let $z \in [R,R]$. Then there exist $m \in \mathbf{Z}^+$, $x_i,y_i,z_i \in R$ and $r_i,r'_i \in R \cup \mathbf{Z}$ for every $i \in \{1,\ldots,m\}$ such that $z = \sum_{i=1}^{m} (z_i + r_i(x_i+y_i-x_i-y_i)r'_i - z_i) = \sum_{i=1}^{m} (z_i + r_ix_ir'_i + r_iy_ir'_i - r_ix_ir'_i - r_iy_ir'_i - z_i)$. If $r_i \in R$ or $r'_i \in R$ for some i, by Remark 1.5 (2), $z_i + r_ix_ir'_i + r_iy_ir'_i - r_ix_ir'_i - r_iy_ir'_i - z_i = 0$. Thus $[R,R] = \{\sum_{i=1}^{m} (z_i + r_i(x_i+y_i-x_i-y_i) - z_i) \,/\, m \in \mathbf{Z}^+, x_i,y_i,z_i \in R, r_i, \in \mathbf{Z}$ for every $i \in \{1,\ldots,m\}\}$.

Theorem 2.25. *For every skewring R, [R,R] is a normal ideal with the trivial multiplication. Moreover,* $R/[R,R]$ *is a ring.*

Proof First, we shall show that [R,R] is a normal ideal with the trivial multiplication.

Claim1. $(x_1+y_1-x_1-y_1)(x_2+y_2-x_2-y_2) = 0$ for all $x_1,x_2,y_1,y_2 \in R$.

Let $x_1,x_2,y_1,y_2 \in R$. Then $(x_1+y_1-x_1-y_1)(x_2+y_2-x_2-y_2) = x_1(x_2+y_2-x_2-y_2) +$

$y_1(x_2+y_2-x_2-y_2) - x_1(x_2+y_2-x_2-y_2) - y_1(x_2+y_2-x_2-y_2) = 0$, by Remark 1.5 (2).

Hence we have Claim 1.

Claim2. $zz' = 0$ for all $z,z' \in [R,R]$.

Let $z,z' \in [R,R]$. Then there exist $m,n \in \mathbb{Z}^+$, $x_i,x'_j,z_i,z'_j \in R$, $r_i,s_j \in \mathbb{Z}$ where $x_i = a_i+b_i-a_i-b_i$, $x'_j = a'_j+b'_j-a'_j-b'_j$ for some $a_i,a'_j,b_i,b'_j \in R$ for all $i \in \{1,\ldots,m\}, j \in \{1,\ldots,n\}$ be such that $z = \sum_{i=1}^{m} (z_i+r_ix_i-z_i)$ and $z' = \sum_{j=1}^{n} (z'_j+s_jx'_j-z'_j)$. Then

$$zz' = \sum_{i=1}^{m} (z_i+r_ix_i-z_i) \sum_{j=1}^{n} (z'_j+s_jx'_j-z'_j)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (z_iz'_j+ z_is_jx'_j- z_iz'_j+ r_ix_iz'_j+ r_ix_is_jx'_j- r_ix_iz'_j-z_iz'_j-z_is_jx'_j+z_iz'_j), \text{ by Remark1.5(2)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (r_ix_is_jx'_j), \text{ by Remark 1.5 (2)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} r_i(a_i+b_i-a_i-b_i)s_j(a'_j+b'_j-a'_j-b'_j)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (r_ia_i+ r_ib_i- r_ia_i- r_ib_i)(s_ja'_j+ s_jb'_j- s_ja'_j- s_jb'_j) = 0, \text{ by Claim1.}$$

Hence we have Claim2. Therefore $[R,R]$ is a normal ideal with the trivial multiplication and $R/[R,R]$ is a ring. #

Corollary 2.26. *If R is a skewring which is not a ring, then R contains a normal ideal with the trivial multiplication of order>1.*

Proof Let R be a skewring which is not a ring. Then there exist $x,y \in R$ such that $x+y \neq y+x$. Therefore $x+y-x-y = (x+y)-(y+x) \neq 0$. By Theorem 2.25, $\langle\{x+y-x-y\}\rangle_n$ is a nonzero normal ideal of R with the trivial multiplication. #

Theorem 2.27. *[R,R] is the smallest normal ideal in a skewring R such that the quotient is a ring.*

Proof. By Theorem 2.25, $R/[R,R]$ is a ring. Let I be a normal ideal of

R such that $R/_I$ is a ring. Then $(R/_I ,+)$ is an abelian group. Let $x,y \in R$. Then $(x+I)+(y+I) = (y+I)+(x+I)$ and $(x+y)-(y+x)+I = I$ which implies that $x+y-x-y \in I$. Therefore $\{x+y-x-y / x,y \in R\} \subseteq I$ and hence $[R,R] \subseteq I$. #

Definition 2.28. *A ring S is called a* ***quotient ring*** *of a skewring R if and only if there exists an epimporphism $f:R \to S$. (i.e. $R/_{Ker(f)} \cong S$.)*

Corollary 2.29. *Let R be a skewring. If $R/_{[R,R]} = 0$, then R has the trivial multiplication.*

Proof. Consider the exact sequence

$0 \longrightarrow [R,R] \xrightarrow{i} R \xrightarrow{\pi} R/_{[R,R]} \longrightarrow 0$ where i is the natural injection map and π is a canonical epimorphism. By Theorem 2.25, $R/_{[R,R]}$ is a quotient ring of R. By assumption, $R/_{[R,R]} = 0$ which implies that $0 \longrightarrow [R,R] \xrightarrow{i} R \xrightarrow{\pi} 0$ is an exact sequence. Therefore [R,R] is isomorphic to R. Since [R,R] has the trivial multiplication, R has the trivial multiplication. #

Theorem 2.30. *Every quotient ring S of a skewring R is a quotient ring of the ring $R/_{[R,R]}$.*

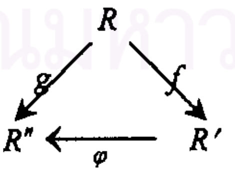
Proof. Let S be a quotient ring of a skewring R. Then there exists an epimorphism $f:R \to S$ and so $R/_{Ker(f)} \cong S$. Since S is a ring, $R/_{Ker(f)}$ is a ring. By Theorem 2.27, $[R,R] \subseteq Ker(f)$. Define $\varphi: R/_{[R,R]} \to R/_{Ker(f)}$ by $\varphi(x+[R,R]) = x+Ker(f)$ for every $x+[R,R] \in R/_{[R,R]}$. By Proposition 2.3, φ is an epimorphism. Since $S \cong R/_{Ker(f)}$, there exists an isomorphism $\psi: R/_{Ker(f)} \to S$. Hence $\psi \circ \varphi: R/_{[R,R]} \to S$ is an epimorphism. Therefore S is a quotient ring of ring $R/_{[R,R]}$.

Moreover, we shall show that $f = (\psi \circ \varphi) \circ \pi$ where $\pi: R \to R/_{[R,R]}$ is the canonical epimorphism. Let $x \in R$. Then $(\psi \circ \varphi)(\pi(x)) = \psi(\varphi(x+[R,R])) = \psi(x+Ker(f)) = f(x)$, by the proof of First Isomorphism Theorem. Hence $f = (\psi \circ \varphi) \circ \pi$.#

Remark 2.31. *By the proof of Theorem 2.30, there exists a unique epimorphism φ from $R/_{[R,R]}$ to S such that $f = \varphi \circ \pi$.*

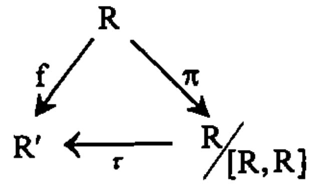
Proof. Let φ and ψ be epimorphism of to S such that $\psi \circ \pi = f = \varphi \circ \pi$. Let $x+[R,R] \in R/_{[R,R]}$. Then $\psi(x+[R,R]) = \psi(\pi(x)) = f(x) = \varphi(\pi(x)) = \varphi(x+[R,R])$. Hence $\psi = \varphi$. #

Theorem 2.32. *Let R be a skewring. Let R' be a quotient ring of R by an epimorphism f. Suppose that for every quotient ring R'' of R by an epimorphism g, there exists a unique epimorphism $\varphi: R' \to R''$ such that the the following diagram is commutative.*

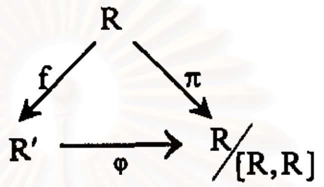


Then $R' \cong R/_{[R,R]}$.

Proof. By Theorem 2.30 and Remark 2.31, there exists a unique epimorphism τ such that the following diagram is commutative.



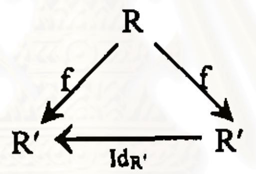
By assumption, there exists a unique epimorphism $\varphi: R' \to R/_{[R,R]}$ such that the following diagram is commutative.



That is $\tau\circ\pi = f$ and $\varphi\circ f = \pi$. Then $(\tau\circ\varphi)\circ f = \tau\circ(\varphi\circ f) = \tau\circ\pi = f$ ....(i)

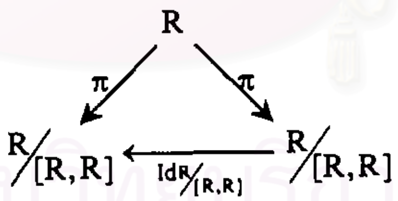
and $(\varphi\circ\tau)\circ\pi = \varphi\circ(\tau\circ\pi) = \varphi\circ f = \pi$ ....(ii)

Consider the commutative diagram,



By assumption and (i), $Id_{R'} = \tau\circ\varphi$. ....(iii)

Consider the commutative diagram,



Consider Remark 2.31 and (ii), $Id_{R/_{[R,R]}} = \varphi\circ\tau$. ....(iv)

By (iii) and (iv), φ and τ are isomorphisms. Hence $R' \cong R/_{[R,R]}$. #

For any skewring R, define $(R,R) = \langle\{xy-yx / x,y\in R\}\rangle_n$.

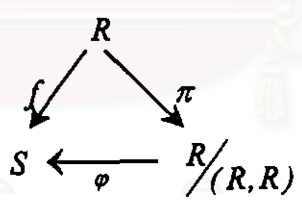
Remark 2.33. $R/_{(R,R)}$ *is a skewring with commutative multiplication.*

Proof. Let $x,y\in R$. Then $xy-yx+(R,R)=(R,R)$ which implies that $(x+(R,R))(y+(R,R))=xy+(R,R)=yx+(R,R)=(y+(R,R))(x+(R,R))$. Hence $R/(R,R)$ has commutative multiplication. #

Remark 2.34. *(R,R) is the smallest normal ideal in a skewring R such that its quotient skewring has a commutative multiplication.*

Proof. Suppose that I is a normal ideal of a skewring R such that R/I has the commutative multiplication. Let $x,y\in R$. Then $xy+I=(x+I)(y+I)=(y+I)(x+I)=yx+I$, so that $xy-yx\in I$. Thus $(R,R)\subseteq I$. By Remark 2.33, this remark is true. #

Theorem 2.35. *Let R be a skewring and S be a skewring which has commutative multiplication. If there exists an epimorphism f of R to S, then there exists a unique epimorphism* $\varphi: R/(R,R)\to S$ *such that the following diagram is commutative.*

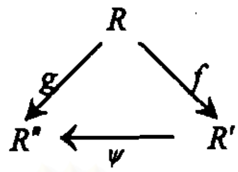


where π is the canonical epimorphism.

Proof. Suppose that there exists an epimorphism $f:R\to S$. By the First Isomorphism Theorem, $S\cong R/\mathrm{Ker}(f)$. Similarly the proof of Theorem 2.30 and Remark 2.31, there exists a unique $\varphi: R/(R,R)\to S$ such that $f=\varphi\circ\pi$.#

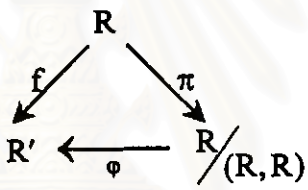
Theorem 2.36. *Let R be a skewring and (R′,f) be a quotient skewring of R with commutative multiplication. Suppose that for every quotient skewring*

(R'',g) of R with the commutative multiplication, there exists a unique epimorphism $\psi: R' \to R''$ such that the following diagram is commutative.



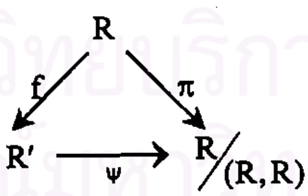
Then $R' \cong R/(R,R)$.

Proof. Since (R',f) is a quotient skewring of R with a commutative multiplication, by Theorem 2.35, there exists a unique epimorphism φ: $R/(R,R) \to R'$ such that the following diagram is commutative.



where π is a canonical epimorphism.

By Remark 2.33, $R/(R,R)$ is a quotient skewring of R with a commutative multiplication. By assumption, there exists a unique epimorphism $\psi: R' \to R/(R,R)$ such that the following diagram is commutative.



That is $\varphi \circ \pi = f$ and $\psi \circ f = \pi$. Similar to the proof of Theorem 2.32, φ and ψ are isomorphisms. Hence $R' \cong R/(R,R)$. #

Definition 2.37. *A subnormal series of a skewring R to $\{0\}$ is a finite chain of subskewrings*

$R = R_0 \geq R_1 \geq \ldots \geq R_n = \{0\}$ such that for each $i \in \{0, 1, \ , n-1\}$, $R_{i+1} \lhd_n R_i$ ……….(*)

A subnormal series is a ***normal series*** *if and only if* $R_i \triangleleft_n R$ *for all i.*

The quotient skewring R_i/R_{i+1} *are called the* ***factors*** *of the series.*

A ***length*** *of the series is the number of strict inclusion in the series (equal the number of nontrivial factors* R_i/R_{i+1}*).*

A ***refinement*** *of (*) is a subnormal series obtained by inserting a finite number of skewrings.*

A refinement is ***proper*** *if the length is larger.*

A subnormal series $R = R_0 > R_1 > \dots > R_n = \{0\}$ *is a* ***composition series*** *(CS) if each factor is simple.*

Definition 2.38. *Two subnormal series of a skewring R,* $R = R_0 \geq R_1 \geq \dots \geq R_n = \{0\}$ *and* $R = P_0 \geq P_1 \geq \dots \geq P_m = \{0\}$ *are called* ***equivalent*** *if there exists a bijective correspondence between* $\{R_i/R_{i+1} \mid R_i/R_{i+1}$ *is nontrivial\} and* $\{P_i/P_{i+1} \mid P_i/P_{i+1}$ *is nontrivial\} such that the corresponding factors are isomorphic.*

Theorem 2.39. *A subnormal series* $R = R_0 \geq R_1 \geq \dots \geq R_n = \{0\}$ *is a composition series if and only if it has no proper refinement.*

Proof. Suppose that $R = R_0 \geq R_1 \geq \dots \geq R_n = \{0\}$ is a composition series and has a proper refinement. Then there exists $i \in \{0,\dots,n-1\}$ and a subskewring P of R such that $R_{i+1} \subset P \subset R_i$ and $R_{i+1} \triangleleft_n P \triangleleft_n R_i$. Therefore P/R_{i+1} is a proper nontrivial normal ideal of R_i/R_{i+1} which contradicts to the simplicity of R_i/R_{i+1}.

Conversely, suppose that $R = R_0 \geq R_1 \geq \dots \geq R_n = \{0\}$ is not a composition.

Then there exists $i\in\{0,\ldots,n-1\}$ such that $R_i/_{R_{i+1}}$ is not simple. Then there exists a subskewring P of R such that $R_{i+1}\subset P\subset R_i$ and $P/_{R_{i+1}}$ is a nontrivial subskewring of $R_i/_{R_{i+1}}$. So that $R = R_0\geq R_1\geq\ldots\geq R_i\geq P\geq R_{i+1}\geq \ldots \geq R_n = \{0\}$ is a proper refinement of $R = R_0\geq R_1\geq \ldots \geq R_n = \{0\}$. #

Theorem 2.40. *If $R = R_0\geq R_1\geq \ldots \geq R_n = \{0\}$ is a composition series of a skewring R, then any refinement is equivalent to itself.*

Proof. It follows from Theorem 2.39. #

The following theorem is generalized from Schreier's refinement Theorem

Theorem 2.41 *Any two subnormal series for a skewring R have equivalent refinement.*

Proof. Let $R = R_0\geq R_1\geq \ldots \geq R_{n+1} = \{0\}$ and $R = P_0\geq P_1\geq \ldots \geq P_{m+1} = \{0\}$ be subnormal series for skewring R. For all $k\in\{0,\ldots,m+1\}$, $i\in\{0,\ldots,n\}$, we set $R(i,k) = (R_i\cap P_k)+R_{i+1}$ and for all $k\in\{0,\ldots,m\}$, $i\in\{0,\ldots,n+1\}$, we set $P(k,i) = (R_i\cap P_k)+P_{k+1}$.

Claim that $R = R(0,0)\,{}_n\!\triangleright R(0,1)\,{}_n\!\triangleright \ldots {}_n\!\triangleright R(0,m+1) = R(1,0)\,{}_n\!\triangleright R(1,1)\,{}_n\!\triangleright \ldots {}_n\!\triangleright R(n,0)\,{}_n\!\triangleright \ldots {}_n\!\triangleright R(n,m+1) = \{0\}$. ....(i)

and $R = P(0,0)\,{}_n\!\triangleright P(0,1)\,{}_n\!\triangleright \ldots {}_n\!\triangleright P(0,n+1) = P(1,0)\,{}_n\!\triangleright P(1,1)\,{}_n\!\triangleright \ldots {}_n\!\triangleright P(m,0)\,{}_n\!\triangleright \ldots {}_n\!\triangleright P(m,n+1) = \{0\}$. ....(ii)

Consider, $R(0,0) = (R_0\cap P_0)+R_1 = (R\cap R)+R_1 = R+R_1 = R$ and $R(n,m+1) = (R_n\cap P_{m+1})+R_{n+1} = (R_n\cap\{0\})+\{0\} = \{0\}+\{0\} = \{0\}$. Let $k\in\{0,\ldots,m\}, i\in\{0,\ldots,n\}$, so we get that $R(i,k+1) = (R_i\cap P_{k+1})+R_{i+1}\subseteq(R_i\cap P_k)+R_{i+1} = R(i,k)$. Let $i\in\{0,\ldots,n-1\}$, so we get that $R(i,m+1) = (R_i\cap P_{m+1})+R_{i+1} = (R_i\cap\{0\})+R_{i+1} = R_{i+1} = (R_{i+1}\cap R)+R_{i+1}) = (R_{i+1}\cap P_0)+R_{i+2} = R(i+1,0)$. By the Fifth Isomorphism Theorem, $R(i,k+1) =$

$(R_i \cap P_{k+1})+R_{i+1} \lhd_n (R_i \cap P_k)+R_{i+1} = R(i,k)$. Hence we have **(i)**. Similarly, we have **(ii)**. So we have the claim.

By Theorem 2.18, $R(i,k)/R(i,k+1) \cong P(k,i)/P(k,i+1)$. Hence we have the proof. #

The following theorem is generalized from Theorem Jordan-Holder Theorem.

Theorem 2.42. *Any two composition series of a skewring R are equivalent.*

Proof. By Theorem 2.41, any two composition series have equivalent refinements. By Theorem 2.40, every refinement of a composition is equivalent to itself. Hence any two composition series are equivalent. #