การหาพลังงานสถานะพื้นและมวลยังผลของระบบโบสในพลังงานศักย์แบบสุ่มโดยวิธีอินทิเกรตตามวิถี

นาย วัฒนา แซ่ลิ่ม

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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DETERMINATION OF GROUND STATE ENERGY AND EFFECTIVE MASS OF BOSE SYSTEMS IN RANDOM POTENTIALS BY PATH INTEGRATION METHOD

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วัฒนา แซ่ลิ้ม : การหาพลังงานสถานะพื้นและมวลยังผลของระบบโบสในพลังงานศักย์แบบสุ่มโดยวิธีอินทิเกรตตามวิถี (DETERMINATION OF GROUND STATE ENERGY AND EFFECTIVE MASS OF BOSE SYSTEMS IN RANDOM POTENTIALS BY PATH INTEGRATION METHOD) อ. ที่ปรึกษา: รศ. ดร. วิชิต ศรีตระกูล, อ. ที่ปรึกษาร่วม : ศ. ดร. วิรุฬห์ สายคณิต, 94 หน้า.

เราประยุกต์ทฤษฎีการอินทิเกรตตามวิถีของฟายน์แมนเพื่อที่จะศึกษาการควบแน่นโบส-ไอน์สไตน์ในบ่อศักย์แบบคู่และในศักย์แบบสุ่ม แนวคิดหลักของวิทยานิพนธ์นี้คือทำการประมาณสนามเฉลี่ยในอันตรกิริยาสองตัว โดยการอินทิเกรตตามวิถี อนุภาคโบสถูกกักในบ่อศักย์แบบคู่ซึ่งเป็นศักย์แบบฮาร์มอนิกกับกำแพงเกาส์เซียนหรือกำแพงโคไซน์ โดยใช้การคำนวณแบบแปรผันเราได้ผลเชิงวิเคราะห์ของพลังงานสถานะ พื้นฟังก์ชันคลื่นและอินทิเกรตซ้อนทับ ซึ่งทำให้เราสามารถคำนวณอัตราการทะลุผ่านซึ่งสอดคล้องกับค่าเชิงตัวเลขสำหรับแบบจำลองสองส่วนของสมการกรอส-ปิทาเอบสกี

เรายังได้พิจารณาแบบจำลองของระบบโบสที่ประกอบด้วย N อนุภาค ที่มีค่าศักย์คู่อันตรกิริยาแบบอันตะ ภายใต้อิทธิพลของศักย์แบบสุ่ม ความพรุนของระบบหรือความไม่เป็นเอกพันธ์สามารถแทนได้ด้วยความหนาแน่นของความพรุนกับค่าของการแกว่งและค่าความยาวสหสัมพันธ์ เราได้ค่าเชิงวิเคราะห์ของมวลยังผลและพลังงานสถานะพื้น ซึ่งสอดคล้องกันดีกับผลจากวิธีของบูโกลิบอฟ ตัวแผ่กระจายของระบบถูกใช้ในการคำนวณฟังก์ชันแบ่งส่วน ค่าความจุความร้อนจำเพาะ อุณหภูมิวิกฤติ ความหนาแน่นของการควบแน่น และความหนาแน่นของของไหลยวดยิ่งของระบบ

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We apply Feynman's path integral theory to study Bose-Einstein condensates in a double well potential and in a random potential. The main idea of this thesis is to perform the mean field approximation in two body-interaction by the path integral approach. Bose particles are confined in the double well which is taken as a harmonic potential with Gaussian or cosine barrier. Performing the variational calculations we obtain analytical results of the ground state energy, wave functions, and overlap integrals. The overlap integrals enable us to calculate the tunneling rates which are in good agreement with numerical values for the two-mode model of Gross-Pitaevskii's equation.

We also consider the model of a Bose system consisting of N particles with finite interacting pair potentials under the influence of random potentials. The porosity or nonhomogeneity of the system can be represented by the density of porosity with amplitude of the fluctuation, and a correlation length. We obtain analytical results of the effective mass and the ground state energy which are in good agreement with that of Bugoliubov approach. The propagator of the system is used to calculate the partition function, specific heat, the critical temperature, condensate density, and the superfluid density of the system.

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List of Symbols

Ω	harmonic oscillator frequency
V_b	height of the Gaussian barrier
σ	width of the Gaussian barrier
g	coupling constant
ω	variational parameter
N	number of interacting particles
n	number of impurities
a	scattering length of interacting particle
b	scattering length of impurity
l	correlation length of interacting particle
L	correlation length of impurity
χ	concentration of impurity
R	strength of disorder
$\overline{N}$	density of interacting particle
$\overline{n}$	density of impurity
ξ_a	mean field energy of interacting particle
ξ_b	mean field energy of impurity
F	free energy
C_v	specific heat
T_c	critical temperature
T_0	critical temperature of ideal gas

$\overline{N}_c$	condensate density
$\overline{N}_0$	condensate density of ideal gas
ρ_s	superfluid density
ρ	total density
m^*	effective mass
t	time
${}_2F_1$	regularized hypergeometric function
${}_pF_q$	generalized hypergeometric function
γ	Euler's constant, $\gamma \simeq 0.577216$
$\mathrm{Ci}[x]$	cosine integral function
$\Gamma\,[x]$	Euler gamma function

CHAPTER I

INTRODUCTION

The phenomenon of Bose-Einstein condensation (BEC) was predicted by Bose [1] and Einstein in 1924 [2]. They predicted that at a finite temperature, almost all the particles of a bosonic system would occupy the ground state as soon as the quantum wave functions of the particles start to overlap. In a Bose-Einstein condensate millions of atoms occupy a single quantum state. In 1995, BEC was reported by scientists at JILA (Boulder) [3], followed by similar reports from Rice University (Texas) [4] and MIT (Cambridge) [5]. The breakthrough was made possible by combining laser cooling with evaporative cooling in a magnetic trap achieved a temperature on the order of 10^{-6} K. The first evidence for condensation emerged from time of flight measurements. A sharp peak in the velocity distribution was observed below a critical temperature.

In this thesis we study two main topics: Bose-Einstein condensation in a double well potential and Bose-Einstein condensation in a disordered system.

1.1 Bose-Einstein Condensation in a Double Well Potential

In 2005, Albiez *et al.* [6] reported the first realization of a single bosonic Josephson junction, implemented by two weakly linked Bose-Einstein condensates in a double-well potential. In the experiment, ^{87}Rb BEC is loaded into an optical effective double well potential, which can be modeled by a harmonic trap superimposed by a cosine barrier. They used a two-mode model based on symmetric and antisymmetric wave functions of the Gross-Pitaevskii equation. In 2006, Ananikian *et al.* improved the two-mode model for BEC in a double well potential [7]. This double well potential is taken as a harmonic potential with Gaussian barrier. In general, these problems cannot be solved analytically. However, we can solve this problem by using Feynman's path integral theory. All calculations can be performed analytically with the help of the generating functional associated with the trial action containing one variational parameter. The ground state energy and wave function are readily derived from the propagator in the path integral formulation. The advantage of this method is that all calculations can be done analytically.

In this topic, we use Feynman's path integral theory to study the BEC in a double well potential of 2 types: a harmonic potential superimposed by a Gaussian and cosine barrier. The two body interaction is assumed to be a delta-shaped potential having the scattering length a. The mean field approximation is performed by replacing the pair potential into a one body potential by neglecting the fluctuation. The ground state and the first excited state wave functions are used to calculate the overlap integral γ_{ij} and compare with the Thomas-Fermi approximation and the two-mode model of the Gross-Pitaevskii equation.

1.2 Bose-Einstein Condensation in a Disordered System

In the last few years great attention has been devoted to the investigation of disordered Bose systems. The experimental realizations of these systems are liquid ^{4}He adsorbed in various types of porous media such as vycor and aerogel. These systems exhibit many interesting properties, which have not yet been fully understood theoretically, such as the suppression of superfluidity [8], a rich variety of elementary excitations [9, 10] a critical behavior near the phase transition different from the bulk [11-15] and condensate-non condensate interaction generated by the nonhomogeneity of the matter.

We consider a model of a system consisting of N Bose particles with two body interactions confined within a volume V. The two body interaction can be assumed to have the scattering length a and the correlation length l. The porosity of the system or the nonhomogeneity can be represented by the density of porosity (n/V) and with amplitude of the fluctuation b and an autocorrelation length L. The main idea of this approach is to perform the mean field approximation in the Feynman approach. This approximation is equivalent to replacing the two body potential into a one body potential interacting with the effective random potential. Performing the random potential due to the mean field approximation of pair potential and the random potential of the nonhomogeneous system we obtained the one body effective propagator. We will show that this effective one body propagator allows us to determine all physical properties such as the ground state energy, the wave functions, the effective masses, and the condensate as well as the superfluid density.

1.3 Outline of Thesis

This thesis is divided as follows. Chapter 2 provides a detailed description of the review of the concepts of path integral theory. Chapter 3 describes Bose-Einstein condensation in a double well potential. We will explain the BEC in both Gaussian and cosine barrier. We also obtain the analytic expressions and use them to calculate the ground state energy and the wave function. Chapter 4 describes the ground state properties of Bose-Einstein condensation in a disordered system. Finally, conclusion and discussion are drawn in Chapter 5.

CHAPTER II

FEYNMAN'S PATH INTEGRAL THEORY

In this chapter we review the concepts of the path integral theory which are related to this thesis. The first section gives the detailed description of the harmonic oscillator and the second section provides the detailed calculations of the non-local harmonic oscillator. These two propagators are used to study the systems in the variational calculation in Chapter 3 and Chapter 4.

In classical mechanics, the principle of the least action is a way of expressing the condition that determines the particular path or classical path $\overline{x}(\tau)$ out of all the possible paths [16]. The action S is defined as

$$S = \int_{t_a}^{t_b} L(\dot{x}, x, \tau) d\tau, \tag{2.1}$$

where L is the Lagrangian of the system, t_a is an initial time and t_b is a final time. For a particle of mass m moving in a potential $V(x, \tau)$, which is a function of position and time, the Lagrangian is

$$L = \frac{m}{2}\dot{x}^2 - V(x, \tau). \tag{2.2}$$

In quantum mechanics, we can not exactly know in which paths the particle go from a to b. Consequently, the total amplitude to go from a to b must be contributed by all paths. Feynman found that they contribute equal amounts to the total amplitude, but contribute at different phases. The phase of the contribution is equal to $S/\hbar$ [16]. The probability $P(b,a)$ to go from x_a at the time τ_a to x_b at the time τ_b can be calculated as follows:

$$P(b,a) = |K(b,a)|^2 \tag{2.3}$$

where $K(b,a)$ is the amplitude to go from a to b. This amplitude is the sum of contribution $\phi\,[x(\tau)]$ from each path.

$$K(b,a) = \sum_{\text{over all paths from a to b}} \phi[x(\tau)]. \tag{2.4}$$

The contribution of a path has a phase proportional to the action S.

$$\phi[x(\tau)] = (\text{const}) \exp\left[\frac{i}{\hbar} S\{x(\tau)\}\right] \tag{2.5}$$

The action is that for the corresponding classical system. The constant will be chosen to normalize K.

We separate the time into small interval ϵ. This gives a set of times $\tau_1, \tau_2, \tau_3, \ldots$ between the values τ_a and τ_b, where $\tau_{i+1} = \tau_i + \epsilon$. At each time, τ_i, we select some special point x_i and construct a path by connecting consecutive points with a straight line. These processes are shown in Figure 2.1. It is possible to define a sum over all paths constructed in this manner by taking a multiple integral over all values of x_i for i from 1 to $N-1$, where

$$\begin{aligned} N\epsilon &= \tau_b - \tau_a \\ \epsilon &= \tau_{i+1} - \tau_i \\ \tau_0 &= \tau_a \quad , \quad \tau_N = \tau_b \\ x_0 &= x_a \quad , \quad x_N = x_b. \end{aligned} \tag{2.6}$$

The resulting equation is

$$K(b,a) \approx \int\int \ldots \int (\text{const}) \exp\left[\frac{i}{\hbar} S\{x(\tau)\}\right] dx_1 dx_2 \ldots dx_{N-1} \tag{2.7}$$

We do not integrate x_0 or x_N because these are the fixed end points x_a and x_b. In order to achieve the correct measure, Eq.(2.7) must be taken in the limit of $\epsilon \to 0$

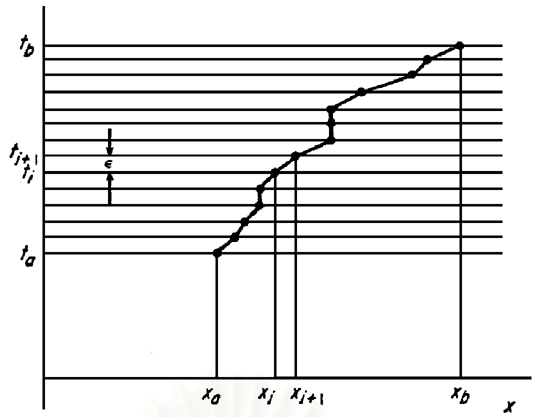

Figure 2.1: Diagram showing the sum over paths which is defined as a limit of large number of specified times separated by very small intervals ϵ [16].

and some normalizing factor A^{-N} which depends on ϵ must be provided in order that the limit of Eq.(2.7) becomes

$$K(b,a) = \lim_{\epsilon\to 0} \frac{1}{A} \int\int \ldots \int (\text{const}) \exp\left[\frac{i}{\hbar} S\{x(\tau)\}\right] \frac{dx_1}{A}\frac{dx_2}{A}\ldots\frac{dx_{N-1}}{A}. \tag{2.8}$$

This equation can also be written in a less restrictive notation as

$$K(b,a) = \int \exp\left[\frac{i}{\hbar} S\{x(\tau)\}\right] D(x(\tau)). \tag{2.9}$$

This is called a path integral and the amplitude $K(b,a)$ is known as the Feynman propagator.

2.1 Harmonic Oscillator

We consider the one dimensional harmonic oscillator described by the Lagrangian

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2. \tag{2.10}$$

Thus the propagator can be written as

$$P\left(x_2, x_1; t\right) = \int_{x_1}^{x_2} \exp\left[\frac{i}{\hbar}\int_0^t \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2\right) d\tau\right] Dx(\tau). \tag{2.11}$$

The integral over all paths goes from $(x_1, 0)$ to (x_2, t). The classical path $\overline{x}$ of the least action satisfies

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\overline{x}}}\right) - \frac{\partial L}{\partial \overline{x}} = 0. \tag{2.12}$$

For a harmonic oscillator, we can write Eq.(2.12) as

$$\ddot{\overline{x}} + \omega^2 \overline{x} = 0. \tag{2.13}$$

The solution of Eq.(2.13) is

$$\overline{x}(\tau) = A\sin\omega\tau + B\cos\omega\tau, \tag{2.14}$$

where A and B are constants. By applying the boundary conditions $\overline{x}(0) = x_1$ and $\overline{x}(t) = x_2$ to Eq.(2.14), we can obtain the constants A and B,

$$\begin{aligned} A &= \frac{x_2 - x_1\cos\omega t}{\sin\omega t} \\ B &= x_1. \end{aligned} \tag{2.15}$$

Now

$$\begin{aligned} S_{cl} &= \int_0^t \frac{m}{2}\left(\dot{\overline{x}}^2(\tau) - \omega^2\overline{x}^2(\tau)\right)d\tau \\ &= \frac{m}{2}\left[\left[\dot{\overline{x}}(\tau)\overline{x}(\tau)\right]_0^t - \int_0^t \overline{x}(\tau)\ddot{\overline{x}}(\tau)d\tau - \omega^2\int_0^t \overline{x}^2(\tau)d\tau\right] \\ &= \frac{m}{2}\left[\dot{\overline{x}}(t)\overline{x}(t) - \dot{\overline{x}}(0)\overline{x}(0) - \int_0^t \overline{x}\left(\ddot{\overline{x}}(\tau) + \omega^2\overline{x}^2(\tau)\right)d\tau\right]. \end{aligned} \tag{2.16}$$

We find that the second term of Eq.(2.16) is equal to zero so we obtain

$$S_{cl} = \frac{m}{2}\left[\dot{\overline{x}}(t)\overline{x}(t) - \dot{\overline{x}}(0)\overline{x}(0)\right]. \tag{2.17}$$

Differentiating Eq.(2.14) with respect to τ, we obtain

$$\dot{\overline{x}}(\tau) = A\omega\cos\omega\tau - B\omega\sin\omega\tau. \tag{2.18}$$

Substituting Eq.(2.14), Eq.(2.15) and Eq.(2.18) into Eq.(2.17), we can write the classical action of the harmonic oscillator as

$$S_{cl} = \frac{m\omega}{2\sin\omega t}\left[\cos\omega t(x_1^2 + x_2^2) - 2x_1x_2\right]. \tag{2.19}$$

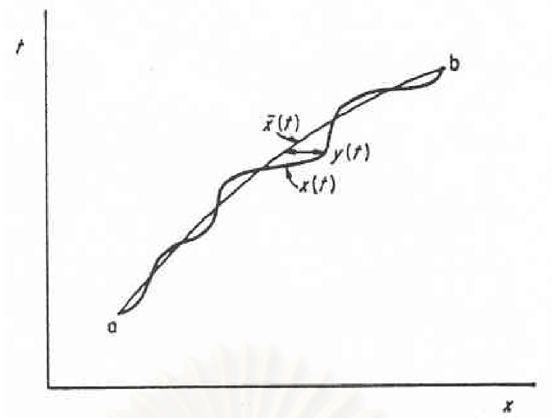

Figure 2.2: Diagram showing the difference between the classical path $\overline{x}(\tau)$ and some possible alternative path $x(\tau)$ as the function $y(\tau)$ [16].

Let $\overline{x}(\tau)$ be the classical path between the specified end points. This is the path which is an extremum for the action S. In the notation we have been using

$$S_{cl}[x_2, x_1] = S[\overline{x}(\tau)]. \tag{2.20}$$

We can represent x in terms of $\overline{x}$ and a new variable $x = \overline{x} + y$

$$x(\tau) = \overline{x}(\tau) + y(\tau). \tag{2.21}$$

This is to say, instead of defining a points on the path by its distance $x(\tau)$ from an arbitrary coordinate axis, we measure instead the deviation $y(\tau)$ from the classical path, as shown in Figure 2.2. Thus we can write the action as

$$\begin{aligned} S\left[x(\tau)\right] &= \int_0^t \left\{ \frac{m}{2}\left[\dot{\overline{x}}(\tau) + \dot{y}(\tau)\right]^2 - \frac{m\omega^2}{2}\left[\overline{x}(\tau) + y(\tau)\right]^2 \right\} d\tau \\ &= \int_0^t \left(\begin{array}{c} \frac{m}{2}\left[\dot{\overline{x}}^2(\tau) + 2\dot{\overline{x}}(\tau)\dot{y}(\tau) + \dot{y}^2(\tau)\right] \\ -\frac{m\omega^2}{2}\left[\overline{x}^2(\tau) + 2\overline{x}^2(\tau)y^2(\tau) + y^2(\tau)\right] \end{array} \right) d\tau \end{aligned} \tag{2.22}$$

or

$$S\left[x(\tau)\right] = S_{cl}\left[\overline{x}(\tau)\right] + \int_0^t \left[\frac{m}{2}\dot{y}^2(\tau) - \frac{m\omega^2}{2}y^2(\tau)\right] d\tau. \tag{2.23}$$

Substituting Eq.(2.23) into Eq.(2.11).

$$P\left(x\left(t\right),x\left(0\right);t\right)=\exp\frac{i}{\hbar}S_{cl}\left[\overline{x}(\tau)\right]\int_{0}^{0}\exp\frac{i}{\hbar}\int_{0}^{t}\left[\frac{m}{2}\dot{y}^{2}(\tau)-\frac{m\omega^{2}}{2}y^{2}(\tau)\right]d\tau D(y(\tau)). \tag{2.24}$$

We find that the integral over $y(\tau)$ does not depend on the classical path and $y(\tau)=0$ at t_a and t_b (See Figure 2.2) so we use symbol $\int_0^0$ for integrating a closed contour. We may write the propagator as

$$P\left(x\left(t\right),x\left(0\right);t\right)=F(t,0)\exp\frac{i}{\hbar}S_{cl}\left[\overline{x}(\tau)\right], \tag{2.25}$$

where

$$F(t,0)=\int_{0}^{0}\exp\frac{i}{\hbar}\int_{0}^{t}\left[\frac{m}{2}\dot{y}^{2}(\tau)-\frac{m\omega^{2}}{2}y^{2}(\tau)\right]d\tau D(y(\tau)). \tag{2.26}$$

We can calculate $F(t,0)$ by expanding $y(\tau)$ as a Fourier series

$$y(\tau)=\sum_{n}a_n\sin\frac{n\pi\tau}{t} \tag{2.27}$$

and then consider the paths as a function of the coefficient a_n instead of functions of $y(\tau)$. The details for calculating $F(t,0)$ is given by Feynman and Hibbs [16]. The result is

$$F(t)=\left(\frac{m\omega}{2\pi i\hbar\sin\omega t}\right)^{1/2}. \tag{2.28}$$

Therefore the propagator of harmonic oscillator is

$$P\left(x\left(t\right),x\left(0\right);t\right)=\left(\frac{m\omega}{2\pi i\hbar\sin\omega t}\right)^{1/2}\exp\left[\frac{im\omega}{2\hbar\sin\omega t}\left[\cos\omega t(x_1^2+x_2^2)-2x_1x_2\right]\right]. \tag{2.29}$$

This propagator can be expanded in exponential function of time multiplied by products of energy eigen function. That is,

$$\begin{aligned}&\left(\frac{m\omega}{2\pi i\hbar\sin\omega t}\right)^{1/2}\exp\left[\frac{im\omega}{2\hbar\sin\omega t}\left[\cos\omega t(x_1^2+x_2^2)-2x_1x_2\right]\right]\\ =&\sum_{n=o}^{\infty}e^{-(i/\hbar)E_nt}\phi_n(x_2)\phi_n^*(x_1).\end{aligned} \tag{2.30}$$

Using the relations

$$\begin{aligned}i\sin\omega t&=\frac{1}{2}e^{i\omega t}(1-e^{-2i\omega t})\\ \cos\omega t&=\frac{1}{2}e^{i\omega t}(1+e^{-2i\omega t}).\end{aligned} \tag{2.31}$$

We can write the left-hand side of Eq.(2.30) as

$$\begin{aligned} & P\left(x\left(t\right),x\left(0\right);t\right) \\ = \; & \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-i\omega t/2}(1-e^{-2i\omega t})^{-1/2} \\ & \times \exp\left\{-\frac{m\omega}{\pi\hbar}\left[(x_1^2+x_2^2)\left(\frac{1+e^{-2i\omega t}}{1-e^{-2i\omega t}}\right)-\frac{4x_1x_2e^{-i\omega t}}{1-e^{-2i\omega t}}\right]\right\}. \end{aligned} \tag{2.32}$$

We can obtain a series having the form of the right-hand side of Eq.(2.30) in the power of $e^{-i\omega t}$. Because of the initial factor $e^{-i\omega t/2}$, it is clear that all terms in the exponential will be of the form $e^{-i\omega t/2}e^{-i\omega t}$ for $n = 0, 1, 2, ...$ This means the energy levels are given by $E_n = \hbar\omega(n+1/2)$. To find the wave functions, we have to carry out the expression completely. We illustrate the method by going only as far as $n = 2$. Expanding the left-hand side of Eq.(2.30) to this order we have

$$\begin{aligned} & P\left(x\left(t\right),x\left(0\right);t\right) \\ = \; & \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{2\hbar}(x_1^2+x_2^2)}e^{-i\omega t/2}(1+\frac{1}{2}e^{-2i\omega t}+...) \\ & \times\left[1+\frac{2m\omega}{\hbar}x_1x_2e^{-i\omega t}+\frac{2m^2\omega^2}{\hbar^2}x_1^2x_2^2e^{-2i\omega t}-\frac{m\omega}{\hbar}(x_1^2+x_2^2)e^{-2i\omega t}+...\right]. \end{aligned} \tag{2.33}$$

From this we pick out the coefficient of the lowest term. It is

$$\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{2\hbar}(x_1^2+x_2^2)}e^{-i\omega t/2} = e^{-(i/\hbar)E_0t}\phi_0(x_2)\phi_0^*(x_1). \tag{2.34}$$

This means that $E_0 = \frac{1}{2}\hbar\omega$ and

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-(m\omega x^2/2\hbar)}. \tag{2.35}$$

The next-order term in the expansion is

$$e^{-i\omega t/2}e^{-i\omega t/}\frac{m\omega}{\pi\hbar}e^{-\frac{m\omega}{2\hbar}(x_1^2+x_2^2)}\frac{2m\omega}{\hbar}x_1x_2 = e^{-(i/\hbar)E_1t}\phi_1(x_2)\phi_1^*(x_1), \tag{2.36}$$

which implies that $E_1 = \frac{3}{2}\hbar\omega$, and

$$\phi_1(x) = \frac{2m\omega}{\hbar}x\phi_0(x). \tag{2.37}$$

The next term corresponds to $E_2 = \frac{5}{2}\hbar\omega$. The part of the term depending on x_1 and x_2 is

$$\begin{aligned} &\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{2\hbar}(x_1^2+x_2^2)} \left[\frac{2m^2\omega^2}{\hbar^2}x_1^2x_2^2 - \frac{m\omega}{\hbar}(x_1^2+x_2^2) + \frac{1}{2}\right] \\ = &\ \frac{1}{2}\left(\frac{2m\omega}{\hbar}x_1^2 - 1\right)\left(\frac{2m\omega}{\hbar}x_2^2 - 1\right)\phi_0(x_2)\phi_0^*(x_1). \end{aligned} \tag{2.38}$$

We find that

$$\phi_2(x) = \frac{1}{\sqrt{2}}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)\phi_0(x). \tag{2.39}$$

From these results, we obtain the energy levels of the harmonic oscillator.

$$E_n = \hbar\omega(n + \frac{1}{2}), \tag{2.40}$$

where n is an integer $0, 1, 2....$ and all of the wave functions can be written in Hermite polynomials [16].

$$\phi_n = (2^n n!)\left(\frac{m\omega}{\pi\hbar}\right) H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right) e^{-(m\omega x^2/2\hbar)}. \tag{2.41}$$

Therefore we can obtain energy levels and wave functions from the harmonic oscillator propagator. This propagator will be used to study the Bose-Einstein condensation in a double well potential in Chapter 3.

2.2 Non-local Harmonic Oscillator

In this section we give a detailed calculation of the non-local harmonic oscillator in three dimensions. The propagator can be written as

$$P\left(\overrightarrow{r}(t), \overrightarrow{r}(0); t; \omega\right) = \int_{\overrightarrow{r}(0)}^{\overrightarrow{r}(t)} D\left(\overrightarrow{r}(\tau)\right) \exp\left[\begin{array}{c} \frac{i}{\hbar}\int_0^t d\tau \frac{1}{2} m \dot{\overrightarrow{r}}^2(\tau) \\ -\frac{i}{\hbar}\frac{m\omega^2}{4t}\int_0^t\int_0^t d\tau d\sigma \left(\overrightarrow{r}(\tau) - \overrightarrow{r}(\sigma)\right)^2 \end{array}\right]. \tag{2.42}$$

Rewrite the translational non-local quadratic harmonic oscillator term by expanding in terms of the local harmonic oscillator plus the nonlocal quadratic harmonic oscillator.

$$P\left(\overrightarrow{r}(t), \overrightarrow{r}(0); t; \omega\right) = \int_{\overrightarrow{r}(0)}^{\overrightarrow{r}(t)} D\left(\overrightarrow{r}(\tau)\right) \exp\left[\begin{array}{c} \frac{i}{\hbar}\int_0^t d\tau \frac{1}{2} m \dot{\overrightarrow{r}}^2(\tau) \\ -\frac{i}{\hbar}\frac{1}{2} m\omega^2 \int_0^t d\tau \overrightarrow{r}(\tau)^2 + \frac{i}{\hbar}\frac{m\omega^2}{2t}\left[\int_0^t d\tau \overrightarrow{r}(\tau)\right]^2 \end{array}\right] \quad (2.43)$$

The Stratonovich transformation is a generalization of the Gaussian integral which is given as

$$\int_{-\infty}^{\infty} dx \exp\left(bx - ax^2\right) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right), \quad (2.44)$$

where in this case

$$\begin{aligned} a &= \frac{i}{\hbar}\frac{t}{2m\omega^2} \\ b &= -\frac{i}{\hbar}\left[\int_0^t d\tau \overrightarrow{r}(\tau)\right]. \end{aligned} \quad (2.45)$$

Physically, this transformation is equivalent to transform the translation quadratic terms into the linear term. Then we can write

$$\exp\left[\frac{i}{\hbar}\frac{m\omega^2}{2t}\left[\int_0^t d\tau \overrightarrow{r}(\tau)\right]^2\right] = \left(\frac{i}{\hbar}\frac{t}{2\pi m\omega^2}\right)^{\frac{3}{2}} \int d\overrightarrow{X} \exp\left[-\frac{i}{\hbar}\left[\int_0^t d\tau \overrightarrow{X}\cdot\overrightarrow{r}(\tau)\right] + \frac{i}{\hbar}\frac{t}{2m\omega^2}\overrightarrow{X}^2\right]. \quad (2.46)$$

We can rewrite the propagator as

$$\begin{aligned} &P\left(\overrightarrow{r}(t), \overrightarrow{r}(0); t; \omega\right) \\ =& \left\langle P_{eff}\left(\overrightarrow{r}(t), \overrightarrow{r}(0); t; \omega\right)\right\rangle_{\overrightarrow{X}} \\ =& \int_{\overrightarrow{r}(0)}^{\overrightarrow{r}(t)} D\left(\overrightarrow{r}(\tau)\right) \exp\left[\begin{array}{c} \frac{i}{\hbar}\int_0^t d\tau \frac{1}{2} m \dot{\overrightarrow{r}}^2(\tau) \\ -\frac{i}{\hbar}\frac{1}{2} m\omega^2 \int_0^t d\tau \overrightarrow{r}(\tau)^2 + \frac{i}{\hbar}\int_0^t d\tau \overrightarrow{X}\cdot\overrightarrow{r}(\tau) \end{array}\right] \end{aligned} \quad (2.47)$$

and the average is defined by

$$\langle A\rangle_{\overrightarrow{X}} = \frac{\int d\overrightarrow{X} A \exp\left(\frac{i}{\hbar}\frac{t}{2m\omega^2}\right)\overrightarrow{X}^2}{\int d\overrightarrow{X} \exp\left(\frac{i}{\hbar}\frac{t}{2m\omega^2}\right)\overrightarrow{X}^2}. \quad (2.48)$$

The effective propagator can be easily evaluated exactly and the result is given in Feynman and Hibbs [16].

$$P_{eff}\left(\overrightarrow{r}(t), \overrightarrow{r}(0); t; \omega\right) = F_{eff}(t;\omega) \exp\left[\frac{i}{\hbar} S_{cl,eff}\left(\overrightarrow{r}(t), \overrightarrow{r}(0); t; \omega\right)\right], \quad (2.49)$$

where

$$F_{eff}(t;\omega) = \left(\frac{m\omega}{2\pi i\hbar\sin\omega t}\right)^{\frac{3}{2}} \tag{2.50}$$

and

$$\begin{aligned}
& S_{cl,eff}\left(\overrightarrow{r}(t), \overrightarrow{r}(0); t, \omega\right) \\
= \; & \frac{m\omega}{2\sin(\omega t)}\left(\overrightarrow{r}^2(t) + \overrightarrow{r}^2(0)\right)\cos\omega t - 2\overrightarrow{r}(t)\cdot\overrightarrow{r}(0) \\
& -\frac{2}{m\omega}\overrightarrow{X}\cdot\overrightarrow{r}(t)\int_0^t d\tau\sin\omega\tau - \frac{2}{m\omega}\overrightarrow{X}\cdot\overrightarrow{r}(0)\int_0^t d\tau\sin(\omega(t-\tau)) \\
& -\frac{2}{m^2\omega^2}\overrightarrow{X}^2\int_0^t\int_0^\tau d\tau d\sigma\sin(\omega(t-\tau))\sin\omega\sigma.
\end{aligned} \tag{2.51}$$

Inserting the $P_{eff}\left(\overrightarrow{r}(t), \overrightarrow{r}(0); t; \omega\right)$ into the above expression and carrying out the $\overrightarrow{X}$-integration, we can write

$$\begin{aligned}
& P\left(\overrightarrow{r}(t), \overrightarrow{r}(0); t; \omega\right) \\
= \; & \left(\frac{m\omega}{2\pi i\hbar\sin\omega t}\right)^{\frac{3}{2}}\left(\frac{i}{\hbar}\frac{t}{2\pi m\omega^2}\right)^{\frac{3}{2}} \\
& \int d\overrightarrow{X}\exp\left[\begin{array}{c}\frac{i}{\hbar}\left\{\begin{array}{c}\frac{m\omega}{2\sin(\omega t)}\left\{\left(\overrightarrow{r}^2(t) + \overrightarrow{r}^2(0)\right)\cos\omega t - 2\overrightarrow{r}(t)\cdot\overrightarrow{r}(0)\right\} \\ -\frac{m\omega}{2\sin(\omega t)}\frac{2}{m\omega}\overrightarrow{X}\cdot\overrightarrow{r}(t)\int_0^t d\tau\sin\omega\tau \\ -\frac{m\omega}{2\sin(\omega t)}\frac{2}{m\omega}\overrightarrow{X}\cdot\overrightarrow{r}(0)\int_0^t d\tau\sin(\omega(t-\tau)) \\ -\frac{m\omega}{2\sin\omega t}\frac{2}{m^2\omega^2}\overrightarrow{X}^2\int_0^t\int_0^\tau d\tau d\sigma\sin(\omega(t-\tau))\sin\omega\sigma\end{array}\right\} \\ +\frac{i}{\hbar}\frac{t}{2m\omega^2}\overrightarrow{X}^2\end{array}\right].
\end{aligned} \tag{2.52}$$

Consider the following integrations

$$\begin{aligned}
\int_0^t d\tau\frac{\sin(\omega(t-\tau))}{\sin\omega t} &= \left(\frac{1}{\omega\sin\omega t} - \frac{\cos\omega t}{\omega\sin\omega t}\right) \\
\int_0^t d\tau\frac{\sin\omega\tau}{\sin\omega t} &= \left(\frac{1}{\omega\sin(\omega t)} - \frac{\cos\omega t}{\omega\sin\omega t}\right) \\
\int_0^t\int_0^\tau d\tau d\sigma\frac{\sin(\omega(t-\tau))\sin\omega\sigma}{\sin\omega t} &= \frac{t}{2\omega} + \frac{1}{\omega}\left(\frac{1}{\omega\sin\omega t}(1-\cos\omega t)\right).
\end{aligned} \tag{2.53}$$

Substituting Eq.(2.53) into Eq.(2.52), we have the propagator

$$
\begin{aligned}
& P\left(\overrightarrow{r}(t), \overrightarrow{r}(0); t; \omega\right) \\
= & \left(\frac{m\omega}{2\pi i\hbar \sin(\omega t)}\right)^{\frac{3}{2}} \left(\frac{i}{\hbar}\frac{t}{2\pi m\omega^2}\right)^{\frac{3}{2}} \\
& \times \exp\left[\frac{i}{\hbar}\frac{m\omega}{2\sin\omega t}\left\{\left(\overrightarrow{r}^2(t) + \overrightarrow{r}^2(0)\right)\cos\omega t - 2\overrightarrow{r}(t)\cdot\overrightarrow{r}(0)\right\}\right] \\
& \times \int d\overrightarrow{X} \exp\left[\begin{array}{c} -\frac{i}{\hbar}\overrightarrow{X}\cdot\left(\frac{1}{\omega\sin\omega t}(1-\cos\omega t)\right)\left(\overrightarrow{r}(t) + \overrightarrow{r}(0)\right) \\ -\frac{i}{\hbar}\frac{1}{m\omega}\overrightarrow{X}^2\left[\frac{1}{\omega}\left(\frac{1}{\omega\sin\omega t}(1-\cos\omega t)\right)\right] \end{array}\right].
\end{aligned} \tag{2.54}
$$

Performing the $\overrightarrow{X}$-integration we obtain

$$
\begin{aligned}
& P\left(\overrightarrow{r}(t), \overrightarrow{r}(0); t; \omega\right) \\
= & \left(\frac{m\omega}{2\pi i\hbar \sin\omega t}\right)^{\frac{3}{2}} \left(\frac{\frac{i}{\hbar}\frac{t}{2m\omega^2}}{\pi}\right)^{\frac{3}{2}} \left(\frac{i}{\hbar}\frac{1}{\pi m\omega^2}\left(\frac{1}{\omega\sin\omega t}(1-\cos\omega t)\right)\right)^{-\frac{3}{2}} \\
& \times \exp\left[\frac{i}{\hbar}\frac{m\omega}{2\sin\omega t}\left\{\left(\overrightarrow{r}^2(t) + \overrightarrow{r}^2(0)\right)\cos\omega t - 2\overrightarrow{r}(t)\cdot\overrightarrow{r}(0)\right\}\right] \\
& \times \exp\left[\frac{\left[\frac{i}{\hbar}\left(\frac{1}{\omega\sin\omega t}(1-\cos\omega t)\right)\right]^2}{\frac{i}{\hbar}\frac{4}{m\omega^2}\left(\frac{1}{\omega\sin\omega t}(1-\cos\omega t)\right)}\left(\overrightarrow{r}(t) + \overrightarrow{r}(0)\right)^2\right].
\end{aligned} \tag{2.55}
$$

Rewrite

$$
\begin{aligned}
& P\left(\overrightarrow{r}(t), \overrightarrow{r}(0); t; \omega\right) \\
= & \left(\frac{m\omega}{2\pi i\hbar \sin(\omega t)}\right)^{\frac{3}{2}} \left(\frac{i}{\hbar}\frac{t}{2\pi m\omega^2}\right)^{\frac{3}{2}} \left(\frac{i}{\hbar}\frac{1}{\pi m\omega^2}\left(\frac{1}{\omega\sin\omega t}(1-\cos\omega t)\right)\right)^{-\frac{3}{2}} \\
& \times \exp\left[\frac{i}{\hbar}\frac{m\omega}{2}\frac{1}{2}\left(\overrightarrow{r}(t) - \overrightarrow{r}(0)\right)^2 \cot\left(\frac{\omega t}{2}\right)\right].
\end{aligned} \tag{2.56}
$$

We now consider the prefactor

$$
F(\omega;t) = F_{eff}(t;\omega)\left(\frac{i}{\hbar}\frac{t}{2\pi m\omega^2}\right)^{\frac{3}{2}}\left[\frac{\pi}{\frac{1}{m\omega^2}\left(\frac{i}{\hbar}\frac{1}{\omega\sin\Omega t}(1-\cos\omega t)\right)}\right]^{\frac{3}{2}}. \tag{2.57}
$$

Using the relation $1-\cos(\omega(t)) = 2\left(\sin\frac{\omega t}{2}\right)^2$, we obtain

$$
F(\omega;t) = \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{3}{2}}\left[\frac{\omega t}{2\left(\sin\frac{\omega t}{2}\right)}\right]^3. \tag{2.58}
$$

Finally, we obtain the propagator

$$P\left(\overrightarrow{r}(t),\overrightarrow{r}(0);t;\omega\right)=\left(\frac{m}{2\pi i\hbar t}\right)^{\frac{3}{2}}\left(\frac{\omega t}{2\sin\frac{\omega t}{2}}\right)^{3}\exp\left[\frac{i}{\hbar}\frac{m}{2}\frac{\omega}{2}\cot\left(\frac{\omega t}{2}\right)\frac{\left(\overrightarrow{r}(t)-\overrightarrow{r}(0)\right)^{2}}{t}\right]. \tag{2.59}$$

We obtain the exact solution of non-local harmonic oscillator propagator. This propagator will be used to study the system which is translational invariant in the topic of Bose-Einstein condensation in a disordered system in Chapter 4.

CHAPTER III

BOSE-EINSTEIN CONDENSATION IN A DOUBLE WELL POTENTIAL

The first experimental implementation of a Josephson junction for Bose-Einstein condensates was reported by Albiez *et al.* [6]. (Josephson junction is weakly-coupled superconductors that are separated by a thin insulating barrier.) For small initial population imbalances of the two wells, they observe Josephson tunneling oscillations which are characterized by an oscillating population and relative phase. If the initial population imbalance is chosen above a critical value, resonant tunneling between the two wells is prohibited because the difference between the on-site particle interaction energies in the two wells exceeds the tunneling energy splitting. As a consequence, the atomic distribution becomes self-locked which is called "macroscopic quantum self-trapping" as shown in Figure 3.1.

The experiment setup and procedure to create the ^{87}Rb BEC is as follows. The thermal cloud is loaded into an optical dipole trap consisting of two crossed, focussed laser beams. The higher energy atoms are evaporated by lowering the intensities. Subsequently, they adiabatically ramp up a periodic one-dimensional light shift potential in x direction. Therefore, the external effective double well potential can be approximated by

$$V\left(x,y,z\right)=\frac{m}{2}(\Omega_x^2x^2+\Omega_y^2y^2+\Omega_z^2z^2)+V_0\left(\cos\left(\pi x/q_0\right)\right)^2, \qquad (3.1)$$

where the mass of ^{87}Rb is 1.44×10^{-25}kg, $\Omega_x = 2\pi78$ Hz, $\Omega_y = 2\pi66$ Hz and $\Omega_z = 2\pi90$ Hz are the harmonic trapping frequencies, $q_0 = 5.2\mu$m is the spacing

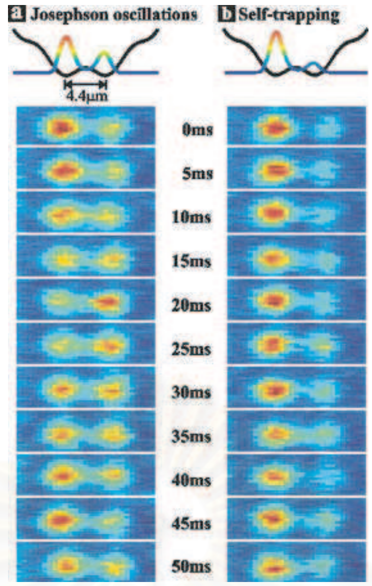

Figure 3.1: The schematics show the tunneling dynamics of two weakly linked BEC in a symmetric double well potential. (a) Josephson oscillations are observed when the initial population imbalance is below the critical value. (b) The initial population imbalance greater than the critical value.

and a barrier height $V_0/h = 412$ Hz, where h is Planck's constant. They observe the dynamical evolution of the population imbalance, relative phase in a double well potential, the tunneling oscillation frequency, etc. The experimental results can be understood by going beyond the two mode model which based on symmetric and antisymmetric wave functions of the Gross-Pitaevskii equation.

In 2006, Ananikian and Bergeman [7] explored the range of validity of two mode model for Bose-Einstein condensation in a double well potential was called as "the improved two mode model". The derivation, like others, used symmetric and antisymmetic basis functions for Gross-Pitaevskii equation. The external effective double well potential was modeled as

$$V\left(x,y,z\right) = \frac{m}{2}(\Omega_x^2 x^2 + \Omega_y^2 y^2 + \Omega_z^2 z^2) + V_b \exp\left[-\left(\frac{x}{\sigma}\right)^2\right], \tag{3.2}$$

where V_b is a height and σ width of the Gaussian barrier. Actually the Gross-Pitaevskii equation with a double well potential cannot be solved analytically.

However we can solve analytically by using the Feynman path integral theory. The advantage of this method is that all calculations can be evaluated analytically.

In this chapter, we study using the Feynman path integral theory Bose-Einstein condensation in two types of double well potentials: harmonic potential with Gaussian barrier and harmonic potential with cosine barrier. We obtain analytical results of the ground state energy and wave functions which are used to calculate the overlap integral γ_{ij} and compare with the numerical results of Ananikian *et al.* [7] and the result of Albiez *et al.* [6] for the first and the second types of the double well potential, respectively. We will show that our results are in good agreement with the two-mode model of the Gross-Pitaevskii equation. An outline of this chapter is as follows. The first part is the model Lagrangian in one dimension. We present the calculations, including comparisons with the Thomas-Fermi approximation and the two-mode model of the Gross-Pitaevskii equation. In the second part, we present a formalism of a Bose system in three dimensions and the final part is devoted to the BEC in a harmonic potential with cosine barriers and comparison with the two-mode model of the Gross-Pitaevskii equation.

3.1 Bose-Einstein Condensation with Gaussian Barrier in One Dimension

We consider a system consisting of N Bose particles with two body interactions confined within a double well potential with Gaussian barrier. The Lagrangian of the system in one dimension can be written as

$$L = \frac{m}{2}\sum_{i=1}^{N}\dot{x}_i^2 - \frac{m}{2}\sum_{i=1}^{N}\Omega^2 x_i^2 - \sum_{i=1}^{N} V_b \exp\left[-\left(\frac{x_i}{\sigma}\right)^2\right] - g\sum_{i<j}^{N}\delta(x_i - x_j). \tag{3.3}$$

Here Ω is the harmonic oscillator frequency, g is a coupling constant, V_b, σ are height and width of the Gaussian barrier, respectively. We introduce the harmonic trial action and can solve for the propagator exactly.

$$S_0 = \int_0^t \left(\frac{m}{2} \sum_{i=1}^{N} \dot{x}_i^2 - \frac{m}{2} \sum_{i=1}^{N} \omega^2 x_i^2 \right) d\tau, \tag{3.4}$$

where ω is treated as a variational parameter. The propagator associated with this action is

$$P_0 \left(x_N \left(t \right), x_N \left(0 \right), \tau \right) = \int_{x(0)}^{x(t)} D^N \left(x \left(\tau \right) \right) \exp \left(\frac{i}{\hbar} S_0 \left(x_N \left(t \right), x_N \left(0 \right) \right) \right), \tag{3.5}$$

where the path integral $\int_{x(0)}^{x(t)} D^N \left(x \left(\tau \right) \right)$ symbol is defined as

$$\int_{x(0)}^{x(t)} D^N \left(x \left(\tau \right) \right) = \int_{x_1(0)}^{x_1(t)} D \left(x_1 \left(\tau \right) \right) \int_{x_2(0)}^{x_2(t)} D \left(x_2 \left(\tau \right) \right) ..... \int_{x_N(0)}^{x_N(t)} D \left(x_N \left(\tau \right) \right). \tag{3.6}$$

The propagator can be rewritten in terms of the trial propagator S_0 as

$$\begin{aligned} P \left(x_N \left(t \right), x_N \left(0 \right), \tau \right) &= P_0 \left(x_N \left(t \right), x_N \left(0 \right), \tau \right) \\ &\times \frac{\int_{x_N(0)}^{x_N(t)} D \left(x_N \left(\tau \right) \right) \exp \left[\frac{i}{\hbar} \left(S - S_0 + S_0 \right) \right]}{\int_{x_N(0)}^{x_N(t)} D \left(x_N \left(\tau \right) \right) \exp \left[\frac{i}{\hbar} S_0 \right]} \\ &= P_0 \left(x_N \left(t \right), x_N \left(0 \right), \tau \right) \left\langle \exp \left[\frac{i}{\hbar} \left(S - S_0 \right) \right] \right\rangle_{S_0}. \end{aligned} \tag{3.7}$$

Expanding the above average in terms of the cumulants, we can write the above equation as

$$\begin{aligned} P \left(x_N \left(t \right), x_N \left(0 \right), \tau \right) &= P_0 \left(x_N \left(t \right), x_N \left(0 \right), \tau \right) \exp \left[\frac{i}{\hbar} \left\langle \left(S - S_0 \right) \right\rangle_{S_0} \right] \\ &\times \exp \left[\frac{1}{2} \left(\frac{i}{\hbar} \right)^2 \left\{ \left\langle \left(S - S_0 \right)^2 \right\rangle_{S_0} - \left\langle \left(S - S_0 \right) \right\rangle_{S_0}^2 \right\} \right]. \end{aligned} \tag{3.8}$$

Keeping only the first cumulant, we have

$$P \left(x_N \left(t \right), x_N \left(0 \right), \tau \right) \simeq P_0 \left(x_N \left(t \right), x_N \left(0 \right), \tau \right) \exp \left[\frac{i}{\hbar} \left\langle \left(S - S_0 \right) \right\rangle_{S_0} \right], \tag{3.9}$$

and obtain

$$\begin{aligned} & P\left(x_N\left(t\right), x_N\left(0\right), \tau\right) \\ \simeq\ & P_0 \exp\left[\int_0^t d\tau \frac{i}{\hbar}\left(\begin{array}{c} \frac{m}{2}\sum_{i=1}^{N}\left(\omega^2-\Omega^2\right)\left\langle x_i^2\right\rangle_{S_0} - \sum_{i=1}^{N} V_b\left\langle \exp\left[-\left(\frac{x_i}{\sigma}\right)^2\right]\right\rangle_{S_0} \\ -g\sum_{i<j}^{N}\left\langle \delta(x_i-x_j)\right\rangle_{S_0} \end{array}\right)\right]. \end{aligned} \tag{3.10}$$

To evaluate the exponent in Eq.(3.10), we proceed as follows.

3.1.1 Generating Functional

We first evaluate $\langle x^2\rangle_{S_0}$. To do so, we consider the generating functional [16]. The generating functional or the characteristic functional is defined as

$$\left\langle x(\tau)\exp\left[\frac{i}{\hbar}\int f(\tau)x(\tau)d\tau\right]\right\rangle = \frac{\delta S'_{Cl}}{\delta f(\tau)}\left\{\exp\left[\frac{i}{\hbar}(S'_{Cl}-S_{Cl})\right]\right\}. \tag{3.11}$$

Thus, by evaluating both sides when $f(\tau)=0$, we obtain

$$\langle x(\tau)\rangle = \left[\frac{\delta S'_{Cl}}{\delta f(\tau)}\right]_{f(\tau)=0}. \tag{3.12}$$

We can continue this process to get the second derivative as

$$\left\langle x(\tau)^2\right\rangle = \left[-\frac{i}{\hbar}\frac{\delta^2 S'_{Cl}}{\delta f(\tau)^2} + \left(\frac{\delta S'_{Cl}}{\delta f(\tau)}\right)^2\right]_{f(\tau)=0}. \tag{3.13}$$

We use the action S'_{Cl} from the harmonic oscillator which is driven by an external force $f(\tau)$. The action is

$$S'_{Cl} = \frac{m\omega}{2\sin\omega t}\left[\begin{array}{c} \cos\omega T(x_2^2+x_1^2) - 2x_1x_2 + \frac{2x_2}{m\omega}\int_0^t f(\tau)\sin\omega\tau d\tau \\ +\frac{2x_1}{m\omega}\int_0^t f(\tau)\sin\omega(t-\tau)d\tau \\ -\frac{1}{m^2\omega^2}\int_0^t\int_0^t f(\tau)f(s)\sin\omega(t-\tau)\sin\omega s ds d\tau \end{array}\right]. \tag{3.14}$$

This result is given in [16]. Replacing Eq.(3.14) by Eq.(3.12) and Eq.(3.13), we obtain

$$\langle x(\tau)\rangle_{S_0} = \frac{x_2\sin\omega\tau + x_1\sin\omega(t-\tau)}{\sin\omega t} \tag{3.15}$$

and

$$\left\langle x(\tau)^2 \right\rangle_{S_0} = \frac{i\hbar}{m\omega}\frac{\sin\omega(t-\tau)\sin\omega\tau}{\sin\omega t} + \left(\frac{x_2\sin\omega\tau + x_1\sin\omega(t-\tau)}{\sin\omega t}\right)^2. \qquad (3.16)$$

3.1.2 N-body Propagator

In order to consider conveniently, we rescale all the physical scales by using the relation $\tau = \frac{\widetilde{\tau}}{\Omega}, x = l\widetilde{x}, k = \frac{\widetilde{k}}{l}, \omega = \Omega\widetilde{\omega}, l = \sqrt{\frac{\hbar}{m\Omega}}, V_b = \hbar\Omega\widetilde{V}_b$ and $g = \hbar\Omega\widetilde{g}l$. Therefore we can write the model Lagrangian in the dimensionless version.

$$\begin{aligned} L &= \frac{m}{2}\sum_{i=1}^{N}\dot{x}_i^2 - \frac{m}{2}\sum_{i=1}^{N}\Omega^2 x_i^2 - \sum_{i=1}^{N}V_b\exp\left[-\left(\frac{x_i}{\sigma}\right)^2\right] - g\sum_{i<j}^{N}\delta(x_i - x_j) \\ &= \hbar\Omega\left[\frac{1}{2}\sum_{i=1}^{N}\dot{\widetilde{x}}_i^2 - \frac{1}{2}\sum_{i=1}^{N}\widetilde{x}_i^2 - \widetilde{V}_b\sum_{i=1}^{N}\exp\left[-\left(\frac{\widetilde{x}_i}{\widetilde{\sigma}}\right)^2\right] - \widetilde{g}\sum_{i<j}^{N}\delta(\widetilde{x}_i - \widetilde{x}_j)\right]. \end{aligned} \qquad (3.17)$$

We use the trial Lagrangian

$$L_0 = \hbar\Omega\left[\frac{1}{2}\sum_{i=1}^{N}\dot{\widetilde{x}}_i^2 - \frac{1}{2}\sum_{i=1}^{N}\widetilde{\omega}^2\widetilde{x}_i^2\right]. \qquad (3.18)$$

Thus the propagator is

$$\begin{aligned} & P\left(\widetilde{x}_N\left(\widetilde{t}\right), \widetilde{x}_N(0), \widetilde{\tau}\right) \\ \simeq\ & P_0\left(\widetilde{x}_N\left(\widetilde{t}\right), \widetilde{x}_N(0), \widetilde{\tau}\right) \\ & \times\exp\left[i\int_0^{\widetilde{t}} d\widetilde{\tau}\left(\begin{array}{c} \frac{1}{2}\sum\limits_{i=1}^{N}\left(\widetilde{\omega}^2 - 1\right)\left\langle\widetilde{x}_i^2\right\rangle_{S_0} - \widetilde{V}_b\sum\limits_{i=1}^{N}\left\langle\exp\left[-\left(\frac{\widetilde{x}_i}{\widetilde{\sigma}}\right)^2\right]\right\rangle_{S_0} \\ -\widetilde{g}\sum\limits_{i<j}^{N}\left\langle\delta(\widetilde{x}_i - \widetilde{x}_j)\right\rangle_{S_0} \end{array}\right)\right]. \end{aligned} \qquad (3.19)$$

The first term of the propagator in Eq.(3.19) can be calculated exactly and the result is

$$\begin{aligned}
&\exp\left[i\int_0^{\widetilde{t}} d\widetilde{\tau}\frac{1}{2}\sum_{i=1}^{N}\left(\widetilde{\omega}^2-1\right)\left\langle\widetilde{x}_i^2\right\rangle_{S_0}\right] \\
=\;&\exp\left[-\left(\frac{1}{4\widetilde{\omega}}-\frac{\widetilde{\omega}}{4}\right)\widetilde{t}\cot\widetilde{\omega}\widetilde{t}-i\left(\frac{1}{4\widetilde{\omega}}-\frac{\widetilde{\omega}}{4}\right)\left(\widetilde{x}_1^2+\widetilde{x}_2^2\right)\cot\widetilde{\omega}\widetilde{t}\right] \\
&\times\exp\left[\frac{i}{2}\left(\widetilde{\omega}^2-1\right)\left(\begin{array}{c}\frac{i}{2\widetilde{\omega}^2}+\frac{\widetilde{x}_1\widetilde{x}_2}{\widetilde{\omega}}\csc\widetilde{\omega}\widetilde{t}+\frac{\widetilde{t}}{2}(\widetilde{x}_1^2+\widetilde{x}_2^2)\csc^2\widetilde{\omega}\widetilde{t} \\ -\widetilde{t}\widetilde{x}_1\widetilde{x}_2\cot\widetilde{\omega}\widetilde{t}\csc\widetilde{\omega}\widetilde{t}\end{array}\right)\right].
\end{aligned} \tag{3.20}$$

Let us consider the Gaussian terms in Eq.(3.19) and perform the Fourier transform

$$\left\langle\exp\left[-\left(\frac{\widetilde{x}_i}{\widetilde{\sigma}}\right)^2\right]\right\rangle_{S_0}=\left(\frac{4\pi}{\widetilde{\sigma}^2}\right)^{-1/2}\int_{-\infty}^{\infty}dk\left\langle\exp\left[ik\widetilde{x}_i\right]\right\rangle_{S_0}\exp\left[-\frac{\widetilde{\sigma}^2k^2}{4}\right]. \tag{3.21}$$

Consider the factor

$$\left\langle\exp\left[ik\widetilde{x}_i\right]\right\rangle_{S_0}=\exp\left[\kappa_i^1\left(\widetilde{\tau}\right)+\kappa_i^2\left(\widetilde{\tau}\right)\right]. \tag{3.22}$$

Here, κ_i^1 denotes the first cumulant and κ_i^2 is the second cumulant. Because of the quadratic action, only the first and second cumulants survive [17]. Therefore,

$$\kappa_i^1\left(\tau\right)=ik\left\langle\widetilde{x}_i\left(\widetilde{\tau}\right)\right\rangle_{S_0} \tag{3.23}$$

and κ_i^2

$$\kappa_i^2\left(\tau\right)=-\frac{1}{2}k^2\left(\left\langle\widetilde{x}_i\left(\widetilde{\tau}\right)^2\right\rangle_{S_0}-\left\langle\widetilde{x}_i\left(\widetilde{\tau}\right)\right\rangle_{S_0}^2\right)=-k^2\frac{g\left(\widetilde{\tau},\widetilde{\tau}\right)}{2}, \tag{3.24}$$

where $g\left(\widetilde{t},\widetilde{t}\right)$ is the Green function defined as

$$g\left(\widetilde{\tau},\widetilde{\tau}\right)=\frac{i}{\widetilde{\omega}}\frac{\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})\sin\widetilde{\omega}\widetilde{\tau}}{\sin\widetilde{\omega}\widetilde{t}}. \tag{3.25}$$

Substituting Eq.(3.22), Eq.(3.23) and Eq.(3.24) into Eq.(3.21), we obtain

$$\left\langle\exp\left[-\left(\frac{\widetilde{x}_i}{\widetilde{\sigma}}\right)^2\right]\right\rangle_{S_0}=\left(\frac{4\pi}{\widetilde{\sigma}^2}\right)^{-1/2}\int_{-\infty}^{\infty}dk\exp\left[ik\cdot\left\langle\widetilde{x}_i\right\rangle_{S_0}\right]\exp\left[-k^2\left(\frac{g\left(\widetilde{\tau},\widetilde{\tau}\right)}{2}+\frac{\widetilde{\sigma}^2}{4}\right)\right]. \tag{3.26}$$

Using the formula $\int_{-\infty}^{\infty} dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$, we obtain

$$\begin{aligned}
&\left\langle \exp\left[-\left(\frac{\widetilde{x}_i}{\widetilde{\sigma}}\right)^2\right]\right\rangle_{S_0} \\
&= \frac{1}{\left(1+\frac{2}{\widetilde{\sigma}^2} g\left(\widetilde{\tau},\widetilde{\tau}\right)\right)^{1/2}} \exp\left[-\frac{\left\langle \widetilde{x}_i\left(\tau\right)\right\rangle_{S_0}^2}{\widetilde{\sigma}^2\left(1+\frac{2}{\widetilde{\sigma}^2} g\left(\widetilde{\tau},\widetilde{\tau}\right)\right)}\right] \\
&= \frac{1}{\left(1+\frac{2i}{\widetilde{\sigma}^2\widetilde{\omega}}\frac{\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})\sin\widetilde{\omega}\widetilde{\tau}}{\sin\widetilde{\omega}\widetilde{t}}\right)^{1/2}} \exp\left[-\frac{\left(\frac{\widetilde{x}_2\sin\widetilde{\omega}\widetilde{\tau}+\widetilde{x}_1\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})}{\sin\widetilde{\omega}\widetilde{T}}\right)^2}{\widetilde{\sigma}^2\left(1+\frac{2i}{\widetilde{\sigma}^2\widetilde{\omega}}\frac{\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})\sin\widetilde{\omega}\widetilde{\tau}}{\sin\widetilde{\omega}\widetilde{t}}\right)}\right]. \quad (3.27)
\end{aligned}$$

This average is too complicated. We cannot integrate directly so we rewrite in the form $\frac{1}{(b+c)^{1/2}}\exp\left[-\frac{a}{\widetilde{\sigma}^2(b+c)}\right]$, where $a = \left(\frac{\widetilde{x}_2\sin\widetilde{\omega}\widetilde{t}+\widetilde{x}_1\sin\widetilde{\omega}(\widetilde{T}-\widetilde{t})}{\sin\widetilde{\omega}\widetilde{T}}\right)^2$, $b = 1-\frac{i}{\widetilde{\sigma}^2\widetilde{\omega}}\cot\left[\widetilde{\omega}\widetilde{t}\right]$ and $c = \frac{i}{\widetilde{\sigma}^2\widetilde{\omega}}\cos\left(2\widetilde{\omega}\widetilde{\tau}-\widetilde{\omega}\widetilde{t}\right)\csc\widetilde{\omega}\widetilde{t}$. For large $\widetilde{t}$, $\cot\widetilde{\omega}\widetilde{t} \to i, \csc\widetilde{\omega}\widetilde{t} \to 0$ and $\frac{c}{b}$, $\frac{a}{\widetilde{\sigma}^2(b+c)} \ll 1$. We expand $\frac{1}{(b+c)^{1/2}}$ and $\exp\left[-\frac{a}{\widetilde{\sigma}^2(b+c)}\right]$ in the powers of $\frac{c}{b}$ and $-\frac{a}{\widetilde{\sigma}^2(b+c)}$, respectively and then integrate each term. After integrating, the infinite terms can be written in a close form as (See Appendix A for more details)

$$\begin{aligned}
&\int_0^{\widetilde{t}} \left\langle \exp\left[-\left(\frac{\widetilde{x}_i\left(\tau\right)}{\widetilde{\sigma}}\right)^2\right]\right\rangle d\widetilde{\tau} \\
&= \frac{\widetilde{t}}{\left(1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}\right)^{1/2}} - i\left(\frac{2\ln\left[2\right]}{\widetilde{\omega}\sqrt{1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}}} - \frac{2\ln\left[1+\sqrt{1-\frac{1}{1+\widetilde{\sigma}^2\widetilde{\omega}}}\right]}{\widetilde{\omega}\sqrt{1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}}}\right) \\
&\quad -i\sum_{n=1}^{\infty}\frac{\left(-1\right)^n\left(\widetilde{x}_1^{2n}+\widetilde{x}_2^{2n}\right)}{\left(2n-1\right)n!}\sum_{i=1}^{\infty}\frac{1}{\widetilde{\omega}^i\widetilde{\sigma}^{2i+2n-2}\left(1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}\right)^{n+i-\frac{1}{2}}}\frac{i\prod_{j=0}^{i}\left(j+\frac{2n-1}{2}\right)}{\left(i+n-1\right)\left(i+\frac{2n-1}{2}\right)i!} \\
&= \frac{\widetilde{t}}{\left(1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}\right)^{1/2}} - i\left(\frac{2\ln\left[2\right]}{\widetilde{\omega}\sqrt{1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}}} - \frac{2\ln\left[1+\sqrt{1-\frac{1}{1+\widetilde{\sigma}^2\widetilde{\omega}}}\right]}{\widetilde{\omega}\sqrt{1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}}}\right) \\
&\quad -i\sum_{n=1}^{\infty}\frac{\left(-1\right)^n\left(\widetilde{x}_1^{2n}+\widetilde{x}_2^{2n}\right)}{2n\widetilde{\omega}\widetilde{\sigma}^{2n}\left(1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}\right)^{\left(n+\frac{1}{2}\right)}}\ {}_2F_1\left(n,n+\frac{1}{2},n+1,\frac{1}{1+\widetilde{\sigma}^2\widetilde{\omega}}\right). \quad (3.28)
\end{aligned}$$

Here, ${}_2F_1$ is the regularized hypergeometric function. This result is exact for the limit $\widetilde{t} \to \infty$. The next task is to find $\left\langle\delta(\widetilde{x}_i-\widetilde{x}_j)\right\rangle_{S_0}$. Performing the Fourier transform

$$\left\langle\delta(x_i-x_j)\right\rangle_{S_0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\left\langle\exp\left[ik\left(\widetilde{x}_i-\widetilde{x}_j\right)\right]\right\rangle_{S_0}. \quad (3.29)$$

Consider the exponent in Eq.(3.29)

$$\langle \exp [ik\widetilde{x}_i] \rangle_{S_0} = \exp \left[\kappa_i^1 (\widetilde{\tau}) + \kappa_i^2 (\widetilde{\tau}) \right]. \tag{3.30}$$

Therefore

$$\kappa_i^1 (\tau) = ik \left(\langle \widetilde{x}_i (\widetilde{\tau}) \rangle_{S_0} - \langle \widetilde{x}_j (\widetilde{\tau}) \rangle_{S_0} \right) \tag{3.31}$$

and κ_i^2

$$\begin{aligned} \kappa_i^2 (\tau) &= -k^2 \frac{1}{2} \left(\left\langle \left(\langle \widetilde{x}_i \rangle_{S_0} - \langle \widetilde{x}_j \rangle_{S_0} \right)^2 \right\rangle_{S_0} - \langle \widetilde{x}_i - \widetilde{x}_j \rangle_{S_0}^2 \right) \\ &= -k^2 \frac{1}{2} \begin{pmatrix} \langle \widetilde{x}_i^2 \rangle_{S_0} - \langle \widetilde{x}_i \rangle_{S_0}^2 + \langle \widetilde{x}_j^2 \rangle_{S_0} - \langle \widetilde{x}_j \rangle_{S_0}^2 \\ +2 \langle \widetilde{x}_i \rangle_{S_0} \langle \widetilde{x}_j \rangle_{S_0} - 2 \langle \widetilde{x}_i \widetilde{x}_j \rangle_{S_0} \end{pmatrix} \\ &= -k^2 \left(g (\widetilde{\tau}, \widetilde{\tau}) + \langle \widetilde{x}_i \rangle_{S_0} \langle \widetilde{x}_j \rangle_{S_0} - \langle \widetilde{x}_i \widetilde{x}_j \rangle_{S_0} \right). \end{aligned} \tag{3.32}$$

Using the Mean Field approximation $\langle \widetilde{x}_i \widetilde{x}_j \rangle_{S_0} = \langle \widetilde{x}_i \rangle_{S_0} \langle \widetilde{x}_j \rangle_{S_0}$. Substituting Eq.(3.31) and Eq.(3.32) into Eq.(3.29) and using the formula $\int dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$, we obtain

$$\begin{aligned} \langle \delta(x_i - x_j) \rangle_{S_0} &= \sqrt{\frac{1}{4\pi g (\widetilde{\tau}, \widetilde{\tau})}} \exp \left[- \frac{\left(\langle \widetilde{x}_i \rangle_{S_0} - \langle \widetilde{x}_j \rangle_{S_0} \right)^2}{4 g (\widetilde{\tau}, \widetilde{\tau})} \right] \\ &= \frac{1}{\left(\frac{4\pi i}{\widetilde{\omega}} \frac{\sin \widetilde{\omega}(\widetilde{t} - \widetilde{\tau}) \sin \widetilde{\omega} \widetilde{\tau}}{\sin \widetilde{\omega} \widetilde{t}} \right)^{1/2}} \exp \left[- \frac{\left(\frac{\left(\widetilde{x}_{2_i} - \widetilde{x}_{2_j} \right) \sin \widetilde{\omega} \widetilde{\tau} + \left(\widetilde{x}_{1_i} - \widetilde{x}_{1_j} \right) \sin \widetilde{\omega} (\widetilde{t} - \widetilde{\tau})}{\sin \widetilde{\omega} \widetilde{t}} \right)^2}{\left(\frac{4i}{\widetilde{\omega}} \frac{\sin \widetilde{\omega} (\widetilde{t} - \widetilde{\tau}) \sin \widetilde{\omega} \widetilde{\tau}}{\sin \widetilde{\omega} \widetilde{t}} \right)} \right]. \end{aligned} \tag{3.33}$$

We can write $\langle \delta(x_i - x_j) \rangle_{S_0} = \frac{1}{\sqrt{\pi}(b+c)^{1/2}} \exp \left[- \frac{a}{(b+c)} \right]$, where $b = -\frac{2i}{\widetilde{\omega}} \cot \widetilde{\omega} \widetilde{t}$ and $c = \frac{i}{\widetilde{\omega}} \cos \left(2\widetilde{\omega}\widetilde{\tau} - \widetilde{\omega}\widetilde{t} \right) \csc \widetilde{\omega}\widetilde{t}$. We expand $\frac{1}{(b+c)^{1/2}}$ and $\exp \left[- \frac{a}{(b+c)} \right]$ in the power of $\frac{c}{b}$ and $-\frac{a}{(b+c)}$, respectively. Integrating each term, we can write all infinite terms in the closed form (See Appendix B for more details). The result is

$$\begin{aligned}
&\widetilde{g}\int_0^{\widetilde{t}} \left\langle \delta(x_i - x_j)\right\rangle_{S_0} d\widetilde{\tau} \\
= \;& \widetilde{g}\sqrt{\frac{\widetilde{\omega}}{2\pi}}\widetilde{t} - i\widetilde{g}\sqrt{\frac{2}{\pi\widetilde{\omega}}}\ln 2 \\
&-i\widetilde{g}\sum_{n=1}^{\infty}\frac{(-1)^n\left((\widetilde{x}_{i_1}-\widetilde{x}_{j_1})^{2n}+(\widetilde{x}_{i_2}-\widetilde{x}_{j_2})^{2n}\right)}{(2n-1)\,n!}\sum_{i=1}^{\infty}\frac{1}{\widetilde{\omega}^{\left(i+n-\frac{1}{2}\right)}}\frac{i\prod_{j=0}^{i}\left(j+\frac{2n-1}{2}\right)}{(i+n-1)\left(i+\frac{2n-1}{2}\right)i!} \\
= \;& \widetilde{g}\sqrt{\frac{\widetilde{\omega}}{2\pi}}\widetilde{t} - \widetilde{g}\sqrt{\frac{2}{\pi\widetilde{\omega}}}\ln 2 - i\widetilde{g}\frac{(\widetilde{x}_{i_1}-\widetilde{x}_{j_1})^{2n}}{2\sqrt{2\pi}}\sqrt{\widetilde{\omega}}\;{}_pF_q\left(\{1,1\},\{\frac{3}{2},2\},\frac{(\widetilde{x}_{i_1}-\widetilde{x}_{j_1})^2\,\widetilde{\omega}}{2}\right) \\
&-i\widetilde{g}\frac{(\widetilde{x}_{i_2}-\widetilde{x}_{j_2})^{2n}}{2\sqrt{2\pi}}\sqrt{\widetilde{\omega}}\;{}_pF_q\left(\{1,1\},\{\frac{3}{2},2\},\frac{(\widetilde{x}_{i_2}-\widetilde{x}_{j_2})^2\,\widetilde{\omega}}{2}\right).
\end{aligned} \tag{3.34}$$

Here, ${}_pF_q$ is the generalized hypergeometric function. Correcting all contributions we obtain the propagator of the system.

$$\begin{aligned}
P \;\sim\; & \exp\left[-iN\left(\frac{\widetilde{\omega}}{4}+\frac{1}{4\widetilde{\omega}}+\frac{\widetilde{V}_b}{\sqrt{1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}}}+\frac{\widetilde{g}\,(N-1)}{2}\sqrt{\frac{\widetilde{\omega}}{2\pi}}\right)\widetilde{t}\right] \\
&\times\exp\left[\begin{array}{c}
-\sum_{i=1}^{N}\left(\frac{\widetilde{\omega}}{4}+\frac{1}{4\widetilde{\omega}}\right)\left(\widetilde{x}_{1_i}^2+\widetilde{x}_{2_i}^2\right) \\
+\widetilde{g}\sum_{i<j}^{N}\left[\frac{\left(\widetilde{x}_{1_i}^2-\widetilde{x}_{1_j}^2\right)^{2n}}{2\sqrt{2\pi}}\sqrt{\widetilde{\omega}}\;{}_pF_q\left(\{1,1\},\{\frac{3}{2},2\},\frac{\left(\widetilde{x}_{i_1}-\widetilde{x}_{j_1}\right)^2\widetilde{\omega}}{2}\right)\right] \\
+\widetilde{g}\sum_{i<j}^{N}\left[\frac{\left(\widetilde{x}_{2_i}^2-\widetilde{x}_{2_j}^2\right)^{2n}}{2\sqrt{2\pi}}\sqrt{\widetilde{\omega}}\;{}_pF_q\left(\{1,1\},\{\frac{3}{2},2\},\frac{\left(\widetilde{x}_{i_2}-\widetilde{x}_{j_2}\right)^2\widetilde{\omega}}{2}\right)\right] \\
-\widetilde{V}_b\sum_{i=1}^{N}\left[\sum_{n=1}^{\infty}\frac{(-i)^n\left(\widetilde{x}_{1_i}^{2n}+\widetilde{x}_{2_i}^{2n}\right)}{2n\widetilde{\omega}\widetilde{\sigma}^{2n}\left(1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}\right)^{n+\frac{1}{2}}}\;{}_2F_1\left(n,n+\frac{1}{2},n+1,\frac{1}{1+\widetilde{\sigma}^2\widetilde{\omega}}\right)\right]
\end{array}\right]
\end{aligned} \tag{3.35}$$

This means that the ground state energy of the entire system has the energy

$$\frac{\widetilde{E}_0\left(\widetilde{\omega}\right)}{N}=\frac{\widetilde{\omega}}{4}+\frac{1}{4\widetilde{\omega}}+\frac{\widetilde{V}_b}{\sqrt{1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}}}+\frac{\widetilde{g}\,(N-1)}{2}\sqrt{\frac{\widetilde{\omega}}{2\pi}}. \tag{3.36}$$

This ground state energy is exactly the same as that of Baym and Pethick [18] for 1D when $\widetilde{V}_b = 0$. The ground state wave function is

$$\phi_0\left(\widetilde{x}_1,\widetilde{x}_2,...,\widetilde{x}_N\right) = \sim \exp\left[\begin{array}{c} -\sum\limits_{i=1}^{N}\left(\frac{\widetilde{\omega}}{4}+\frac{1}{4\widetilde{\omega}}\right)\widetilde{x}_i^2 \\ +\widetilde{g}\sum\limits_{i<j}^{N}\left[\frac{1}{2}\sqrt{\frac{\widetilde{\omega}}{2\pi}}\left(\widetilde{x}_i-\widetilde{x}_j\right)^{2n}\ {}_pF_q\left(\{1,1\},\{\frac{3}{2},2\},\frac{(\widetilde{x}_i-\widetilde{x}_j)^2\widetilde{\omega}}{2}\right)\right] \\ -\widetilde{V}_b\sum\limits_{i=1}^{N}\left[\sum\limits_{n=1}^{\infty}\frac{(-i)^n\widetilde{x}_i^{2n}}{2n\widetilde{\omega}\widetilde{\sigma}^{2n}\left(1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}\right)^{n+\frac{1}{2}}}\ {}_pF_1\left(n,n+\frac{1}{2},n+1,\frac{2}{1+\widetilde{\sigma}^2\widetilde{\omega}}\right)\right]\end{array}\right]. \tag{3.37}$$

Assuming $\widetilde{x}_j = 0$, we can write the wave function in this form.

$$\phi_0\left(\widetilde{x}_1,\widetilde{x}_2,...,\widetilde{x}_N\right) = \phi_0\left(\widetilde{x}_1\right)\phi_0\left(\widetilde{x}_2\right)...\phi_0\left(\widetilde{x}_N\right) \tag{3.38}$$

Therefore

$$\phi_0\left(\widetilde{x}\right) \sim \exp\left[\begin{array}{c} -\left(\frac{\widetilde{\omega}}{4}+\frac{1}{4\widetilde{\omega}}\right)\widetilde{x}^2 \\ +\frac{\widetilde{g}(N-1)}{4}\sqrt{\frac{\widetilde{\omega}}{2\pi}}\widetilde{x}^{2n}\ {}_pF_q\left(\{1,1\},\{\frac{3}{2},2\},\frac{\widetilde{x}^2\widetilde{\omega}}{2}\right) \\ -\widetilde{V}_b\sum\limits_{n=1}^{\infty}\frac{(-1)^n\widetilde{x}^{2n}}{2n\widetilde{\omega}\widetilde{\sigma}^{2n}\left(1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}\right)^{n+\frac{1}{2}}}\ {}_2F_1\left(n,n+\frac{1}{2},n+1,\frac{1}{1+\widetilde{\sigma}^2\widetilde{\omega}}\right)\end{array}\right]. \tag{3.39}$$

We can normalize this wave function by using the condition $\int\limits_{-\infty}^{\infty}\left|\phi_0\left(\widetilde{x}\right)\right|^2 d\widetilde{x} = 1$. The first excited state wave function can be approximated in this form. (See Appendix E for more details.)

$$\phi_1\left(\widetilde{x}\right) \sim \widetilde{x}\phi_0\left(\widetilde{x}\right) \tag{3.40}$$

The overlap integral γ_{++}, γ_{+-} and γ_{--} are defined as

$$\begin{aligned}\gamma_{++} &= \widetilde{g}N\int\phi_0^4\left(\widetilde{x}\right)d\widetilde{x} \\ \gamma_{+-} &= \widetilde{g}N\int\phi_0^2\left(\widetilde{x}\right)\phi_1^2\left(\widetilde{x}\right)d\widetilde{x} \\ \gamma_{--} &= \widetilde{g}N\int\phi_1^4\left(\widetilde{x}\right)d\widetilde{x}.\end{aligned} \tag{3.41}$$

Physically, the γ_{ij} is the transitional probability which can be used to calculate the tunneling rate as shown in [7].

3.1.3 The Calculated Results

We minimize the ground state energy by solving $\frac{dE_0(\widetilde{\omega})}{d\widetilde{\omega}} = 0$. The parameters $\widetilde{\sigma} = 1.5$, $\widetilde{V}_b = 4$, $\widetilde{g}N = 1$ [7]. We find a curve which has a minimum point. From Figure 3.2, we get

$$\begin{aligned} \widetilde{E}_0 &= 3.555 \\ \widetilde{\omega} &= 0.282. \end{aligned} \tag{3.42}$$

We find that the ground energy is in good agreement with the Thomas-Fermi approximation and the two-mode model of the Gross-Pitaevskii equation as shown in Figure 3.3. Replacing $\widetilde{\omega}$ into Eq.(3.37) and Eq.(3.40) and normalizing the wave function, we obtain the normalized ground state and excited state wave functions which correspond to the potential as shown in Figure 3.4. Substituting ϕ_0 and ϕ_1 into Eq.(3.41), we obtain γ_{ij}.

$$\begin{aligned} \gamma_{++} &= 0.230 \\ \gamma_{+-} &= 0.224 \\ \gamma_{--} &= 0.303 \end{aligned} \tag{3.43}$$

For small $\widetilde{g}N$, we find that γ_{++} are in good agreement with the two-mode model of the Gross-Pitaevskii equation. In this case, the Thomas-Fermi approximation is different from the two-mode model of the Gross-Pitaevskii equation and path integral as shown in Figure 3.5 and Figure 3.6. Therefore the Thomas-Fermi approximation is not valid in this regime. We can continue calculating the γ_{ij} with various $\widetilde{V}_b$ as shown in Figure 3.7 and Figure 3.8. We find that for large enough $\widetilde{V}_b$, all γ_{ij} are equal.

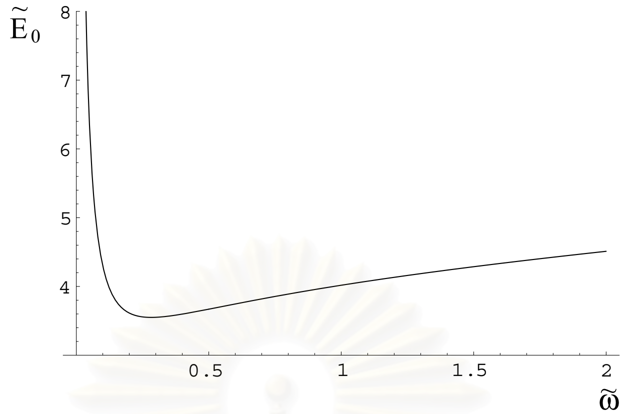

Figure 3.2: The ground state energy per particle plotted against $\widetilde{\omega}$.

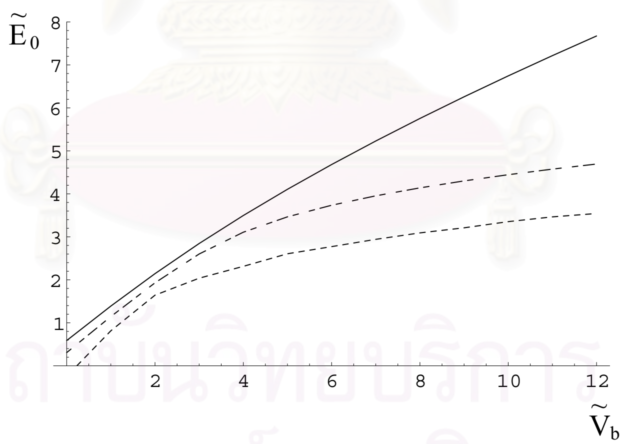

Figure 3.3: The ground state energy plotted against $\widetilde{V}_b$. The dash line is calculated by the Thomas-Fermi, the dot-dash line is calculated by Gross-Pitaevskii and the solid line is calculated by the path integral. ($\widetilde{g}N = 0.5, \widetilde{\sigma} = 1.5$)

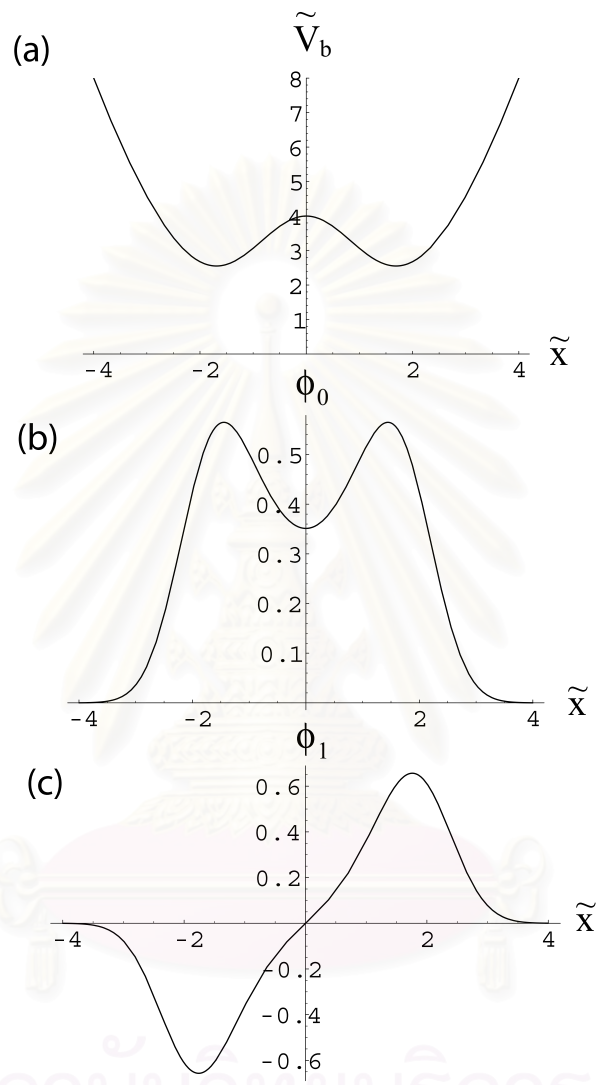

Figure 3.4: (a) The external potential $\frac{1}{2}\widetilde{x}^2 + \widetilde{V}_b \exp\left[-\left(\frac{\widetilde{x}}{\widetilde{\sigma}}\right)^2\right]$ plotted against $\widetilde{x}$, (b) the ground state and (c) excited state wave function plotted against $\widetilde{x}$ $\left(\widetilde{g}N = 1, \widetilde{V}_b = 4, \widetilde{\sigma} = 1.5\right)$.

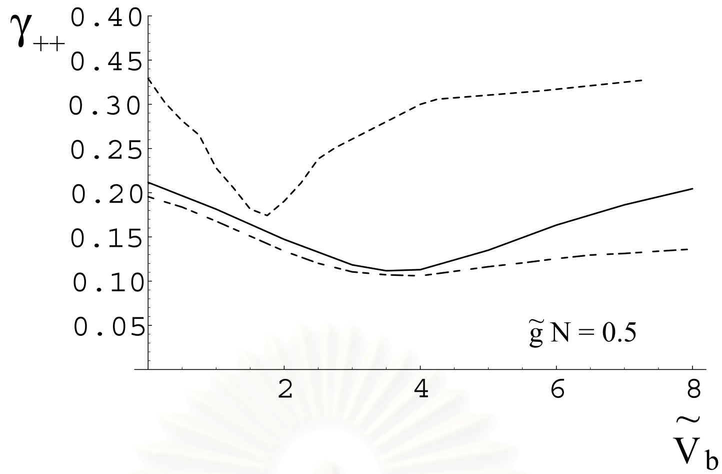

Figure 3.5: The γ_{++} plotted against $\widetilde{V}_b$. The dash line is calculated by the Thomas-Fermi approximation, the dot-dash line is calculated by the two-mode model of Gross-Pitaevskii equation and the solid line is calculated by path integral theory. ($\widetilde{g}N = 0.5, \widetilde{\sigma} = 1.5$)

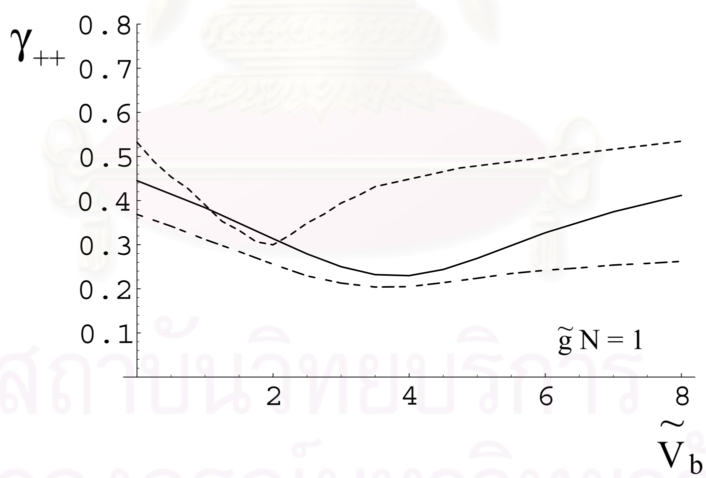

Figure 3.6: The γ_{++} plotted against $\widetilde{V}_b$. The dash line is calculated by Thomas-Fermi approximation, the dot-dash line is calculated by two-mode model of Gross-Pitaevskii equation and the solid line is calculated by path integral theory. ($\widetilde{g}N = 1, \widetilde{\sigma} = 1.5$)

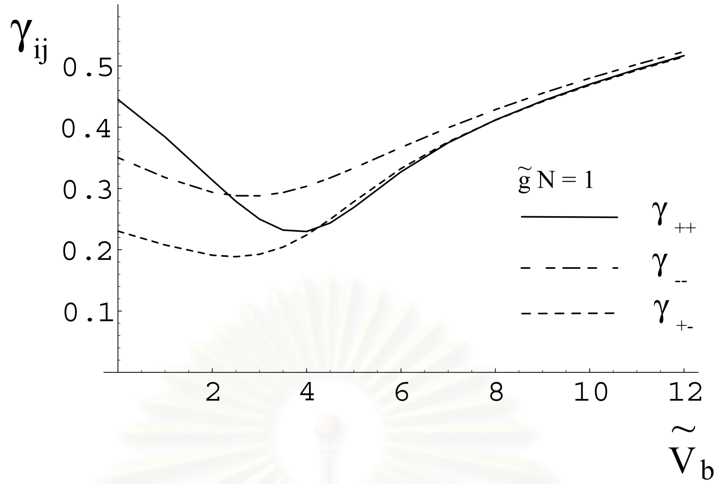

Figure 3.7: The γ_{ij} plotted against $\widetilde{V}_b$. The dash line is γ_{+-}, the dot-dash line is γ_{--} and the solid line is γ_{++}. ($\widetilde{g}N = 1, \widetilde{\sigma} = 1.5$)

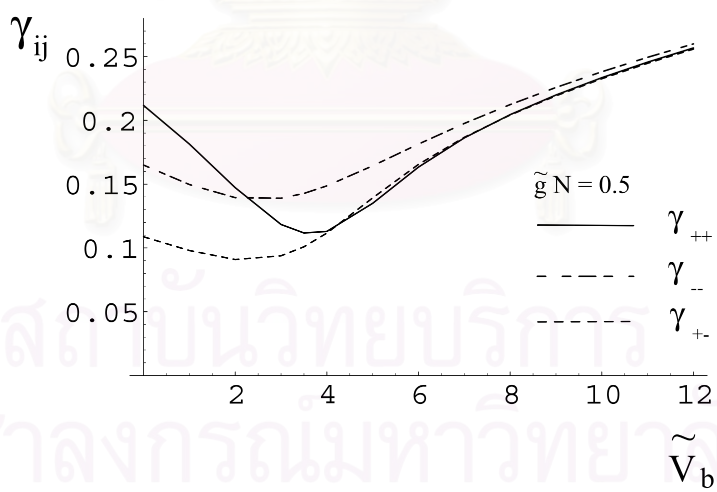

Figure 3.8: The γ_{ij} plotted against $\widetilde{V}_b$. The dash line is γ_{+-}, the dot-dash line is γ_{--} and the solid line is γ_{++}. ($\widetilde{g}N = 0.5, \widetilde{\sigma} = 1.5$)

3.2 Bose-Einstein Condensation with Gaussian Barrier in Three Dimensions

We consider the N interacting Bosons with mean field repulsive energy which is a s-wave scattering lengths $a > 0$. The N Bosons are confined in the 3D harmonic trap having a Gaussian barrier in x direction. The Lagrangian for entire system is

$$\begin{aligned} L &= \frac{m}{2}\sum_{i=1}^{N}(\dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2)-\frac{m}{2}\sum_{i=1}^{N}(\Omega_x^2x_i^2+\Omega_y^2y_i^2+\Omega_z^2z_i^2)-\sum_{i=1}^{N}V_b\exp\left[-\left(\frac{x_i}{\sigma}\right)^2\right] \\ &\quad -g\sum_{i<j}^{N}\delta(\overrightarrow{r}_i-\overrightarrow{r}_j). \end{aligned} \tag{3.44}$$

Using the relation $t=\frac{\widetilde{t}}{\overline{\Omega}}, x=l\widetilde{x}, \sigma=l\widetilde{\sigma}, \omega=\overline{\Omega}\widetilde{\omega}, \Omega_i=\overline{\Omega}\widetilde{\Omega}_i, l=\sqrt{\frac{\hbar}{m\overline{\Omega}}}, V_b=\hbar\overline{\Omega}\widetilde{V}_b$, $\widetilde{g}=\frac{4\pi a}{l}$ and $\overline{\Omega}=(\Omega_x\Omega_y\Omega_z)^{1/3}$, we obtain the model Lagrangian in the dimensionless version.

$$L=\hbar\overline{\Omega}\left[\begin{array}{c} \frac{1}{2}\sum\limits_{i=1}^{N}(\dot{\widetilde{x}}_i^2+\dot{\widetilde{y}}_i^2+\dot{\widetilde{z}}_i^2)-\frac{1}{2}\sum\limits_{i=1}^{N}(\widetilde{\Omega}_x^2\widetilde{x}_i^2+\widetilde{\Omega}_y^2\widetilde{y}_i^2+\widetilde{\Omega}_z^2\widetilde{z}_i^2) \\ -\widetilde{V}_b\sum\limits_{i=1}^{N}\exp\left[-\left(\frac{\widetilde{x}_i}{\widetilde{\sigma}}\right)^2\right]-\widetilde{g}\sum\limits_{i<j}^{N}\delta(\overrightarrow{\widetilde{r}}_i-\overrightarrow{\widetilde{r}}_j) \end{array}\right]. \tag{3.45}$$

We use the trial Lagrangian

$$L_0=\hbar\overline{\Omega}\left[\frac{1}{2}\sum_{i=1}^{N}(\dot{\widetilde{x}}_i^2+\dot{\widetilde{y}}_i^2+\dot{\widetilde{z}}_i^2)-\frac{1}{2}\sum_{i=1}^{N}\widetilde{\omega}^2(\widetilde{x}_i^2+\widetilde{y}_i^2+\widetilde{z}_i^2)\right]. \tag{3.46}$$

Here, $\widetilde{\omega}$ is the variational parameter. The propagator of the system is

$$\begin{aligned} & P\left(\overrightarrow{\widetilde{r}}_N\left(\widetilde{t}\right),\overrightarrow{\widetilde{r}}_N(0),\widetilde{\tau}\right) \\ \simeq\ & P_0\left(\overrightarrow{\widetilde{r}}_N\left(\widetilde{t}\right),\overrightarrow{\widetilde{r}}_N(0),\widetilde{\tau}\right) \\ & \times\exp\left[i\int_0^{\widetilde{t}}d\widetilde{\tau}\left(\begin{array}{c} \frac{1}{2}\sum\limits_{i=1}^{N}\left(\widetilde{\omega}^2-\widetilde{\Omega}_x^2\right)\langle\widetilde{x}_i^2\rangle_{S_0}+\frac{1}{2}\sum\limits_{i=1}^{N}\left(\widetilde{\omega}^2-\widetilde{\Omega}_y^2\right)\langle\widetilde{y}_i^2\rangle_{S_0} \\ +\frac{1}{2}\sum\limits_{i=1}^{N}(\widetilde{\omega}^2-\widetilde{\Omega}_z^2)\langle\widetilde{z}_i^2\rangle_{S_0}-\widetilde{V}_b\sum\limits_{i=1}^{N}\left\langle\exp\left[-\left(\frac{\widetilde{x}_i}{\widetilde{\sigma}}\right)^2\right]\right\rangle_{S_0} \\ -\widetilde{g}\sum\limits_{i<j}^{N}\left\langle\delta(\overrightarrow{\widetilde{r}}_i-\overrightarrow{\widetilde{r}}_j)\right\rangle_{S_0} \end{array}\right)\right]. \end{aligned} \tag{3.47}$$

The next step is to find $\left\langle \delta(\overrightarrow{\widetilde{r}}_i - \overrightarrow{\widetilde{r}}_j)\right\rangle_{S_0}$ in three dimensions. Performing $\left\langle \delta(\overrightarrow{\widetilde{r}}_i - \overrightarrow{\widetilde{r}}_j)\right\rangle_{S_0}$ in the Fourier transform, expanding the exponent in the first and second cumulant and using the Mean Field approximation $\left\langle \overrightarrow{\widetilde{r}}_i \overrightarrow{\widetilde{r}}_j \right\rangle_{S_0} = \left\langle \overrightarrow{\widetilde{r}}_i \right\rangle_{S_0} \left\langle \overrightarrow{\widetilde{r}}_j \right\rangle_{S_0}$, we obtain

$$
\begin{aligned}
& \left\langle \delta(\overrightarrow{\widetilde{r}}_i - \overrightarrow{\widetilde{r}}_j)\right\rangle_{S_0} \\
= \; & \left\langle \delta(\widetilde{x}_i - \widetilde{x}_j)\right\rangle_{S_0} \left\langle \delta(\widetilde{y}_i - \widetilde{y}_j)\right\rangle_{S_0} \left\langle \delta(\widetilde{z}_i - \widetilde{z}_j)\right\rangle_{S_0} \\
= \; & \left(\frac{1}{4\pi g\left(\widetilde{\tau},\widetilde{\tau}\right)}\right)^{3/2} \exp\left[- \left(\begin{array}{c} \left(\frac{\left(\widetilde{x}_{2_i}-\widetilde{x}_{2_j}\right)\sin\widetilde{\omega}\widetilde{\tau}+\left(\widetilde{x}_{1_i}-\widetilde{x}_{1_j}\right)\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})}{\sin\widetilde{\omega}\widetilde{t}}\right)^2 \\ +\left(\frac{\left(\widetilde{y}_{2_i}-\widetilde{y}_{2_j}\right)\sin\widetilde{\omega}\widetilde{\tau}+\left(\widetilde{y}_{1_i}-\widetilde{y}_{1_j}\right)\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})}{\sin\widetilde{\omega}\widetilde{t}}\right)^2 \\ +\left(\frac{\left(\widetilde{z}_{2_i}-\widetilde{z}_{2_j}\right)\sin\widetilde{\omega}\widetilde{\tau}+\left(\widetilde{z}_{1_i}-\widetilde{z}_{1_j}\right)\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})}{\sin\widetilde{\omega}\widetilde{t}}\right)^2 \end{array} \right) /4g\left(\widetilde{\tau},\widetilde{\tau}\right) \right].
\end{aligned}
\tag{3.48}
$$

Calculating the same as delta function in $\widetilde{x}$ direction (See Appendix C for more details), we obtain

$$
\begin{aligned}
& \widetilde{g}\int_0^{\widetilde{t}} \left\langle \delta(\overrightarrow{\widetilde{r}}_i - \overrightarrow{\widetilde{r}}_j)\right\rangle_{S_0} d\widetilde{\tau} \\
= \; & \widetilde{g}\left(\frac{\widetilde{\omega}}{2\pi}\right)^{3/2} \widetilde{t} \\
& -i\widetilde{g}\frac{\left(\widetilde{r}_{1_i}-\widetilde{r}_{1_j}\right)^2}{12\sqrt{2}} \left(\frac{\widetilde{\omega}}{\pi}\right)^{3/2} {}_pF_q\left(\{1,1\},\{\frac{5}{2},2\},\frac{\left(\widetilde{r}_{1_i}-\widetilde{r}_{1_j}\right)^2\widetilde{\omega}}{2}\right) \\
& -i\widetilde{g}\frac{\left(\widetilde{r}_{2_i}-\widetilde{r}_{2_j}\right)^2}{12\sqrt{\pi}} \left(\frac{\widetilde{\omega}}{\pi}\right)^{3/2} {}_pF_q\left(\{1,1\},\{\frac{5}{2},2\},\frac{\left(\widetilde{r}_{2_i}-\widetilde{r}_{2_j}\right)^2\widetilde{\omega}}{2}\right),
\end{aligned}
\tag{3.49}
$$

where

$$
\begin{aligned}
\left(\widetilde{r}_{1_i}-\widetilde{r}_{1_j}\right)^2 &= \left(\widetilde{x}_{1_i}-\widetilde{x}_{1_j}\right)^2+\left(\widetilde{y}_{1_i}-\widetilde{y}_{1_j}\right)^2+\left(\widetilde{z}_{1_i}-\widetilde{z}_{1_j}\right)^2 \\
\left(\widetilde{r}_{2_i}-\widetilde{r}_{2_j}\right)^2 &= \left(\widetilde{x}_{2_i}-\widetilde{x}_{2_j}\right)^2+\left(\widetilde{y}_{2_i}-\widetilde{y}_{2_j}\right)^2+\left(\widetilde{z}_{2_i}-\widetilde{z}_{2_j}\right)^2 .
\end{aligned}
\tag{3.50}
$$

The propagator of the system in three dimensions is

$$
\begin{aligned}
P \sim & \exp\left[-iN\left(\frac{3\widetilde{\omega}}{4}+\frac{\widetilde{\Omega}_x^2}{4\widetilde{\omega}}+\frac{\widetilde{\Omega}_y^2}{4\widetilde{\omega}}+\frac{\widetilde{\Omega}_z^2}{4\widetilde{\omega}}+\frac{\widetilde{V}_b}{\sqrt{1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}}}+\frac{\widetilde{g}\left(N-1\right)}{2}\left(\frac{\widetilde{\omega}}{2\pi}\right)^{3/2}\right)\widetilde{t}\right] \\
& \times\exp\left[\begin{array}{c}
-\sum_{i=1}^{N}\left(\frac{\widetilde{\omega}}{4}+\frac{\widetilde{\Omega}_x^2}{4\widetilde{\omega}}\right)\left(\widetilde{x}_{1_i}^2+\widetilde{x}_{2_i}^2\right)-\sum_{i=1}^{N}\left(\frac{\widetilde{\omega}}{4}+\frac{\widetilde{\Omega}_y^2}{4\widetilde{\omega}}\right)\left(\widetilde{y}_{1_i}^2+\widetilde{y}_{2_i}^2\right) \\
-\sum_{i=1}^{N}\left(\frac{\widetilde{\omega}}{4}+\frac{\widetilde{\Omega}_z^2}{4\widetilde{\omega}}\right)\left(\widetilde{z}_{1_i}^2+\widetilde{z}_{2_i}^2\right) \\
-\widetilde{V}_b\sum_{i=1}^{N}\left[\sum_{n=1}^{\infty}\frac{(-i)^n\left(\widetilde{x}_{1_i}^{2n}+\widetilde{x}_{2_i}^{2n}\right)}{2n\widetilde{\omega}\widetilde{\sigma}^{2n}\left(1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}\right)^{n+\frac{1}{2}}}\;{}_2\widetilde{F}_1\left(n,n+\frac{1}{2},n+1,\frac{2}{1+\widetilde{\sigma}^2\widetilde{\omega}}\right)\right] \\
+\widetilde{g}\sum_{i<j}^{N}\left[\frac{\left(\widetilde{r}_{1_i}-\widetilde{r}_{1_j}\right)^2}{12\sqrt{2}}\left(\frac{\widetilde{\omega}}{\pi}\right)^{3/2}\;{}_pF_q\left(\{1,1\},\{\frac{5}{2},2\},\frac{\left(\widetilde{r}_{1_i}-\widetilde{r}_{1_j}\right)^2\widetilde{\omega}}{2}\right)\right] \\
+\widetilde{g}\sum_{i<j}^{N}\left[\frac{\left(\widetilde{r}_{2_i}-\widetilde{r}_{2_j}\right)^2}{12\sqrt{2}}\left(\frac{\widetilde{\omega}}{\pi}\right)^{3/2}\;{}_pF_q\left(\{1,1\},\{\frac{5}{2},2\},\frac{\left(\widetilde{r}_{i_2}-\widetilde{r}_{j_2}\right)^2\widetilde{\omega}}{2}\right)\right]
\end{array}\right].
\end{aligned}
\tag{3.51}
$$

Thus, the ground state energy is

$$
\frac{\widetilde{E}_0\left(\widetilde{\omega}\right)}{N}=\frac{3\widetilde{\omega}}{4}+\frac{\widetilde{\Omega}_x^2}{4\widetilde{\omega}}+\frac{\widetilde{\Omega}_y^2}{4\widetilde{\omega}}+\frac{\widetilde{\Omega}_z^2}{4\widetilde{\omega}}+\frac{\widetilde{V}_b}{\sqrt{1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}}}+\frac{\widetilde{g}\left(N-1\right)}{2}\left(\frac{\widetilde{\omega}}{2\pi}\right)^{3/2} \tag{3.52}
$$

and the ground state wave function is

$$
\begin{aligned}
&\phi_0\left(\widetilde{x},\widetilde{y},\widetilde{z}\right) \\
&\sim \exp\left[\begin{array}{c}
-\left(\frac{\widetilde{\omega}}{4}+\frac{\widetilde{\Omega}_x^2}{4\widetilde{\omega}}\right)\widetilde{x}^2-\left(\frac{\widetilde{\omega}}{4}+\frac{\widetilde{\Omega}_y^2}{4\widetilde{\omega}}\right)\widetilde{y}^2-\left(\frac{\widetilde{\omega}}{4}+\frac{\widetilde{\Omega}_z^2}{4\widetilde{\omega}}\right)\widetilde{z}^2 \\
+\frac{\widetilde{g}(N-1)}{2}\left[\frac{\widetilde{x}^2}{12\sqrt{2}}\left(\frac{\widetilde{\omega}}{\pi}\right)^{3/2}\;{}_pF_q\left(\{1,1\},\{\frac{5}{2},2\},\frac{\widetilde{x}^2\widetilde{\omega}}{2}\right)\right] \\
+\frac{\widetilde{g}(N-1)}{2}\left[\frac{\widetilde{y}^2}{12\sqrt{2}}\left(\frac{\widetilde{\omega}}{\pi}\right)^{3/2}\;{}_pF_q\left(\{1,1\},\{\frac{5}{2},2\},\frac{\widetilde{y}^2\widetilde{\omega}}{2}\right)\right] \\
+\frac{\widetilde{g}(N-1)}{2}\left[\frac{\widetilde{z}^2}{12\sqrt{2}}\left(\frac{\widetilde{\omega}}{\pi}\right)^{3/2}\;{}_pF_q\left(\{1,1\},\{\frac{5}{2},2\},\frac{\widetilde{z}^2\widetilde{\omega}}{2}\right)\right] \\
-\widetilde{V}_b\sum_{n=1}^{\infty}\frac{(-i)^n\widetilde{x}^{2n}}{2n\widetilde{\omega}\widetilde{\sigma}^{2n}\left(1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}}\right)^{n+\frac{1}{2}}}\;{}_2\widetilde{F}_1\left(n,n+\frac{1}{2},n+1,\frac{2}{1+\widetilde{\sigma}^2\widetilde{\omega}}\right)
\end{array}\right].
\end{aligned}
\tag{3.53}
$$

The first excited state of the system can be approximated as

$$
\phi_1\left(\widetilde{x},\widetilde{y},\widetilde{z}\right)\sim\widetilde{x}\phi_0\left(\widetilde{x}\right)\phi_0\left(\widetilde{y}\right)\phi_0\left(\widetilde{z}\right). \tag{3.54}
$$

We can normalize this wave function by using the condition

$$
\int\widetilde{x}^2\phi_0^2\left(\widetilde{x}\right)d\widetilde{x}\int\phi_0^2\left(\widetilde{y}\right)d\widetilde{y}\int\phi_0^2\left(\widetilde{z}\right)d\widetilde{z}=1. \tag{3.55}
$$

The γ_{ij} in three dimensions can be written as

$$\begin{aligned}
\gamma_{++} &= \widetilde{g}N\int\phi_0^4(\widetilde{x})\,d\widetilde{x}\int\phi_0^4(\widetilde{y})\,d\widetilde{y}\int\phi_0^4(\widetilde{z})\,d\widetilde{z}\\
\gamma_{+-} &= \widetilde{g}N\int\phi_0^2(\widetilde{x})\,\phi_1^2(\widetilde{x})\,d\widetilde{x}\int\phi_0^2(\widetilde{y})\,\phi_1^2(\widetilde{y})\,d\widetilde{y}\int\phi_0^2(\widetilde{z})\,\phi_1^2(\widetilde{z})\,d\widetilde{z}\\
\gamma_{--} &= \widetilde{g}N\int\phi_1^4(\widetilde{x})\,d\widetilde{x}\int\phi_1^4(\widetilde{y})\,d\widetilde{y}\int\phi_1^4(\widetilde{z})\,d\widetilde{z}. \qquad (3.56)
\end{aligned}$$

3.2.1 The Calculated Results

We minimize the ground state energy by using the parameters $\widetilde{\Omega}_x=\widetilde{\Omega}_y=\widetilde{\Omega}_z=1$, $\widetilde{\sigma}=1.5$, $\widetilde{V}_b=4$, $\widetilde{g}N=1$. We find one curve which has a minimum point as shown in Figure 3.9. Therefore we obtain the value of the ground state energy and the value of the minimized parameter.

$$\begin{aligned}
\widetilde{E}_0 &= 4.740\\
\widetilde{\omega} &= 0.657 \qquad (3.57)
\end{aligned}$$

Replacing $\widetilde{\omega}$ in to Eq.(3.53) and Eq.(3.54) and normalizing the wave function, we obtain the normalized ground state and excited state wave functions as shown in Figure 3.10 and Figure 3.11, respectively. We can continue calculating the γ_{ij} with various $\widetilde{V}_b$. The results are shown in Figure 3.12.

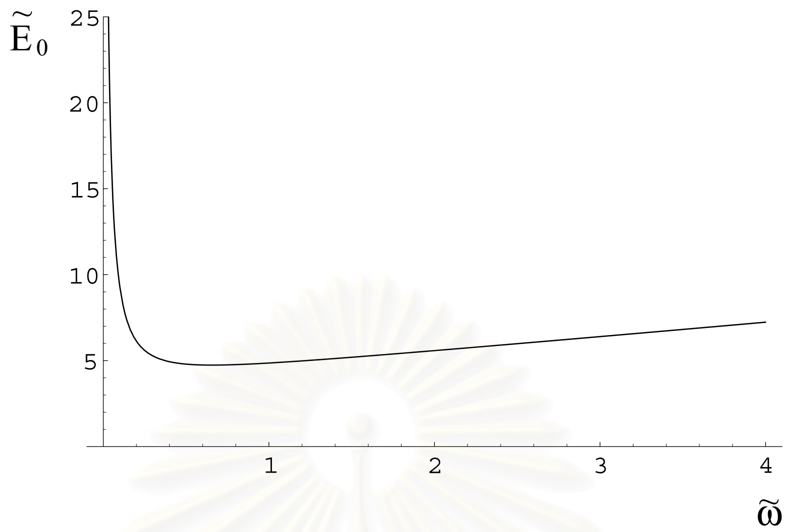

Figure 3.9: The ground state energy per particle plotted against $\widetilde{\omega}$.

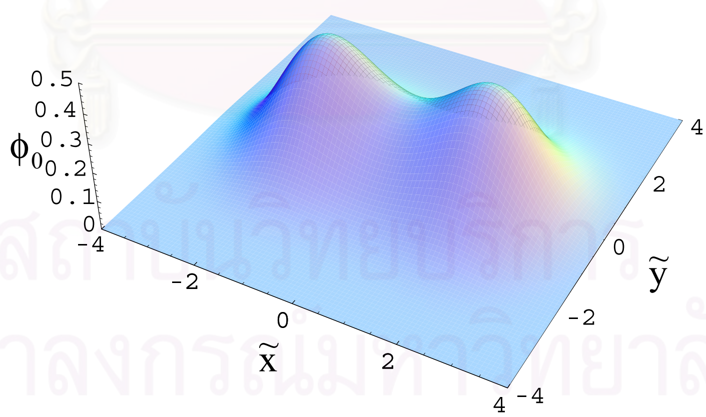

Figure 3.10: The ground state wave function plotted in three dimensions. ($\widetilde{g}N = 1$, $\widetilde{\sigma} = 1.5$)

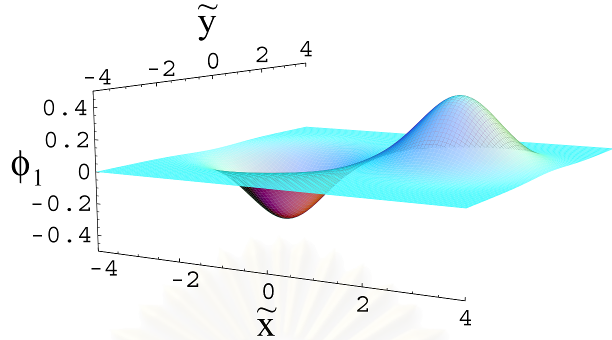

Figure 3.11: The excited state wave function plotted in three dimensions. ($\widetilde{g}N = 1$, $\widetilde{\sigma} = 1.5$)

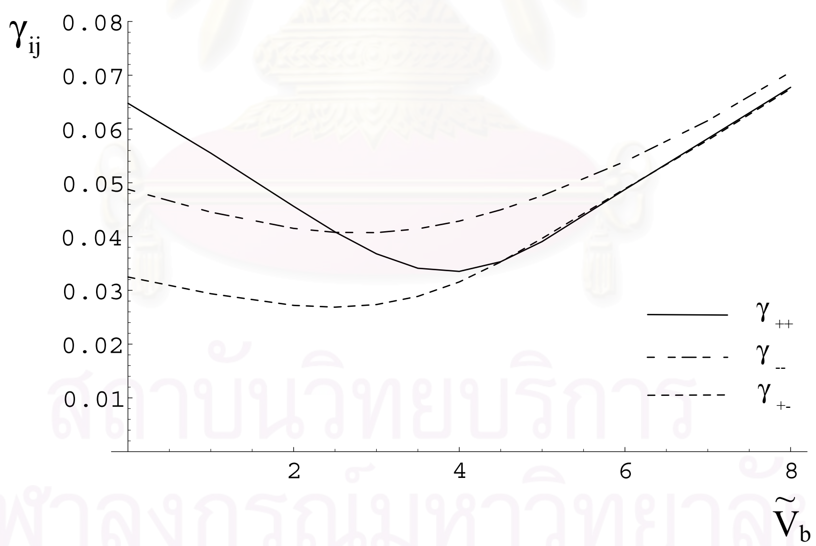

Figure 3.12: The γ_{ij} plotted against $\widetilde{V}_b$. The dash line is γ_{+-}, the dot-dash line is γ_{--} and the solid line is γ_{++}. $(\widetilde{g}N = 1, \widetilde{\sigma} = 1.5)$

3.3 Bose-Einstein Condensation with Cosine Barrier in Three Dimensions

We consider the N Bosons confining in the 3D harmonic trap with a cosine barrier in x direction. The Lagrangian for the entire system is

$$\begin{aligned} L &= \frac{m}{2}\sum_{i=1}^{N}(\dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2)-\frac{m}{2}\sum_{i=1}^{N}(\Omega_x^2x_i^2+\Omega_y^2y_i^2+\Omega_z^2z_i^2)-\frac{A}{2}\sum_{i=1}^{N}(\cos(2kx_i)+1) \\ &\quad -\frac{4\pi\hbar^2 a}{m}\sum_{i<j}^{N}\delta(\overrightarrow{r}_i-\overrightarrow{r}_j). \end{aligned} \tag{3.58}$$

Using the relation $t=\frac{\widetilde{t}}{\overline{\Omega}}, x=l\widetilde{x}, k=\frac{\widetilde{k}}{l}, \omega_i=\overline{\Omega}\widetilde{\omega}_i, \Omega_i=\overline{\Omega}\widetilde{\Omega}_i, l=\sqrt{\frac{\hbar}{m\overline{\Omega}}}, A=\hbar\overline{\Omega}\widetilde{A}$ and $\widetilde{g}=\frac{4\pi a}{l}$, the model Lagrangian in the dimensionless version is

$$L=\hbar\overline{\Omega}\left[\begin{array}{c} \frac{1}{2}\sum\limits_{i=1}^{N}(\dot{\widetilde{x}}_i^{2}+\dot{\widetilde{y}}_i^{2}+\dot{\widetilde{z}}_i^{2})-\frac{1}{2}\sum\limits_{i=1}^{N}(\widetilde{\Omega}_x^2\widetilde{x}_i^2+\widetilde{\Omega}_y^2\widetilde{y}_i^2+\widetilde{\Omega}_z^2\widetilde{z}_i^2) \\ -\frac{\widetilde{A}}{2}\sum\limits_{i=1}^{N}\left(\cos\left(2\widetilde{k}\widetilde{x}_i\right)+1\right)-\widetilde{g}\sum\limits_{i<j}^{N}\delta(\overrightarrow{\widetilde{r}}_i-\overrightarrow{\widetilde{r}}_j) \end{array}\right]. \tag{3.59}$$

We use the trial Lagrangian

$$L_0=\hbar\overline{\Omega}\left[\frac{1}{2}\sum_{i=1}^{N}(\dot{\widetilde{x}}_i^{2}+\dot{\widetilde{y}}_i^{2}+\dot{\widetilde{z}}_i^{2})-\frac{1}{2}\sum_{i=1}^{N}\widetilde{\omega}^2(\widetilde{x}_i^2+\widetilde{y}_i^2+\widetilde{z}_i^2)\right]. \tag{3.60}$$

Thus the propagator is

$$\begin{aligned} & P\left(\overrightarrow{\widetilde{r}}_N\left(\widetilde{t}\right),\overrightarrow{\widetilde{r}}_N(0),\widetilde{\tau}\right) \\ \simeq\ & P_0\left(\overrightarrow{\widetilde{r}}_N\left(\widetilde{t}\right),\overrightarrow{\widetilde{r}}_N(0),\widetilde{\tau}\right) \\ & \times\exp\left[i\int_0^{\widetilde{t}}d\widetilde{\tau}\left(\begin{array}{c} \frac{1}{2}\sum\limits_{i=1}^{N}\left(\widetilde{\omega}^2-\widetilde{\Omega}_x^2\right)\langle\widetilde{x}_i^2\rangle_{S_0}+\frac{1}{2}\sum\limits_{i=1}^{N}\left(\widetilde{\omega}^2-\widetilde{\Omega}_y^2\right)\langle\widetilde{y}_i^2\rangle_{S_0} \\ +\frac{1}{2}\sum\limits_{i=1}^{N}(\widetilde{\omega}^2-\widetilde{\Omega}_z^2)\langle\widetilde{z}_i^2\rangle_{S_0}-\frac{\widetilde{A}}{2}\sum\limits_{i=1}^{N}\left\langle\cos\left(2\widetilde{k}\widetilde{x}_i\right)+1\right\rangle_{S_0} \\ -\widetilde{g}\sum\limits_{i<j}^{N}\left\langle\delta(\overrightarrow{\widetilde{r}}_i-\overrightarrow{\widetilde{r}}_j)\right\rangle_{S_0} \end{array}\right)\right]. \end{aligned} \tag{3.61}$$

Thus we may write $\left\langle\cos\left(2\widetilde{k}\widetilde{x}_i\right)\right\rangle_{S_0}$ as

$$\left\langle\cos\left(2\widetilde{k}\widetilde{x}_i\right)\right\rangle_{S_0}=\left\langle e^{2i\widetilde{k}\widetilde{x}_i}+e^{-2i\widetilde{k}\widetilde{x}_i}\right\rangle/2=\left(\left\langle e^{2i\widetilde{k}\widetilde{x}_i}\right\rangle+\left\langle e^{-2i\widetilde{k}\widetilde{x}_i}\right\rangle\right)/2. \tag{3.62}$$

We expand the average up to the first and second cumulants. Because S_0 is quadratic, only the first two cumulants are non-zero [17], i.e.

$$\left\langle \exp\left[2i\widetilde{k}\widetilde{x}_i\right]\right\rangle_{S_0} = \exp\left[2i\widetilde{k}\left\langle\widetilde{x}_i\right\rangle_{S_0} - 2\widetilde{k}^2\left(\left\langle\widetilde{x}_i^2\right\rangle_{S_0} - \left\langle\widetilde{x}_i\right\rangle_{S_0}^2\right)\right] \tag{3.63}$$

$$\left\langle \exp\left[-2i\widetilde{k}\widetilde{x}_i\right]\right\rangle_{S_0} = \exp\left[-2i\widetilde{k}\left\langle\widetilde{x}_i\right\rangle_{S_0} - 2\widetilde{k}^2\left(\left\langle\widetilde{x}_i^2\right\rangle_{S_0} - \left\langle\widetilde{x}_i\right\rangle_{S_0}^2\right)\right]. \tag{3.64}$$

The Green function $g(\widetilde{\tau},\widetilde{\tau})$ can be defined as

$$g(\widetilde{\tau},\widetilde{\tau}) = \left\langle\widetilde{x}_i^2\right\rangle_{S_0} - \left\langle\widetilde{x}_i\right\rangle_{S_0}^2. \tag{3.65}$$

Thus

$$\left\langle \exp\left[2i\widetilde{k}\widetilde{x}_i\right]\right\rangle_{S_0} = \exp\left[2i\widetilde{k}\left\langle\widetilde{x}_i\right\rangle_{S_0} - 2\widetilde{k}^2 g(\widetilde{\tau},\widetilde{\tau})\right]. \tag{3.66}$$

Expanding $\left\langle \exp\left[2i\widetilde{k}\widetilde{x}_i\right]\right\rangle_{S_0}$ in the series by using the relation $e^x = \sum\limits_{n=0}^{\infty}\frac{x^n}{n!}$, integrating each term and taking the limit at $\widetilde{t}\to\infty$, we find that the terms which depend on $\widetilde{t}$ and end point term can be written in the closed form. (See Appendix D for more details.) Thus we obtain

$$\begin{aligned}
& -\frac{i\widetilde{A}}{2}\int_0^{\widetilde{t}} d\widetilde{\tau}\left(\sum_{i=1}^{N}\left\langle\cos\left(2\widetilde{k}\widetilde{x}_i\right)+1\right\rangle_{S_0}\right) \\
= & -\frac{iN\widetilde{A}}{2}\exp\left(-\frac{k^2}{\widetilde{\omega}}\right)\widetilde{t} - \frac{iN\widetilde{A}}{2}\widetilde{t} \\
& -\sum_{i=1}^{N}\frac{1}{4\widetilde{\omega}}\left(-2\left(\gamma - \mathrm{Ci}\left[2\widetilde{k}\widetilde{x}_{1_i}\right] + \ln 2\widetilde{k}\widetilde{x}_{1_i}\right) - 2\left(\gamma - \mathrm{Ci}\left[2\widetilde{k}\widetilde{x}_{2_i}\right] + \ln 2\widetilde{k}\widetilde{x}_{2_i}\right)\right) \\
& -\sum_{i=1}^{N}\frac{\widetilde{A}}{2\widetilde{\omega}^2}\sum_{j=1}^{\infty}\frac{(-1)^{(j+1)}(2j+1)\left(2\widetilde{k}\right)^{2(j+1)}}{(2j+2)!j}\left(\widetilde{x}_{1_i}^j + \widetilde{x}_{2_i}^j\right)^{2j} \\
& \times e^{-\frac{\widetilde{k}^2}{\widetilde{\omega}}}(j+1)\left(-\frac{\widetilde{k}^2}{\widetilde{\omega}}\right)^{-(1+j)}\left(\Gamma(1+j) - \Gamma\left(1+j, -\frac{\widetilde{k}^2}{\widetilde{\omega}}\right)\right),
\end{aligned} \tag{3.67}$$

where γ is Euler's constant, with numerical value $\simeq 0.577216$, Ci$[x]$ is the cosine integral function, $\Gamma[x]$ is the Euler gamma function. Thus we obtain the propagator of the system

$$
\begin{aligned}
P \quad \sim \quad & \exp\left[-iN\left(\frac{3\widetilde{\omega}}{4}+\frac{\widetilde{\Omega}_x^2}{4\widetilde{\omega}}+\frac{\widetilde{\Omega}_y^2}{4\widetilde{\omega}}+\frac{\widetilde{\Omega}_z^2}{4\widetilde{\omega}}+\frac{\widetilde{A}}{2}+\frac{\widetilde{A}}{2}\exp\left(-\frac{\widetilde{k}^2}{\widetilde{\omega}}\right)+\frac{\widetilde{g}\,(N-1)}{2}\left(\frac{\widetilde{\omega}}{2\pi}\right)^{3/2}\right)\widetilde{t}\right] \\
& \times\exp\left[\begin{array}{c}
+\widetilde{g}\sum_{i<j}^{N}\left[\frac{\left(\widetilde{r}_{1_i}-\widetilde{r}_{1_j}\right)^2}{12\sqrt{2}}\left(\frac{\widetilde{\omega}}{\pi}\right)^{3/2}\ {}_pF_q\left(\{1,1\},\{\frac{5}{2},2\},\frac{\left(\widetilde{r}_{1_i}-\widetilde{r}_{1_j}\right)^2\widetilde{\omega}}{2}\right)\right] \\
+\widetilde{g}\sum_{i<j}^{N}\left[\frac{\left(\widetilde{r}_{2_i}-\widetilde{r}_{2_j}\right)^2}{12\sqrt{2}}\left(\frac{\widetilde{\omega}}{\pi}\right)^{3/2}\ {}_pF_q\left(\{1,1\},\{\frac{5}{2},2\},\frac{\left(\widetilde{r}_{i_2}-\widetilde{r}_{j_2}\right)^2\widetilde{\omega}}{2}\right)\right] \\
-\sum_{i=1}^{N}\frac{1}{4\widetilde{\omega}}\left(\begin{array}{c}
-2\left(\gamma-\mathrm{Ci}\left[2\widetilde{k}\widetilde{x}_{1_i}\right]+\ln 2\widetilde{k}\widetilde{x}_{1_i}\right) \\
-2\left(\gamma-\mathrm{Ci}\left[2\widetilde{k}\widetilde{x}_{2_i}\right]+\ln 2\widetilde{k}\widetilde{x}_{2_i}\right)
\end{array}\right) \\
-\frac{\widetilde{A}}{2\widetilde{\omega}^2}\sum_{j=1}^{\infty}\frac{(-1)^{(j+1)}(2j+1)\left(2\widetilde{k}\right)^{2(j+1)}}{(2j+2)!j}\left(\widetilde{x}_{1_i}^{2j}+\widetilde{x}_{2_i}^{2j}\right) \\
\times e^{-\frac{\widetilde{k}^2}{\widetilde{\omega}}}(j+1)\left(-\frac{\widetilde{k}^2}{\widetilde{\omega}}\right)^{-(1+j)}\left(\Gamma\left[1+j\right]-\Gamma\left[1+j,-\frac{\widetilde{k}^2}{\widetilde{\omega}}\right]\right)
\end{array}\right].
\end{aligned}
\tag{3.68}
$$

This means that the ground state energy of the entire system has the energy

$$
\frac{\widetilde{E}_0}{N}=\frac{3\widetilde{\omega}}{4}+\frac{\widetilde{\Omega}_x^2}{4\widetilde{\omega}}+\frac{\widetilde{\Omega}_y^2}{4\widetilde{\omega}}+\frac{\widetilde{\Omega}_z^2}{4\widetilde{\omega}}+\frac{\widetilde{A}}{2}+\frac{\widetilde{A}}{2}\exp\left(-\frac{\widetilde{k}^2}{\widetilde{\omega}}\right)+\frac{\widetilde{g}\,(N-1)}{2}\left(\frac{\widetilde{\omega}}{2\pi}\right)^{3/2}. \tag{3.69}
$$

and the ground state wave function in three dimensions

$$
\phi_0\left(\overrightarrow{\widetilde{r}}\right) \quad \sim \quad \exp\left[\begin{array}{c}
-\left(\frac{\widetilde{\omega}}{4}+\frac{\widetilde{\Omega}_x^2}{4\widetilde{\omega}}\right)\widetilde{x}^2-\left(\frac{\widetilde{\omega}}{4}+\frac{\widetilde{\Omega}_y^2}{4\widetilde{\omega}}\right)\widetilde{y}^2-\left(\frac{\widetilde{\omega}}{4}+\frac{\widetilde{\Omega}_z^2}{4\widetilde{\omega}}\right)\widetilde{z}^2 \\
+\frac{\widetilde{g}(N-1)}{2}\left[\frac{\widetilde{x}^2}{12\sqrt{2}}\left(\frac{\widetilde{\omega}}{\pi}\right)^{3/2}\ {}_pF_q\left(\{1,1\},\{\frac{5}{2},2\},\frac{\widetilde{x}^2\widetilde{\omega}}{2}\right)\right] \\
+\frac{\widetilde{g}(N-1)}{2}\left[\frac{\widetilde{y}^2}{12\sqrt{2}}\left(\frac{\widetilde{\omega}}{\pi}\right)^{3/2}\ {}_pF_q\left(\{1,1\},\{\frac{5}{2},2\},\frac{\widetilde{y}^2\widetilde{\omega}}{2}\right)\right] \\
+\frac{\widetilde{g}(N-1)}{2}\left[\frac{\widetilde{z}^2}{12\sqrt{2}}\left(\frac{\widetilde{\omega}}{\pi}\right)^{3/2}\ {}_pF_q\left(\{1,1\},\{\frac{5}{2},2\},\frac{\widetilde{z}^2\widetilde{\omega}}{2}\right)\right] \\
+\frac{\widetilde{A}}{4\widetilde{\omega}}\left(\gamma-\mathrm{Ci}\left[2\widetilde{k}\widetilde{x}\right]+\ln 2\widetilde{k}\widetilde{x}\right)-\frac{\widetilde{A}}{2\widetilde{\omega}^2}\sum_{j=1}^{\infty}\frac{(-1)^{(j+1)}(2j+1)\left(2\widetilde{k}\right)^{2(j+1)}}{(2j+2)!j}\widetilde{x}^{2j} \\
\times e^{-\frac{\widetilde{k}^2}{\widetilde{\omega}}}(j+1)\left(-\frac{\widetilde{k}^2}{\widetilde{\omega}}\right)^{-(1+j)}\left(\Gamma\left[1+j\right]-\Gamma\left[1+j,-\frac{\widetilde{k}^2}{\widetilde{\omega}}\right]\right)
\end{array}\right]. \tag{3.70}
$$

The first excited state of the system can be approximated as $\phi_1\left(\overrightarrow{\widetilde{r}}\right)\sim\widetilde{x}\phi_0\left(\widetilde{x}\right)\phi_0\left(\widetilde{y}\right)\phi_0\left(\widetilde{z}\right)$.

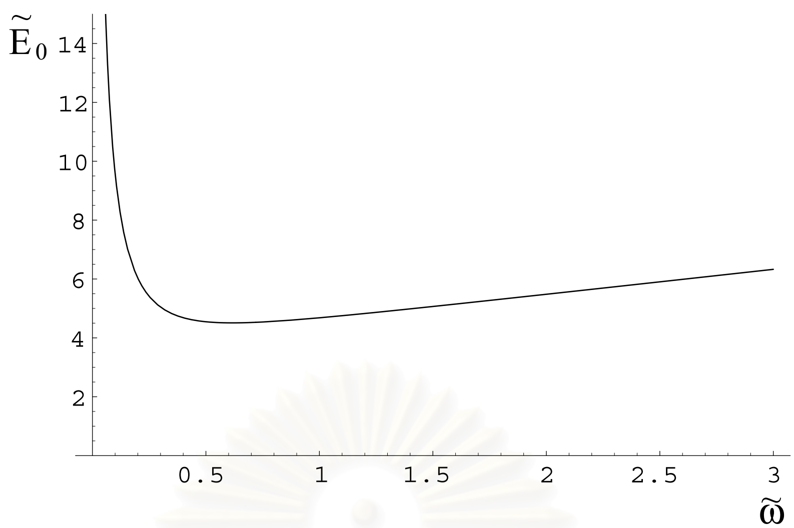

Figure 3.13: The ground state energy per particle plotted against $\widetilde{\omega}$.

3.3.1 The Calculated Results

We minimize the ground state energy using the parameters $\widetilde{\Omega}_x = \widetilde{\Omega}_y = \widetilde{\Omega}_z = 1$, $\widetilde{g} = 0.05$, $\widetilde{A} = 4$, $N = 20$. We find one curve which has a minimum point as shown in Figure 3.13.

$$\begin{aligned} \widetilde{E}_0 &= 4.508 \\ \widetilde{\omega} &= 0.617 \end{aligned} \tag{3.71}$$

Replacing $\widetilde{\omega}$ into $\phi_0(\overrightarrow{\widetilde{r}})$ and $\phi_1(\overrightarrow{\widetilde{r}})$ and normalizing the wave function. We obtain the normalized ground state and excited state wave functions as shown in Figure 3.14 and Figure 3.15, respectively. We find that γ_{++} are in good agreement with the two-mode model of Gross-Pitaevskii equation as shown in Figure 3.16. We can continue calculating the γ_{ij} with various $\widetilde{A}$. The results are shown in Figure 3.16.

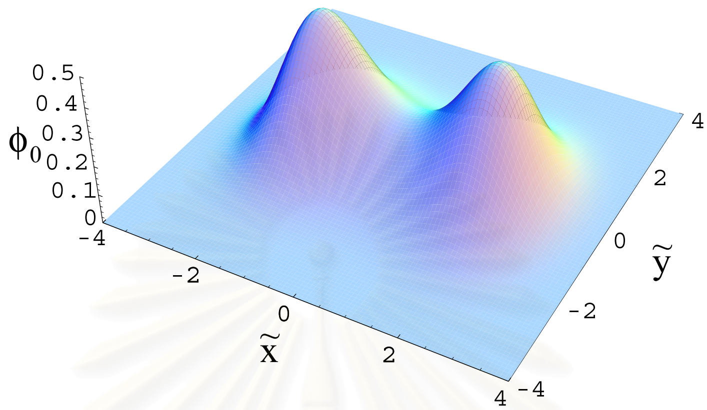

Figure 3.14: The ground state wave function plotted in three dimensions ($\tilde{g}N = 1$).

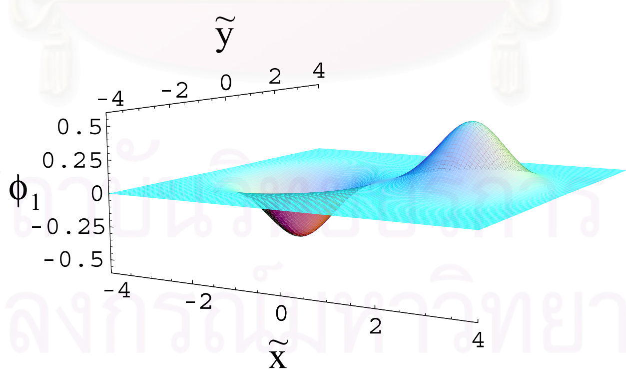

Figure 3.15: The excited state wave function plotted in three dimensions. ($\tilde{g}N = 1$)

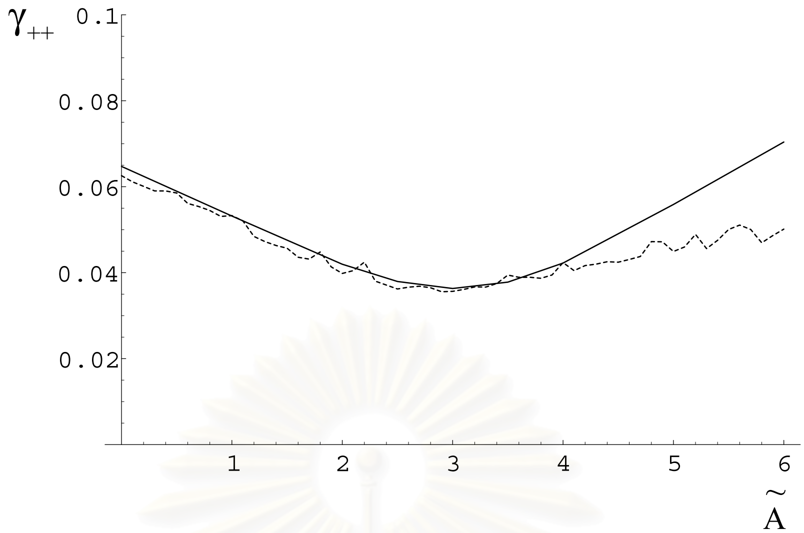

Figure 3.16: The γ_{++} plotted against $\widetilde{A}$. The dash line is calculated by two-mode model of Gross-Pitaevskii equation and the solid line is calculated by the path integral theory. ($\widetilde{g}N = 1$).

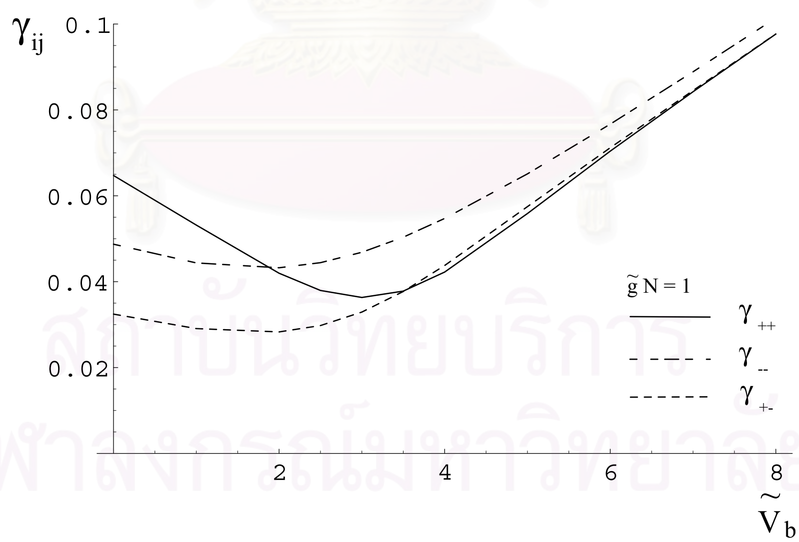

Figure 3.17: The γ_{ij} plotted against $\widetilde{A}$. The dash line is γ_{--}, the dot-dash line is γ_{+-} and the solid line is γ_{++}. ($\widetilde{g}N = 1$)

We have studied Bose-Einstein condensation in a double well potential in two types of double well potentials. We find that the ground state energy is exactly the same as that of Baym and Pethick [18] for the case of the BEC in harmonic potential $\left(\widetilde{V}_b = \widetilde{A} = 0\right)$. The wave functions of the system are used to calculated γ_{ij}. We find that γ_{ij} are in good agreement with the two-mode model of Gross-Pitaevskii equation for small $\widetilde{g}N$ in both types of the double well potentials.

CHAPTER IV

BOSE-EINSTEIN CONDENSATION IN A DISORDERED SYSTEM

Bose-Einstein condensation in a disordered system or "dirty Bose system" recently has attracted many researchers, both theoretically and experimentally. In the experiment, liquid ^{4}He adsorbed in porous media, such as Vycor or silica gel (aerogel, xerogel). The suppression of the superfluidity and the critical behavior at the phase transition have been investigated by Reppy [8], and the elementary excitations of liquid ^{4}He have been studied by Glyde *et al.* [9] using neutron inelastic scattering. The system exhibited many interesting properties such as the suppression of the superfluid and critical behavior near the phase transition.

In a dilute Bose gas, the transition temperature T_c will be an increasing function of the interaction parameter, $\overline{N}a^3$ [22-26] where a is the hard sphere diameter and $\overline{N}$ the particle density. In the case of liquid ^{4}He, the transition temperature would be reduced as a consequence of interparticle interaction. In the experiment of Reppy *et al.* [12], they study the liquid ^{4}He in a Vycor glass for the low density regime. The ^{4}He-Vycor system offers advantages over the BEC systems of trapped atomic gases because in the Vycor case the interaction parameter can be varied continuously from the low density to high density regime. The ^{4}He-Vycor system allows sample sizes on the order of a cubic centimeter. The interior channels of the porous Vycor glass used for measurements range in diameter from 4 to 8 nm and form a highly interconnected 3D network. The superfluid helium atoms are constrained by van der Waals forces to move over the complex 3D-connected surface provided by the pores. The system is cooled by the cryogenic techniques and thermometry methods [12]. They used a torsional oscillator technique to obtain a signal proportional to the superfluid particle density. In the low density limit, thermodynamic properties similar to an ideal Bose gas with an effective mass, m^*. From the experiment, the critical temperature increases by the effect of Vycor system.

The dilute Bose gas in the presence of quenched impurities can be worked out analytically within the Bogoliubov model by treating the random external potential as a perturbation. The effect of disorder on the ground state energy, superfluid and condensate fraction are calculated by Astrakharchik *et al.* [15]. They have found that the superfluid and condensate components of the system are suppressed by the disorder.

We study Bose-Einstein condensation in a disordered system in finite correlation length using the Feynman path integral. The advantage of this method is that we can study the system in both white noise limit and long length correlation. In this Chapter, we consider the model of Bose system consisting of N particles with two body interactions confined within a volume V under the nonhomogeneity of the system. The two body interaction is assumed to have the scattering length a and the correlation length l. The nonhomogeneity of the system can be represented as porosity of the system. The main idea of this approach is to perform the mean field approximation in the Feynman approach. This approximation is equivalent to replacing the two body potential into a one body potential interacting with the effective random potential. Performing the random potential due to the mean field approximation of two body potential and the random potential of the nonhomogeneous system we obtained the one body effective propagator. For high density and weak scattering potentials, the effective propagator can be assumed to be under the influence of Gaussian random potential. We consider the correlation function arising from random potentials are Gaussian functions with different scattering strengths. The calculation is carried out within the first cumulant approximation measured with respect to a non-local harmonic action containing one variational parameter. Taking the trace of the propagator and performing the variational calculations we obtained analytical result of the effective mass and the ground state energy. Several limiting cases, both for short and long ranges of the interacting Bose gas and the correlation function of the impurity. The propagator of the system is used to calculate the partition function, specific heat, the critical temperature condensate density and superfluid density.

As a consequence of the above assumptions, we will show that all the above mentioned physical quantities of the dimensionless parameters are: $\overline{N}a^3$ is the gas parameter, a/l and b/L are the ratio of the scattering length and the correlation length of the interacting particles and impurities, respectively, $\chi = n/N$ is the concentration of the impurity, and $R = \chi\,(b/a)^2$ is the strength of disorder. An outline of this Chapter is as follows. Section 4.1 is the model Lagrangian of the system. Section 4.2, we introduce the non-local harmonic trial action. Section 4.3, we present the single propagator which leads to calculating the ground state energy in both short and long correlation lengths. In the final section we study the statistical properties of BEC in disordered systems such as condensate density, critical temperature, superfluid density, etc.

4.1 The Model Lagrangian

We consider the N Bosons interacting with the pair potential $u\left(\overrightarrow{r}_i - \overrightarrow{r_j}\right)$ under the influence of n external impurities potential $v\left(\overrightarrow{r}_i - \overrightarrow{R_j}\right)$ distributed randomly. The Lagrangian in this model system is given as

$$L=\sum_{i=1}^{N}\frac{1}{2}m\dot{\overrightarrow{r}}_i^{\,2}-\sum_{i<j}^{N}u\left(\overrightarrow{r}_i-\overrightarrow{r_j}\right)-\sum_{i=1}^{N}\sum_{k=1}^{n}v\left(\overrightarrow{r}_i-\overrightarrow{R_k}\right), \tag{4.1}$$

where $\overrightarrow{R_i}$ is the positions of impurities which are assumed to be completely random. The action associated with this Lagrangian is

$$S\left[\overrightarrow{r}_N(t),\overrightarrow{r}_N(0);\left\{\overrightarrow{R}_n\right\}\right]$$
$$=\int_0^t d\tau\left[\sum_{i=1}^N\frac{1}{2}m\dot{\overrightarrow{r}}_i^2(\tau)-\sum_{i<j}^N u\left(\overrightarrow{r}_i(\tau)-\overrightarrow{r_j}(\tau)\right)-\sum_{i=1}^N\sum_{k=1}^n v\left(\overrightarrow{r}_i-\overrightarrow{R_k}\right)\right] \tag{4.2}$$

and the propagator associated with this action is

$$P\left(\overrightarrow{r}_N(t),\overrightarrow{r}_N(0),t;\left\{\overrightarrow{R}_n\right\}\right)$$
$$=\int_{\overrightarrow{r}_N(0)}^{\overrightarrow{r}_N(t)}D^N\left(\overrightarrow{r}_N(\tau)\right)\exp\left(\frac{i}{\hbar}S\left[\overrightarrow{r}_N(t),\overrightarrow{r}_N(0);\left\{\overrightarrow{R}_n\right\}\right]\right). \tag{4.3}$$

Carrying out the random average, we can write the averaged propagator as

$$\begin{aligned}P\left(\overrightarrow{r}_N(t),\overrightarrow{r}_N(0),t\right) &= \left\langle P\left(\overrightarrow{r}_N(t),\overrightarrow{r}_N(0),t;\left\{\overrightarrow{R}_n\right\}\right)\right\rangle_{\{\overrightarrow{R}_n\}}\\ &= \int\frac{d\overrightarrow{R}_1}{V}\int\frac{d\overrightarrow{R}_2}{V}\int\frac{d\overrightarrow{R}_3}{V}...\frac{d\overrightarrow{R}_n}{V}\int_{\overrightarrow{r}_N(0)}^{\overrightarrow{r}_N(t)}D^N\left(\overrightarrow{r}_N(\tau)\right)\\ &\quad\times\exp\left(\frac{i}{\hbar}S\left[\overrightarrow{r}_N(t),\overrightarrow{r}_N(0);\left\{\overrightarrow{R}_n\right\}\right]\right),\end{aligned} \tag{4.4}$$

where the symbol

$$\langle A\rangle_{\{\overrightarrow{R}_n\}}=\int\frac{d\overrightarrow{R}_1}{V}\int\frac{d\overrightarrow{R}_2}{V}\int\frac{d\overrightarrow{R}_3}{V}...\frac{d\overrightarrow{R}_n}{V}A. \tag{4.5}$$

We average the propagator with respect to the impurities as shown in the details below.

$$\begin{aligned}&\left\langle P\left(\overrightarrow{r}_N(t),\overrightarrow{r}_N(0),t;\left\{\overrightarrow{R}_n\right\}\right)\right\rangle_{\{\overrightarrow{R}_n\}}\\ &= \int_{\overrightarrow{r}_N(0)}^{\overrightarrow{r}_N(t)}D^N\left(\overrightarrow{r}_N(\tau)\right)\exp\left[\frac{i}{\hbar}\int_0^t d\tau\left[\sum_{i=1}^N\frac{1}{2}m\dot{\overrightarrow{r}}_i^2(\tau)-\sum_{i<j}^N u\left(\overrightarrow{r}_i(\tau)-\overrightarrow{r_j}(\tau)\right)\right]\right]\\ &\quad\times\int\frac{d\overrightarrow{R}_1}{V}\int\frac{d\overrightarrow{R}_2}{V}...\frac{d\overrightarrow{R}_n}{V}\exp\left[\frac{i}{\hbar}\int_0^t d\tau\left[-\sum_{i=1}^N\sum_{k=1}^n v\left(\overrightarrow{r}_i(\tau)-\overrightarrow{R_k}\right)\right]\right]\\ &= \int_{\overrightarrow{r}_N(0)}^{\overrightarrow{r}_N(t)}D^N\left(\overrightarrow{r}_N(\tau)\right)e^O\left[1+\frac{n\int\frac{d\overrightarrow{R}}{V}\exp\left(\frac{i}{\hbar}\int_0^t d\tau\left[-\sum_{i=1}^N v\left(\overrightarrow{r}_i(\tau)-\overrightarrow{R}\right)\right]\right)-n}{n}\right]^n,\end{aligned} \tag{4.6}$$

where $O = \frac{i}{\hbar}\int_0^t d\tau \left[\sum_{i=1}^{N}\frac{1}{2}m\dot{\overrightarrow{r}}_i^{\,2}(\tau) - \sum_{i<j}^{N} u\left(\overrightarrow{r}_i(\tau) - \overrightarrow{r_j}(\tau)\right)\right]$. Using the formula $e^x \simeq \left(1 + \frac{x}{n}\right)^n$, which is valid for large n, we obtain

$$
\begin{aligned}
&\left\langle P\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t; \left\{\overrightarrow{R}_n\right\}\right)\right\rangle_{\left\{\vec{R}_n\right\}} \\
= &\int_{\overrightarrow{r}_N(0)}^{\overrightarrow{r}_N(t)} D^N\left(\overrightarrow{r}_N(\tau)\right) \\
&\times e^O \exp\left[\overline{n}\int d\overrightarrow{R}\left[\exp\left(\frac{i}{\hbar}\int_0^t d\tau\left[-\sum_{i=1}^{N} v\left(\overrightarrow{r}_i(\tau) - \overrightarrow{R}\right)\right]\right) - 1\right]\right],
\end{aligned}
\tag{4.7}
$$

where $n = \overline{n}\int d\overrightarrow{R}$ and $\overline{n} = \frac{n}{V}$. Then the averaged propagator with respect the impurities or disorder becomes

$$
\begin{aligned}
P\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right) = &\int_{\overrightarrow{r}_N(0)}^{\overrightarrow{r}_N(t)} D^N\left(\overrightarrow{r}_N(\tau)\right)\exp\left[\sum_{i=1}^{N}\frac{i}{\hbar}\int_0^t d\tau\left(\frac{1}{2}m\dot{\overrightarrow{r}}_i^{\,2}(\tau)\right)\right] \\
&\times\exp\left[\begin{array}{c} -\sum_{i<j}^{N}\frac{i}{\hbar}\int_0^t d\tau u\left(\overrightarrow{r}_i(\tau) - \overrightarrow{r_j}(\tau)\right) \\ +\sum_{i=1}^{N}\overline{n}\int_0^t d\tau\int d\overrightarrow{R}\left(e^{-\frac{i}{\hbar}\int_0^t d\tau v\left(\overrightarrow{r}_i(\tau)-\overrightarrow{R}\right)} - 1\right)\end{array}\right].
\end{aligned}
\tag{4.8}
$$

For Gaussian approximation we can expand the $\left(e^{-\frac{i}{\hbar}\int_0^t d\tau v\left(\overrightarrow{r}_i(\tau)-\overrightarrow{R}\right)} - 1\right)$ in power series and keep only second orders in the random potential $v\left(\overrightarrow{r}_i(\tau) - \overrightarrow{R}\right)$.

$$
\begin{aligned}
&P\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right) \\
= &\int_{\overrightarrow{r}_N(0)}^{\overrightarrow{r}_N(t)} D^N\left(\overrightarrow{r}_N(\tau)\right)\times\exp\left[\frac{i}{\hbar}\int_0^t d\tau\left[\sum_{i=1}^{N}\frac{1}{2}m\dot{\overrightarrow{r}}_i^{\,2}(\tau) - \sum_{i<j}^{N} u\left(\overrightarrow{r}_i(\tau) - \overrightarrow{r_j}(\tau)\right)\right]\right] \\
&\times\exp\left[\begin{array}{c} -\sum_{i=1}^{N}\overline{n}\int d\overrightarrow{R}\frac{i}{\hbar}\int_0^t d\tau v\left(\overrightarrow{r}_i(\tau) - \overrightarrow{R}\right) \\ +\sum_{i=1}^{N}\frac{1}{2}\overline{n}\left(-\frac{i}{\hbar}\right)^2\int_0^t\int_0^t d\tau d\sigma d\overrightarrow{R} v\left(\overrightarrow{r}_i(\tau) - \overrightarrow{R}\right) v\left(\overrightarrow{r}_i(\sigma) - \overrightarrow{R}\right)\end{array}\right].
\end{aligned}
\tag{4.9}
$$

The next step, we make the mean field approximation which is equivalent to average over the j particles. Physically, this approximation is equivalent to replacing one of the dynamics variable $\overrightarrow{r_j}(\tau)$ of the two-body interaction into a static parameter $\overrightarrow{R}_j$. We can write the the propagator as

$$\begin{aligned} P\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right) &= \left\langle P\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t; \left\{\overrightarrow{R}_N\right\}\right)\right\rangle_{\left\{\overrightarrow{R}_{jN}\right\}} \\ &= \int \frac{d\overrightarrow{R}_{j1}}{V} \int \frac{d\overrightarrow{R}_{j2}}{V} \int \frac{d\overrightarrow{R}_{j3}}{V} \ldots \frac{d\overrightarrow{R}_{jN}}{V} \int_{\overrightarrow{r}_N(0)}^{\overrightarrow{r}_N(t)} D^N\left(\overrightarrow{R}_N(\tau)\right) \\ &\quad \times \exp\left(\frac{i}{\hbar} S\left[\overrightarrow{r}_N(t), \overrightarrow{r}_N(0); \left\{\overrightarrow{R}_{jN}\right\}\right]\right). \end{aligned} \tag{4.10}$$

We average the propagator with respect to the static parameters $\overrightarrow{R}_j$. For high density and weak scattering of the interacting particles, we can write the propagator as

$$\begin{aligned} & P\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right) \\ = & \int_{\overrightarrow{r}_N(0)}^{\overrightarrow{r}_N(T)} D^N\left(\overrightarrow{r}_N(\tau)\right) \times \exp\left[\sum_{i=1}^{N} \frac{i}{\hbar} \int_0^t d\tau \frac{1}{2} m \dot{\overrightarrow{r}}_i^{\,2}(\tau)\right] \\ & \times \exp\left[\begin{array}{c} -\frac{\overline{N}}{2}\sum\limits_{i=1}^{N} \int d\overrightarrow{R}_j \frac{i}{\hbar} \int_0^t d\tau u\left(\overrightarrow{r}_i(\tau) - \overrightarrow{R}_j\right) - \sum\limits_{i=1}^{N} \overline{n} \int d\overrightarrow{R} \frac{i}{\hbar} \int_0^t d\tau v\left(\overrightarrow{r}_i(\tau) - \overrightarrow{R}\right) \\ +\frac{\overline{N}}{4}\sum\limits_{i=1}^{N} \left(-\frac{i}{\hbar}\right)^2 \int_0^t \int_0^t d\tau d\sigma \int d\overrightarrow{R}_j u\left(\overrightarrow{r}_i(\tau) - \overrightarrow{R}_j\right) u\left(\overrightarrow{r}_i(\sigma) - \overrightarrow{R}_j\right) \\ +\sum\limits_{i=1}^{N} \frac{\overline{n}}{2} \left(-\frac{i}{\hbar}\right)^2 \int_0^t \int_0^t d\tau d\sigma \int d\overrightarrow{R} v\left(\overrightarrow{r}_i(\tau) - \overrightarrow{R}\right) v\left(\overrightarrow{r}_i(\sigma) - \overrightarrow{R}\right) \end{array}\right]. \end{aligned} \tag{4.11}$$

The two-particle interacting potential is assumed to be a Gaussian function of the form

$$u\left(\overrightarrow{r}_i(\tau) - \overrightarrow{R}_j\right) = \left(\frac{4\pi\hbar^2 a}{m}\right)\left(\pi l^2\right)^{-3/2} \exp\left[-\left(\frac{\overrightarrow{r}_i(\tau) - \overrightarrow{R}_j}{l}\right)^2\right]. \tag{4.12}$$

The random potential is taken as

$$v\left(\overrightarrow{r}_i(\tau) - \overrightarrow{R}\right) = \left(\frac{2\pi\hbar^2 b}{m}\right)\left(\pi L^2\right)^{-3/2} \exp\left[-\left(\frac{\overrightarrow{r}_i(\tau) - \overrightarrow{R}}{L}\right)^2\right]. \tag{4.13}$$

Here a and b are the scattering lengths of the interacting particles and impurities, l and L are the correlation length of interacting particle and of impurity, respectively. The first term of the impurities in Eq.(4.11) is easily calculated and the result is

$$\begin{aligned}
&\frac{n}{V}\int d\overrightarrow{R}\left(-\frac{i}{\hbar}\int_0^t d\tau v\left(\overrightarrow{r}_i(\tau)-\overrightarrow{R}\right)\right) \\
=&\ -\frac{i}{\hbar}\overline{n}\int_0^t d\tau\frac{2\pi\hbar^2 b}{m\left(\pi L^2\right)^{3/2}}\int d\overrightarrow{R}\exp\left[-\left(\frac{\overrightarrow{r}_i(\tau)-\overrightarrow{R}}{L}\right)^2\right] \\
=&\ -\frac{i}{\hbar}\int_0^t d\tau\frac{2\overline{n}\pi\hbar^2 b}{m}=-\frac{i}{\hbar}\frac{\xi_b}{2}t,
\end{aligned} \tag{4.14}$$

where $\xi_b=\frac{4\overline{n}\pi\hbar^2 b}{m}$. The second order or the correlation term is

$$\sum_{i=1}^{N}\left(-\frac{i}{\hbar}\right)^2\overline{n}\int_0^t\int_0^t d\tau d\sigma\int d\overrightarrow{R}v\left(\overrightarrow{r}_i(\tau)-\overrightarrow{R}\right)v\left(\overrightarrow{r}_i(\sigma)-\overrightarrow{R}\right). \tag{4.15}$$

Performing the $\overrightarrow{R}$-integration we obtain

$$\begin{aligned}
&\int d\overrightarrow{R}v\left(\overrightarrow{r}_i(\tau)-\overrightarrow{R}\right)v\left(\overrightarrow{r}_i(\sigma)-\overrightarrow{R}\right) \\
=&\ \left(\frac{2\pi\hbar^2 b}{m}\right)^2\int_{-\infty}^{\infty}d\overrightarrow{R}\frac{1}{\left(\pi L^2\right)^3}\exp\left[-\left(\frac{\overrightarrow{r}_i(\tau)-\overrightarrow{R}}{L}\right)^2\right]\exp\left[-\left(\frac{\overrightarrow{r}_i(\sigma)-\overrightarrow{R}}{L}\right)^2\right].
\end{aligned} \tag{4.16}$$

Using the formula

$$\int dx\exp\left[a\left(x_2-x\right)^2\right]\exp\left[b\left(x-x_1\right)^2\right]=\sqrt{\frac{-\pi}{a+b}}\exp\left[\frac{ab}{a+b}\left(x_2-x_1\right)^2\right], \tag{4.17}$$

we obtain

$$\begin{aligned}
&\overline{n}\int d\overrightarrow{R}v\left(\overrightarrow{r}_i(\tau)-\overrightarrow{R}\right)v\left(\overrightarrow{r}_i(\sigma)-\overrightarrow{R}\right) \\
=&\ \overline{n}\left(\frac{2\pi\hbar^2 b}{m}\right)^2\left(\frac{1}{2\pi L^2}\right)^{3/2}\exp\left[-\frac{\left(\overrightarrow{r}_i(\tau)-\overrightarrow{r}_i(\sigma)\right)^2}{2L^2}\right].
\end{aligned} \tag{4.18}$$

Similarly, we can calculate the interaction term. The results are

$$\frac{1}{2}\overline{N}\int d\overrightarrow{R}_j\left(-\frac{i}{\hbar}\int_0^t d\tau u\left(\overrightarrow{r}_i(\tau)-\overrightarrow{R}_j\right)\right)=-\frac{i}{\hbar}\int_0^t d\tau\frac{2\overline{N}\pi\hbar^2 a}{m}=-\frac{i}{\hbar}\frac{\xi_a}{2}t, \tag{4.19}$$

where $\xi_a=\frac{4\overline{N}\pi\hbar^2 a}{m}$ and $\overline{N}=\frac{N}{V}$. The second order is

$$\begin{aligned}
&\frac{1}{4}\overline{N}\int d\overrightarrow{R}_j u\left(\overrightarrow{r}_i(\tau)-\overrightarrow{R}_j\right)u\left(\overrightarrow{r}_i(\sigma)-\overrightarrow{R}_j\right) \\
=&\ \frac{1}{4}\overline{N}\left(\frac{4\pi\hbar^2 a}{m}\right)^2\left(\frac{1}{2\pi l^2}\right)^{3/2}\exp\left[-\frac{\left(\overrightarrow{r}_i(\tau)-\overrightarrow{r}_i(\sigma)\right)^2}{2l^2}\right].
\end{aligned} \tag{4.20}$$

Collecting all terms we have the average propagator as

$$
\begin{aligned}
& P\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right) \\
= & \int_{\overrightarrow{r}_N(0)}^{\overrightarrow{r}_N(t)} D^N\left(\overrightarrow{r}_N(\tau)\right) \exp\left[\frac{i}{\hbar}\int_0^t d\tau \sum_{i=1}^{N}\frac{1}{2}m\dot{\overrightarrow{r}}_i^{\,2}(\tau)\right] \\
& \times \exp\left[-\frac{i}{\hbar}N\frac{\xi_a}{2}t + \sum_{i=1}^{N}\left(-\frac{i}{\hbar}\right)^2\int_0^t\int_0^t d\tau d\sigma \xi_l \exp\left[-\frac{\left(\overrightarrow{r}_i(\tau)-\overrightarrow{r}_i(\sigma)\right)^2}{2l^2}\right]\right] \\
& \times \exp\left[-\frac{i}{\hbar}N\frac{\xi_b}{2}t + \sum_{i=1}^{N}\left(-\frac{i}{\hbar}\right)^2\int_0^t\int_0^t d\tau d\sigma \xi_L \exp\left[-\frac{\left(\overrightarrow{r}_i(\tau)-\overrightarrow{r}_i(\sigma)\right)^2}{2L^2}\right]\right],
\end{aligned}
\tag{4.21}
$$

where we define parameters $\xi_l = \frac{1}{4}\overline{N}\left(\frac{4\pi\hbar^2 a}{m}\right)^2\left(\frac{1}{2\pi l^2}\right)^{3/2}$ and $\xi_L = \frac{1}{2}\overline{n}\left(\frac{2\pi\hbar^2 b}{m}\right)^2\left(\frac{1}{2\pi L^2}\right)^{3/2}$.

4.2 The Trial Action

After averaging over the random potentials, the system becomes translational invariant. Therefore, it is reasonable to model the trial action with translational invariance action given as

$$
S_0 = \sum_{i=1}^{N}\int_0^t d\tau \frac{1}{2}m\dot{\overrightarrow{r}}_i^{\,2}(\tau) - \sum_{i=1}^{N}\frac{m\omega^2}{4t}\int_0^t\int_0^t d\tau d\sigma\left(\overrightarrow{r}_i(\tau) - \overrightarrow{r_i}(\sigma)\right)^2, \tag{4.22}
$$

here ω is a variational parameter. The averaged propagator can be rewritten in terms of the trial propagator as

$$
P\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right) = P_0\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right)\left\langle \exp\left[\frac{i}{\hbar}(S - S_0)\right]\right\rangle_{S_0}. \tag{4.23}
$$

Expanding the above average in terms of the cumulants

$$
\begin{aligned}
P\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right) = & P_0\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right)\exp\left[\frac{i}{\hbar}\langle(S - S_0)\rangle_{S_0}\right] \\
& \times \exp\left[+\frac{1}{2}\left(\frac{i}{\hbar}\right)^2\left\{\left\langle(S - S_0)^2\right\rangle_{S_0} - \langle(S - S_0)\rangle_{S_0}^2\right\}\right],
\end{aligned}
\tag{4.24}
$$

keeping only the first cumulant we have

$$
P_1\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right) \simeq P_0\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right)\exp\left[\frac{i}{\hbar}\langle(S - S_0)\rangle_{S_0}\right]. \tag{4.25}
$$

Therefore, our calculations reduce to determine the trial propagator $P_0\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right)$ and the action difference $\frac{i}{\hbar}\left\langle (S - S_0)\right\rangle_{S_0}$.

4.2.1 The Trial Propagator

This propagator can be obtained in a closed form and the result is given in [20]. The exact expression is

$$\begin{aligned} P_0\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right) &= \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{3}{2}N}\left(\frac{\omega t}{2\sin\frac{\omega t}{2}}\right)^{3N} \\ &\quad \times \exp\left[\frac{i}{\hbar}\sum_{i=1}^{N}\frac{m\omega}{4}\cot\left(\frac{\omega t}{2}\right)\left(\overrightarrow{r_i}(t) - \overrightarrow{r_i}(0)\right)^2\right]. \end{aligned} \tag{4.26}$$

This expression is reduced to free particle propagator for $\omega = 0$.

$$P_0\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right) = \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{3}{2}N}\exp\left[\frac{i}{\hbar}\sum_{i=1}^{N}\frac{m}{2}\frac{\left(\overrightarrow{r_i}(t) - \overrightarrow{r_i}(0)\right)^2}{t}\right] \tag{4.27}$$

The next step is to evaluate the action difference $\frac{i}{\hbar}\left\langle S - S_0\right\rangle_{S_0}$which is written as

$$\begin{aligned} \frac{i}{\hbar}\left\langle S - S_0\right\rangle_{S_0} &= -\frac{i}{\hbar}N\left(\frac{\xi_a}{2} + \frac{\xi_b}{2}\right)t \\ &\quad + \sum_{i=1}^{N}\left(-\frac{i}{\hbar}\right)^2\int_0^t\int_0^\tau d\tau d\sigma\xi_l\left\langle\exp\left[-\frac{\left(\overrightarrow{r}_i(\tau) - \overrightarrow{r}_i(\sigma)\right)^2}{2l^2}\right]\right\rangle_{S_0} \\ &\quad + \sum_{i=1}^{N}\left(-\frac{i}{\hbar}\right)^2\int_0^t\int_0^\tau d\tau d\sigma\xi_L\left\langle\exp\left[-\frac{\left(\overrightarrow{r}_i(\tau) - \overrightarrow{r}_i(\sigma)\right)^2}{2L^2}\right]\right\rangle_{S_0} \\ &\quad - \sum_{i=1}^{N}\frac{m\omega^2}{4t}\int_0^t\int_0^t d\tau d\sigma\left\langle\left(\overrightarrow{r}_i(\tau) - \overrightarrow{r_i}(\sigma)\right)^2\right\rangle_{S_0}. \end{aligned} \tag{4.28}$$

Let us consider $\left\langle\exp\left[-\frac{\left(\overrightarrow{r}_i(\tau) - \overrightarrow{r}_i(\sigma)\right)^2}{2L^2}\right]\right\rangle_{S_0}$ and perform the Fourier transform

$$\begin{aligned} &\left\langle\exp\left[-\frac{\left(\overrightarrow{r}_i(\tau) - \overrightarrow{r}_i(\sigma)\right)^2}{2L^2}\right]\right\rangle_{S_0} \\ &= \left(\frac{2\pi}{L^2}\right)^{-3/2}\int_{-\infty}^{\infty}d\overrightarrow{k}\left\langle\exp\left[i\overrightarrow{k}\cdot\left(\overrightarrow{r}_i(\tau) - \overrightarrow{r_i}(\sigma)\right) - \frac{L^2\overrightarrow{k}^2}{2}\right]\right\rangle_{S_0}. \end{aligned} \tag{4.29}$$

We expand the exponent in the cumulant expansion

$$\left\langle \exp\left[i\overrightarrow{k}\cdot\left(\overrightarrow{r}_i(\tau)-\overrightarrow{r_i}(\sigma)\right)\right]\right\rangle_{S_0} = \exp\left[\kappa_i^1(\tau,\sigma)+\kappa_i^2(\tau,\sigma)\right]. \tag{4.30}$$

Here κ_i^1 denotes the first cumulant and κ_i^2 denotes the second cumulant. Because of the quadratic action, only the first and second cumulants survive. Therefore,

$$\kappa_i^1(\tau,\sigma) = i\overrightarrow{k}\cdot\left\langle\overrightarrow{r}_i(\tau)-\overrightarrow{r_i}(\sigma)\right\rangle_{S_0} \tag{4.31}$$

and the second cumulant $\kappa_i^2(\tau,\sigma)$

$$\kappa_i^2(\tau,\sigma) = -\overrightarrow{k}^2 G(\tau,\sigma), \tag{4.32}$$

where the Green function $G(\tau,\sigma)$ is given by

$$G(\tau,\sigma) = \frac{1}{2}\left[\frac{1}{3}\left\langle\left(\overrightarrow{r}_i(\tau)-\overrightarrow{r_i}(\sigma)\right)^2\right\rangle_{S_0} - \left\langle\overrightarrow{r}_i(\tau)-\overrightarrow{r_i}(\sigma)\right\rangle_{S_0}^2\right]. \tag{4.33}$$

The k-integration can be calculated by using the formula $\int dx \exp\left[-ax^2+bx\right] = \sqrt{\frac{\pi}{a}}\exp\left[\frac{b^2}{4a}\right]$. Thus we obtain

$$\begin{aligned}
&\left\langle \exp\left[-\frac{\left(\overrightarrow{r}_i(\tau)-\overrightarrow{r}_i(\sigma)\right)^2}{2L^2}\right]\right\rangle_{S_0} \\
&= \left(\frac{2\pi}{L^2}\right)^{-3/2}\int_{-\infty}^{\infty} d\overrightarrow{k}\exp\left[i\overrightarrow{k}\cdot\left\langle\overrightarrow{r}_i(\tau)-\overrightarrow{r_i}(\sigma)\right\rangle_{S_0} - \overrightarrow{k}^2\left(G(\tau,\sigma)+\frac{L^2}{2}\right)\right] \\
&= \frac{1}{\left(1+\frac{2}{L^2}G(\tau,\sigma)\right)^{3/2}}\exp\left[-\frac{\left\langle\overrightarrow{r}_i(\tau)-\overrightarrow{r_i}(\sigma)\right\rangle_{S_0}^2}{2L^2\left(1+\frac{2}{L^2}G(\tau,\sigma)\right)}\right].
\end{aligned} \tag{4.34}$$

We have

$$\begin{aligned}
&\sum_{i=1}^{N}\left(-\frac{i}{\hbar}\right)^2\int_0^t\int_0^\tau d\tau d\sigma \xi_L\left\langle\exp\left[-\frac{\left(\overrightarrow{r}_i(\tau)-\overrightarrow{r}_i(\sigma)\right)^2}{2L^2}\right]\right\rangle_{S_0} \\
&= \sum_{i=1}^{N}\left(-\frac{i}{\hbar}\right)^2\int_0^t\int_0^\tau d\tau d\sigma \xi_L\frac{1}{\left(1+\frac{2}{L^2}G(\tau,\sigma)\right)^{3/2}}\exp\left[-\frac{\left\langle\overrightarrow{r}_i(\tau)-\overrightarrow{r_i}(\sigma)\right\rangle_{S_0}^2}{2L^2\left(1+\frac{2}{L^2}G(\tau,\sigma)\right)}\right].
\end{aligned} \tag{4.35}$$

Therefore we obtain the first averaged action $\frac{i}{\hbar}\left\langle S\right\rangle_{S_0}$

$$
\begin{aligned}
&\frac{i}{\hbar}\left\langle S\right\rangle_{S_0} \\
= \; & -\frac{i}{\hbar}N\frac{\xi_a}{2}t+\sum_{i=1}^{N}\left(-\frac{i}{\hbar}\right)^2\int_0^t\int_0^\tau d\tau d\sigma\xi_l\frac{1}{\left(1+\frac{2}{l^2}G\left(\tau,\sigma\right)\right)^{\frac{3}{2}}}\exp\left[-\frac{\left\langle\overrightarrow{r}_i\left(\tau\right)-\overrightarrow{r_i}\left(\sigma\right)\right\rangle_{S_0}^2}{2l^2\left(1+\frac{2}{l^2}G\left(\tau,\sigma\right)\right)}\right] \\
& -\frac{i}{\hbar}N\frac{\xi_b}{2}t+\sum_{i=1}^{N}\left(-\frac{i}{\hbar}\right)^2\int_0^t\int_0^\tau d\tau d\sigma\xi_L\frac{1}{\left(1+\frac{2}{L^2}G\left(\tau,\sigma\right)\right)^{\frac{3}{2}}}\exp\left[-\frac{\left\langle\overrightarrow{r}_i\left(\tau\right)-\overrightarrow{r_i}\left(\sigma\right)\right\rangle_{S_0}^2}{2L^2\left(1+\frac{2}{L^2}G\left(\tau,\sigma\right)\right)}\right].
\end{aligned}
\tag{4.36}
$$

Finally $\frac{i}{\hbar}\left\langle S_0\right\rangle_{S_0}$is given in the paper in [20]

$$
\begin{aligned}
\frac{i}{\hbar}\left\langle S_0\right\rangle_{S_0} &= -\sum_{i=1}^{N}\frac{i}{\hbar}\frac{m\omega^2}{4t}\int_0^t\int_0^t d\tau d\sigma\left\langle\left(\overrightarrow{r}_i\left(\tau\right)-\overrightarrow{r}_i\left(\sigma\right)\right)^2\right\rangle_{S_0} \\
&= -\frac{3}{2}\left(\frac{1}{2}\omega t\cot\frac{1}{2}\omega t-1\right) \\
&\quad -\frac{i}{\hbar}m\left[\frac{1}{4}\omega t\cot\frac{1}{2}\omega t-\left(\frac{1}{2}\omega t\csc\frac{1}{2}\omega t\right)^2\right]\frac{\left(\overrightarrow{r_i}\left(t\right)-\overrightarrow{r_i}\left(0\right)\right)^2}{2t}.
\end{aligned}
\tag{4.37}
$$

4.2.2 Approximated Propagator

Collecting all contributions, we have the approximated propagator in the first cumulant approximation as

$$
\begin{aligned}
P\left(\overrightarrow{r}_N\left(t\right),\overrightarrow{r}_N\left(0\right),t\right) &= \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{3}{2}N}\left(\frac{\omega t}{2\sin\frac{\omega t}{2}}\right)^{3N} \\
&\times\exp\left[\frac{3N}{2}\left(\frac{1}{2}\omega t\cot\frac{1}{2}\omega t-1\right)\right]\exp\left[-\frac{i}{\hbar}N\left(\frac{\xi_a}{2}+\frac{\xi_b}{2}\right)t\right] \\
&\times\exp\left[\sum_{i=1}^{N}\left(\frac{i}{\hbar}\right)^2\int_0^t\int_0^\tau d\tau d\sigma\xi_l\frac{1}{\left(1+\frac{2}{l^2}G\left(\tau,\sigma\right)\right)^{3/2}}\exp\left[-\frac{\left\langle\overrightarrow{r}_i\left(\tau\right)-\overrightarrow{r_i}\left(\sigma\right)\right\rangle_{S_0}^2}{2l^2\left(1+\frac{2}{l^2}G\left(\tau,\sigma\right)\right)}\right]\right] \\
&\times\exp\left[\sum_{i=1}^{N}\left(\frac{i}{\hbar}\right)^2\int_0^t\int_0^\tau d\tau d\sigma\xi_L\frac{1}{\left(1+\frac{2}{L^2}G\left(\tau,\sigma\right)\right)^{3/2}}\exp\left[-\frac{\left\langle\overrightarrow{r}_i\left(\tau\right)-\overrightarrow{r_i}\left(\sigma\right)\right\rangle_{S_0}^2}{2L^2\left(1+\frac{2}{L^2}G\left(\tau,\sigma\right)\right)}\right]\right] \\
&\times\exp\left[\sum_{i=1}^{N}\frac{i}{\hbar}m\left[\frac{1}{4}\omega t\cot\frac{1}{2}\omega t-\left(\frac{1}{2}\omega t\csc\frac{1}{2}\omega t\right)^2\right]\frac{\left(\overrightarrow{r_i}\left(t\right)-\overrightarrow{r_i}\left(0\right)\right)^2}{2t}\right].
\end{aligned}
\tag{4.38}
$$

The averaged propagator contains only the following averages $\langle \overrightarrow{r}(\tau)\rangle_{S_0}$, $\left\langle \overrightarrow{r}(\tau)^2\right\rangle_{S_0}$, $\langle \overrightarrow{r}(\tau)\cdot\overrightarrow{r}(\sigma)\rangle_{S_0}$ and $\left\langle (\overrightarrow{r}(\tau)-\overrightarrow{r}(\sigma))^2\right\rangle_{S_0}$. These averages can be obtained from the generating functional.

4.2.3 Generating Functional

Generating functional or the characteristic functional is defined as

$$\begin{aligned}\left\langle \exp\left[\frac{i}{\hbar}\int_0^t \overrightarrow{f}\cdot\overrightarrow{r}(\tau)\,d\tau\right]\right\rangle_{S_0} &= \left\langle \exp\left[\frac{i}{\hbar}\left(S_0+\int_0^t d(\tau)\,\overrightarrow{f}\cdot\overrightarrow{r}(\tau)\right)\right]\right\rangle_{S_0} \\ &= \left\langle \exp\left[\frac{i}{\hbar}S'\right]\right\rangle, \end{aligned} \tag{4.39}$$

where

$$S' = S_0 + \int_0^t \overrightarrow{f}\cdot\overrightarrow{r}(\tau)\,d\tau. \tag{4.40}$$

The classical action $S'_{0,cl}$ associated with this action S' can be determined exactly. The result is given in [20]

$$\begin{aligned} S'_{0,cl} &= \frac{m\omega}{4}\cot\left(\frac{\omega t}{2}\right)(\overrightarrow{r}(t)-\overrightarrow{r}(0))^2 \\ &+\overrightarrow{r}(0)\cdot\int_0^t d(\tau)\,\overrightarrow{f}(\tau)\cdot\left(\frac{\sin(\omega(t-\tau))}{\sin\omega t}+\frac{\sin\left(\frac{\omega}{2}(t-\tau)\right)\sin\frac{\omega}{2}\tau}{\cos\left(\frac{\omega}{2}T\right)}\right) \\ &+\overrightarrow{r}(t)\cdot\int_0^t d(\tau)\,\overrightarrow{f}(\tau)\cdot\left(\frac{\sin\omega\tau}{\sin\omega t}+\frac{\sin\left(\frac{\omega}{2}(t-\tau)\right)\sin\frac{\omega}{2}\tau}{\cos\frac{\omega}{2}t}\right) \\ &+\int_0^t\int_0^t d\tau d\sigma\,\overrightarrow{f}(\tau)\cdot\overrightarrow{f}(\sigma)\left(\begin{array}{c}\frac{\sin(\omega(t-\tau))\sin\omega\tau}{m\omega\sin\omega t} \\ -4\frac{\sin\left(\frac{\omega}{2}(t-\tau)\right)\sin\left(\frac{\omega}{2}(\tau)\right)\sin\left(\frac{\omega}{2}(t-\sigma)\right)\sin\frac{\omega}{2}\sigma}{m\omega\sin\omega t}\end{array}\right). \end{aligned} \tag{4.41}$$

Using this generating functional and using the following expressions given in [16].

$$\langle \overrightarrow{r}(\tau)\rangle_{S_0} = \left[\frac{\delta S'_{0,cl}}{\delta\overrightarrow{f}(\tau)}\right]_{\overrightarrow{f}(\tau)=0} \tag{4.42}$$

$$\langle \overrightarrow{r}(\tau)\cdot\overrightarrow{r}(\sigma)\rangle_{S_0} = \left[\left(\frac{\hbar}{i}\right)\frac{\delta^2 S'_{0,cl}}{\delta\overrightarrow{f}(\tau)\cdot\delta\overrightarrow{f}(\sigma)}+\frac{\delta S'_{0,cl}}{\delta\overrightarrow{f}(\tau)}\cdot\frac{\delta S'_{0,cl}}{\delta\overrightarrow{f}(\sigma)}\right]_{\overrightarrow{f}(\tau)=0} \tag{4.43}$$

One can obtain the averages by differentiating the classical action once or twice with respect to $\overrightarrow{f}(\tau)$ and setting $\overrightarrow{f}(\tau)=0$. Following these procedures we obtain the linear terms

$$\begin{aligned}\langle\overrightarrow{r}(\tau)\rangle_{S_0} &= \overrightarrow{r}(t)\cdot\left(\frac{\sin\omega\tau}{\sin\omega t}-\frac{\sin\left(\frac{\omega}{2}(t-\tau)\right)\sin\frac{\omega}{2}\tau}{\cos\frac{\omega}{2}t}\right)\\ &\quad+\overrightarrow{r}(0)\cdot\left(\frac{\sin(\omega(t-\tau))}{\sin\omega t}-\frac{\sin\left(\frac{\omega}{2}(t-\tau)\right)\sin\frac{\omega}{2}\tau}{\cos\frac{\omega}{2}t}\right).\end{aligned} \tag{4.44}$$

Then the difference of $\langle\overrightarrow{r}(\tau)\rangle_{S_0}-\langle\overrightarrow{r}(\sigma)\rangle_{S_0}$ is

$$\langle\overrightarrow{r}(\tau)\rangle_{S_0}-\langle\overrightarrow{r}(\sigma)\rangle_{S_0}=\frac{\cos\left(\frac{\omega}{2}(t-|\tau+\sigma|)\right)\sin\left(\frac{\omega}{2}(\tau-\sigma)\right)}{\sin\frac{\omega}{2}t}\left[\overrightarrow{r}(t)-\overrightarrow{r}(0)\right]. \tag{4.45}$$

The second cumulant or Green Function is

$$\begin{aligned}g(\tau,\tau) &= \frac{1}{2}\left[\frac{1}{3}\left\langle\overrightarrow{r}_i(\tau)^2\right\rangle_{S_0}-\langle\overrightarrow{r}_i(\tau)\rangle_{S_0}^2\right]\\ &= \frac{i\hbar}{2}\left[\frac{\sin(\omega(t-\tau))\sin\omega\tau}{m\omega\sin\omega t}-4\frac{\left(\sin\left(\frac{\omega}{2}(t-\tau)\right)\sin\frac{\omega}{2}\tau\right)^2}{m\omega\sin\omega t}\right]\end{aligned} \tag{4.46}$$

and

$$\begin{aligned}G(\tau,\sigma) &= \frac{1}{2}\left[\frac{1}{3}\left\langle(\overrightarrow{r}_i(\tau)-\overrightarrow{r_i}(\sigma))^2\right\rangle_{S_0}-\langle\overrightarrow{r}_i(\tau)-\overrightarrow{r_i}(\sigma)\rangle_{S_0}^2\right]\\ &= \frac{i\hbar\sin\left(\frac{\omega(\tau-\sigma)}{2}\right)\sin\left(\frac{\omega(t-(\tau-\sigma))}{2}\right)}{m\omega\sin\left(\frac{\omega t}{2}\right)}.\end{aligned} \tag{4.47}$$

4.3 Single Particle Propagator

Since every term is in the power of the order N, therefore we can consider only the single particle propagator.

$$P(\overrightarrow{r}_N(t),\overrightarrow{r}_N(0),t)=\left(\frac{m}{2\pi i\hbar t}\right)^{3/2}\left(\frac{\omega t}{2\sin\frac{\omega t}{2}}\right)^3$$

$$\times\exp\left[\frac{3}{2}\left(\frac{1}{2}\omega t\cot\frac{1}{2}\omega t-1\right)\right]\exp\left[-\frac{i}{\hbar}\left(\frac{\xi_a}{2}+\frac{\xi_b}{2}\right)t\right]$$

$$\times \exp\left[\left(\frac{i}{\hbar}\right)^2 \int_0^t \int_0^\tau d\tau d\sigma \xi_l \frac{1}{\left(1+\frac{2}{l^2}G(\tau,\sigma)\right)^{3/2}} \exp\left[-\frac{\langle \overrightarrow{r}_i(\tau) - \overrightarrow{r_i}(\sigma)\rangle_{S_0}^2}{2l^2\left(1+\frac{2}{l^2}G(\tau,\sigma)\right)}\right]\right]$$

$$\times \exp\left[\left(\frac{i}{\hbar}\right)^2 \int_0^t \int_0^\tau d\tau d\sigma \xi_L \frac{1}{\left(1+\frac{2}{L^2}G(\tau,\sigma)\right)^{3/2}} \exp\left[-\frac{\langle \overrightarrow{r}_i(\tau) - \overrightarrow{r_i}(\sigma)\rangle_{S_0}^2}{2L^2\left(1+\frac{2}{L^2}G(\tau,\sigma)\right)}\right]\right]$$

$$\times \exp\left[\frac{i}{\hbar}m\left[\frac{1}{4}\omega t \cot\frac{1}{2}\omega t - \left(\frac{1}{2}\omega t \csc\frac{1}{2}\omega t\right)^2\right]\frac{\left(\overrightarrow{r_i}(t) - \overrightarrow{r_i}(0)\right)^2}{2t}\right]. \qquad (4.48)$$

The single particle propagator contains all information which we will try to work out from this propagator. To obtain the ground state energy, we expand the exponential term in a series and keep only the first order. Let us consider the first order

$$\int_0^t \int_0^\tau d\tau d\sigma \frac{1}{\left(1+\frac{2}{L^2}G(\tau,\sigma)\right)^{3/2}} = \int_0^t \int_0^\tau d\tau d\sigma \frac{1}{(B+C)^{3/2}}, \qquad (4.49)$$

where $B = 1 - \frac{i\hbar\mu\cot\left(\frac{1}{2}\omega t\right)}{L^2m^2\nu}$ and $C = \frac{i\hbar\cos\left(\frac{1}{2}\omega t - 2\omega(\sigma-\tau)\right)\csc\left(\frac{1}{2}\omega t\right)}{L^2m\omega}$. For large $\widetilde{t}$, $\cot\widetilde{\omega}\widetilde{t} \to i$, $\csc\left[\widetilde{\omega}\widetilde{t}\right] \to 0$ and $\frac{C}{B} \ll 1$. Expanding $\frac{1}{(B+C)^{3/2}}$ in the power of $\frac{C}{B}$ and then integrating each term. After integrating, the infinite terms can be written in a close form (See Appendix F for more details)

$$\int_0^t \int_0^\tau d\tau d\sigma \frac{1}{\left(1+\frac{2}{L^2}G(\tau,\sigma)\right)^{3/2}}$$
$$= -\frac{t^2}{\left(1+\frac{2E_L}{E_\omega}\right)^{3/2}} + t\frac{2i\hbar}{\left(1+\frac{2E_L}{E_\omega}\right)^{3/2}E_\omega}\left(1 - \ln 2 - \sqrt{1+\frac{2E_L}{E_\omega}} + \ln\left(1+\sqrt{\frac{1}{1+\frac{2E_L}{E_\omega}}}\right)\right). \qquad (4.50)$$

This result is exact solution for the limit $t \to \infty$. The first term is the width of the ground state energy due to fluctuation and the second term is the ground state energy term. Similarly, we can calculate the interaction part.

4.3.1 Ground State Energy

Collecting all terms, we obtain ground state energy in any correlation lengths of the interaction and impurity.

$$\begin{aligned} E_0 &= \frac{3}{4}E_\omega + 4\pi a^3\overline{N}E_a\left(1+\frac{b}{a}\frac{n}{N}\right) \\ &+\frac{2\xi_l}{\left(1+\frac{2E_l}{E_\omega}\right)^{3/2}E_\omega}\left(1-\ln 2-\sqrt{1+\frac{2E_l}{E_\omega}}+\ln\left(1+\sqrt{\frac{1}{1+\frac{2E_l}{E_\omega}}}\right)\right) \\ &+\frac{2\xi_L}{\left(1+\frac{2E_L}{E_\omega}\right)^{3/2}E_\omega}\left(1-\ln 2-\sqrt{1+\frac{2E_L}{E_\omega}}+\ln\left(1+\sqrt{\frac{1}{1+\frac{2E_L}{E_\omega}}}\right)\right), \end{aligned} \tag{4.51}$$

where $E_\omega = \hbar\omega, E_a = \frac{\hbar^2}{2ma^2}, E_l = \frac{\hbar^2}{2ml^2}$ and $E_L = \frac{\hbar^2}{2mL^2}$.

White noise limit

Taking the small l and L limit so $\frac{2E_l}{E_\omega} >> 1$ and $\frac{2E_L}{E_\omega} >> 1$. Therefore we have

$$E_0 = 4\pi a^3\overline{N}E_a\left(1+\frac{b}{a}\frac{n}{N}\right)+\frac{3}{4}E_\omega+\frac{2\xi_l\left(1-\ln 2\right)\sqrt{E_\omega}}{\left(2E_l\right)^{3/2}}+\frac{2\xi_L\left(1-\ln 2\right)\sqrt{E_\omega}}{\left(2E_L\right)^{3/2}}-\frac{\xi_l}{E_l}-\frac{\xi_L}{E_L}. \tag{4.52}$$

Consider the 4^{th} term in Eq.(4.52)

$$\begin{aligned} \frac{2\xi_L\left(1-\ln 2\right)\sqrt{E_\omega}}{\left(2E_L\right)^{3/2}} &= 2\left(1-\ln 2\right)\frac{\frac{1}{2}\overline{n}\left(\frac{2\pi\hbar^2 b}{m}\right)^2\left(\frac{1}{\pi 2L^2}\right)^{3/2}\sqrt{E_\omega}}{\left(\frac{2\hbar^2}{2mL^2}\right)^{3/2}} \\ &= 2\sqrt{\pi}a^3\overline{N}\sqrt{E_a}\sqrt{E_\omega}R\left(1-\ln 2\right) \end{aligned} \tag{4.53}$$

and the last term in Eq.(4.52)

$$\begin{aligned} -\frac{\xi_L}{E_L} &= -\frac{\frac{1}{2}\overline{n}\left(\frac{2\pi\hbar^2 b}{m}\right)^2\left(\frac{1}{\pi 2L^2}\right)^{3/2}}{\frac{\hbar^2}{2mL^2}} \\ &= -\frac{\overline{n}\hbar^2 b^2\sqrt{2\pi}}{Lm} = -4\pi a^3\overline{N}E_a\frac{b}{a}\frac{n}{N}\frac{1}{\sqrt{2\pi}}\frac{b}{L}. \end{aligned}$$

We can write the ground state energy in term of the dimensionless parameters: $\overline{N}a^3, \frac{b}{a}, \frac{a}{l}, \frac{b}{L}$ and R.

$$\begin{aligned} E_0 &= 4\pi a^3\overline{N}E_a\left(1+\frac{b}{a}\frac{n}{N}\right) - 4\pi a^3\overline{N}E_a\left(\sqrt{\frac{2}{\pi}}\frac{a}{l}+\frac{b}{a}\frac{n}{N}\frac{1}{\sqrt{2\pi}}\frac{b}{L}\right) \\ &\quad +\frac{3}{4}E_\omega + 2a^3\overline{N}\sqrt{E_a}\sqrt{E_\omega}\sqrt{\pi}\,(2+R)\,(1-\ln 2) \end{aligned} \tag{4.54}$$

In order to avoid the divergency in taking the "White noise limit", we consider $l \geqslant a, L \geqslant 2b-a$. The physical meaning of the hard-sphere model is that the correlation length of the interacting particle and the impurity cannot be less than diameter of particle a and diameter of impurity $2b-a$ [15]. Minimizing the ground state energy by solving $dE_0/dE_\omega = 0$, we get

$$E_\omega = \left(\frac{4}{3}\sqrt{\pi}a^3\overline{N}\,(2+R)\,(1-\ln 2)\right)^2 E_a, \tag{4.55}$$

where $R = \frac{n}{N}\left(\frac{b}{a}\right)^2$. Substituting Eq.(4.55) into Eq.(4.54) and setting $l = a, L = 2b-a$, we obtain

$$\begin{aligned} \frac{E_0}{E_a} &= 4\pi a^3\overline{N}\left(1+\frac{b}{a}\frac{n}{N}\right) - 4\pi a^3\overline{N}\left(\sqrt{\frac{2}{\pi}}-\frac{1}{\sqrt{2\pi}}\frac{1}{2-\frac{a}{b}}\frac{a}{b}\frac{n}{N}\right) \\ &\quad +4\pi\left(a^3\overline{N}\,(2+R)\,(1-\ln 2)\right)^2. \end{aligned} \tag{4.56}$$

This ground state energy is lower than the ground energy from the Bogoliubov theory which is in agreement with Feynman path integral theory as shown in Figure 4.1.

Long length limit

Taking the large l and L limit so $\frac{2E_l}{E_\nu} << 1$ and $\frac{2E_L}{E_\nu} << 1$. Therefore we can expand in the power of $\frac{2E_l}{E_\nu}$ and $\frac{2E_L}{E_\nu}$ as $\sqrt{1+\frac{2E_l}{E_\omega}} \simeq 1+\frac{E_l}{E_\omega}$ and $\ln\left(1+\sqrt{\frac{1}{1+\frac{2E_l}{E_\omega}}}\right) \simeq \ln 2 - \frac{E_l}{2E_\omega}$. Thus we can write Eq.(4.51) as

$$E_0 \simeq \frac{3}{4}E_\omega + 4\pi a^3\overline{N}E_a\left(1+\frac{b}{a}\frac{n}{N}\right) - \frac{12a^3\overline{N}\sqrt{E_a}\sqrt{2\pi}E_l^{5/2}}{E_\omega^2} - \frac{6a^3\overline{N}\sqrt{E_a}\sqrt{2\pi}E_L^{5/2}R}{E_\omega^2}. \tag{4.57}$$

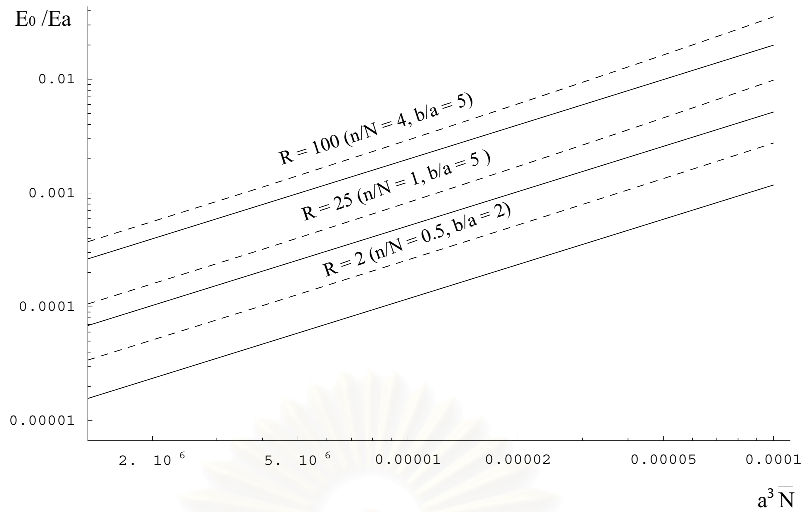

Figure 4.1: The ground energy in the unit of E_a against $a^3\overline{N}$. The dash line is calculated by Bogoliubov theory and the solid line is calculated by path integral theory for the white noise limit.

Minimizing the ground state energy by solving $dE_0/dE_\omega = 0$, we get

$$E_\omega = -2\sqrt{2}\left(a^3\overline{N}\sqrt{\pi E_a}\left(2E_l^{5/2} + E_L^{5/2}R\right)\right)^{1/3}. \tag{4.58}$$

Substituting Eq.(4.58) into Eq.(4.57), we obtain

$$\frac{E_0}{E_a} = 4\pi a^3\overline{N}\left(1 + \frac{b}{a}\frac{n}{N}\right) - \frac{9}{2\sqrt{2}}\left(\sqrt{\pi}a^3\overline{N}\left(\frac{a}{l}\right)^5\left(2 + \left(\frac{l}{L}\right)^5 R\right)\right)^{1/3}. \tag{4.59}$$

This ground state energy for long length limit is shown in Figure 4.2. We find that the second term in Eq.(4.59) is so small. Therefore we can approximate the ground state energy for long length limit as the mean-field energy.

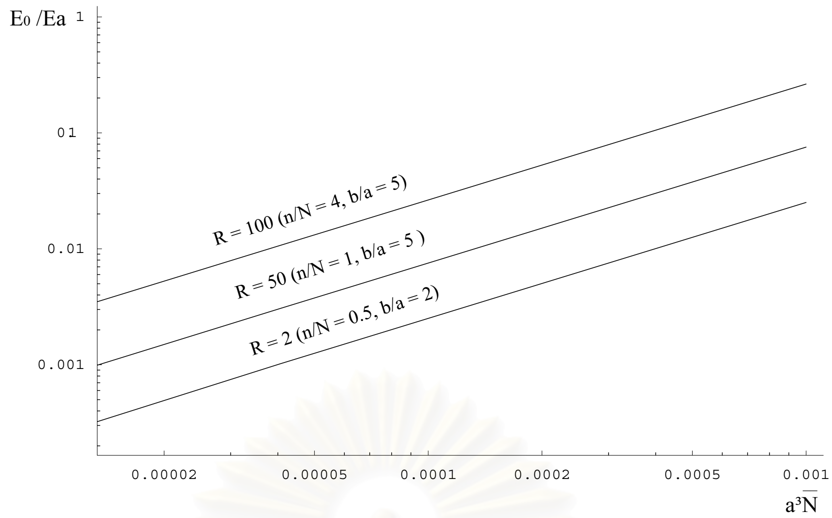

Figure 4.2: The ground state energy in the unit of E_a for long correlation length limit against $a^3\overline{N}$. $(a/l = 0.0001, l/L = 0.5)$

4.4 Condensate Density

The propagator of the system is reduced to the free particle propagator when $\omega = 0$ (no harmonic trap). This means that we measure the system by free particle propagator. To obtain the condensate contributions and contact with the Bugoliubov approach, let us consider the single particle propagator in Eq.(4.48) again. Taking the limit $\omega = 0$, we have

$$\langle \overrightarrow{r}(\tau)\rangle_{S_0} - \langle \overrightarrow{r}(\sigma)\rangle_{S_0} \underset{\omega\to 0}{=} \frac{(\tau-\sigma)}{2}\left(\overrightarrow{r}(t) - \overrightarrow{r}(0)\right) \tag{4.60}$$

and

$$G(\tau,\sigma) \underset{\omega\to 0}{=} \frac{i\hbar(\tau-\sigma)(t-(\tau-\sigma))}{2mt} \tag{4.61}$$

$$P_0\left(\overrightarrow{r}_N(t), \overrightarrow{r}_N(0), t\right) \underset{\omega\to 0}{=} \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{3}{2}N} \exp\left[\frac{i}{\hbar}\sum_{i=1}^{N}\frac{m}{2}\frac{\left(\overrightarrow{r_i}(t) - \overrightarrow{r_i}(0)\right)^2}{t}\right]. \tag{4.62}$$

The single particle propagator can be written as

$$P_1\left(\overrightarrow{r}\left(t\right),\overrightarrow{r}\left(0\right),\tau\right)=P_0\left(\overrightarrow{r}_N\left(t\right),\overrightarrow{r}_N\left(0\right),\tau\right)\exp\left[\frac{i}{\hbar}\left\langle S-S_0\right\rangle_{S_0}\right]$$

$$=\left(\frac{m}{2\pi i\hbar t}\right)^{3/2}\exp\left[\frac{i}{\hbar}\frac{m}{2t}\left(\overrightarrow{r}_i\left(t\right)-\overrightarrow{r}_i\left(0\right)\right)^2\right]\exp\left[-\frac{i}{\hbar}\left(\frac{\xi_a}{2}+\frac{\xi_b}{2}\right)t\right]$$

$$\times\exp\left[-\frac{\xi_l}{\hbar^2}\int_0^t\int_0^\tau d\tau d\sigma\frac{1}{\left(1+\frac{2}{l^2}\frac{i\hbar(\tau-\sigma)(t-(\tau-\sigma))}{2mt}\right)^{3/2}}\exp\left[-\frac{\left(\frac{\tau-\sigma}{2}\right)^2\left(\overrightarrow{r}\left(t\right)-\overrightarrow{r}\left(0\right)\right)^2}{2l^2\left(1+\frac{2}{l^2}\frac{i\hbar(\tau-\sigma)(t-(\tau-\sigma))}{2mt}\right)}\right]\right]$$

$$\times\exp\left[-\frac{\xi_L}{\hbar^2}\int_0^t\int_0^\tau d\tau d\sigma\frac{1}{\left(1+\frac{2}{L^2}\frac{i\hbar(\tau-\sigma)(t-(\tau-\sigma))}{2mt}\right)^{3/2}}\exp\left[-\frac{\left(\frac{\tau-\sigma}{2}\right)^2\left(\overrightarrow{r}\left(t\right)-\overrightarrow{r}\left(0\right)\right)^2}{2L^2\left(1+\frac{2}{L^2}\frac{i\hbar(\tau-\sigma)(t-(\tau-\sigma))}{2mt}\right)}\right]\right].\tag{4.63}$$

Expanding the exponential term up to second orders, we can calculate the first and the second terms exactly. The results are

$$-\frac{\xi_l}{\hbar^2}\int_0^t\int_0^\tau d\tau d\sigma\frac{1}{\left(1+\frac{2}{l^2}G\left(\tau,\sigma\right)\right)^{3/2}}=-\frac{\xi_l}{\hbar^2}\frac{2l^2mt^2}{i\hbar t\left(1-\frac{4il^2m}{\hbar t}\right)}\tag{4.64}$$

and the second term is

$$\frac{\xi_l}{\hbar^2}\int_0^t\int_0^\tau d\tau d\sigma\frac{\left(\frac{\tau-\sigma}{2}\right)^2\left(\overrightarrow{r}\left(t\right)-\overrightarrow{r}\left(0\right)\right)^2}{2l^2\left(1+\frac{2}{l^2}G\left(\tau,\sigma\right)\right)^{5/2}}=\frac{\xi_l}{\hbar^2}\frac{2l^2m^2t^2}{3\left(i\hbar t\right)^2\left(1-\frac{4il^2m}{\hbar t}\right)^2}\left(\overrightarrow{r}\left(t\right)-\overrightarrow{r}\left(0\right)\right)^2.\tag{4.65}$$

For large t limit, we can expand Eq.(4.64) and Eq.(4.65) in the power of $\frac{1}{t}$ and keep only the first two terms. Thus the propagator can be written in the simple form as

$$P_1\left(\overrightarrow{r}\left(t\right),\overrightarrow{r}\left(0\right),t\right)=\left(\frac{m^*}{2\pi i\hbar t}\right)^{3N/2}e^{-\frac{i}{\hbar}NE_0t}\times\exp\left[\begin{array}{c}\frac{i}{\hbar}\frac{m}{2t}\left(1-\frac{16\sqrt{2\pi}}{3}a^3\overline{N}\left(\frac{2l}{a}+\frac{L}{a}R\right)\right)\\\times\sum_{i=1}^{N}\left(\overrightarrow{r}_i\left(t\right)-\overrightarrow{r}_i\left(0\right)\right)^2\end{array}\right].\tag{4.66}$$

In order to obtain the normalized wave function, we set

$$\frac{m^*}{m\left(1-\frac{16\sqrt{2\pi}}{3}a^3\overline{N}\left(\frac{2l}{a}+\frac{L}{a}R\right)\right)}=1.\tag{4.67}$$

Therefore the propagator of the system can be written in the simple form.

$$P_1\left(\overrightarrow{r}(t),\overrightarrow{r}(0),t\right)=\left(\frac{m^*}{2\pi i\hbar t}\right)^{3N/2}e^{-\frac{i}{\hbar}NE_0t}\exp\left[\frac{i}{\hbar}\frac{m^*}{2t}\sum_{i=1}^{N}\left(\overrightarrow{r}_i(t)-\overrightarrow{r}_i(0)\right)^2\right] \tag{4.68}$$

where the ground state energy is

$$E_0=4\pi a^3\overline{N}E_a\left(1+\frac{b}{a}\frac{n}{N}\right)-4\pi a^3\overline{N}E_a\sqrt{\frac{2}{\pi}}\frac{a}{l}-4\pi a^3\overline{N}E_a\frac{b}{a}\frac{n}{N}\frac{1}{\sqrt{2\pi}}\frac{b}{L}. \tag{4.69}$$

This ground state energy is the same as the ground state energy in Eq.(4.54) in the limit of $E_\omega\to 0$. The effective mass ratio is

$$\frac{m^*}{m}=1-\frac{16\sqrt{2\pi}}{3}a^3\overline{N}\left(\frac{2l}{a}+\frac{L}{a}R\right). \tag{4.70}$$

Thus we obtain the ground state energy and effective mass for any correlation lengths of the interacting particles and of impurity. We replace t by $-i\hbar\beta$. Therefore the N-body density matrix can be written as

$$\rho_1\left(\overrightarrow{r}(t),\overrightarrow{r}(0),t\right)=\left(\frac{m^*kT}{2\pi\hbar^2}\right)^{3N/2}e^{-NE_0\beta}\exp\left[-\frac{m^*}{2\hbar^2\beta}\sum_{i=1}^{N}\left(\overrightarrow{r}_i(t)-\overrightarrow{r}_i(0)\right)^2\right]. \tag{4.71}$$

Suppose a permutation contains C_v cycles of length v and $\sum_{i=1}^{\infty}vC_v=N$ for $v>1$. Thus the partition function is

$$Q^{(\mu)}=\exp\left[-F\beta\right]=\exp\left[\sum_{\nu=1}^{N}h_\nu\frac{\alpha^\nu}{\nu}\right], \tag{4.72}$$

where $\alpha=e^{\mu\beta}$ and

$$\begin{aligned}h_\nu &= \frac{V}{\nu^{3/2}}\left(\frac{m^*}{2\pi\hbar\beta}\right)^{3\nu/2}\left(\frac{2\pi\hbar^2\beta}{m^*}\right)^{3(\nu-1)/2}\\ &= V\left(\frac{m^*}{2\pi\hbar^2\beta\nu}\right)^{3/2}.\end{aligned} \tag{4.73}$$

Therefore

$$\exp\left[-F\beta\right]=\exp\left[V\left(\frac{m^*}{2\pi\hbar^2\beta}\right)^{3/2}\zeta_{5/2}(\alpha)\right], \tag{4.74}$$

where F is the free energy, $\alpha=e^{\mu\beta},\zeta_{5/2}(\alpha)=\sum_{\nu=1}^{N}\frac{\alpha^\nu}{\nu^{5/2}}$. For temperature $T<T_c$ so $\alpha=1$ and $\zeta_{5/2}(1)=1.341$. Thus the free energy of the system is

$$F=-\frac{\zeta_{5/2}(1)V}{\beta}\left(\frac{m^*}{2\pi\hbar^2\beta}\right)^{3/2}. \tag{4.75}$$

The specific heat is

$$\begin{aligned} C_v &= -\frac{T\partial^2 F}{\partial T^2} = \zeta_{5/2}(1)\frac{15Vk}{4}\left(\frac{km^*T}{2\pi\hbar^2}\right)^{3/2} \\ &= \zeta_{5/2}(1)\frac{15Vk}{4}\left(\frac{kmT}{2\pi\hbar^2}\right)^{3/2}\left(1-\frac{16\sqrt{2\pi}}{3}a^3\overline{N}\left(\frac{2l}{a}+\frac{L}{a}R\right)\right)^{3/2}. \end{aligned} \tag{4.76}$$

The critical temperature T_c is

$$T_c = \frac{2\pi\hbar^2}{m\left(1-\frac{16\sqrt{2\pi}}{3}a^3\overline{N}\left(\frac{2l}{a}+\frac{L}{a}R\right)\right)k}\left(\frac{\overline{N}_c}{\zeta_{3/2}(1)}\right)^{2/3}. \tag{4.77}$$

Where $\overline{N}_c$ is the condensate density of the system. For dilute gas $a^3\overline{N} << 1$, we obtain the critical temperature

$$T_c = T_0\left(1+\frac{16\sqrt{2\pi}}{3}a^3\overline{N}\left(\frac{2l}{a}+\frac{L}{a}R\right)\right), \tag{4.78}$$

where T_0 is the critical temperature of ideal gas $T_0 = \frac{2\pi\hbar^2}{mk}\left(\frac{\overline{N}_c}{\zeta_{3/2}(1)}\right)^{2/3}$. We find that the critical temperature is increased by the repulsive interaction and the strength of disorder which is in agreement with [22] as shown in Figure 4.3. The condensate density is

$$\begin{aligned} \overline{N}_c &= \frac{1}{\beta Q^{(\mu)}}\frac{\partial Q^{(\mu)}}{\partial \mu} = \left(\frac{m^*kT_c}{2\pi\hbar^2}\right)^{3/2}\zeta_{3/2}(1) \\ &= \left(\frac{mkT_c}{2\pi\hbar^2}\left(1-\frac{16\sqrt{2\pi}}{3}a^3\overline{N}\left(\frac{2l}{a}+\frac{L}{a}R\right)\right)\right)^{3/2}\zeta_{3/2}(1) \\ &\simeq \overline{N}_0\left(1-8\sqrt{2\pi}a^3\overline{N}\left(\frac{2l}{a}+\frac{L}{a}R\right)\right), \end{aligned} \tag{4.79}$$

where $\overline{N}_0$ is condensate density of ideal gas $\overline{N}_0 = \left(\frac{mkT_c}{2\pi\hbar^2}\right)^{3/2}\zeta_{3/2}(1)$. We find that the condensate density is depleted by the repulsive interaction and the strength of disorder (see Figure. 4.4). If there is no interaction and impurity, the system is the ideal Bose gas. The condensate density is also depleted by the correlation length of interaction and the strength of disorder. Therefore the long correlation length can suppress the condensate density.

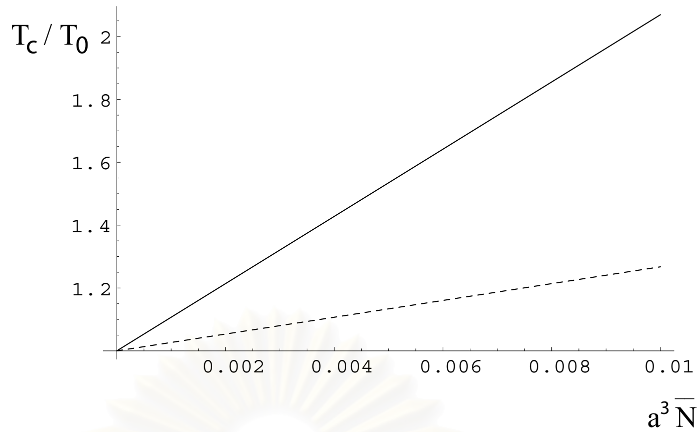

Figure 4.3: The critical temperature against $a^3\overline{N}$. The dash line is critical temperature of Bose gas with no impurity and the solid line is critical temperature in the presence of the impurity for white noise limit. $(l = a, L = 2b - a, R = 2$ and $b/a = 2)$

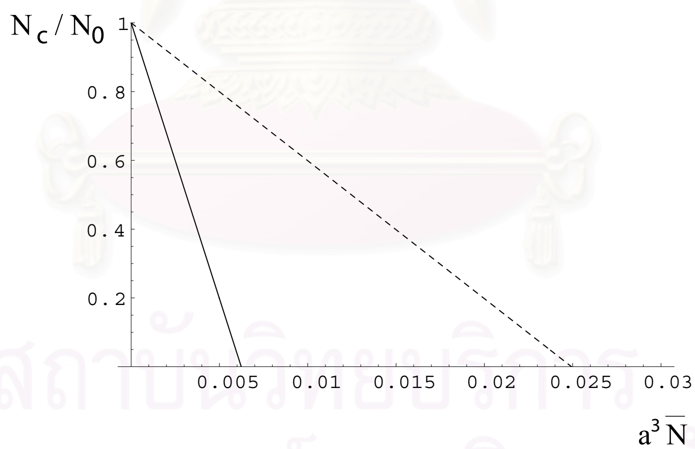

Figure 4.4: The condensate density against $a^3\overline{N}$. The dash line is condensate density with no impurity and the solid line is condensate density in the presence of the impurity for white noise limit.
$(l = a, L = 2b - a, R = 2$ and $b/a = 2)$

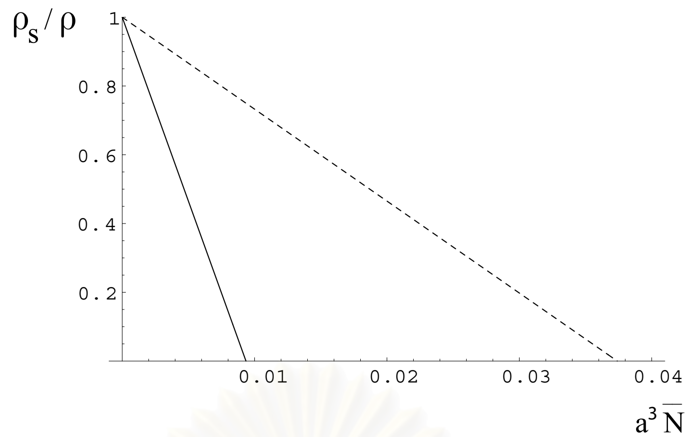

Figure 4.5: The superfluid fraction against $a^3\overline{N}$. The dash line is superfluid fraction with no impurity and the solid line is superfluid fraction in the presence of the impurity for white noise limit.
$(l = a, L = 2b - a,\ R = 2$ and $b/a = 2)$

4.4.1 Superfluid Fraction

The superfluid fraction can be defined as the difference between the energies E_v and ground state energy E_0.

$$\frac{Nmv^2}{2}\frac{\rho_s}{\rho} = E_v - E_0 \tag{4.80}$$

where ρ_s is the superfluid density, ρ is the total density of the system, E_v is the ground state energy of the system moving with velocity v and E_0 is the ground state of the rest system. Using this definition we have

$$\frac{Nmv^2}{2}\frac{\rho_s}{\rho} = \frac{Nm^*v^2}{2}. \tag{4.81}$$

Therefore

$$\frac{\rho_s}{\rho} = \frac{m^*}{m} = 1 - \frac{16\sqrt{2\pi}}{3}a^3\overline{N}\left(\frac{2l}{a} + \frac{L}{a}R\right). \tag{4.82}$$

We find that the repulsive interaction, the strength of disorder, the correlation length of interaction, and impurity deplete the superfluid fraction, as well as the condensate density.

Bose-Einstein condensation in a disordered system was studied using Feynman's path integral theory. The correlation function of the interacting particle and impurity are assumed to be a Gaussian function with different scattering strengths. The mean field approximation on the two body interaction is equivalent to replacing the two body interaction into a one body interaction by averaging over the interacting particles. After averaging over the impurities and the interacting particles, the system became translation invariant so we chose the non-local harmonic action to be the trial action. Performing the variation calculations we obtained the analytical result of the ground state energy. We studied the ground state energy into 2 main cases: short and long correlation lengths. Short correlation length or white noise limit is in good agreement with Bugoliubov approach. We also considered the statistical properties. We have found that when $\omega \to 0$, the system reduces to the free particle system with the effective mass m^*. We summed over all permutations so we obtained the partition function which led to the critical temperature, condensate density and superfluid fraction, etc. We have found that the impurity suppresses the condensate density and superfluid fraction.

CHAPTER V

CONCLUSION

5.1 Bose-Einstein Condensation in a Double Well Potential

We have studied the Bose-Einstein condensation in a double well potential of 2 types: harmonic potential superimposed with a Gaussian or with a cosine barrier using the Feynman path integral theory. The mean field approximation is performed by replacing the pair potential into a one body potential by path integral approach. The calculations are carried out within the first cumulant approximation measured with respect to a local harmonic action containing one variational parameter. The advantage of this method is that all calculations can be evaluated analytically. We obtained analytical results of the ground state energy which is exactly the same as ref. [18] for $\widetilde{V}_b = 0$ or $\widetilde{A} = 0$. The ground state and the first excited state wave functions used to calculate the overlap integral γ_{ij} are in good agreement with that from two-mode model of Gross-Pitaevskii equation for small $\widetilde{g}N$.

5.2 Bose-Einstein Condensation in a Disordered System

We have studied Bose-Einstein condensation in a disodered system by Feynman's path integral theory. This leads to determining the propagator in the limit of very low temperatures. The propagator contains all information of the system such as the ground state energy and the wave function which is similar to the free particle wave function with an effective mass m^*. We study the system in two main cases: white noise limit and long correlation length of interacting particles and of impurities.

For white noise limit, the system go beyond to the limit of Bugoliubov approach which is a special case of our model. The interacting potential and impurity potential are reduced to the Dirac delta function. In dilute Bose gas $a^3\overline{N} \ll 1$, we obtain the ground state energy which is in good agreement with Bugoliubov theory. Moreover Feynman path integral theory can be used to study the long correlation length of interacting particles and impurities by taking the limit of long correlation length of interacting particles and impurities, Therefore, the Gaussian potential of interacting particles and impurities is broad and flat. This implies that these potentials are too weak.

In the limit $\omega \to 0$, the propagator of the system reduces to the free particle propagator. We obtain the density matrix. After summing over all permutations of N-body density matrix, we obtained the partition function, specific heat, critical temperature and condensate density. We found that the repulsive interacting particles, the strength of disorder, and the correlation length of both interacting particles and impurities, suppress the condensate density and the superfluid fraction. The critical temperature is increased by the repulsive interacting particles, the strength of disorder and the correlation length.

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Appendices

Appendix A

Average of Gaussian Potential

We show the detailed calculations of the integral of the Gaussian function of the propagator in Eq.(3.19). Let us consider

$$\left\langle \exp\left[-\left(\frac{\widetilde{x}_i}{\widetilde{\sigma}}\right)^2\right]\right\rangle_{S_0} = \frac{1}{\left(1+\frac{2i}{\widetilde{\sigma}^2\widetilde{\omega}}\frac{\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})\sin\widetilde{\omega}\widetilde{t}}{\sin\widetilde{\omega}\widetilde{t}}\right)^{1/2}}\exp\left[-\frac{\left(\frac{\widetilde{x}_2\sin\widetilde{\omega}\widetilde{t}+\widetilde{x}_1\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})}{\sin\widetilde{\omega}\widetilde{t}}\right)^2}{\widetilde{\sigma}^2\left(1+\frac{2i}{\widetilde{\sigma}^2\widetilde{\omega}}\frac{\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})\sin\widetilde{\omega}\widetilde{\tau}}{\sin\widetilde{\omega}\widetilde{t}}\right)}\right]. \tag{A.1}$$

We let

$$\begin{aligned} a &= \left(\frac{\widetilde{x}_2\sin\widetilde{\omega}\widetilde{\tau}+\widetilde{x}_1\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})}{\sin\widetilde{\omega}\widetilde{t}}\right)^2 \\ b &= 1+\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}} \\ c &= \frac{i}{\widetilde{\sigma}^2\widetilde{\omega}}\cos\left[2\widetilde{\omega}\widetilde{\tau}-\widetilde{\omega}\widetilde{t}\right]\csc\left[\widetilde{\omega}\widetilde{t}\right]. \end{aligned} \tag{A.2}$$

Therefore we can write in the form

$$\int_0^{\widetilde{t}}\left\langle \exp\left[-\left(\frac{\widetilde{x}_i(\tau)}{\widetilde{\sigma}}\right)^2\right]\right\rangle d\widetilde{\tau} = \int_0^{\widetilde{t}}\frac{1}{(b+c)^{1/2}}\exp\left[-\frac{a}{\widetilde{\sigma}^2(b+c)}\right]d\widetilde{t}. \tag{A.3}$$

For large $\widetilde{t}$, $\cot\widetilde{\omega}\widetilde{t}\to i, \csc\left[\widetilde{\omega}\widetilde{t}\right]\to 0,\ \frac{c}{b}\ ,\ \frac{a}{\widetilde{\sigma}^2(b+c)}\ll 1$. Therefore we can expand $\exp\left[-\frac{a}{\widetilde{\sigma}^2(b+c)}\right]$ in the series.

$$\int_0^{\widetilde{t}}\left\langle \exp\left[-\left(\frac{\widetilde{x}_i(\tau)}{\widetilde{\sigma}}\right)^2\right]\right\rangle d\widetilde{\tau} = \int_0^{\widetilde{t}}\frac{1}{(b+c)^{1/2}}\left(\begin{array}{c} 1-\frac{a}{\widetilde{\sigma}^2(b+c)}+\frac{1}{2}\left(\frac{a}{\widetilde{\sigma}^2(b+c)}\right)^2 \\ -\frac{1}{6}\left(\frac{a}{\widetilde{\sigma}^2(b+c)}\right)^3+\frac{1}{24}\left(\frac{a}{\widetilde{\sigma}^2(b+c)}\right)^4+... \end{array}\right)d\widetilde{\tau}. \tag{A.4}$$

We consider the first term in Eq.(A.4) and expand in the series

$$\frac{1}{(b+c)^{1/2}} = \sqrt{\frac{1}{b}}-\frac{c}{2b^{3/2}}+\frac{3c^2}{8b^{5/2}}-\frac{15c^3}{16b^{7/2}}+\frac{35c^4}{128b^{9/2}}+.... \tag{A.5}$$

Integrating in each term, we have

$$1^{\text{st}} \quad : \quad \int_0^{\widetilde{t}} \sqrt{\frac{1}{b}}\, d\widetilde{\tau} = \frac{\widetilde{t}}{b^{1/2}}$$

$$2^{\text{nd}} \quad : \quad -\int_0^{\widetilde{t}} \frac{c}{2b^{3/2}}\, d\widetilde{\tau} = -\frac{i}{2\widetilde{\sigma}^2\widetilde{\omega}^2 b^{3/2}}$$

$$3^{\text{rd}} \quad : \quad \int_0^{\widetilde{t}} \frac{3c^2}{8b^{5/2}}\, d\widetilde{\tau} = -\frac{3\cot\left[\widetilde{\omega}\widetilde{t}\right]}{16\widetilde{\sigma}^4\widetilde{\omega}^3 b^{5/2}} - \frac{3\widetilde{t}\csc^2\left[\widetilde{\omega}\widetilde{t}\right]}{16\widetilde{\sigma}^4\widetilde{\omega}^3 b^{5/2}}$$

$$4^{\text{th}} \quad : \quad -\int_0^{\widetilde{t}} \frac{15c^3}{16b^{7/2}}\, d\widetilde{\tau} = -\frac{5i}{96\widetilde{\sigma}^6\widetilde{\omega}^4 b^{7/2}} + \frac{5\cot^2\left[\widetilde{\omega}\widetilde{t}\right]}{96\widetilde{\sigma}^6\widetilde{\omega}^4 b^{7/2}} + \frac{25i\csc^2\left[\widetilde{\omega}\widetilde{t}\right]}{96\widetilde{\sigma}^6\widetilde{\omega}^4 b^{7/2}}. \tag{A.6}$$

In order to study the Bose-Einstein condensation, we have to consider in the large $\widetilde{t}$ limit $\left(\widetilde{t} \to \infty\right)$, we find that $\cot\left[\widetilde{\omega}\widetilde{t}\right] \to i$ and $\csc\left[\widetilde{\omega}\widetilde{t}\right] \to 0$. We continue integrating in every terms in the large $\widetilde{t}$ limit, we get

$$1^{\text{st}} \quad : \quad \int_0^{\widetilde{t}} \sqrt{\frac{1}{b}}\, d\widetilde{\tau} = \frac{\widetilde{t}}{b^{1/2}}$$

$$2^{\text{nd}} \quad : \quad -\int_0^{\widetilde{t}} \frac{c}{2b^{3/2}}\, d\widetilde{\tau} = -\frac{i}{2\widetilde{\sigma}^2\widetilde{\omega}^2 b^{3/2}}$$

$$3^{\text{rd}} \quad : \quad \int_0^{\widetilde{t}} \frac{3c^2}{8b^{5/2}}\, d\widetilde{\tau} = -\frac{3i}{16\widetilde{\sigma}^4\widetilde{\omega}^3 b^{5/2}}$$

$$i^{\text{th}} \quad : \quad \sqrt{\frac{1}{b}} \int_0^{\widetilde{t}} \frac{\left(-\frac{c}{b}\right)^i}{\left(i+\frac{1}{2}\right) i!} \prod_{j=0}^{i} \left(j+\frac{1}{2}\right) d\widetilde{\tau} = -\frac{1}{\widetilde{\sigma}^{2i}\widetilde{\omega}^{i+1} b^{i+\frac{1}{2}}} \frac{\prod_{j=0}^{i}\left(j+\frac{1}{2}\right)}{\left(i+\frac{1}{2}\right) i!}. \tag{A.7}$$

Thus

$$\begin{aligned} \int_0^{\widetilde{t}} \frac{1}{(b+c)^{1/2}}\, d\widetilde{\tau} &= \frac{\widetilde{t}}{b^{1/2}} - i\sum_{i=1}^{\infty} \frac{1}{\widetilde{\sigma}^{2i}\widetilde{\omega}^{i+1} b^{i+\frac{1}{2}}} \frac{\prod_{j=0}^{i}\left(j+\frac{1}{2}\right)}{\left(i+\frac{1}{2}\right) i!} \\ &= \frac{\widetilde{t}}{b^{1/2}} - i\left(\frac{2\ln 2}{\sqrt{b}\widetilde{\omega}} - \frac{2\ln\left[1+\sqrt{1-\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}b}}\right]}{\sqrt{b}\widetilde{\omega}}\right). \end{aligned} \tag{A.8}$$

The first term is the ground state energy and the second term is the normalized factor. Consider the second term in Eq.(A.4)

$$\int_0^{\widetilde{t}} \frac{1}{(b+c)^{1/2}} \frac{-a}{\widetilde{\sigma}^2 (b+c)} d\widetilde{\tau} = \int_0^{\widetilde{t}} \frac{-a}{\widetilde{\sigma}^2 (b+c)^{3/2}} d\widetilde{\tau}. \tag{A.9}$$

For $\widetilde{t} \to \infty$, $\frac{c}{b} << 1$, we can expand $\frac{-a}{\widetilde{\sigma}^2 (b+c)^{3/2}}$ in the power of $\frac{c}{b}$.

$$\frac{-a}{\widetilde{\sigma}^2 (b+c)^{3/2}} = -\frac{a}{\widetilde{\sigma}^2 b^{3/2}} + \frac{3ac}{\widetilde{\sigma}^2 b^{5/2}} - \frac{15ac^2}{8\widetilde{\sigma}^2 b^{7/2}} + \frac{35ac^3}{16\widetilde{\sigma}^2 b^{9/2}} - \frac{315ac^4}{128\widetilde{\sigma}^2 b^{11/2}} + ... \tag{A.10}$$

Considering the integration of the first term, we can calculate exactly as shown below.

$$1^{\text{st}} \quad : \quad \int_0^{\widetilde{t}} -\frac{a}{\widetilde{\sigma}^2 b^{3/2}} d\widetilde{\tau} = \frac{\cot\left[\widetilde{\omega}\widetilde{t}\right]}{2\widetilde{\sigma}^2 \widetilde{\omega} b^{3/2}} \left(\widetilde{x}_1^2 + \widetilde{x}_2^2\right) - \frac{\csc\left[\widetilde{\omega}\widetilde{t}\right] \widetilde{x}_1^2 \widetilde{x}_2^2}{\widetilde{\sigma}^2 \widetilde{\omega} b^{3/2}} - \frac{\widetilde{t}\cot\left[\widetilde{\omega}\widetilde{t}\right] \csc\left[\widetilde{\omega}\widetilde{t}\right] \widetilde{x}_1^2 \widetilde{x}_2^2}{\widetilde{\sigma}^2 \widetilde{\omega} b^{3/2}}$$
$$- \frac{\widetilde{t}\csc^2\left[\widetilde{\omega}\widetilde{t}\right]}{2\widetilde{\sigma}^2 \widetilde{\omega} b^{3/2}} \left(\widetilde{x}_1^2 + \widetilde{x}_2^2\right). \tag{A.11}$$

Integrating all terms in the large $\widetilde{t}$ limit, we get

$$1^{\text{st}} \quad : \quad \int_0^{\widetilde{t}} -\frac{a}{\widetilde{\sigma}^2 b^{3/2}} d\widetilde{\tau} = \frac{i}{2\widetilde{\sigma}^2 \widetilde{\omega} b^{3/2}} \left(\widetilde{x}_1^2 + \widetilde{x}_2^2\right)$$

$$2^{\text{nd}} \quad : \quad \int_0^{\widetilde{t}} \frac{3ac}{2\widetilde{\sigma}^2 b^{5/2}} d\widetilde{\tau} = \frac{3i}{8\widetilde{\sigma}^4 \widetilde{\omega}^2 b^{5/2}} \left(\widetilde{x}_1^2 + \widetilde{x}_2^2\right)$$

$$3^{\text{rd}} \quad : \quad \int_0^{\widetilde{t}} -\frac{15ac^2}{8\widetilde{\sigma}^2 b^{7/2}} d\widetilde{\tau} = \frac{5i}{16\widetilde{\sigma}^6 \widetilde{\omega}^3 b^{7/2}} \left(\widetilde{x}_1^2 + \widetilde{x}_2^2\right)$$

$$i^{\text{th}} \quad : \quad \int_0^{\widetilde{t}} \frac{\left(-\frac{c}{b}\right)^i}{b^{3/2}\left(i+\frac{3}{2}\right) i!} \prod_{j=0}^{i} \left(j + \frac{3}{2}\right) d\widetilde{\tau} = \frac{1}{\widetilde{\sigma}^{2i} \widetilde{\omega}^i b^{\left(i+\frac{1}{2}\right)}} \frac{\prod_{j=0}^{i} \left(j+\frac{1}{2}\right)}{\left(i+\frac{1}{2}\right) i!} \left(\widetilde{x}_1^2 + \widetilde{x}_2^2\right). \tag{A.12}$$

Collecting the infinite term, we can solve the second term exactly for large $\widetilde{t}$ limit. The result is

$$\int_0^{\widetilde{t}} \frac{-a}{\widetilde{\sigma}^2 (b+c)^{3/2}} d\widetilde{\tau} = \sum_{i=1}^{\infty} \frac{1}{\widetilde{\sigma}^{2i} \widetilde{\omega}^i b^{\left(i+\frac{1}{2}\right)}} \frac{\prod_{j=0}^{i} \left(j+\frac{1}{2}\right)}{\left(i+\frac{1}{2}\right) i!} \left(\widetilde{x}_1^2 + \widetilde{x}_2^2\right). \tag{A.13}$$

Consider the third term in Eq.(A.4).

$$\int_0^{\widetilde{t}} \frac{1}{(b+c)^{1/2}} \frac{1}{2} \left(\frac{a}{\widetilde{\sigma}^2 (b+c)} \right)^2 d\widetilde{\tau} = \int_0^{\widetilde{t}} \frac{1}{2} \frac{a^2}{\widetilde{\sigma}^4 (b+c)^{5/2}} d\widetilde{\tau} \tag{A.14}$$

Expanding $\frac{-a}{\widetilde{\sigma}^2 (b+c)^{3/2}}$ in the series, we obtain

$$\frac{1}{2} \frac{a^2}{\widetilde{\sigma}^4 (b+c)^{5/2}} = \frac{a^2}{2\widetilde{\sigma}^4 b^{5/2}} - \frac{5a^2 c}{4\widetilde{\sigma}^4 b^{7/2}} + \frac{35a^2 c^2}{16\widetilde{\sigma}^4 b^{9/2}} - \frac{105a^2 c^3}{32\widetilde{\sigma}^4 b^{11/2}} + \frac{1155a^2 c^4}{256\widetilde{\sigma}^4 b^{13/2}} + ... \tag{A.15}$$

We integrate all terms in the large $\widetilde{t}$ limit. Therefore

$$1^{\text{st}} \quad : \quad \int_0^{\widetilde{t}} \frac{a^2}{2\widetilde{\sigma}^4 b^{5/2}} d\widetilde{\tau} = -\frac{i}{8\widetilde{\sigma}^4 \widetilde{\omega} b^{5/2}} \left(\widetilde{x}_1^4 + \widetilde{x}_2^4 \right)$$

$$2^{\text{nd}} \quad : \quad \int_0^{\widetilde{t}} -\frac{5a^2 c}{4\widetilde{\sigma}^4 b^{7/2}} d\widetilde{\tau} = -\frac{5i}{24\widetilde{\sigma}^6 \widetilde{\omega}^2 b^{7/2}} \left(\widetilde{x}_1^4 + \widetilde{x}_2^4 \right)$$

$$3^{\text{rd}} \quad : \quad \int_0^{\widetilde{t}} \frac{35a^2 c^2}{16\widetilde{\sigma}^4 b^{9/2}} d\widetilde{\tau} = -\frac{35i}{128\widetilde{\sigma}^8 \widetilde{\omega}^3 b^{9/2}} \left(\widetilde{x}_1^4 + \widetilde{x}_2^4 \right)$$

$$i^{\text{th}} \quad : \quad \int_0^{\widetilde{t}} \frac{\left(-\frac{c}{b}\right)^i}{b^{5/2} \left(i + \frac{5}{2}\right) i!} \prod_{j=0}^{i} \left(j + \frac{5}{2} \right) d\widetilde{\tau} = -\frac{i}{\widetilde{\sigma}^{2i+2} \widetilde{\omega}^i b^{\left(i+\frac{3}{2}\right)}} \frac{\prod_{j=0}^{i} \left(j + \frac{3}{2} \right)}{6 \left(i + \frac{3}{2} \right) i!} \left(\widetilde{x}_1^4 + \widetilde{x}_2^4 \right). \tag{A.16}$$

Collecting the infinite terms, we have

$$\int_0^{\widetilde{t}} \frac{1}{2} \frac{a^2}{\widetilde{\sigma}^4 (b+c)^{5/2}} d\widetilde{\tau} \underset{\widetilde{t}\to\infty}{=} -\sum_{i=1}^{\infty} \frac{i}{\widetilde{\sigma}^{2i+2} \widetilde{\omega}^i b^{\left(i+\frac{3}{2}\right)}} \frac{\prod_{j=0}^{i} \left(j + \frac{3}{2} \right)}{6 \left(i + \frac{3}{2} \right) i!} \left(\widetilde{x}_1^4 + \widetilde{x}_2^4 \right). \tag{A.17}$$

Similarly, we can solve the N^{th} order terms, the results are

$$2^{\text{nd}} \quad : \quad \int_0^{\widetilde{t}} \frac{-a}{\widetilde{\sigma}^2 (b+c)^{3/2}} d\widetilde{\tau} = \sum_{i=1}^{\infty} \frac{i}{\widetilde{\sigma}^{2i} \widetilde{\omega}^i b^{\left(i+\frac{1}{2}\right)}} \frac{\prod_{j=0}^{i} \left(j + \frac{1}{2} \right)}{\left(i + \frac{1}{2} \right) i!} \left(\widetilde{x}_1^2 + \widetilde{x}_2^2 \right)$$

$$3^{\text{rd}} \quad : \quad \int_0^{\widetilde{t}} \frac{a^2}{2\widetilde{\sigma}^4 (b+c)^{5/2}} d\widetilde{\tau} = -\sum_{i=1}^{\infty} \frac{i}{\widetilde{\sigma}^{2i+2} \widetilde{\omega}^i b^{\left(i+\frac{3}{2}\right)}} \frac{i \prod_{j=0}^{i} \left(j + \frac{3}{2} \right)}{6 (i+1) \left(i + \frac{3}{2} \right) i!} \left(\widetilde{x}_1^4 + \widetilde{x}_2^4 \right)$$

$$4^{\text{th}} \quad : \quad \int_0^{\widetilde{t}} \frac{a^3}{6\widetilde{\sigma}^6 (b+c)^{7/2}} d\widetilde{\tau} = \sum_{i=1}^{\infty} \frac{i}{\widetilde{\sigma}^{2i+4} \widetilde{\omega}^i b^{\left(i+\frac{5}{2}\right)}} \frac{i \prod_{j=0}^{i} \left(j + \frac{5}{2} \right)}{30 (i+2) \left(i + \frac{5}{2} \right) i!} \left(\widetilde{x}_1^6 + \widetilde{x}_2^6 \right)$$

$$n^{\text{th}} \quad : \quad \int_0^{\widetilde{t}} \frac{1}{n!\,(b+c)^{1/2}} \left(-\frac{a}{\widetilde{\sigma}^2\,(b+c)}\right)^n d\widetilde{\tau}$$

$$= \quad -\frac{(-1)^n\,(\widetilde{x}_1^{2n}+\widetilde{x}_2^{2n})}{(2n-1)\,n!} \sum_{i=1}^{\infty} \frac{i}{\widetilde{\sigma}^{2i+(2n-2)}\widetilde{\omega}^i b^{\left(i+\frac{2n-1}{2}\right)}} \frac{i\prod_{j=0}^{i}\left(j+\frac{2n-1}{2}\right)}{(i+n-1)\left(i+\frac{2n-1}{2}\right)i!}. \tag{A.18}$$

Thus we can calculate $\int_0^{\widetilde{t}} \left\langle \exp\left[-\left(\frac{\widetilde{x}_i(\tau)}{\widetilde{\sigma}}\right)^2\right]\right\rangle d\widetilde{\tau}$ in the large $\widetilde{t}$ limit.

$$\begin{aligned}
&\int_0^{\widetilde{t}} \left\langle \exp\left[-\left(\frac{\widetilde{x}_i(\tau)}{\widetilde{\sigma}}\right)^2\right]\right\rangle d\widetilde{\tau} \\
= \quad & \frac{\widetilde{t}}{b^{1/2}} - i\left(\frac{2\ln[2]}{\sqrt{b\widetilde{\omega}}} - \frac{2\ln\left[1+\sqrt{1-\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}b}}\right]}{\sqrt{b\widetilde{\omega}}}\right) \\
& -i\sum_{n=1}^{\infty} \frac{(-1)^n\,(\widetilde{x}_1^{2n}+\widetilde{x}_2^{2n})}{(2n-1)\,n!} \sum_{i=1}^{\infty} \frac{1}{\widetilde{\sigma}^{2i+(2n-2)}\widetilde{\omega}^i b^{\left(i+\frac{2n-1}{2}\right)}} \frac{i\prod_{j=0}^{i}\left(j+\frac{2n-1}{2}\right)}{(i+n-1)\left(i+\frac{2n-1}{2}\right)i!} \\
= \quad & \frac{\widetilde{t}}{b^{1/2}} - i\left(\frac{2\ln[2]}{\sqrt{b\widetilde{\omega}}} - \frac{2\ln\left[1+\sqrt{1-\frac{1}{\widetilde{\sigma}^2\widetilde{\omega}b}}\right]}{\sqrt{b\widetilde{\omega}}}\right) \\
& -i\sum_{n=1}^{\infty} \frac{(-1)^n\,(\widetilde{x}_1^{2n}+\widetilde{x}_2^{2n})}{2n\widetilde{\omega}\widetilde{\sigma}^{2n}b^{\left(n+\frac{1}{2}\right)}}\ {}_2F_1\left[n, n+\frac{1}{2}, n+1, \frac{1}{b\widetilde{\sigma}^2\widetilde{\omega}}\right].
\end{aligned} \tag{A.19}$$

Here ${}_2F_1$ is the regularized hypergeometric function.

Appendix B

Average of Delta Interaction in One Dimension

In this appendix we show the detailed calculation of the delta function in Eq.(3.19). The average of Dirac delta function can be written as

$$\langle\delta(x_i - x_j)\rangle_{S_0} = \frac{1}{\left(\frac{4\pi i}{\widetilde{\omega}}\frac{\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})\sin\widetilde{\omega}\widetilde{\tau}}{\sin\widetilde{\omega}\widetilde{t}}\right)^{1/2}} \exp\left[-\frac{\left(\frac{\left(\widetilde{x}_{2_i}-\widetilde{x}_{2_j}\right)\sin\widetilde{\omega}\widetilde{\tau}+\left(\widetilde{x}_{1_i}-\widetilde{x}_{1_j}\right)\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})}{\sin\widetilde{\omega}\widetilde{t}}\right)^2}{\left(\frac{4i}{\widetilde{\omega}}\frac{\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})\sin\widetilde{\omega}\widetilde{\tau}}{\sin\widetilde{\omega}\widetilde{t}}\right)}\right]. \tag{B.1}$$

We let

$$a = \left(\frac{\widetilde{X}_2\sin\widetilde{\omega}\widetilde{\tau} + \widetilde{X}_1\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})}{\sin\widetilde{\omega}\widetilde{t}}\right)^2 \tag{B.2}$$

$$b = -\frac{2i}{\widetilde{\omega}}\cot\widetilde{\omega}\widetilde{t} \underset{\widetilde{t}\to\infty}{=} \frac{2}{\widetilde{\omega}}$$

$$c = \frac{i}{\widetilde{\omega}}\cos\left(2\widetilde{\omega}\widetilde{\tau} - \widetilde{\omega}\widetilde{t}\right)\csc\widetilde{\omega}\widetilde{t}, \tag{B.3}$$

here $\widetilde{X}_2 = \left(\widetilde{x}_{2_i} - \widetilde{x}_{2_j}\right)$ and $\widetilde{X}_1 = \left(\widetilde{x}_{1_i} - \widetilde{x}_{1_j}\right)$. Therefore

$$\int_0^{\widetilde{t}} \langle\delta(x_i - x_j)\rangle_{S_0}\, d\widetilde{\tau} = \int_0^{\widetilde{t}} \frac{1}{\sqrt{\pi}\,(b+c)^{1/2}} \exp\left[-\frac{a}{(b+c)}\right] d\widetilde{\tau} \tag{B.4}$$

Expand $\exp\left[-\frac{a}{(b+c)}\right]$ in the series.

$$\int_0^{\widetilde{t}} \langle\delta(x_i - x_j)\rangle_{S_0}\, d\widetilde{\tau} = \int_0^{\widetilde{t}} \frac{1}{\sqrt{\pi}\,(b+c)^{1/2}} \left(\begin{array}{c} 1 - \frac{a}{b+c} + \frac{1}{2}\left(\frac{a}{b+c}\right)^2 \\ -\frac{1}{6}\left(\frac{a}{b+c}\right)^3 + \frac{1}{24}\left(\frac{a}{b+c}\right)^4 + \ldots \end{array}\right) d\widetilde{\tau} \tag{B.5}$$

We consider the first term in Eq.(B.5) and expand in the series

$$\frac{1}{\sqrt{\pi}\,(b+c)^{1/2}}=\sqrt{\frac{1}{\pi b}}-\frac{c}{2\sqrt{\pi}b^{3/2}}+\frac{3c^2}{8\sqrt{\pi}b^{5/2}}-\frac{15c^3}{16\sqrt{\pi}b^{7/2}}+\frac{35c^4}{128\sqrt{\pi}b^{9/2}}+.... \quad \text{(B.6)}$$

In the large $\widetilde{t}$ limit $\left(\widetilde{t}\to\infty\right)$, we find that $\cot\widetilde{\omega}\widetilde{t}\to i$ and $\csc\widetilde{\omega}\widetilde{t}\to 0$. We continue integrating every terms in the large $\widetilde{t}$ limit. We get

$$1^{\text{st}} \;:\; \int_0^{\widetilde{t}}\sqrt{\frac{1}{\pi b}}d\widetilde{\tau}=\frac{\widetilde{t}}{\sqrt{\pi b}}$$

$$2^{\text{nd}} \;:\; -\int_0^{\widetilde{t}}\frac{c}{2\sqrt{\pi}b^{3/2}}d\widetilde{\tau}=-\frac{i}{2\sqrt{\pi}\widetilde{\omega}^2b^{3/2}}$$

$$3^{\text{rd}} \;:\; \int_0^{\widetilde{t}}\frac{3c^2}{8\sqrt{\pi}b^{5/2}}d\widetilde{\tau}=-\frac{3i}{4\sqrt{\pi}\widetilde{\omega}^3b^{5/2}}$$

$$i^{\text{th}} \;:\; \sqrt{\frac{1}{\pi b}}\int_0^{\widetilde{t}}\frac{\left(-\frac{c}{b}\right)^i}{\left(i+\frac{1}{2}\right)i!}\prod_{j=0}^{i}\left(j+\frac{1}{2}\right)d\widetilde{\tau}=-\frac{2^i i}{\sqrt{\pi}\widetilde{\omega}^{i+1}b^{i+\frac{1}{2}}}\frac{\prod_{j=0}^{i}\left(j+\frac{1}{2}\right)}{i\left(i+\frac{1}{2}\right)i!}. \quad \text{(B.7)}$$

Thus

$$\begin{aligned}\int_0^{\widetilde{t}}\frac{1}{\sqrt{\pi}\,(b+c)^{1/2}}d\widetilde{\tau} &= \frac{\widetilde{t}}{\sqrt{\pi b}}-i\sum_{i=1}^{\infty}\frac{2^i}{\sqrt{\pi}\widetilde{\omega}^{i+1}b^{i+\frac{1}{2}}}\frac{\prod_{j=0}^{i}\left(j+\frac{1}{2}\right)}{i\left(i+\frac{1}{2}\right)i!}\\ &= \frac{\widetilde{t}}{\sqrt{\pi b}}-i\sqrt{\frac{2}{\pi\widetilde{\omega}}}\ln 2. \end{aligned} \quad \text{(B.8)}$$

The first term is the ground state energy and the second term is the normalized factor. Consider the second term in Eq.(B.5).

$$\int_0^{\widetilde{t}}\frac{1}{\sqrt{\pi}\,(b+c)^{1/2}}\frac{-a}{(b+c)}d\widetilde{\tau}=\int_0^{\widetilde{t}}\frac{-a}{\sqrt{\pi}\,(b+c)^{3/2}}d\widetilde{\tau}. \quad \text{(B.9)}$$

Expanding $\frac{-a}{\widetilde{\sigma}^2(b+c)^{3/2}}$ in the series, we obtain

$$\frac{-a}{\sqrt{\pi}\,(b+c)^{3/2}}=-\frac{a}{\sqrt{\pi}b^{3/2}}+\frac{3ac}{2\sqrt{\pi}b^{5/2}}-\frac{15ac^2}{8\sqrt{\pi}b^{7/2}}+\frac{35ac^3}{16\sqrt{\pi}b^{9/2}}-\frac{315ac^4}{128\sqrt{\pi}b^{11/2}}+... \quad \text{(B.10)}$$

Integrating all terms in the large $\widetilde{t}$ limit, we get

$$1^{\text{st}} \quad : \quad \int_0^{\widetilde{t}} -\frac{a}{\sqrt{\pi}b^{3/2}}d\widetilde{\tau} = \frac{i}{2\sqrt{\pi}\widetilde{\omega}b^{3/2}}\left(\widetilde{X}_1^2+\widetilde{X}_2^2\right)$$

$$2^{\text{nd}} \quad : \quad \int_0^{\widetilde{t}} \frac{3ac}{2\sqrt{\pi}b^{5/2}}d\widetilde{\tau} = \frac{3i}{4\sqrt{\pi}\widetilde{\omega}^2b^{5/2}}\left(\widetilde{X}_1^2+\widetilde{X}_2^2\right)$$

$$3^{\text{rd}} \quad : \quad \int_0^{\widetilde{t}} -\frac{15ac^2}{8\sqrt{\pi}b^{7/2}}d\widetilde{\tau} = \frac{5i}{4\sqrt{\pi}\widetilde{\omega}^3b^{7/2}}\left(\widetilde{X}_1^2+\widetilde{X}_2^2\right)$$

$$i^{\text{th}} \quad : \quad \int_0^{\widetilde{t}} \frac{\left(-\frac{c}{b}\right)^i}{\sqrt{\pi}b^{3/2}\left(i+\frac{3}{2}\right)i!}\prod_{j=0}^{i}\left(j+\frac{3}{2}\right)d\widetilde{\tau} = \frac{2^i i}{\sqrt{\pi}\widetilde{\omega}^i b^{\left(i+\frac{1}{2}\right)}}\frac{\prod_{j=0}^{i}\left(j+\frac{1}{2}\right)}{2\left(i+\frac{1}{2}\right)i!}\left(\widetilde{X}_1^2+\widetilde{X}_2^2\right). \tag{B.11}$$

Collecting the infinite terms, we have

$$\int_0^{\widetilde{t}} \frac{-a}{\sqrt{\pi}\left(b+c\right)^{3/2}}d\widetilde{\tau} = i\sum_{i=1}^{\infty}\frac{2^i}{\sqrt{\pi}\widetilde{\omega}^i b^{\left(i+\frac{1}{2}\right)}}\frac{\prod_{j=0}^{i}\left(j+\frac{1}{2}\right)}{2\left(i+\frac{1}{2}\right)i!}\left(\widetilde{X}_1^2+\widetilde{X}_2^2\right). \tag{B.12}$$

Consider the third term in Eq.(A.4)

$$\int_0^{\widetilde{t}} \frac{1}{\sqrt{\pi}\left(b+c\right)^{1/2}}\frac{1}{2}\left(\frac{a}{\left(b+c\right)}\right)^2 d\widetilde{\tau} = \int_0^{\widetilde{t}} \frac{1}{2}\frac{a^2}{\sqrt{\pi}\left(b+c\right)^{5/2}}d\widetilde{\tau}. \tag{B.13}$$

Expanding $\frac{-a}{\widetilde{\sigma}^2(b+c)^{3/2}}$ in the series, we obtain

$$\frac{1}{2}\frac{a^2}{\sqrt{\pi}\left(b+c\right)^{5/2}} = \frac{a^2}{2\sqrt{\pi}b^{5/2}} - \frac{5a^2c}{4\sqrt{\pi}b^{7/2}} + \frac{35a^2c^2}{16\sqrt{\pi}b^{9/2}} - \frac{105a^2c^3}{32\sqrt{\pi}b^{11/2}} + \frac{1155a^2c^4}{256\sqrt{\pi}b^{13/2}} + ... \tag{B.14}$$

We integrate all terms in the large $\widetilde{t}$ limit. Therefore

$$1^{\text{st}} \quad : \quad \int_0^{\widetilde{t}} \frac{a^2}{2\sqrt{\pi}b^{5/2}} d\widetilde{\tau} = -\frac{i}{8\sqrt{\pi}\widetilde{\omega}b^{5/2}} \left(\widetilde{X}_1^4 + \widetilde{X}_2^4\right)$$

$$2^{\text{nd}} \quad : \quad \int_0^{\widetilde{t}} -\frac{5a^2c}{4\sqrt{\pi}b^{7/2}} d\widetilde{\tau} = -\frac{5i}{12\sqrt{\pi}\widetilde{\omega}^2 b^{7/2}} \left(\widetilde{X}_1^4 + \widetilde{X}_2^4\right)$$

$$3^{\text{rd}} \quad : \quad \int_0^{\widetilde{t}} \frac{35a^2c^2}{16\sqrt{\pi}b^{9/2}} d\widetilde{\tau} = -\frac{35i}{32\sqrt{\pi}\widetilde{\omega}^3 b^{9/2}} \left(\widetilde{X}_1^4 + \widetilde{X}_2^4\right)$$

$$i^{\text{th}} \quad : \quad \int_0^{\widetilde{t}} \frac{\left(-\frac{c}{b}\right)^i}{\sqrt{\pi}b^{5/2}\left(i+\frac{5}{2}\right)i!} \prod_{j=0}^{i} \left(j+\frac{5}{2}\right) d\widetilde{\tau} = -\frac{i}{\sqrt{\pi}\widetilde{\omega}^i b^{\left(i+\frac{3}{2}\right)}} \frac{i\prod_{j=0}^{i}\left(j+\frac{3}{2}\right)}{12\left(i+\frac{3}{2}\right)(i+1)\, i!} \left(\widetilde{X}_1^4 + \widetilde{X}_2^4\right). \tag{B.15}$$

Collecting the infinite term, we have

$$\int_0^{\widetilde{t}} \frac{1}{2} \frac{a^2}{\sqrt{\pi}\,(b+c)^{5/2}} d\widetilde{\tau} = -i \sum_{i=1}^{\infty} \frac{1}{\sqrt{\pi}\widetilde{\omega}^i b^{\left(i+\frac{3}{2}\right)}} \frac{i\prod_{j=0}^{i}\left(j+\frac{3}{2}\right)}{12\left(i+\frac{3}{2}\right)(i+1)\, i!} \left(\widetilde{X}_1^4 + \widetilde{X}_2^4\right) \tag{B.16}$$

Similarly, we can solve the other terms, the results are

$$2^{\text{nd}} \quad : \quad \int_0^{\widetilde{t}} \frac{-a}{\sqrt{\pi}\,(b+c)^{3/2}} d\widetilde{\tau} = i \sum_{i=1}^{\infty} \frac{2^i}{\sqrt{\pi}\widetilde{\omega}^i b^{\left(i+\frac{1}{2}\right)}} \frac{\prod_{j=0}^{i}\left(j+\frac{1}{2}\right)}{2\left(i+\frac{1}{2}\right) i!} \left(\widetilde{X}_1^2 + \widetilde{X}_2^2\right)$$

$$3^{\text{rd}} \quad : \quad \int_0^{\widetilde{t}} \frac{a^2}{2\sqrt{\pi}\,(b+c)^{5/2}} d\widetilde{\tau} = -i \sum_{i=1}^{\infty} \frac{2^i}{\sqrt{\pi}\widetilde{\omega}^i b^{\left(i+\frac{3}{2}\right)}} \frac{i\prod_{j=0}^{i}\left(j+\frac{3}{2}\right)}{12\,(i+1)\left(i+\frac{3}{2}\right) i!} \left(\widetilde{X}_1^4 + \widetilde{X}_2^4\right)$$

$$4^{\text{th}} \quad : \quad \int_0^{\widetilde{t}} \frac{-a^3}{6\sqrt{\pi}\,(b+c)^{7/2}} d\widetilde{\tau} = i \sum_{i=1}^{\infty} \frac{2^i}{\sqrt{\pi}\widetilde{\omega}^i b^{\left(i+\frac{5}{2}\right)}} \frac{i\prod_{j=0}^{i}\left(j+\frac{5}{2}\right)}{60\,(i+2)\left(i+\frac{5}{2}\right) i!} \left(\widetilde{X}_1^6 + \widetilde{X}_2^6\right)$$

$$5^{\text{th}} \quad : \quad \int_0^{\widetilde{t}} \frac{a^4}{24\sqrt{\pi}\,(b+c)^{9/2}} d\widetilde{\tau} = -i\sum_{i=1}^{\infty} \frac{2^i}{\sqrt{\pi}\widetilde{\omega}^i b^{\left(i+\frac{7}{2}\right)}} \frac{i\prod_{j=0}^{i}\left(j+\frac{7}{2}\right)}{336\,(i+3)\left(i+\frac{7}{2}\right) i!}\left(\widetilde{X}_1^8+\widetilde{X}_2^8\right)$$

$$n^{\text{th}} \quad : \quad \int_0^{\widetilde{t}} \frac{1}{n!\sqrt{\pi}\,(b+c)^{1/2}}\left(-\frac{a}{(b+c)}\right)^n d\widetilde{\tau}$$

$$= \quad -i\frac{(-1)^n\left(\widetilde{X}_1^{2n}+\widetilde{X}_2^{2n}\right)}{2\,(2n-1)\,n!}\sum_{i=1}^{\infty}\frac{2^i}{\sqrt{\pi}\widetilde{\omega}^i b^{\left(i+\frac{2n-1}{2}\right)}}\frac{i\prod_{j=0}^{i}\left(j+\frac{2n-1}{2}\right)}{(i+n-1)\left(i+\frac{2n-1}{2}\right)i!}. \tag{B.17}$$

Thus we can calculate $\int_0^{\widetilde{t}} \langle\delta(x_i-x_j)\rangle_{S_0}\, d\widetilde{\tau}$ in the large $\widetilde{t}$ limit.

$$\begin{aligned}
&\int_0^{\widetilde{t}} \langle\delta(x_i-x_j)\rangle_{S_0}\, d\widetilde{\tau} \\
= \quad & \sqrt{\frac{\widetilde{\omega}}{2}}\widetilde{t} - i\sqrt{\frac{2}{\pi\widetilde{\omega}}}\ln[2] \\
& -i\sum_{n=1}^{\infty}\frac{(-1)^n\left(\widetilde{x}_{1_i}-\widetilde{x}_{1_j}\right)^{2n}+\left(\widetilde{x}_{2_i}-\widetilde{x}_{2_j}\right)^{2n}}{(2n-1)\,n!}\sum_{i=1}^{\infty}\frac{2^i}{\sqrt{\pi}\widetilde{\omega}^i b^{\left(i+\frac{2n-1}{2}\right)}}\frac{i\prod_{j=0}^{i}\left(j+\frac{2n-1}{2}\right)}{(i+n-1)\left(i+\frac{2n-1}{2}\right)i!} \\
= \quad & \sqrt{\frac{\widetilde{\omega}}{2}}\widetilde{t} - i\sqrt{\frac{2}{\pi\widetilde{\omega}}}\ln[2] \\
& -i\frac{\sqrt{\widetilde{\omega}}\left(\widetilde{x}_{1_i}-\widetilde{x}_{1_j}\right)^2}{2\sqrt{2\pi}}\; {}_pF_q\left[\{1,1\},\{\frac{3}{2},2\},\frac{\left(\widetilde{x}_{1_i}-\widetilde{x}_{1_j}\right)^2\widetilde{\omega}}{2}\right] \\
& -i\frac{\sqrt{\widetilde{\omega}}\left(\widetilde{x}_{2_i}-\widetilde{x}_{2_j}\right)^2}{2\sqrt{2\pi}}\; {}_pF_q\left[\{1,1\},\{\frac{3}{2},2\},\frac{\left(\widetilde{x}_{2_i}-\widetilde{x}_{2_j}\right)^2\widetilde{\omega}}{2}\right],
\end{aligned} \tag{B.18}$$

where ${}_pF_q$ is generalized hypergeometric function.

Appendix C

Average of Delta Interaction in Three Dimensions

We show the detailed calculation of the integral of the delta function in three dimensions in Eq.(3.47). The delta function in three dimensions can be written as

$$\begin{aligned}
&\left\langle \delta(\overrightarrow{\widetilde{r}}_i - \overrightarrow{\widetilde{r}}_j) \right\rangle_{S_0} \\
&= \left(\frac{1}{4\pi g\left(\widetilde{\tau}, \widetilde{\tau}\right)} \right)^{3/2} \exp \left[- \left(\begin{array}{c} \left(\frac{\left(\widetilde{x}_{2_i} - \widetilde{x}_{2_j}\right) \sin \widetilde{\omega}\widetilde{\tau} + \left(\widetilde{x}_{1_i} - \widetilde{x}_{1_j}\right) \sin \widetilde{\omega}(\widetilde{t} - \widetilde{\tau})}{\sin \widetilde{\omega}\widetilde{t}} \right)^2 \\ + \left(\frac{\left(\widetilde{y}_{2_i} - \widetilde{y}_{2_j}\right) \sin \widetilde{\omega}\widetilde{\tau} + \left(\widetilde{y}_{1_i} - \widetilde{y}_{1_j}\right) \sin \widetilde{\omega}(\widetilde{t} - \widetilde{\tau})}{\sin \widetilde{\omega}\widetilde{t}} \right)^2 \\ + \left(\frac{\left(\widetilde{z}_{2_i} - \widetilde{z}_{2_j}\right) \sin \widetilde{\omega}\widetilde{\tau} + \left(\widetilde{z}_{1_i} - \widetilde{z}_{1_j}\right) \sin \widetilde{\omega}(\widetilde{t} - \widetilde{\tau})}{\sin \widetilde{\omega}\widetilde{t}} \right)^2 \end{array} \right) / 4g\left(\widetilde{\tau}, \widetilde{\tau}\right) \right].
\end{aligned} \tag{C.1}$$

We let

$$\begin{aligned} a = &\left(\frac{\widetilde{X}_2 \sin \widetilde{\omega}\widetilde{\tau} + \widetilde{X}_1 \sin \widetilde{\omega}(\widetilde{t} - \widetilde{\tau})}{\sin \widetilde{\omega}\widetilde{t}} \right)^2 + \left(\frac{\widetilde{Y}_2 \sin \widetilde{\omega}\widetilde{\tau} + \widetilde{Y}_1 \sin \widetilde{\omega}(\widetilde{t} - \widetilde{\tau})}{\sin \widetilde{\omega}\widetilde{t}} \right)^2 \\ &+ \left(\frac{\widetilde{Z}_2 \sin \widetilde{\omega}\widetilde{\tau} + \widetilde{Z}_1 \sin \widetilde{\omega}(\widetilde{t} - \widetilde{\tau})}{\sin \widetilde{\omega}\widetilde{t}} \right)^2. \end{aligned} \tag{C.2}$$

Here $\widetilde{Y}_2 = \left(\widetilde{y}_{2_i} - \widetilde{y}_{2_j}\right)$, $\widetilde{Y}_1 = \left(\widetilde{y}_{1_i} - \widetilde{y}_{1_j}\right)$ and $\widetilde{Z}_2 = \left(\widetilde{z}_{2_i} - \widetilde{z}_{2_j}\right)$, $\widetilde{Z}_1 = \left(\widetilde{z}_{1_i} - \widetilde{z}_{1_j}\right)$.

$$\int_0^{\widetilde{t}} \left\langle \delta(\overrightarrow{\widetilde{r}}_i - \overrightarrow{\widetilde{r}}_j) \right\rangle_{S_0} d\widetilde{\tau} = \int_0^{\widetilde{t}} \frac{1}{\pi^{3/2} \left(b + c\right)^{3/2}} \exp \left[- \frac{a}{(b + c)} \right] d\widetilde{\tau}. \tag{C.3}$$

Expanding $\exp\left[-\frac{a}{(b+c)}\right]$ in a series and integrating all terms for the large $\widetilde{t}$ limit, therefore we obtain

$$1^{\text{st}} \quad : \quad \int_0^{\widetilde{t}} \frac{1}{\pi^{3/2}(b+c)^{3/2}} d\widetilde{\tau} = \frac{1}{\pi^{3/2} b^{3/2}}$$

$$2^{\text{nd}} \quad : \quad \int_0^{\widetilde{t}} \frac{-a}{\pi^{3/2}(b+c)^{5/2}} d\widetilde{\tau} = \sum_{i=1}^{\infty} \frac{i\left(\widetilde{R}_1^2 + \widetilde{R}_2^2\right)}{6\pi^{3/2}\widetilde{\omega}^i b^{\left(i+\frac{3}{2}\right)}} \frac{\prod_{j=0}^{i}\left(j+\frac{3}{2}\right)}{\left(i+\frac{3}{2}\right) i!} 2^i$$

$$3^{\text{rd}} \quad : \quad \int_0^{\widetilde{t}} \frac{a^2}{\pi^{3/2}(b+c)^{7/2}} d\widetilde{\tau} = -\sum_{i=1}^{\infty} \frac{i\left(\widetilde{R}_1^4 + \widetilde{R}_2^4\right)}{20\pi^{3/2}\widetilde{\omega}^i b^{\left(i+\frac{5}{2}\right)}} \frac{i\prod_{j=0}^{i}\left(j+\frac{5}{2}\right)}{(i+1)\left(i+\frac{5}{2}\right) i!} 2^i$$

$$4^{\text{th}} \quad : \quad \int_0^{\widetilde{t}} \frac{-a^3}{\pi^{3/2}(b+c)^{9/2}} d\widetilde{\tau} = \sum_{i=1}^{\infty} \frac{i\left(\widetilde{R}_1^6 + \widetilde{R}_2^6\right)}{84\pi^{3/2}\widetilde{\omega}^i b^{\left(i+\frac{7}{2}\right)}} \frac{i\prod_{j=0}^{i}\left(j+\frac{7}{2}\right)}{(i+2)\left(i+\frac{7}{2}\right) i!} 2^i.$$

Therefore the integral of delta function in 3 dimension is

$$\begin{aligned}
&\int_0^{\widetilde{t}} \left\langle \delta(\overrightarrow{\widetilde{r}}_i - \overrightarrow{\widetilde{r}}_j) \right\rangle_{S_0} d\widetilde{\tau} \\
&= \sum_{n=1}^{\infty} \frac{\left(\left(\widetilde{R}_1^{2n} + \widetilde{R}_2^{2n}\right)\right)}{2(2n+1)\, n!} \sum_{i=1}^{\infty} \frac{i}{\pi^{3/2}\widetilde{\omega}^i b^{\left(i+\frac{2n+1}{2}\right)}} \frac{i\prod_{j=0}^{i}\left(j+\frac{2n+1}{2}\right)}{(i+2)\left(i+\frac{2n+1}{2}\right) i!} 2^i \\
&= \frac{1}{\pi^{3/2} b^{3/2}} - \frac{i\widetilde{\omega}^{3/2}\widetilde{R}_1^2}{12\sqrt{2}\pi^{3/2}}\ {}_pF_q\left[\{1,1\},\{2,\frac{5}{2}\},\frac{\widetilde{R}_1^2\widetilde{\omega}}{2}\right] \\
&\quad - \frac{i\widetilde{\omega}^{3/2}\widetilde{R}_2^2}{12\sqrt{2}\pi^{3/2}}\ {}_pF_q\left[\{1,1\},\{2,\frac{5}{2}\},\frac{\widetilde{R}_2^2\widetilde{\omega}}{2}\right],
\end{aligned} \tag{C.4}$$

where

$$\begin{aligned}
\left(\widetilde{r}_{1_i} - \widetilde{r}_{1_j}\right)^2 &= \widetilde{R}_1^2 = \widetilde{X}_1^2 + \widetilde{Y}_1^2 + \widetilde{Z}_1^2 \\
\left(\widetilde{r}_{2_i} - \widetilde{r}_{2_j}\right)^2 &= \widetilde{R}_2^2 = \widetilde{X}_2^2 + \widetilde{Y}_2^2 + \widetilde{Z}_2^2.
\end{aligned} \tag{C.5}$$

Appendix D

Average of Cosine Function

We show the detailed calculation of the integral of the cosine in Eq. (3.61). We can write $\langle \cos(2kx_i) \rangle_{S_0}$ as

$$\left\langle \cos\left(2\widetilde{k}\widetilde{x}_i\right)\right\rangle_{S_0} = \left\langle e^{2i\widetilde{k}\widetilde{x}_i} + e^{-2i\widetilde{k}\widetilde{x}_i}\right\rangle /2 = \left(\left\langle e^{2i\widetilde{k}\widetilde{x}_i}\right\rangle + \left\langle e^{-2i\widetilde{k}\widetilde{x}_i}\right\rangle\right)/2. \tag{D.1}$$

We expand $e^{2i\widetilde{k}\widetilde{x}_i}$ in the first and second cumulants. Because S_0 is quadratic, only the first two cumulants are non-zero (Kubo 1962) [17].

$$\begin{aligned} \left\langle \exp\left[2i\widetilde{k}\widetilde{x}_i\right]\right\rangle_{S_0} &= \exp\left[2i\widetilde{k}\left\langle \widetilde{x}_i\right\rangle_{S_0} - \widetilde{k}^2 2\left(\left\langle \widetilde{x}_i^2\right\rangle_{S_0} - \left\langle \widetilde{x}_i\right\rangle_{S_0}^2\right)\right] \\ &= \exp\left[2i\widetilde{k}\left(\frac{\widetilde{x}_2\sin\widetilde{\omega}\widetilde{\tau} + \widetilde{x}_1\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})}{\sin\widetilde{\omega}\widetilde{t}}\right) - 2\widetilde{k}^2 g\left(\widetilde{\tau},\widetilde{\tau}\right)\right]. \end{aligned} \tag{D.2}$$

We let

$$p = 2ik\left(\frac{\widetilde{x}_2\sin\widetilde{\omega}\widetilde{\tau} + \widetilde{x}_1\sin\widetilde{\omega}(\widetilde{t}-\widetilde{\tau})}{\sin\widetilde{\omega}\widetilde{t}}\right) - 2\widetilde{k}^2 g\left(\widetilde{\tau},\widetilde{\tau}\right). \tag{D.3}$$

For large $\widetilde{t}$, $p << 1$. Therefore we can expand e^p in the series by using the relation $e^p = \sum\limits_{n=0}^{\infty}\frac{p^n}{n!} = 1 + p + \frac{1}{2}p^2 + \frac{1}{6}p^3 + ...$ Let us consider the 2$^{\text{nd}}$ order of e^p

$$\int_0^{\widetilde{t}} p\,d\widetilde{\tau} = -\frac{i\widetilde{k}^2}{\widetilde{\omega}^2} + \frac{i\widetilde{k}^2\widetilde{t}\cot\widetilde{\omega}\widetilde{t}}{\widetilde{\omega}} + \frac{2i\widetilde{k}}{\widetilde{\omega}}\left(\widetilde{x}_1 + \widetilde{x}_2\right)\tan\widetilde{\omega}\widetilde{t}. \tag{D.4}$$

the 3$^{\text{rd}}$ order

$$\begin{aligned} \frac{1}{2}\int_0^{\widetilde{t}} p^2 d\widetilde{\tau} &= \frac{3\widetilde{k}^4\cot\widetilde{\omega}\widetilde{t}}{4\widetilde{\omega}^3} + \frac{\widetilde{k}^4\widetilde{t}}{4\widetilde{\omega}^2} - \frac{\widetilde{k}^4\widetilde{t}\cot^2\widetilde{\omega}\widetilde{t}}{4\widetilde{\omega}^2} - \frac{4\widetilde{k}^3}{3\widetilde{\omega}^2}\left(\widetilde{x}_1 + \widetilde{x}_2\right)\tan^2\widetilde{\omega}\widetilde{t} \\ &\quad + \frac{\widetilde{k}^2}{\widetilde{\omega}}\left(\widetilde{x}_1^2 + \widetilde{x}_2^2\right)\cot\widetilde{\omega}\widetilde{t} + ... \end{aligned} \tag{D.5}$$

the 4$^{\text{th}}$ order

$$\frac{1}{6}\int\limits_0^{\widetilde{t}} p^3 d\widetilde{\tau} = -\frac{11i\widetilde{k}^6}{144\widetilde{\omega}^4}+\frac{11i\widetilde{k}^6\widetilde{t}\cot^2\widetilde{\omega}\widetilde{t}}{48\widetilde{\omega}^4}+\frac{i\widetilde{k}^6\widetilde{t}\cot\widetilde{\omega}\widetilde{t}}{8\widetilde{\omega}^3}-\frac{i\widetilde{k}^6\widetilde{t}\cot^3\widetilde{\omega}\widetilde{t}}{24\widetilde{\omega}^3}$$
$$-\frac{2i\widetilde{k}^5\cot\widetilde{\omega}\widetilde{t}}{5\widetilde{\omega}^3}\left(\widetilde{x}_1+\widetilde{x}_2\right)+\frac{2i\widetilde{k}^5\cot^3\widetilde{\omega}\widetilde{t}}{15\widetilde{\omega}^3}\left(\widetilde{x}_1+\widetilde{x}_2\right)-\frac{i\widetilde{k}^4}{8\widetilde{\omega}^2}\left(\widetilde{x}_1^2+\widetilde{x}_2^2\right)$$
$$+\frac{3i\widetilde{k}^4\cot^2\widetilde{\omega}\widetilde{t}}{8\widetilde{\omega}^2}\left(\widetilde{x}_1^2+\widetilde{x}_2^2\right)+\frac{i\widetilde{k}^3\cot\widetilde{\omega}\widetilde{t}}{3\widetilde{\omega}}\left(\widetilde{x}_1^3+x_2^3\right)$$
$$-\frac{i\widetilde{k}^3\cot^3\widetilde{\omega}\widetilde{t}}{9\widetilde{\omega}}\left(\widetilde{x}_1^3+\widetilde{x}_2^3\right)+\ldots \tag{D.6}$$

Taking limit $\widetilde{t}\to\infty$. Thus $\cot\widetilde{\omega}\widetilde{t}\to i$ and $\csc\widetilde{\omega}\widetilde{t}\to 0$. We find that the term which depend on $\widetilde{t}$ and end point term can be written in the closed form as shown below.

$$\int\limits_0^{\widetilde{t}} p d\widetilde{\tau} = -\frac{i\widetilde{k}^2}{\widetilde{\omega}^2}-\frac{i\widetilde{k}^2\widetilde{t}}{\widetilde{\omega}}+\frac{2i\widetilde{k}}{\widetilde{\omega}}\left(\widetilde{x}_1+\widetilde{x}_2\right)$$

$$\frac{1}{2}\int\limits_0^{\widetilde{t}} p^2 d\widetilde{\tau} = -\frac{3i\widetilde{k}^4}{4\widetilde{\omega}^3}+\frac{\widetilde{k}^4}{2\widetilde{\omega}^2}\widetilde{t}-\frac{4\widetilde{k}^3}{3\widetilde{\omega}^2}\left(\widetilde{x}_1+\widetilde{x}_2\right)+\frac{i\widetilde{k}^2}{\widetilde{\omega}}\left(\widetilde{x}_1^2+\widetilde{x}_2^2\right)$$

$$\frac{1}{6}\int\limits_0^{\widetilde{t}} p^3 d\widetilde{\tau} = -\frac{11i\widetilde{k}^6}{36\widetilde{\omega}^4}-\frac{\widetilde{k}^6}{6\widetilde{\omega}^3}\widetilde{t}+\frac{8\widetilde{k}^5}{15\widetilde{\omega}^3}\left(\widetilde{x}_1+\widetilde{x}_2\right)-\frac{i\widetilde{k}^4}{2\widetilde{\omega}^2}\left(\widetilde{x}_1^2+\widetilde{x}_2^2\right)-\frac{4\widetilde{k}^3}{9\widetilde{\omega}}\left(\widetilde{x}_1^3+\widetilde{x}_2^3\right)$$

$$\frac{1}{24}\int\limits_0^{\widetilde{t}} p^4 d\widetilde{\tau} = \frac{25i\widetilde{k}^8}{288\widetilde{\omega}^5}+\frac{\widetilde{k}^8}{24\widetilde{\omega}^4}\widetilde{t}-\frac{16\widetilde{k}^7}{105\widetilde{\omega}^4}\left(\widetilde{x}_1+\widetilde{x}_2\right)+\frac{i\widetilde{k}^6}{6\widetilde{\omega}^3}\left(\widetilde{x}_1^2+\widetilde{x}_2^2\right)+\frac{8\widetilde{k}^5}{45\widetilde{\omega}^2}\left(\widetilde{x}_1^3+\widetilde{x}_2^3\right)$$
$$-\frac{i\widetilde{k}^4}{6\widetilde{\omega}}\left(\widetilde{x}_1^4+\widetilde{x}_2^4\right)$$

$$\frac{1}{n!}\int\limits_0^{\widetilde{t}} p^n d\widetilde{\tau} = \frac{1}{n!}\left(-\frac{\widetilde{k}^2}{\widetilde{\omega}}\right)^n\widetilde{t}+\frac{1}{\widetilde{\omega}}\frac{\left(2i\widetilde{k}\right)^j}{j!j}\left(\widetilde{x}_1^j+\widetilde{x}_2^j\right)$$
$$+\frac{1}{\widetilde{\omega}^2}\frac{2^{j+1}\left(j+1\right)\left(i\widetilde{k}\right)^{j+2}}{\left(j+2\right)!j}\left(\widetilde{x}_1^j+\widetilde{x}_2^j\right)\sum_{n=0}^{\infty}\frac{\left(-1\right)^n\left(\frac{2\widetilde{k}^2}{\widetilde{\omega}}\right)^n}{\frac{1}{j+2}\prod\limits_{i=0}^{n}\left(2i+j+2\right)}. \tag{D.7}$$

Therefore

$$\int_0^{\widetilde{t}} \exp\left[2i\widetilde{k}\widetilde{x}_i\right] d\widetilde{\tau} = \widetilde{t}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\widetilde{k}^2}{\widetilde{\omega}}\right)^n + \sum_{n=1}^{\infty}\frac{1}{\widetilde{\omega}}\frac{\left(2i\widetilde{k}\right)^j}{j!j}\left(\widetilde{x}_1^j+\widetilde{x}_2^j\right) \tag{D.8}$$

$$+\frac{1}{\widetilde{\omega}^2}\sum_{j=1}^{\infty}\frac{2^{j+1}(j+1)\left(i\widetilde{k}\right)^{j+2}}{(j+2)!j}\left(\widetilde{x}_1^j+\widetilde{x}_2^j\right)\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{2\widetilde{k}^2}{\widetilde{\omega}}\right)^n}{\frac{1}{j+2}\prod_{i=0}^{n}(2i+j+2)}. \tag{D.9}$$

Similarly we can calculate

$$\int_0^{\widetilde{t}} \exp\left[-2i\widetilde{k}\widetilde{x}_i\right] d\widetilde{\tau} = \widetilde{t}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\widetilde{k}^2}{\widetilde{\omega}}\right)^n + \sum_{n=1}^{\infty}\frac{1}{\widetilde{\omega}}\frac{\left(-2i\widetilde{k}\right)^j}{j!j}\left(\widetilde{x}_1^j+\widetilde{x}_2^j\right)$$

$$+\frac{1}{\widetilde{\omega}^2}\sum_{j=1}^{\infty}\frac{2^{j+1}(j+1)\left(-i\widetilde{k}\right)^{j+2}}{(j+2)!j}\left(\widetilde{x}_1^j+\widetilde{x}_2^j\right)\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{2\widetilde{k}^2}{\widetilde{\omega}}\right)^n}{\frac{1}{j+2}\prod_{i=0}^{n}(2i+j+2)}. \tag{D.10}$$

Thus

$$\int_0^{\widetilde{t}} \left\langle \cos\left(2\widetilde{k}\widetilde{x}_i\right)\right\rangle_{S_0} d\widetilde{\tau}$$

$$= \frac{1}{2}\int_0^{\widetilde{t}} \exp\left[2i\widetilde{k}\widetilde{x}_i\right] d\widetilde{\tau} + \frac{1}{2}\int_0^{\widetilde{t}} \exp\left[-2i\widetilde{k}\widetilde{x}_i\right] d\widetilde{\tau} = \exp\left(-\frac{\widetilde{k}^2}{\widetilde{\omega}}\right)\widetilde{t}$$

$$+\frac{1}{2\widetilde{\omega}}\left(\sum_{n=1}^{\infty}\frac{\left(2i\widetilde{k}\right)^j}{j!j}\left(\widetilde{x}_1^j+\widetilde{x}_2^j\right)+\sum_{n=1}^{\infty}\frac{\left(-2i\widetilde{k}\right)^j}{j!j}\left(\widetilde{x}_1^j+\widetilde{x}_2^j\right)\right)$$

$$+\frac{1}{2\widetilde{\omega}^2}\sum_{j=1}^{\infty}\frac{2^{j+1}(j+1)\left(i\widetilde{k}\right)^{j+2}}{(j+2)!j}\left(x_1^j+x_2^j\right)\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{2\widetilde{k}^2}{\widetilde{\omega}}\right)^n}{\frac{1}{j+2}\prod_{i=0}^{n}(2i+j+2)}$$

$$+\frac{1}{2\widetilde{\omega}^2}\sum_{j=1}^{\infty}\frac{2^{j+1}(j+1)\left(-i\widetilde{k}\right)^{j+2}}{(j+2)!j}\left(\widetilde{x}_1^j+\widetilde{x}_2^j\right)\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{2k^2}{\widetilde{\omega}}\right)^n}{\frac{1}{j+2}\prod_{i=0}^{n}(2i+j+2)}. \tag{D.11}$$

Therefore we obtain

$$
\begin{aligned}
& -\frac{i\widetilde{A}}{2}\int\limits_{0}^{\widetilde{t}} d\widetilde{\tau}\left(\sum_{i=1}^{N}\left\langle \cos\left(2\widetilde{k}\widetilde{x}_i\right)+1\right\rangle_{S_0}\right) \\
= \; & -\frac{iN\widetilde{A}}{2}\exp\left(-\frac{k^2}{\widetilde{\omega}}\right)\widetilde{t}-\frac{iN\widetilde{A}}{2}\widetilde{t} \\
& -\sum_{i=1}^{N}\frac{1}{4\widetilde{\omega}}\left(\begin{array}{c} -2\left(\gamma-\mathrm{Ci}\left[2\widetilde{k}\widetilde{x}_{1_i}\right]+\ln 2\widetilde{k}\widetilde{x}_{1_i}\right) \\ -2\left(\gamma-\mathrm{Ci}\left[2\widetilde{k}\widetilde{x}_{2_i}\right]+\ln 2\widetilde{k}\widetilde{x}_{2_i}\right)\end{array}\right) \\
& -\sum_{i=1}^{N}\frac{\widetilde{A}}{2\widetilde{\omega}^2}\sum_{j=1}^{\infty}\frac{(-1)^{(j+1)}(2j+1)\left(2\widetilde{k}\right)^{2(j+1)}}{(2j+2)!j}\left(\widetilde{x}_{1_i}^{j}+\widetilde{x}_{2_i}^{j}\right)^{2j} \\
& \times e^{-\frac{\widetilde{k}^2}{\widetilde{\omega}}}(j+1)\left(-\frac{\widetilde{k}^2}{\widetilde{\omega}}\right)^{-(1+j)}\left(\Gamma\left[1+j\right]-\Gamma\left[1+j,-\frac{\widetilde{k}^2}{\widetilde{\omega}}\right]\right),
\end{aligned}
\tag{D.12}
$$

Here γ is Euler's constant, $\gamma \simeq 0.577216$, $\mathrm{Ci}[x]$ is the cosine integral function and $\Gamma\left[x\right]$ is the Euler gamma function.

Appendix E

The First Excited State

We consider the terms which depend on $\csc\widetilde{\omega}\widetilde{t}$ in the propagator and expand $\csc\widetilde{\omega}\widetilde{t}$ in a series.

$$\csc\widetilde{\omega}\widetilde{t} = \frac{2ie^{-i\widetilde{\omega}\widetilde{t}}}{\left(1-e^{-2i\widetilde{\omega}\widetilde{t}}\right)} = 2ie^{-i\widetilde{\omega}\widetilde{t}}\left(1+e^{-2i\widetilde{\omega}\widetilde{t}}+...\right) \tag{E.1}$$

Consider $\exp\left[i\int\limits_0^{\widetilde{t}} d\widetilde{\tau}\frac{1}{2}\sum\limits_{i=1}^{N}\left(\widetilde{\omega}^2-\widetilde{\Omega}_x^2\right)\langle\widetilde{x}_i^2\rangle_{S_0}\right]$ in the propagator.

$$\begin{aligned}
&\exp\left[i\int\limits_0^{\widetilde{t}} d\widetilde{\tau}\frac{1}{2}\sum_{i=1}^{N}\left(\widetilde{\omega}^2-1\right)\langle\widetilde{x}_i^2\rangle_{S_0}\right] \\
=\ &\exp\left[-N\left(\frac{1}{4\widetilde{\omega}}-\frac{\widetilde{\omega}}{4}\right)\widetilde{t}\cot\widetilde{\omega}\widetilde{t}-iN\left(\frac{1}{4\widetilde{\omega}}-\frac{\widetilde{\omega}}{4}\right)\left(\widetilde{x}_1^2+\widetilde{x}_2^2\right)\cot\widetilde{\omega}\widetilde{t}\right] \\
&\times\exp\left[\frac{iN}{2}\left(\widetilde{\omega}^2-1\right)\frac{\widetilde{x}_1\widetilde{x}_2}{\widetilde{\omega}}\csc\widetilde{\omega}\widetilde{t}\right].
\end{aligned}$$

Using the same method we can calculate the Gaussian term. Keeping only the terms which depend on $\csc\widetilde{\omega}\widetilde{t}$, the result is

$$\begin{aligned}
&\int\limits_0^{\widetilde{t}}\left\langle\exp\left[-\left(\frac{\widetilde{x}_i(\tau)}{\widetilde{\sigma}}\right)^2\right]\right\rangle d\widetilde{\tau} \\
=\ &-2\widetilde{x}_1\widetilde{x}_2\csc\widetilde{\omega}\widetilde{t}\frac{1}{\widetilde{\omega}\widetilde{\sigma}^2\left(1+\frac{1}{\widetilde{\omega}\widetilde{\sigma}^2}\right)^{3/2}}\left(\begin{array}{c} 1-\frac{1}{1+\left(1+\frac{1}{\widetilde{\omega}\widetilde{\sigma}^2}\right)^{-1/2}} \\ -\ln 2+\ln\left[1+\left(1+\frac{1}{\widetilde{\omega}\widetilde{\sigma}^2}\right)^{-1/2}\right]\end{array}\right) \\
&+....
\end{aligned} \tag{E.2}$$

and the interaction term.

$$\int\limits_0^{\widetilde{t}}\langle\delta(x_i-x_j)\rangle_{S_0}\,d\widetilde{\tau} = \widetilde{X}_1\widetilde{X}_2\sqrt{\frac{\widetilde{\omega}}{2\pi}}\ln 2\csc\widetilde{\omega}\widetilde{t}+.... \tag{E.3}$$

Collecting all terms, we can write the propagator as

$$
\begin{aligned}
& P\left(\widetilde{x}_N\left(\widetilde{t}\right),\widetilde{x}_N\left(0\right),\widetilde{\tau}\right) \\
\simeq\; & P_0\left(\widetilde{x}_N\left(\widetilde{t}\right),\widetilde{x}_N\left(0\right),\widetilde{\tau}\right) \\
& \times\exp\left[i\int\limits_0^{\widetilde{t}} d\widetilde{\tau}\left(\begin{array}{c}\frac{1}{2}\sum\limits_{i=1}^{N}\left(\widetilde{\omega}^2-1\right)\left\langle\widetilde{x}_i^2\right\rangle_{S_0}-\widetilde{V}_b\sum\limits_{i=1}^{N}\left\langle\exp\left[-\left(\frac{\widetilde{x}_i}{\widetilde{\sigma}}\right)^2\right]\right\rangle_{S_0} \\ -\widetilde{g}\sum\limits_{i<j}^{N}\left\langle\delta(\widetilde{x}_i-\widetilde{x}_j)\right\rangle_{S_0}\end{array}\right)\right] \\
=\; & \phi_0(\widetilde{x}_2)\phi_0^*(\widetilde{x}_1)\exp\left[-iNE_0\widetilde{t}\right] \\
& \exp\left[\left(\begin{array}{c}\frac{2Nm\widetilde{\omega}}{\hbar}\widetilde{x}_1\widetilde{x}_2-\frac{N}{2}\left(\widetilde{\omega}-\frac{1}{\widetilde{\omega}}\right)\widetilde{x}_1\widetilde{x}_2 \\ +\frac{\widetilde{V}_b 2\widetilde{x}_1\widetilde{x}_2}{\widetilde{\omega}\widetilde{\sigma}^2\left(1+\frac{1}{\widetilde{\omega}\widetilde{\sigma}^2}\right)^{3/2}}\left(\begin{array}{c}1-\frac{1}{1+\left(1+\frac{1}{\widetilde{\omega}\widetilde{\sigma}^2}\right)^{-1/2}} \\ -\ln 2+\ln\left[1+\left(1+\frac{1}{\widetilde{\omega}\widetilde{\sigma}^2}\right)^{-1/2}\right]\end{array}\right) \\ -\widetilde{g}\frac{N(N-1)}{2}\widetilde{x}_1\widetilde{x}_2\sqrt{\frac{\widetilde{\omega}}{2\pi}}\ln 2\end{array}\right)\right]\csc\widetilde{\omega}\widetilde{t}+\ldots
\end{aligned}
\tag{E.4}
$$

Expanding the exponential term in the series and keeping only the first two terms, we obtain

$$
P\left(\widetilde{x}_N\left(\widetilde{t}\right),\widetilde{x}_N\left(0\right),\widetilde{\tau}\right)\sim\phi_0(\widetilde{x}_2)\phi_0^*(\widetilde{x}_1)\exp\left[-iNE_0\widetilde{t}\right]\left(1+(\text{constant})\;\widetilde{x}_1\widetilde{x}_2 e^{-i\widetilde{\omega}\widetilde{t}}\right)+\ldots
\tag{E.5}
$$

Thus the propagator for the first excited state can be written in the simple form:

$$
\phi_1(\widetilde{x})=(\text{constant})\,\widetilde{x}\phi_0(\widetilde{x})
\tag{E.6}
$$

We can also find the constant by using the normalized condition:

$$
\int\limits_{-\infty}^{\infty}\left|\phi_1\left(\widetilde{x}\right)\right|^2 d\widetilde{x}=1.
\tag{E.7}
$$

Appendix F

The Ground State Energy of BEC in a Disordered System

In this appendix we give the detailed calculations of the ground state energy in Eq. (4.51). The Green function $G(\tau,\sigma)$ can be rewritten as

$$\left(1+\frac{2}{L^2}G(\tau,\sigma)\right) = 1-\frac{i\hbar\cot\frac{1}{2}\omega t}{L^2 m\omega}+\frac{i\hbar\cos\left(\frac{1}{2}\omega t-2\omega(\sigma-\tau)\right)\csc\frac{1}{2}\omega t}{L^2 m\omega}. \quad \text{(F.1)}$$

Let us consider the first order

$$\int_0^t\int_0^\tau d\tau d\sigma\frac{1}{\left(1+\frac{2}{L^2}G(\tau,\sigma)\right)^{3/2}} = \int_0^t\int_0^\tau d\tau d\sigma\frac{1}{(B+C)^{3/2}}, \quad \text{(F.2)}$$

where

$$\begin{aligned} B &= 1-\frac{i\hbar\mu\cot\frac{1}{2}\omega t}{L^2m^2\nu} \underset{t\to\infty}{=} 1+\frac{2E_L}{E_\omega} \\ C &= \frac{i\hbar\cos\left(\frac{1}{2}\omega t-2\omega(\sigma-\tau)\right)\csc\frac{1}{2}\omega t}{L^2 m\omega}. \end{aligned} \quad \text{(F.3)}$$

For $t\to\infty, C<<1$ so we can expand $\frac{1}{(B+C)^{3/2}}$ in the series.

$$\frac{1}{(B+C)^{3/2}} = \frac{1}{B^{3/2}}-\frac{3C}{2B^{5/2}}+\frac{15C^2}{8B^{7/2}}-\frac{35C^3}{16B^{9/2}}+\frac{315C^4}{128B^{11/2}}+\ldots. \quad \text{(F.4)}$$

Integrating each term, the results are

$$\begin{aligned} 1^{\text{st}} &: \int_0^t\int_0^\tau \frac{1}{B^{3/2}}d\tau d\sigma = -\frac{t^2}{B^{3/2}} \\ 2^{\text{nd}} &: \int_0^t\int_0^\tau -\frac{3C}{2B^{5/2}}d\tau d\sigma = -\frac{3i\hbar t}{2B^{5/2}L^2m\omega^2} \\ 3^{\text{rd}} &: \int_0^t\int_0^\tau \frac{15C^2}{8B^{7/2}}d\tau d\sigma = -\frac{15\hbar^2t^2\csc^2\frac{t\omega}{2}}{32B^{7/2}L^4m^2\omega^2}-\frac{15\hbar^2 t\cot\frac{t\omega}{2}}{16B^{7/2}L^4m^2\omega^3} \end{aligned}$$

In order to find the ground state energy, we take the limit $t \to \infty$. Integrating all terms, we can write the infinite terms in the closed form as

$$\begin{aligned}
&\int_0^t \int_0^\tau d\tau d\sigma \frac{1}{(B+C)^{3/2}} \\
= &\; -\frac{t^2}{B^{3/2}} - \frac{3i\hbar E_L}{B^{5/2}E_\omega^2}t - \frac{15i\hbar E_L^2}{4B^{7/2}E_\omega^3}t - \frac{35i\hbar E_L^3}{6B^{9/2}E_\omega^5}t + \frac{315i\hbar E_L^4}{32B^{11/2}E_\omega^5}t - \frac{693i\hbar E_L^5}{40B^{13/2}E_\omega^6}t + ... \\
= &\; -\frac{t^2}{B^{3/2}} + t\frac{i\hbar}{B^{3/2}E_\omega}\sum_{i=1}^{\infty}\frac{2^{i+1}}{ii!}\left(\frac{E_L}{BE_\omega}\right)^i \prod_{j=0}^{i}\left(j+\frac{1}{2}\right) \\
= &\; -\frac{t^2}{B^{3/2}} + t\frac{2i\hbar}{B^{3/2}E_\omega}\left(1 - \ln 2 - \frac{1}{\sqrt{1+\frac{2E_L}{BE_\omega}}} + \ln\left(1+\sqrt{1+\frac{2E_L}{BE_\omega}}\right)\right) \\
= &\; -\frac{t^2}{\left(1+\frac{2E_L}{E_\omega}\right)^{3/2}} + t\frac{2i\hbar}{\left(1+\frac{2E_L}{E_\omega}\right)^{3/2}E_\omega}\left(1 - \ln 2 - \sqrt{1+\frac{2E_L}{E_\omega}} + \ln\left(1+\sqrt{\frac{1}{1+\frac{2E_L}{E_\omega}}}\right)\right).
\end{aligned} \tag{F.5}$$

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2. W. Lim, V. Sa-yakanit, H. R. Glyde and W. Sritrakool "*Bose-Einstein Condensation in a Disordered System: Feynman Path Integral Approach*", to be published, *Mod. Phys. Lett. B*, (2007).

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