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ROBUST PID CONTROLLER DESIGN FOR BELT CONVEYOR SYSTEM

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ตัวควบคุมแบบสัดส่วน-ปริพันธ์-อนุพันธ์ (ตัวควบคุมพีไอดี) เป็นตัวควบคุมที่ใช้อย่างแพร่หลายในระบบควบคุมอัตโนมัติ ในวิทยานิพนธ์ฉบับนี้เราพิจารณาปัญหาการออกแบบตัวควบคุมพีไอดีเพื่อประกันเสถียรภาพและสมรรถนะคงทน สำหรับระบบสัญญาณเข้าหนึ่งสัญญาณ-สัญญาณออกหนึ่งสัญญาณ ภายใต้ความไม่แน่นอนของแบบจำลองคณิตศาสตร์ ก่อนหน้านี้ ปัญหาดังกล่าวสามารถแปลงเป็นปัญหาการทำให้เสถียรพร้อมกันระหว่างพหุนามลักษณะเฉพาะของวงปิด กับวงศ์ของพหุนามเชิงซ้อน เมื่อกำหนดอัตราสัดส่วนของตัวควบคุมที่ค่าหนึ่ง อัตราขยายปริพันธ์และอัตราขยายอนุพันธ์สามารถคำนวณได้ทันทีจากการโปรแกรมเชิงเส้น ลักษณะสำคัญของแนงทางนี้คือ การคำนวณได้ผลลัพธ์เป็นเซตของตัวควบคุมทั้งหมดที่ยอมรับได้ นั่นคือสอดคล้องกับเงื่อนไขการออกแบบ

งานวิจัยนี้ยังนำเสนอระเบียบวิธ๊สำหรับการทำให้เสถียรของพหุนาม โดยเฉพาะอย่างยิ่ง โปรแกรม MATLAB ได้รับการพัฒนาขึ้นเพื่อหา อัตราขยายพีไอดีที่เหมาะสม เราได้ทดสอบกับระบบตัวอย่างที่ครอบคลุมปัญหาเสถียรภาพ ณ สภาวะระบุไปจนถึงปัญหาเสถียรภาพและสมรรถนะคงทน โปรแกรมคอมพิวเตอร์ประยุกต์ใช้แนวทางการทำให้เสถียรของพหุนามอย่างมีเอกภาพและได้ขั้นตอนมีระเบียบแบบแผน ต่อมาเราใช้โปรแกรมที่พัฒนาขึ้น ออกแบบตัวควบคุมพีไอดีสำหรับระบบสายพานลำเลียงในขนาดของห้องปฏิบัติการ หลังจากป้อนข้อมูลแบบจำลองพลวัตและความไม่แน่นอน โปรแกรมจะแสดงผลลัพธ์เป็นบริเวณที่ยอบรับได้ของพารามิเตอร์พีไอดีที่สอดคล้องกับเงื่อนไขเสถียรภาพ ณ สภาวะระบุและสมรรถนะคงทน ตามลำดับ บริเวณที่ยอบรับได้ของพารามิเตอร์พีไอดีสามารถแสดงเป็นลักษณะสองมิติ (เมื่อกำหนดให้ k_p เป็นค่าคงที่ค่าหนึ่ง) หรือ สามมิติ (เมื่อกำหนดให้ k_p แปรค่าได้หลายค่า) การจำลองผลด้วยคอมพิวเตอร์ยืนยันว่าตัวควบคุมพีไอดีที่เลือกขึ้นมา ให้ผลตอบสนองเป็นที่น่าพอใจ ทั้งสมรรถนะ ณ สภาวะระบุและสมรรถนะคงทน เมื่อเปรียบเทียบผลลัพธ์ที่ได้กับตัวควบคุมจากวิธี Ziegler-Nichols สังเกตว่าตัวควบคุมจากวิธี Ziegler-Nichols มีค่าอยู่ใกล้กับขอบของบริเวณที่ยอมรับได้ ดังนั้นโปรแกรมคอมพิวเตอร์ที่พัฒนาขึ้นนับเป็น เครื่องมือที่มีประสิทธิผลและเป็นประโยชน์ต่อการปรับจูนค่าพีไอดีคงทน

ภาควิชา ....วิศวกรรมไฟฟ้า.... **ลายมือชื่อนิสิต** ....
สาขาวิชา ....วิศวกรรมไฟฟ้า.... **ลายมือชื่ออาจารย์ที่ปรึกษา** ....
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KEY WORD: PID TUNING / LINEAR PROGRAMMING / ROBUST PERFORMANCE / BELT CONVEYOR SYSTEM

ADDY WAHYUDIE: ROBUST PID CONTROLLER DESIGN FOR BELT CONVEYOR SYSTEM, THESIS ADVISOR: DAVID BANJERDPONGCHAI, Ph.D., 59+xi pp., ISBN 974-17-3488-3

Proportional-integral-derivative (PID) controllers have been widely used in automatic control systems. In this thesis, we consider the problem of synthesizing PID controllers which guarantee robust stability and performance for single-input single-output (SISO) plants in the presence of model uncertainty. It is previously shown that this problem can be translated to simultaneous stabilization of the closed-loop characteristic polynomial and a family of complex polynomials. For a fixed proportional gain, integral and derivative gain values can be constructively determined using linear programming. The most important feature of this method is that it computationally characterizes the entire set of the admissible PID gain values.

This research work also provides an algorithm of polynomial stabilization. In particular, MATLAB programs are developed to design appropriate PID parameters. We verify design results on sample problems spanning from nominal stability to to robust stability and performance. The computer programs employ the unified and systematic approach using polynomial stabilization. Subsequently, we apply the developed programs to design PID controllers for a laboratory-scale belt conveyor system. Based on the dynamical model and its uncertainties, we characterize admissible regions satisfying nominal stability and robust performance. The admissible regions of PID gains are shown both in 2D plot at specified value of k_p, and in 3D plot for various values of k_p. The computer simulations confirm that the chosen PID robust controller yields satisfactory nominal and robust performance. Comparing the design result with the well-known Ziegler-Nichols method. it is observed that the PID controller by Ziegler-Nichols method is quite close to the boundary of the admissible region obtained from this work. Hence, the developed computer programs provide a viable and practical means for robust PID tuning.

Department ....Electrical Engineering....	**Student's signature** ....
Field of study ....Electrical Engineering....	**Advisor's signature** ....
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I am indebted to Professor Manop wongsaisuwan and Professor Naebboon Hoonchareon for being my orals and reading committee. Their inputs, helpful suggestions, and comments on the proposal exam are invaluable for the next step of my research

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Many professors in the department of Electrical Engineering at Gadjah Mada University kindly helped me come to Chulalongkorn university. In this, I am thankful to Professor Thomas Sri Widodo, and T. Haryono. My entire study at Chulalongkorn was financially supported by AUN/SEED-NET scholarship, which is gratefully acknowledged.

I am indebted and deeply thanks to my parent, Muhammad Hanafiah and Fauziah; brothers, Eddy Hariyadie, Yudhi Mahendra, and Hendrawan for their years of unfailling love, caring and support. Mina Aula Dinah, for being the best part in my past and future. It is to them that I dedicate this thesis.

Lastly, I am foremost grateful to ALLAH; for giving me the Islam as the light of my life.

Those who remember ALLAH (always, and in prayers) standing, sitting, and lying down on their side and think deeply about creation of the heavens and the earth, (saying): “Our LORD! You have not created (all) this without purpose, glory to You! (Exalted are You above all that they associate with You as patners). Give us salvation from the torment of the Fire.” [Al Qur’an 3:(191)]

Contents

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List of Symbols

$\mathcal{R}$	The set of real numbers
Δ	Uncertainty block
$\|.\|$	Absolute value
$\|\|.\|\|_\infty$	The $\mathcal{L}_\infty$ norm
H_∞	Hardy space of essentially bounded functions on the imaginary axis, with analytic continuation into right half-plane
j	Imaginary unit
$:=$	“is defined as”
$\in$	“belong to”
$\forall$	“for all”
$\triangleq$	“equal by definition”
$\mapsto$	“maps to”
$\square$	The end of the proof

Chapter 1

Introduction

1.1 Motivation

The majority of control systems in this world are operated by PID controllers. Indeed, it has reported that 98% of the control loops in the pulp and paper industries are controlled by SISO PI controllers [1] and that in the process control applications, more than 95% of the controllers are of the PID type [2]. Similar statistic holds in the motion control and aerospace industries.

Given the widespread industrial use of PID controllers, it is clear that even a small percentage improvement in PID design could have a tremendous impact worldwide. Despite this, it is unfortunate that currently there is not much theory dealing with PID design. Indeed, most of the industrial PID designs are still carried out using only empirical techniques, and the mathematically elegant and sophisticated theories developed in the context of modern optimal control cannot be applied to them. Meanwhile, belt conveyor systems have been used in many industrial applications, especially in manufacture industries for transporting material. Because of these reasons, we are interested in doing research in synthesis of robust PID controllers for conveyor system, by the method which is simple and computationally efficient so that it can be applied in real applications.

1.2 Literature Review and Previous Work

PID controller is the most widely used controller structure in industrial applications. Its structural simplicity and sufficient ability of solving many practical control problem have greatly contributed to this wide acceptance. Over past decades, many PID design techniques have been proposed for industrial use. Collection of these various methods can be found in the book by Åström and Hägglund [2]. Most of these control techniques are based on simple characterization of process dynamics, such as the characterization by a first order model with a time delay. In spite of this, for plants having higher order, there exist few generally accepted design methods.

When dealing with mathematical model, the problem that always facing by the control system engineer is model uncertainty which can bring us to the robust stability and robust performance problem. Robust control methods in [3, 4, 5] can be used for solving this problem. The problem of robust performance design is to synthesize a controller for which the closed-loop system is internally stabilized and the desired performance specifications are satisfied despite of plant model uncertainty. H_∞ and μ-synthesis techniques have been successfully applied to solve the problem of robust performance design. H_∞ control aims to guarantee the worst case performance of the uncertain system, and H_∞ controllers are obtained from the solution of two coupled Riccati equations. One of the methods in the class of H_∞ control is loop shaping method [6]. The idea of H_∞ loop shaping method is to design a controller that minimizes the signal transmission from load disturbances and measurement noise to plant input and output by expressed H_∞ norm. After that, we shape open loop transfer function of the system which will meet our design performance and robustness by using precompensator and post compensator, in high and low frequency, then the final controller is constructed by combining H_∞ controller with the shaping functions of precompensator and post compensator.

Many papers dealt with H_∞ control for their PID parameters design. In [7], they cast problem for finding H_∞ loop shaping into the non-convex optimization problem. In [8], they used H_∞ loop shaping for design PID controller for lower order plant with time delay. In [9], they derived one order system with time delay for finding PID parameter based on H_∞ theory. For application in the power system [10], H_∞ control is used to control a boiler-turbine unit. The paper [11] presented an application of H_∞ theory for a conveyor system with the tracking specification. They setup experiment for various speed of the conveyor system, test their controller, and also suggest a method for compensating the saturation.

In this thesis, we focus on the problem of synthesizing a stabilizing PID controller, if any, for which the disturbance rejection design specification is achieved for a plant with multiplicative uncertainty. In the papers [12, 13, 14], based on the generalized Hermite-Biehler theorem [15, 16, 17], a computational characterization of all stabilizing PID controllers was given for an arbitrary nominal polynomial plant. This solution of the PID stabilization is an essential first step to any rational design of PID controllers. Recently, an extension of PID stabilization to the case of complex polynomials was developed in [18, 19, 20, 21] and it was shown that such an extension could be exploited to carry out many H_∞ robust stability PID design problems. In this thesis, we show that the results from [18, 19, 20, 21] can be also used to provide a computational characterization of all admissible PID controllers for robust performance. Such a characterization for all admissible PID controllers involves the solution of linear programming problems. Accordingly, an efficient algorithm has been developed for generating the parametric space of entire admissible PID gain values. Then in order to implement this algorithm, we develop MATLAB program for synthesizing PID controllers. Finally, we will use this approach for finding PID controller for belt conveyor system.

1.3 Objective

The primary objective of this research is to synthesize proportional-integral-derivative (PID) controllers for single-input single-output (SISO) plants in the presence of model uncertainty. The secondary objective is that these robust PID controllers are designed and applied to belt conveyor systems.

1.4 Scope of Thesis

1. Construct the design method for tuning robust PID controller.
2. Implement a computer program for tuning robust PID controller.

1.5 Research Procedure

1. Study related literature on PID controller and robust PID controller.
2. Study the model of belt conveyor system.
3. Solve the design problem within the framework of robust PID controller.
4. Develop a computer program for tuning robust PID controller.
5. Compare the result with other methods.
6. Apply the PID design technique to belt conveyor system.
7. Write the thesis and organize all documents.

1.6 Contribution

1. Robust PID controllers for belt conveyor systems.
2. A computational tool for robust PID tuning.

1.7 Thesis Outline

The organization of the thesis is as follows. In chapter 2, we briefly summarize the basic system theories which will be used to formulate design problem. We will present the definition of nominal performance. Then, we will present the multiplicative uncertainty which is a model to represent our model uncertainty. We will continue the discussion on robust stability and robust performance. We will also give the graphical interpretation of the Nyquist plot on the important equations.

In chapter 3, we will discuss the algorithm for finding the PID gain which can make a given nominal plant stable. The derivation of the procedure leads into real polynomial stabilization. The outcome of our procedure is the whole region of PID gains which can make the nominal system stable. Next, we consider the uncertainty in the model, and expect that PID controller can stabilize the system and satisfy performance specification. Fortunately, we can still use the basic idea in real polynomial stabilization and generalize our algorithm to cope with robust control. As we will see later, this generalization leads to the problem of complex polynomial stabilization.

Some numerical examples on this method can be found in chapter 4. Step-by-step implemetations of the algorithm given in chapter 3 is presented in this chapter. Examples on PID stabilization for nominal system can be found here. The proof of its nominal stability and the comparison result with the well-known Ziegler-Nichols method will be presented in this chapter. Moreover, we will give an example for PID controller synthesis of nominal system satisfying a performance in form of H_∞ norm. Finally, a numerical example for robust PID controller synthesis satisfying robust performance specification will be given.

In chapter 5, we will present mathematical models of servo-driven belt conveyor system. The discussion starts with a nominal model of servo-driven belt conveyor and model uncertainty of this system. Then, we find the PID controller gains which stabilize the servo-driven belt conveyor. Next, we carry on the procedure for finding robust PID controller satisfying our design performance specification. Some tests and computer simulations are performed.

In chapter 6, we present the conclusions of this research. The approach of polynomial stabilization is applicable to the other open control applications. Moreover, some suggestions on extensions to the algorithm and potential topics for future research are presented.

Chapter 2

Mathematical Preliminaries

This chapter briefly summarizes the basic system theories which are used in this thesis. In the section §2.1, we present the definition of nominal performance. Then in §2.2, we present the multiplicative uncertainty, which is the model to represent model uncertainty of belt conveyor system. We continue discussion to the robust stability in §2.3. Finally, we discuss about robust performance in §2.4. In some sections, we also give the graphical interpretation based on the Nyquist plot for important equations.

2.1 Nominal Performance

Consider the block diagram follow:

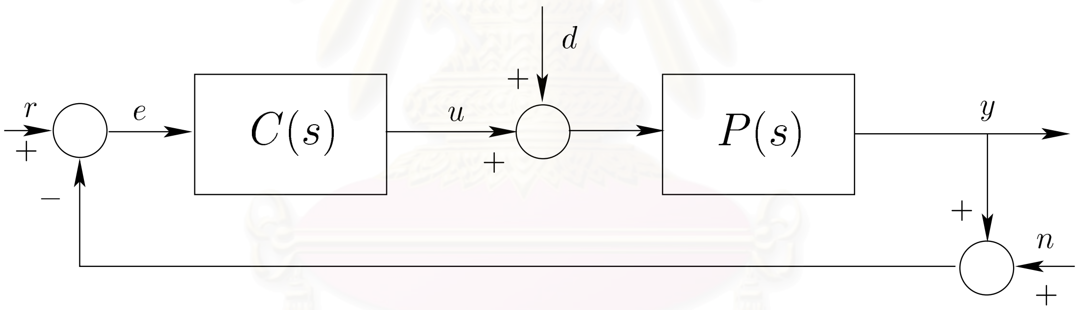

Figure 2.1: Unity-feedback system.

Let L denote the loop transfer function, $L := PC$. The transfer function from the reference input r to tracking error e is

$$S := \frac{1}{1+L},$$

called the *sensitivity function*. The name *sensitivity function* comes from the following idea. Let T denote the transfer function from r to y:

$$T = \frac{PC}{1+PC}$$

One way to quantify how sensitive T is to variations in P is to take the limiting ratio of a relative perturbation in T (i.e., $\Delta T/T$) to a relative perturbation in P (i.e., $\Delta P/P$). Thinking of P as a variable and T as a function of it, we get

$$\lim_{\Delta P \to 0} \frac{\Delta T/T}{\Delta P/P} = \frac{dT}{dP}\frac{P}{T}.$$

The right-hand side is easily evaluated to be S. In this way, S is the sensitivity of the closed-loop transfer function T to an infinitesimal perturbation in P.

Now we have to decide on a performance specification, a measure of tracking performance. This decision depends on two factors: what we know about r and what measure we choose to assign to the tracking error. Usually, r is not known in advance$-$ few control systems are designed for one and only one input. Rather, a set of possible rs will be known or at least postulated for the purpose of design.

Let us first consider sinusoidal inputs. Suppose that r can be any sinusoid of amplitude ≤ 1 and we want e to have amplitude $< \epsilon$. Then the performance specification can be expressed succinctly as

$$||S||_\infty < \epsilon.$$

The maximum amplitude of e equals the H_∞ norm of the transfer function. Or if we define the weighting function $W_1(s) = 1/\epsilon$, then the performance specification is $||W_1 S||_\infty < 1$.

There is a nice graphical interpretation of the norm bound $||W_1 S||_\infty < 1$. Note that

$$\begin{aligned} ||W_1 S||_\infty < 1 &\iff \left| \frac{W_1(j\omega)}{1 + L(j\omega)} \right| < 1, \qquad \forall\omega \\ &\iff | \, W_1(j\omega) \, | < | \, 1 + L(j\omega) \, |, \qquad \forall\omega. \end{aligned}$$

The last inequality says that at every frequency, the point $L(j\omega)$ on the Nyquist plot lies outside the disk of center -1, radius $| \, W_1(j\omega) \, |$. See Fig. 2.2.

2.2 Multiplicative Perturbation

Consider Fig. 2.3, suppose that the nominal plant transfer function is P and consider perturbed plant transfer function of the form $\tilde{P} = (1 + \Delta W_2)P$. Here W_2 is a fixed stable transfer function, named as the uncertainty weight, and Δ is a variable stable transfer function satisfying $||\Delta||_\infty \leq 1$. Furthermore, it is assumed that no unstable poles of P are canceled in forming of $\tilde{P}$ (Thus, P and $\tilde{P}$ have the same unstable poles). Such a perturbation Δ is said to be *allowable*.

The idea behind this uncertainty model is that ΔW_2 is normalized plant perturbation away from 1:

$$\frac{\tilde{P}}{P} - 1 = \Delta W_2.$$

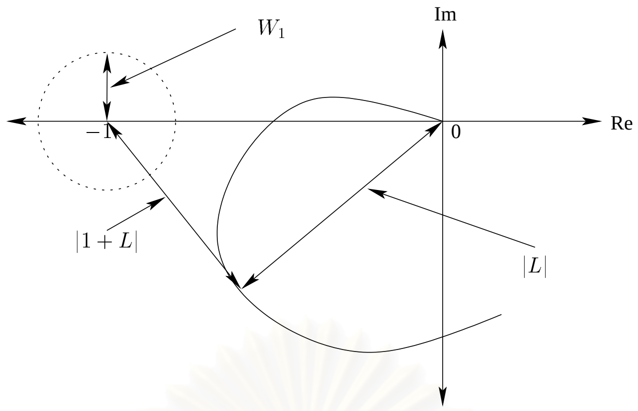

Figure 2.2: Graphical interpretation of nominal performance.

Hence if $||\Delta||_\infty \leq 1$, then

$$\left|\frac{\tilde{P}(j\omega)}{P(j\omega)} - 1\right| \leq |W_2(j\omega)|, \qquad \forall\omega,$$

so, $|W_2(j\omega)|$ provides the uncertainty profile. This inequality describes a disk with center 1, radius $|W_2|$. Typically, $|W_2(j\omega)$ is a (roughly) increasing function of ω, uncertainty increases with increasing frequency. The main purpose of Δ is to account for phase uncertainty and to act as a scaling factor on the magnitude of the perturbation, i.e., $|\Delta|$ is varies between 0 and 1.

2.3 Robust Stability

The notion of the robustness can be described as follows. Suppose that the plant transfer function P belongs to set $\mathcal{P}$. Consider some characteristic of the feedback system, for example, that is internally stable. A controller C is *robust* with respect to this characteristic if this characteristic holds for every plant in $\mathcal{P}$. Notion of robustness therefore requires a controller, a set of plants, and some characteristic of the system. For us, the two most important

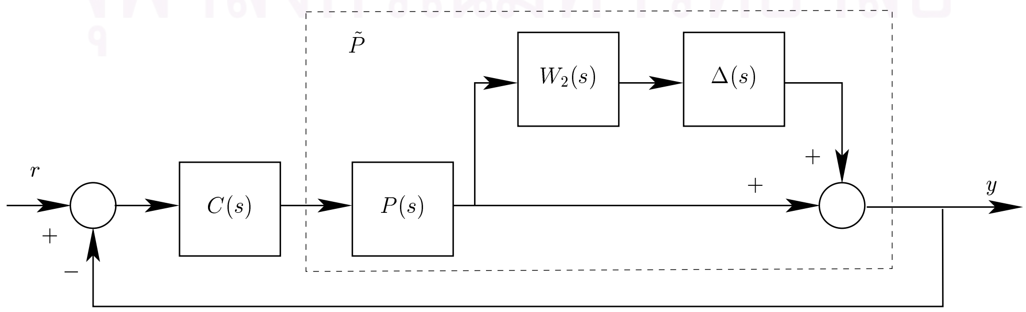

Figure 2.3: Multiplicative perturbation

variations of this notion are robust stability, treated in this section, and robust performance, treated in the next.

A controller C provides *robust stability* if it provides internally stability for every plant in $\mathcal{P}$. We might like to have a test for robust stability, a test involving C and $\mathcal{P}$. Or if $\mathcal{P}$ has an associated size, the maximum size such that C stabilizes all of $\mathcal{P}$ might be a useful notion of stability margin.

The Nyquist plot gives information about stability margin. Note that the distance from the critical point -1 to the nearest point on the Nyquist plot of L equals $1/||S||_\infty$:

$$\begin{aligned}\text{distance from } -1 \text{ to Nyquist plot} &= \inf_\omega \mid -1 - L(j\omega) \mid \\ &= \inf_\omega \mid 1 + L(j\omega) \mid \\ &= \left[\sup_\omega \frac{1}{\mid 1 + L(j\omega) \mid}\right]^{-1} \\ &= ||S||_\infty^{-1}.\end{aligned}$$

Thus if $||S||_\infty \gg 1$, the Nyquist plot comes to the critical point, and the feedback system is nearly unstable.

Now we look at a typical robust stability test, one for the multiplicative perturbation model. Assume that the nominal feedback system (i.e., with $\Delta = 0$) is internally stable for controller C. Bring in again the complementary sensitivity function

$$T = 1 - s = \frac{L}{1+L} = \frac{PC}{1+PC}.$$

Theorem 2.1 *(Multiplicative uncertainty model) C provides robust stability if and only if* $||W_2T||_\infty < 1$.

The proof of this theorem can be found in [3].

The condition $||W_2T||_\infty < 1$ also has a nice graphical interpretation. Note that

$$\begin{aligned}||W_2T||_\infty < \iff & \left|\frac{W_2(j\omega)L(j\omega)}{1+L(j\omega)}\right| < 1, \qquad \forall\omega \\ \iff & \mid W_2(j\omega)L(j\omega) \mid < \mid 1 + L(j\omega) \mid, \qquad \forall\omega.\end{aligned}$$

The last inequality says that at every frequency, the critical point, -1, lies outside the disk of center $L(j\omega)$, radius $\mid W_2(j\omega)L(j\omega) \mid$. See Fig. 2.4.

2.4 Robust Performance

Now, we look into performance of the perturbed plant. Suppose that the plant transfer function belong to a set $\mathcal{P}$. The general notion of the *robust performance* is that internal stability

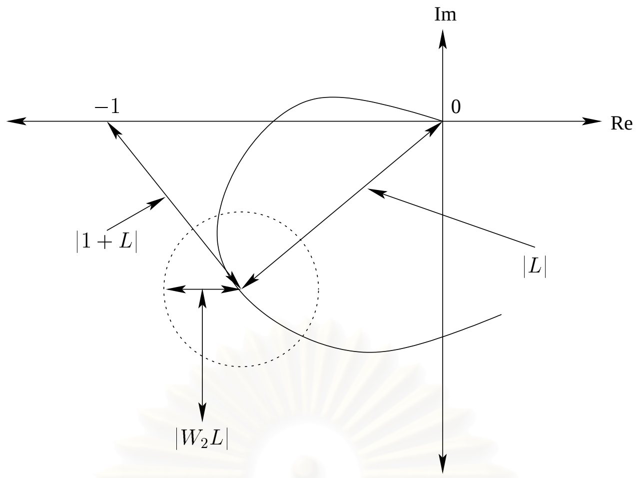

Figure 2.4: Graphical interpretation of robust stability.

and performance, of a specified type, should hold for all plants in $\mathcal{P}$. Again we focus on multiplicative perturbations.

Recall that when the nominal feedback system is internally stable, the *nominal performance* condition is $||W_1S||_\infty < 1$ and the *robust stability* condition is $||W_2T||_\infty < 1$. If P is perturbed to $(1 + \Delta W_2)P$, S is perturbed to

$$\frac{1}{1 + (1 + \Delta W_2 T)L} = \frac{S}{1 + \Delta W_2 T}.$$

Clearly, the robust *robust performance* condition should therefore be

$$||W_2T||_\infty \text{ and } \left|\left| \frac{W_1S}{1 + \Delta W_2 T} \right|\right|_\infty < 1, \quad \forall \Delta.$$

Here Δ must be allowable. The next theorem gives a test for robust performance in terms of the function

$$s \mapsto |W_1(s)S(s)| + |W_2(s)T(s)|,$$

which is denoted as $|W_1S| + |W_2T|$.

Theorem 2.2 *A necessary and sufficient condition for robust performance is*

$$||\ |W_1S| + |W_2T|\ ||_\infty < 1. \tag{2.1}$$

The proof of this theorem can be found in [3].

Test (2.1) also has a nice graphical interpretation. For each frequency ω, construct two disks: one with center -1, radius $|W_1(j\omega)|$; the other with center $L(j\omega)$, radius $|W_2(j\omega)L(j\omega)|$. Then (2.1) holds if and only if for each ω these two disks are disjoint. See Fig. 2.5.

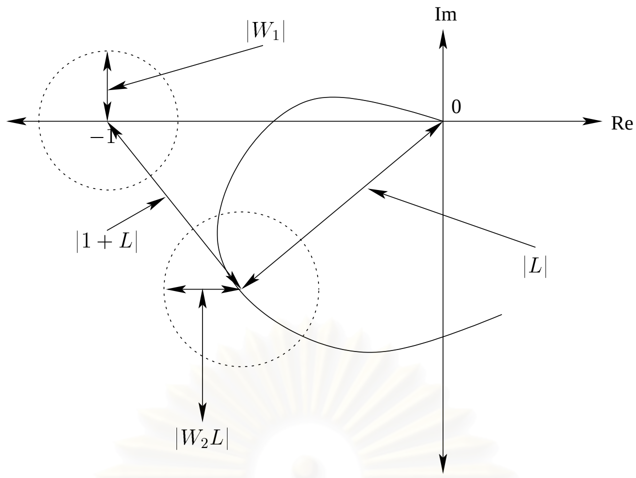

Figure 2.5: Graphical interpretation of robust performance.

2.5 Summary

Chapter 2 outlines some of the key definitions and the mathematical preliminaries which useful for setup our problem formulation. In this thesis, we consider the plant model uncertainty in the form of multiplicative uncertainty. The graphical interpretation norm of bound $||W_1S||_\infty < 1$, representing the nominal performance specification of the system, is that at every frequency, the point $L(j\omega)$ on Nyquist plot lies outside the disk of center -1, radius $|W_1(j\omega)|$. The norm bound of robust stability $||W_2T||_\infty < 1$ has the graphical meaning that at every frequency, the critical point(-1) on the Nyquist plot, lies outside the disk of center $L(j\omega)$, radius $|W_2(j\omega)L(j\omega)|$. The robust performance criterion is the combination of nominal performance and robust stability criterion in the form of $|| \, |W_1S| + |W_2T| \, ||_\infty < 1$ also has graphical interpretation is that disk with center -1, radius $|W_1(j\omega)|$; and the other disk with center $L(j\omega)$, radius $|W_2(j\omega)L(j\omega)|$, are disjoint.

Chapter 3

Controller Design via Simultaneous Polynomial Stabilization

In this chapter, we will discuss the algorithm for finding the PID gain which stabilizes a given nominal plant. The derivation of the procedure can lead us into real polynomial stabilization. The outcome of our procedure is the whole region of PID gains which can make the nominal system stable. If we put some performances in system, and also consider about the uncertainty in the model, and expect that PID controller can make the system stable and satisfy our design specification, then we will deal with robust control problem. Fortunately, we can still use the basic idea in real polynomial stabilization and generalize our algorithm to cope with robust control. As we will see later, this generalization leads us into the problem of complex polynomial stabilization.

The organization of this chapter is as follows. In §3.1, we will discuss about PID stabilization for a nominal plant. Later on, in §3.2 we will extend our works on synthesis PID controller for the plant which has uncertainty in the model, as well as satisfid a given performance. The algorithm for finding PID gain on each case will be also presented.

3.1 PID Stabilization for a Nominal Plant

We would like to present a generalization of the Hermite-Biehler theorem for the case coefficients of the polynomial are real. This theorem is useful later on as stability condition for our PID stabilization. The complete and detail works on this theorem can be found in [22, 15, 17]. First, we introduce the standard signum $\mathcal{R} \to \{-1, 0, 1\}$ defined by

$$\mathrm{sgn}[x] = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Let

$$\delta(s) = \delta_0 + \delta_1 s + \ldots + \delta_a s^d$$

be a given real polynomial of degree d. We write

$$\delta(s) = \delta_e(s^2) + s\delta_o(s^2)$$

where $\delta_e(s^2)$ and $s\delta_o(s^2)$ are the components of $\delta(s)$ made up of even and odd powers of s, respectively. We substitute s with $j\omega$ and decompose $\delta(j\omega)$ as "real-imaginary" decomposition, i.e.,

$$\delta(j\omega) = p(\omega) + jq(\omega)$$

where

$$p(\omega) = \delta_e(-\omega^2) \text{ and, } q(\omega) = \omega\delta_o(-\omega^2).$$

Furthermore, define

$$\delta_f(j\omega) = p_f(\omega) + jq_f(\omega)$$

where

$$p_f = \frac{p(\omega)}{f(\omega)},\ q_f = \frac{q(\omega)}{f(\omega)},\ \text{and } f(\omega) = (1+\omega^2)^{\frac{d}{2}}.$$

Define the number of the open left half plane zeros of $\delta(s)$ as $l(\delta(s))$, and the number of the open right plane zeros of $\delta(s)$ as $r(\delta(s))$. We define the signature of the polynomial $\delta(s)$ by $\sigma(\delta(s))$ as

$$\sigma(\delta(s)) = l(\delta(s)) - r(\delta(s)).$$

Then we can state the following theorem.

Theorem 3.1 *(A generalization of the Hermite-Biehler theorem: real coefficient polynomial case).*
Let $\delta(s)$ be a given real polynomial of degree d with a root at the origin of multiplicity k.
Let $0 = \omega_0 < \omega_1 < \ldots < \omega_{t-1}$ be the real, non-negative, distinct finite zeros of $q_f(\omega)$ with odd multiplicities. Also define $\omega_t = \infty$ and denote $p^{(k)}(\omega_0) = \frac{d^k p(\omega)}{d\omega^k}\Big|_{\omega=\omega_0}$. Then

$$\sigma(\delta) = \begin{cases} \{\mathrm{sgn}[p^{(k)}(\omega_0)] - 2\mathrm{sgn}[p_f(\omega_1)] + 2\mathrm{sgn}[p_f(\omega_2)] + \ldots \\ \quad + (-1)^{t-1}2\mathrm{sgn}[p_f(\omega_{t-1})] + (-1)^t\mathrm{sgn}[p_f(\omega_t)]\} \cdot (-1)^{t-1}\mathrm{sgn}[q(\infty)] \\ \qquad \textit{if d is even.} \\ \{\mathrm{sgn}[p^{(k)}(\omega_0)] - 2\mathrm{sgn}[p_f(\omega_1)] + 2\mathrm{sgn}[p_f(\omega_2)] + \ldots \\ \quad + (-1)^{t-1}2\mathrm{sgn}[p_f(\omega_{t-1})]\} \cdot (-1)^{t-1}\mathrm{sgn}[q(\infty)] \\ \qquad \textit{if d is odd.} \end{cases}$$

Now consider SISO feedback control system shown in Fig. 3.1. Here, r is the command signal and y is output, $G(s) = N(s)/D(s)$ is the plant to be controlled, where $N(s)$ and $D(s)$ are coprime polynomials. $C(s)$ is PID the controller that used for stabilizing the control system, and has a transfer function in the form of $k_p + \frac{k_i}{s} + k_d s$. The closed-loop characteristic polynomial equation of the system is

$$\delta(s, k_p, k_i, k_d) = sD(s) + (k_i + k_d s^2)N(s) + k_p sN(s).$$

The control objective is to determine the values of k_p, k_i and k_d for which the closed-loop characteristic polynomial $\delta(s, k_p, k_i, k_d)$ is Hurwitz. That is, $\delta(s, k_p, k_i, k_d)$ has all roots in the open left half plane.

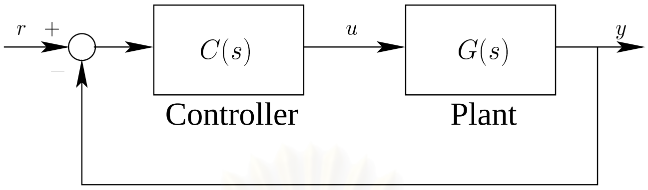

Figure 3.1: Feedback control system.

We observe that for $\delta(s, k_p, k_i, k_d)$, controller parameters k_i, k_d, k_p are distributed on $N(s)$, so that it causes difficulty for finding all stabilizing PID controllers. We will consider the following procedure for solving our design problem. First, we consider the even-odd decompositions of

$$\begin{aligned} N(s) &= N_e(s^2) + sN_o(s^2) \\ D(s) &= D_e(s^2) + sD_o(s^2) \end{aligned}$$

and define

$$N^*(s) = N(-s) = N_e(s^2) - sN_o(s^2).$$

To achieve parameter separation, we multiply $\delta(s, k_p, k_i, k_d)$ by $N^*(s)$ to obtain

$$\begin{aligned} v(s) = \;& \delta(s, k_p, k_i, k_d)N^*(s) \\ = \;& s^2(N_e(s^2)D_o(s^2) - D_e(s^2)N_o(s^2)) + (k_i + k_d s^2)(N_e(s^2)N_e(s^2))s^2N_o(s^2)N_o(s^2)) \\ & + s[D_e(s^2)N_e(s^2) - s^2D_o(s^2)N_o(s^2) + k_p(N_e(s^2)N_e(s^2) - s^2N_o(s^2)N_o(s^2))]. \end{aligned}$$

Let n, m be the degrees of $\delta(s, k_p, k_i, k_d)$ and $N(s)$, respectively. Next, we substitute s with $j\omega$ in $v(s)$, and decompose $v(s)$ into "real-imaginary" decomposition. We get

$$v(j\omega) = \delta(j\omega, k_p, k_i, k_d)N^*(j\omega) = p(\omega, k_i, k_d) + jq(\omega, k_p)$$

where

$$\begin{aligned} p(\omega, k_i, k_d) &= p_1(\omega) + (k_i - k_d\omega^2)p_2(\omega) \\ q(\omega, k_p) &= q_1(\omega) + k_p q_2(\omega) \\ p_1(\omega) &= -\omega^2(N_e(-\omega^2)D_o(-\omega^2) - D_e(-\omega^2)N_o(-\omega^2)) \\ p_2(\omega) &= N_e(-\omega^2)N_e(-\omega^2) + \omega^2N_o(-\omega^2)N_o(-\omega^2)) \\ q_1(\omega) &= \omega(D_e(-\omega^2)N_e(-\omega^2) + \omega^2D_o(-\omega^2)N_o(-\omega^2)) \\ q_2(\omega) &= \omega(N_e(-\omega^2)N_e(-\omega^2) + \omega^2N_o(-\omega^2)N_o(-\omega^2)). \end{aligned}$$

Note that polynomial $v(j\omega)$ exhibits parameter separation, that is, k_p appears in the imaginary term (or q term) only, while k_i and k_d appear in the real term (or p term) only. $\delta(s, k_p, k_i, k_d)$ is Hurwitz if and only if $v(s)$ has exactly the same number of closed RHP zeros as $N^*(s)$.
Let $\omega_0,\ \omega_1,\ \omega_2,\ ...,\ \omega_{t-1}$ denote the real, nonnegative distinct roots of $q(\omega, k_p)$ of odd multiplicity, where

$$0 = \omega_0 < \omega_1 < \omega_2 < ... < \omega_{t-1}$$

and define $\omega_t = \infty$. By using Theorem 3.1, the stability condition is that:

$$n-(l(N(s))-r(N(s))) = \begin{cases} \{\mathrm{sgn}[p^{(k)}(\omega_0)] - 2\mathrm{sgn}[p_f(\omega_1)] + 2\mathrm{sgn}[p_f(\omega_2)] + ... \\ + (-1)^{t-1}2\mathrm{sgn}[p_f(\omega_{t-1})] + (-1)^t\mathrm{sgn}[p_f(\omega_t)]\} \\ \cdot (-1)^{t-1}\mathrm{sgn}[q(\infty)] \\ \qquad \text{if } m+n \text{ is even} \\ \{\mathrm{sgn}[p^{(k)}(\omega_0)] - 2\mathrm{sgn}[p_f(\omega_1)] + 2\mathrm{sgn}[p_f(\omega_2)] + ... \\ + (-1)^{t-1}2\mathrm{sgn}[p_f(\omega_{t-1})]\} \cdot (-1)^{t-1}\mathrm{sgn}[q(\infty)] \\ \qquad \text{if } m+n \text{ is odd} \end{cases} \tag{3.1}$$

From the stability condition in (3.1), a necessary condition is that $q(\omega, k_p)$ has at least

$$\begin{cases} \frac{|n-(l(N(s))-r(N(s)))|}{2} & \text{for } m+n \text{ even} \\ \frac{|n-(l(N(s))-r(N(s)))|+1}{2} & \text{for } m+n \text{ odd} \end{cases} \tag{3.2}$$

real, nonnegative, distinct roots of odd multiplicity. The ranges of k_p satisfying this condition are called *allowable*. For every fixed k_p in q term, we can determine stabilizing values for k_i and k_d in p term, by the following step:
First, define

$$\mathrm{sgn}[p_f(\omega_b)] = i_b \qquad \text{for } b = 0, 1, ..., t.$$

we can construct sequences of number $i_0, i_1, ..., i_t$ by following rule:

- If $N^*(j\omega_b) = 0$ for some $b = 1, 2, ..., t-1$, then define

$$i_b = 0.$$

- for all other, $b = 0, 1, 2, ..., t$,

$$i_b = (-1 \quad \text{or} \quad 1).$$

With $i_0, i_1, ...$ defined in this way, define the set $A_{(k_p)}$ i.e., the set of all admissible string, satisfying stability condition (3.1), as

$$A_{(k_p)} = \begin{cases} \{i_0, i_1, ..., i_t\} & \text{if } m+n \text{ is even} \\ \{i_0, i_1, ..., i_{t-1}\} & \text{if } m+n \text{ is odd} \end{cases}$$

Next, we determine admissible string of $\mathcal{I} = \{i_0, i_1, ..., i_{t-1} \text{or } i_t\}$ in $A_{(k_p)}$ which satisfies the stability condition (3.1). We note that, each member of set in $\mathcal{I}$, i.e., $\{i_0, i_1, ..., i_{t-1} \text{or } i_t\}$, is associated with the sign of p evaluated at associated frequency ω, or mathematically, we can write as the following linear inequality:

$$\begin{aligned} &[p_1(\omega_b) + (k_i - k_d\omega_b^2)p_2(\omega_b)] > 0, \forall b = 0, 1, 2, ..., t-1 \text{ or } t \qquad \text{if } i_b > 0 \\ &[p_1(\omega_b) + (k_i - k_d\omega_b^2)p_2(\omega_b)] < 0, \forall b = 0, 1, 2, ..., t-1 \text{ or } t \qquad \text{if } i_b < 0 \end{aligned} \tag{3.3}$$

The set of stabilizing k_i, k_d is obtained from intersection of the feasible region of k_i and k_d satisfying linear inequalities (3.3) for the admissible string $\mathcal{I}$. In other word, each string $\mathcal{I}$ in $A_{(k_p)}$ is one family of the linear inequality problem which can be solved efficiently by linear programming tools. Suppose that we have admissible strings $\mathcal{I}_1, \mathcal{I}_2, ..., \mathcal{I}_v$ in $A_{(k_p)}$ which is associated with feasible regions of k_i, k_d denoted by $S_1, S_2, ..., S_v$, The set of all stabilizing k_i and k_d corresponding to the fixed k_p is given by

$$S_{(k_p)} = \cup_{x=1}^{v} S_x.$$

As k_p is varied within allowable value of k_p, we will have the whole feasible region of stabilizing k_i, k_d, k_p.

To summarize our discussion, we make the following algorithm for finding PID controllers for stabilization of a given nominal plant.

Step 1. For the given $N(s)$ and $D(s)$, compute the corresponding $p_1(\omega), p_2(\omega), q_1(\omega)$, and $q_2(\omega)$.

Step 2. Determine the allowable ranges of k_p, denote this ranges as P_i for $i = 1, 2, ..., r$. The resulting ranges of k_p are the only ranges of k_p for which stabilizing (k_i, k_d) values may exist.

Step 3. If there is no k_p satisfying 2 then output NO SOLUTION and EXIT.

Step 4. Initialize $j = 1$ and $P = P_j$.

Step 5. Pick a range $[k_l, k_u]$ in P and initialize $k_p = k_l$.

Step 6. Pick the number of the rigid points N and set $\texttt{step} = \frac{1}{N+1}[k_u - k_l]$.

Step 7. Increase k_p as follows: $k_p = k_p + \texttt{step}$. If $k_p > k_u$ the GOTO 14.

Step 8. For fixed k_p in step 7, solve for the real, nonnegative, distinct finite zeros of $q(\omega, k_p)$ with odd multiplicities and denote them by $0 = \omega_0 < \omega_1 < \omega_2 < ... < \omega_{t-1}$. Also define $\omega_t = \infty$.

Step 9. Construct sequences of number i_b for $b = 0, 1, ..., t$ as follows:

- If $N^*(j\omega_b) = 0$ for some $b = 1, 2, ..., t-1$, then define

$$i_b = 0.$$

- for all others, $b = 0, 1, 2, ..., t$,

$$i_b = (-1 \quad \text{or} \quad 1).$$

With $i_0, i_1, ...$ defined in this way, define the set $A_{(k_p)}$ (i.e., the set of all admissible string, satisfying stability condition 3.1) as

$$A_{(k_p)} = \begin{cases} \{i_0, i_1, ..., i_t\} & \text{if } m+n \text{ is even} \\ \{i_0, i_1, ..., i_{t-1}\} & \text{if } m+n \text{ is odd} \end{cases}$$

Step 10. Determine the admissible strings $\mathcal{I} = \{i_0, i_1, ..., i_{t-1}\}$ or $\{i_0, i_1, ..., i_t\}$ in $A_{(k_p)}$. If there is no admissible string then GOTO step 7.

Step 11. For an admissible string $\mathcal{I} = \{i_0, i_1, ..., i_{t-1}\}$ or $\{i_0, i_1, ..., i_t\}$, determine the set of (k_i, k_d) values simultaneously satisfying the following string of linear inequalities:

$$[p_1(\omega_b) + (k_i - k_d\omega_b^2)p_2(\omega_b)]i_b > 0, \forall b = 0, 1, 2, ..., t-1 \text{ or } t$$

Step 12. Repeat 11 for all admissible strings $\mathcal{I}_1, \mathcal{I}_2, ..., \mathcal{I}_v$ in $A_{(k_p)}$, to obtain the corresponding admissible (k_i, k_d) sets $S_1, S_2, ..., S_v$. The set of all stabilizing (k_i, k_d) values corresponding to the fixed k_p is then given by

$$S_{(k_p)} = \cup_{x=1}^{v} S_x.$$

Step 13. GOTO 7.

Step 14. Set $j = j + 1$ and $P = P_j$. If $j \leq r$ GOTO 5;
else, terminate the program.

3.2 Robust PID Tuning for Uncertain System

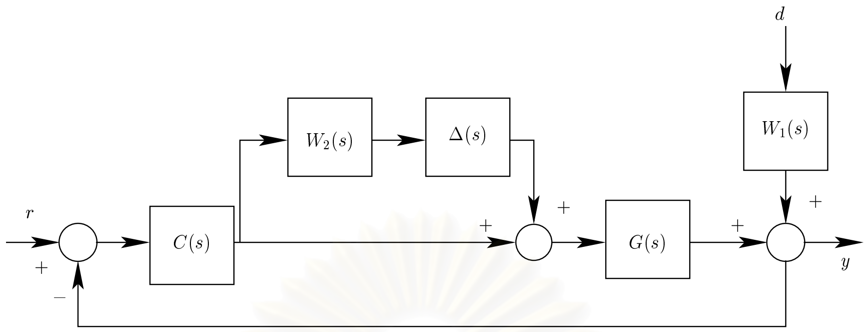

Figure 3.2: Feedback control system with multiplicative uncertainty.

Consider the SISO feedback control system shown in Fig. 3.2. Here, r is the command signal, y is output, and d is energy-bounded disturbance. $G(s) = N(s)/D(s)$ is the plant to be controlled, where $N(s)$ and $D(s)$ are coprime polynomials. $\Delta(s)$ is any stable and proper transfer function with $||\Delta||_\infty \leq 1$. The weights $W_1(s)$ and $W_2(s)$ describe the frequency-domain characteristics of the performance specifications and model uncertainty, respectively. $C(s)$ is PID controller in the form of

$$C(s) = k_p + \frac{k_i}{s} + k_d s = \frac{k_i + k_p s + k_d s^2}{s}.$$

The closed-loop characteristic polynomial is as follows.

$$\rho(s, k_p, k_i, k_d) = sD(s) + (k_i + kp_s + k_d s^2)N(s).$$

Then, the complementary sensitivity function is

$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

and the sensitivity function is

$$S(s) = \frac{1}{1 + C(s)G(s)}.$$

Especially, we consider the problem of the disturbance rejection for the plant with multiplicative uncertainty. This problem can be formulated as the following condition:

$$||| W_1(s)S(s) | + | W_2(s)T(s) |||_\infty < 1 \tag{3.4}$$

where $W_1(s) = N_1(s)/D_1(s)$ and $W_2(s) = N_2(s)/D_2(s)$, and $N_1(s), D_1(s), N_2(s)$ and $D_2(s)$ are some real polynomials.

Our control objective is to design PID controllers that stabilize system and satisfy the robust condition given in (3.4). In order to achieve stability of the nominal system, we need the roots of characteristic equation lies on the left half plane, i.e.,

$$\rho(s, k_p, k_i, k_d) = sD(s) + (k_i + kp_s + k_d s^2)N(s) \quad \text{is Hurwitz.} \tag{3.5}$$

It is obvious that our procedure in previous section could be applied to solve stabilization problem. But, for satisfying robust performance condition in (3.4), we will need the following lemma to convert (3.4) into simultaneous polynomial stabilization.

Lemma 3.1 *Let*

$$\frac{A(s)}{B(s)} = \frac{a_0 + a_1 s + \ldots + a_x s^x}{b_0 + b_1 s + \ldots + b_x s^x}$$

and

$$\frac{E(s)}{F(s)} = \frac{e_0 + e_1 s + \ldots + e_y s^y}{f_0 + f_1 s + \ldots + f_y s^y}$$

be stable and proper rational functions with $b_x \neq 0$ and $f_y \neq 0$. Then

$$\left\| \left| \frac{A(s)}{B(s)} \right| + \left| \frac{E(s)}{F(s)} \right| \right\|_\infty < 1 \tag{3.6}$$

if and only if

a) $|a_x/b_x| + |e_y/f_y| < 1$*;*

b) $B(s)F(s) + e^{j\theta}A(s)F(s) + e^{j\phi}E(s)B(s)$ *is Hurwitz for all θ and $\phi \in [0, 2\pi)$.*

Proof: Necessity. Suppose that (3.6) holds. Then, condition a) is obvious. The necessity of condition b) is established in the following way. Since

$$\left| \frac{A(j\omega)}{B(j\omega)} \right| + \left| \frac{E(j\omega)}{F(j\omega)} \right| = \frac{|e^{j\omega}A(j\omega)F(j\omega)| + |e^{j\phi}E(j\omega)B(j\omega)|}{|B(j\omega)F(j\omega)|}$$

it follows that

$$|B(j\omega)F(j\omega)| > |e^{j\theta}A(j\omega)F(j\omega) + e^{j\phi}E(j\omega)B(j\omega)|.$$

Because $B(s)$ and $F(s)$ are Hurwitz, using Rouche's Theorem [23], we conclude that $B(s)F(s)+e^{j\omega}A(s)B(s) + e^{j\phi}E(s)B(s)$ is Hurwitz for all θ and $\phi \in [0, 2\pi)$.

Sufficiency. Proceeding by contradiction, we assume that conditions a) and b) are true, but

$$\left\| \left| \frac{A(s)}{B(s)} \right| + \left| \frac{E(s)}{F(s)} \right| \right\|_\infty \geq 1.$$

Since $|A(j\omega)/B(j\omega)| + |E(j\omega)/F(j\omega)|$ is a continuous function of ω and

$$\lim_{\omega \to \infty} \left[\left| \frac{A(j\omega)}{B(j\omega)} \right| + \left| \frac{E(j\omega)}{F(j\omega)} \right| \right] = \left| \frac{a_x}{b_x} \right| + \left| \frac{e_y}{f_y} \right| < 1$$

then there must exist at least one $\omega_0 \in \mathcal{R}$ such that

$$\left|\frac{A(j\omega_0)}{B(j\omega_0)}\right| + \left|\frac{E(j\omega_0)}{F(j\omega_0)}\right| = \frac{|A(j\omega_0)F(j\omega_0)| + |E(j\omega_0)B(j\omega_0)|}{|B(j\omega_0)F(j\omega_0)|} = 1.$$

Therefore, it implies that there exist θ_1 and $\theta_2 \in [0, 2\pi)$ such that

$$B(j\omega_0)F(j\omega_0) + e^{j\theta_1}A(j\omega_0)F(j\omega_0) + e^{j\theta_2}E(j\omega_0)B(j\omega_0) = 0$$

and this obviously contradicts condition b). □

If we fit the robust performance condition in (3.4) into the criterion (3.6), we will easily find that the following condition must hold.

$$|W_1(\infty)S(\infty)| + |W_2(\infty)T(\infty)| < 1 \tag{3.7}$$

Define

$$\begin{aligned}\psi(s, k_p, k_i, k_d, \theta, \varnothing) &\triangleq sD_1(s)D_2(s)D(s) + e^{j\theta}sN_1(s)D_2(s)D(s)\\ &\quad + (k_d s^2 + k_p s + k_i)[D_1(s)D_2(s)N(s) + e^{j\phi}D_1(s)N_2(s)N(s)].\end{aligned}$$

The criterion (3.6) also implies that:

$$\psi(s, k_p, k_i, k_d, \theta, \varnothing) \text{ is Hurwitz for all } \theta \text{ and } \varnothing \in [0, 2\pi). \tag{3.8}$$

Then, for solving our robust PID tuning problem, we have to satisfy three conditions simultaneously given in (3.5), (3.7), and (3.8), i.e.,

a) $\rho(s, k_p, k_i, k_d)$ is Hurwitz;

b) $\psi(s, k_p, k_i, k_d, \theta, \varnothing)$ is Hurwitz for all θ and $\varnothing \in [0, 2\pi)$;

c) $\mid W_1(\infty)S(\infty) \mid + \mid W_2(\infty)T(\infty) \mid < 1$.

In condition b), we will have the problem for stabilizing complex polynomial. The following procedure describes how we find stabilizing PID controllers for complex polynomial. Consider the complex polynomial of the form

$$\psi(s, k_p, k_i, k_d, \theta, \varnothing) = L(s) + (k_d s^2 + k_p s + k_i)M(s) \tag{3.9}$$

Set

$$L(s) = sD_1(s)D_2(s)D(s) + e^{j\theta}sN_1(s)D_2(s)D(s)$$

and

$$M(s) = D_1(s)D_2(s)N(s) + e^{j\phi}D_1(s)N_2(s)N(s).$$

Let

$L(s) = (a_0 + jb_0) + (a_1 + jb_1)s + ... + (a_{k-1} + jb_{k-1})s^{k-1} + (a_k + jb_k)s^k,$ with $a_k + jb_k \neq 0$

$M(s) = (c_0+jd_0)+(c_1+jd_1)s+...+(c_{m-1}+jd_{m-1})s^{m-1}+(c_m+jd_m)s^m,$ with $c_m+jd_m \neq 0.$

If $s = j\omega$, we can consider the following “real-imaginary” decomposition of $L(s)$ and $M(s)$:

$$L(s) = L_R(s) + L_I(s)$$

$$M(s) = M_R(s) + M_I(s)$$

where

$$L_R(s) = a_0 + jb_1 s + a_2 s^2 + jb_3 s^3 + ...$$

$$L_I(s) = jb_0 + a_1 s + jb_2 s^2 + a_3 s^3 + ...$$

$$M_R(s) = c_0 + jd_1 s + c_2 s^2 + jd_3 s^3 + ...$$

$$M_I(s) = jd_0 + c_1 s + jd_2 s^2 + c_3 s^3 + ...$$

Define

$$M^*(s) = M_R(s) - M_I(s).$$

Also, let n, m be the degrees of $\psi(s, k_p, k_i, k_d, \theta, \o)$ and $M(s)$, respectively. Multiplying $\psi(s, k_p, k_i, k_d, \theta, \o)$ by $M^*(s)$ and evaluating the resulting polynomial at $s = j\omega$, we obtain

$$\psi(s, k_p, k_i, k_d, \theta, \o)M^*(j\omega) = p(\omega, k_i, k_d) + jq(\omega, k_p)$$

where

$$\begin{aligned}
p(\omega, k_i, k_d) &= p_1(\omega) + (k_i - k_d\omega^2)p_2(\omega)\\
q(\omega, k_p) &= q_1(\omega) + k_p q_2(\omega)\\
p_1(\omega) &= L_R(j\omega)M_R(j\omega) - L_I(j\omega)M_I(j\omega)\\
p_2(\omega) &= M_R^2(j\omega) - M_I^2(j\omega)\\
q_1(\omega) &= \frac{1}{j}[L_I(j\omega)M_R(j\omega) - L_R(j\omega)M_I(j\omega)]\\
q_2(\omega) &= \omega[M_R^2(j\omega) - M_I^2(j\omega)].
\end{aligned}$$

Also, define

$$q_f(\omega, k_p) = \frac{q(\omega, k_p)}{(1 + \omega^2)^{\frac{m+n}{2}}}.$$

Let $m, n, q_f(\omega, k_p)$ be as already defined. Denote ξ as leading coefficient of

$$\psi(s, k_p, k_i, k_d, \theta, \o)M^*(s).$$

For a given fixed k_p, let $\omega_1 < \omega_2 < ..., \omega_{t-1}$ be the real, distinct finite zeros of $q_f(\omega, k_p)$ with odd multiplicities. Also define $\omega_0 = -\infty$ and $\omega_t = +\infty$. Define a sequence of numbers $i_0, i_1, i_2, ..., i_t$ as

$$A_{k_p} := \begin{cases} \{i_0, i_1, ..., i_t\} & \text{if } m+n \text{ is even and } \xi \text{ is purely real, or} \\ & m+n \text{ is odd and } \xi \text{ is purely imaginary.} \\ \{i_1, i_2, ..., i_{t-1}\} & \text{if } m+n \text{ is even and } \xi \text{ is not purely real, or} \\ & m+n \text{ is odd and } \xi \text{ is not purely imaginary.} \end{cases}$$

where

$$\text{for} \quad b = 0, 1, ..., t, \qquad i_b \in \begin{cases} \{-1, 1\}, & \text{if } j\omega_b \text{ is not a } j\omega \text{ axis root of } M^*(s) \\ \{0\}, & \text{otherwise.} \end{cases}$$

Let string $\mathcal{I} = \{i_0, i_1, ..., i_t\}$ or $\{i_1, i_2, ..., i_{t-1}\}$ in $A_{(k_p)}$. Next, we lets $\gamma(\mathcal{I})$ denote the signature associated with the string $\mathcal{I}$ as

$$\gamma(\mathcal{I}) := \begin{cases} \frac{1}{2}\{i_0 \cdot (-1)^{l-1} + 2\sum_{r=1}^{l-1} i_r \cdot (-1)^{l-1-r} - i_l\} \cdot \text{sgn}[q(\infty, k_p)], \\ \text{if } m+n \text{ is even and } \xi \text{ is purely real, or } m+n \text{ is odd and } \xi \text{ is purely imaginary.} \\ \frac{1}{2}\{2\sum_{r=1}^{l-1} i_r \cdot (-1)^{l-1-r}\} \cdot \text{sgn}[q(\infty, k_p)], \\ \text{if } m+n \text{ is even and } \xi \text{ is not purely real, or } m+n \text{ is odd and } \xi \text{ is not purely imaginary.} \end{cases}$$

The set of strings in $A_{(k_p)}$, with a prescribed signature $\gamma = \psi$ is denoted by $A_{(k_p)}(\psi)$. For a given fixed k_p, we also define set of admissible strings for the complex PID stabilization problem as

$$F^*_{(k_p)} = A_{(k_p)}(n + \sigma(m^*(s))).$$

Theorem 3.2 *The extended PID stabilization problem, with a fixed k_p, is solvable for given complex polynomial $L(s)$ and $M(s)$ if and only if the following conditions hold:*

a) $F^*_{(k_p)}$ is not empty, i.e., at least one feasible string exists and

b) There exists a string $\mathcal{I}$, either $\mathcal{I} = \{i_0, i_1, ..., i_l\}$ or $\{i_1, i_2, ..., i_{l-1}\} \in F^*_{(k_p)}$, and values of k_i and k_d such that either $\forall t = 0, 1, ..., l$ or $1, 2, ..., l-1$ for which $M^*(j\omega) \neq 0$

$$[p_1(\omega_t) + (k_i - k_d\omega_t^2)p_2(\omega_t)]i_t > 0. \qquad (3.10)$$

Furthermore, if there exist values of k_i and k_d such that the above condition is satisfied for the feasible strings $\mathcal{I}_1$, $\mathcal{I}_2$,..., $\mathcal{I}_s \in F^*_{(k_p)}$, then the set of stabilizing (k_i, k_p) values corresponding to the fixed (k_p) is the union of the (k_i, k_d) values satisfying (3.10) for $\mathcal{I}_1$, $\mathcal{I}_2$,..., $\mathcal{I}_s$.

From Theorem 3.2, for a fixed k_p, the characterization of all stabilizing (k_i, k_d) values involves the feasible solution of the set of linear inequalities (3.10). Thus, linear programming can be used to characterize the region of all stabilizing (k_i, k_d) values. The set of all stabilizing (k_p, k_i, k_d) values can now be found by simply sweeping over k_p and determining the stabilizing (k_i, k_d) values by using Theorem 3.2 at each k_p. The necessary condition for existence of the feasible string $\mathcal{I}$ such that $\gamma(\mathcal{I}) = n + \sigma(M^*(s))$ can be determined by the number of real and distinct finite zeros of $q_f(\omega, k_p)$. Hence the necessary ranges of k_p over which the sweeping has to be carried out can then be *a priori* narrowed down by determining the ranges of k_p such that $q_f(\omega, k_p)$ has at least

$$\begin{cases} \mid n+\sigma(M^*(s)) \mid -1, & \text{if } m+n \text{ is even and } \xi \text{ is purely real,} \\ & \text{or } m+n \text{ is odd and } \xi \text{ is purely imaginary} \\ \mid n+\sigma(M^*(s)) \mid, & \text{if } m+n \text{ is even and } \xi \text{ is not purely real,} \\ & \text{or } m+n \text{ is odd and } \xi \text{ is not purely imaginary} \end{cases}$$

real, distinct finite zeros with odd multiplicities. The resulting ranges of k_p are the only ranges of k_p for which stabilizing (k_i, k_d) values may exist. Using root locus ideas, the author of [24] proposed a procedure for determining the real roots distribution of $q(\omega, k_p) = q_1(\omega) + k_p q_2(\omega)$ corresponding to the different ranges of k_p.

Our design problem to find PID controllers that stabilize the system, given uncertainty in the model, and satisfy the robust performance, is solvable if and only if there exists gain value (k_p, k_i, k_d) such that conditions (3.5), (3.7), and (3.8) hold. For a given value of k_p, we denote the set of admissible (k_i, k_d) gain values satisfying condition (3.5) to be $\mathcal{X}_{(1,k_p)}$. We also denote he set of admissible (k_p, k_i, k_d) values satisfying condition (3.7) to be $\mathcal{X}_{(2,k_p)}$ where

$$\mathcal{X}_{(2,k_p)} = \cap_{\theta,\phi\in[0,2\pi)}\mathcal{X}_{(2,k_p,\theta,\phi)}.$$

For condition (3.8), we denote the set of admissible (k_i, k_d) values satisfying this condition as $\mathcal{X}_{(3,k_p)}$. Thus, for a fixed k_p, the set of all admissible (k_i, k_d) values for which the robust performance condition (3.4) is satisfied, denoted by $\mathcal{X}_{k_p}$ is given by

$$\mathcal{X}_{k_p} = \cap_{i=1}^{3}\mathcal{X}_{(i,k_p)}.$$

The set of all admissible (k_p, k_i, k_d) values such that the robust performance condition (3.4) is satisfied can now be found by simply sweeping over the necessary range of k_p and determining $\mathcal{X}_{k_p}$ at each stage.

The following algorithm summarizes our discussion for finding PID controllers in the case of complex polynomial.

Step 1. For the given $L(s)$ and $M(s)$, compute the corresponding $p_1(\omega), p_2(\omega), q_1(\omega)$ and $q_2(\omega)$.

Step 2. Determine the range of k_p such that $q(\omega, k_p)$ has at least

$$\begin{cases} \mid n + \sigma(M^*(s)) \mid -1, & \text{if } m+n \text{ is even and } \xi \text{ is purely real,} \\ & \text{or } m+n \text{ is odd and } \xi \text{ is purely imaginary} \\ \mid n + \sigma(M^*(s)) \mid, & \text{if } m+n \text{ is even and } \xi \text{ is not purely real,} \\ & \text{or } m+n \text{ is odd and } \xi \text{ is not purely imaginary} \end{cases}$$

real, distinct finite zeros with odd multiplicities. The resulting ranges of k_p for which stabilizing (k_i, k_d) values may exist. These ranges one called allowable ranges and denote these ranges as P. After that, define these ranges of k_p as $P_i, i = 1, 2, ..., r$.

Step 3. If there is no k_p satisfying Step 2 then output NO SOLUTION and EXIT.

Step 4. Initialize $j = 1$ for $P = P_j$.

Step 5. Pick a range $[k_l, k_u]$ in P and initialize $k_p = k_l$.

Step 6. Pick the number of grid points N and set `step`= $\frac{1}{N+1}[k_u - k_l]$.

Step 7. Increase k_p as follows: $k_p = k_p +$`step`. If $k_p > k_u$ then GOTO Step 14.

Step 8. For fixed k_p in Step 7, solve for the real, distinct finite zeros of $q(\omega, k_p)$ with odd multiplicities and denote them by $\omega_1 < \omega_2 < ..., \omega_{t-1}$ and $\omega_0 = -\infty$ and $\omega_t = \infty$.

Step 9. Construct sequences of number i_b for $b = 0, 1, 2, ..., t$ as follows:

- If $N^*(j\omega_b) = 0$ for some $b = 1, 2, ..., t-1$, then define $i_b = 0$
  else
  $i_b \in \{-1, 1\}$, for all other $b = 1, 2, ..., t-1$
- If $m + n$ is even and ξ is purely real, or $m + n$ is odd and ξ is purely imaginary then define i_0 and $i_t \in \{-1, 1\}$,
  else define $i_0 = 0$ and $i_t = 0$.
  With $i_0, i_1, ..., i_t$ defined in this way, define the set $A_{(k_p)}$ as $A_{(k_p)} := \{i_0, i_1, ..., i_t\}$

Step 10. Determine the *admissible* strings $\mathcal{I} = \{i_0, i_1, ..., i_t\} \in A_{(k_p)}$ such that the following equality holds

$$\gamma(\mathcal{I}) := \begin{cases} \frac{1}{2}\{i_0 \cdot (-1)^{l-1} + 2\sum_{r=1}^{l-1} i_r \cdot (-1)^{l-1-r} - i_l\} \cdot \mathrm{sgn}[q(\infty, k_p)], \\ \text{if } m+n \text{ is even and } \xi \text{ is purely real, or } m+n \text{ is odd and } \xi \text{ is purely imaginary.} \\ \frac{1}{2}\{2\sum_{r=1}^{l-1} i_r \cdot (-1)^{l-1-r}\} \cdot \mathrm{sgn}[q(\infty, k_p)], \\ \text{if } m+n \text{ is even and } \xi \text{ is not purely real, or } m+n \text{ is odd and} \\ \xi \text{ is not purely imaginary.} \end{cases}$$

If there is no admissible string then GOTO Step 7.

Step 11. For an admissible string $\mathcal{I} = \{i_0, i_1, ..., i_t\}$, determine the set of (k_i, k_d) values that simultaneously satisfy the string of linear inequalities

$$[p_1(\omega_b) + (k_i - k_d\omega_b^2)p_2(\omega_b)]i_b > 0$$

$\forall b = 0, 1, ..., t$ for which $i_b \neq 0$.

Step 12. Repeat Step 11 for all admissible strings $\mathcal{I}_1,\ \mathcal{I}_2,\ ...,\ \mathcal{I}_v$ to obtain the corresponding admissible (k_i, k_d) sets $\mathcal{S}_1,\ \mathcal{S}_2,\ ...,\ \mathcal{S}_v$. The set of all stabilizing (k_i, k_d) values coressponding to the fixed k_p is then given by

$$\mathcal{S}_{k_p} = \cup_{x=1}^{v}\mathcal{S}_x.$$

Step 13. GOTO Step 7.

Step 14. Set $j = j + 1$ and $P = P_j$. If $j \leq d$ GOTO Step 5; else, terminate the algorithm.

3.3 Root Locus Technique to Narrow the Sweeping Range for k_p

Consider the problem of determining the root locus of $U(x) + kV(x) = 0$, where $U(x)$ and $V(x)$ are real coprime polynomial and k varies from $-\infty$ to ∞. Then, we make the following observations:

1. The real breakaway points on the root loci of $U(x) + kV(x) = 0$ correspond to real multiple root and must, therefore, satisfy

$$\frac{d\left(\frac{V(x)}{U(x)}\right)}{dx} = 0$$

i.e

$$\frac{U(x)\frac{dV(x)}{dx} - V(x)\frac{dU(x)}{dx}}{U^2(x)} = 0$$

the real breakaway points are real zeros of the above equation.

2. Let $k_1 < k_2 < ... < k_z$ be the distinct, finite values of k corresponding to the real breakaway points x_i, $i = 1, 2, ..., z$ on the root loci of $U(x) + kV(x) = 0$. Also define $k_0 = -\infty$ and $k_{z+1} = \infty$. Then x_i, $i = 1, 2, ..., z$ are multiple real roots of $U(x) + kV(x) = 0$ and the corresponding k's are the k_i's. We note that for $k \in (k_i, k_i + 1)$, the real roots of $U(x) + kV(x) = 0$ are simple and the number of real roots of $U(x) + kV(x) = 0$ is invariant.

3. If $U(x) + kV(x) \neq 0$ for all $k \in (k_i, k_{i+1})$, then the distribution of the real roots of $U(x) + kV(x) = 0$ with respect to the origin is invariant over this range of k values.

The following procedure illustrates how the above observations can be used to determine the distribution of the real roots of $U(x) + kV(x) = 0$ with respect to the origin as k varies from $-\infty$ to ∞.

Let

$$U(x) = -4\omega^9 + 89\omega^7 - 128\omega^5 + 75\omega^3 - \omega$$

and

$$V(x) = \omega^7 + \omega^5 - 3\omega^3 + \omega$$

$$\frac{U(x)\frac{dV(x)}{dx} - V(x)\frac{dU(x)}{dx}}{U^2(x)} = 0$$

The breakaway points x_i, which are the real zeros of the above expression, are:

$$x_1 = 0, x_2 = 0.5621, x_3 = 1.0167, x_4 = 2.7808$$

and the corresponding finite k_i's (arranged in ascending order of magnitude)

$$k_1 = -24.7513, k_2 = -18.5837, k_3 = -6.1855$$

Furthermore, $U(x) + kV(x)$ has a root at the origin when

$$k = k^* = 1$$

Now, for $k \in (k_i, k_{1+1})$ and $k^* \notin (k_i, k_{i+1})$, the distribution of the real roots of $U(x) + kV(x) = 0$ with respect to the origin is invariant. Thus, we can simply check an arbitrary $k \in (k_i, k_{i+1})$ and determine the real root distribution of $U(x) + kV(x) = 0$ with respect to the origin, and this distribution is valid for all k in that interval.

The real root distribution, with respect to the origin of $U(x) + kV(x) = 0$ for k belonging to the different intervals is given below:

$$\begin{aligned}
k \in (-\infty, -24.7513): &\quad \text{1 positive simple root} \\
k \in (-24.7513, -18.5837): &\quad \text{3 positive simple roots} \\
k \in (-18.5837, -6.1855): &\quad \text{5 positive simple roots} \\
k \in (-6.1855, 1): &\quad \text{3 positive simple roots} \\
k \in (1, \infty): &\quad \text{2 positive simple roots}
\end{aligned}$$

3.4 Summary

It has been shown that for finding the stabilizing PID gains for nominal plant, we need to solve the problem in real polynomial stabilization. If we extend our problem for the case of robust control, we need to solve three polynomial equations simultaneously, consist of real polynomial stabilization, complex polynomial stabilization, and one condition equation. By employing the procedure in this chapter, linear programming provides all admissible PID controllers which stabilize system and satisfy robust performance specification. The most important feature of this procedure is that it computationally characterizes the entire set of the admissible PID gain values. The root locus technique can be used for reducing the search are for finding the admissble k_p, so that it will be make the computation time shorter.

Chapter 4

Numerical Examples

In this chapter, we will provide some numerical examples how to implement our procedure in different control setup problem. We will present the numerical examples as consecutive order from nominal control case to the robust control case.

In §4.1, we will show the procedure for finding PID controller to make a given nominal plant stable. In this section, for the reason of clarity, step by step procedure for finding PID controller will be given explicitly. We will also compare the result with the well-known Ziegler-Nichols method. Discussion in this section will be followed by the problem for finding PID controller satisfy performance in the form of H_∞ norm. In §4.2, synthesis of PID controller for a given plant with the presence of uncertainty in the model and satisfy a given performance, will be given. The admissible set of PID gain $(k_p, k_i, k_d,)$ will be shown in 2D and 3D plot.

4.1 PID Stabilization for Nominal Plant

4.1.1 Step by Step Example

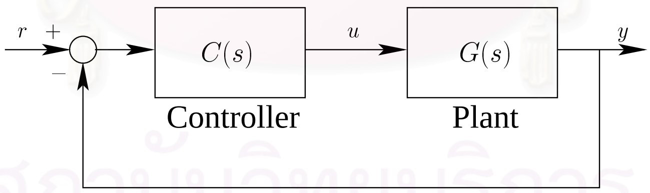

Figure 4.1: Nominal feedback control system.

Consider the problem of determining stabilizing PID gains for the plant $G(s) = \frac{N(s)}{D(s)}$ where

$$\begin{aligned} N(s) &= s^3 - 2s^2 - s - 1 \\ D(s) &= s^6 + 2s^5 + 32s^4 + 26s^3 + 65s^2 - 8s + 1. \end{aligned}$$

The closed-loop characteristic polynomial is

$$\delta(s, k_p, k_i, k_d) = sD(s) + (k_i + k_d s^2)N(s) + k_p s N(s).$$

Thus $n = 7$ and $m = 3$. Also

$N_e(s^2) = -2s^2 - 1, N_o(s^2) = s^2 - 1, D_e(s^2) = s^6 + 32s^4 + 65s^2 + 1, D_o(s^2) = 2s^4 + 26s^2 - 8,$

and

$$N^*(s) = (-2s^2 - 1) - s(s^2 - 1).$$

Multiply $\delta(s, k_p, k_i, k_d)$ with $N^*(s)$. Therefore, we obtain

$$\begin{aligned}\delta(s, k_p, k_i, k_d)N^*s(s) =& [s^2(-s^8 - 35s^6 - 87s^4 + 54s^2 + 9) + (k_i + k_d s^2)(-s^6 + 6s^4 + 3s^2 + 1)] \\ &+ s[(-4s^8 - 89s^6 - 128s^4 - 75s^2 - 1) + k_p(-s^6 + 6s^4 + 3s^2 + 1)]\end{aligned}$$

so that

$$\delta(j\omega, k_p, k_i, k_d)N^*s(j\omega) = [p_1(\omega) + (k_i - k_d\omega^2)p_2(\omega)] + j[q_1(\omega) + k_p q_2(\omega)]$$

where

$$\begin{aligned}p_1(\omega) &= \omega^{10} - 35\omega^8 + 87\omega^6 + 54\omega^4 - 9\omega^2 \\ p_2(\omega) &= \omega^6 + 6\omega^4 - 3\omega^2 + 1 \\ q_1(\omega) &= -4\omega^9 + 89\omega^7 - 128\omega^5 + 75\omega^3 - \omega \\ q_2(\omega) &= \omega^7 + \omega^5 - 3\omega^3 + \omega\end{aligned}$$

For instance, $k_p = -18$, we have

$$\begin{aligned}q(\omega, -18) &= q_1(\omega) - 18q_2(\omega) \\ &= -4\omega^9 + 71\omega^7 - 236\omega^5 + 129\omega^3 - 19\omega.\end{aligned}$$

Then the real, non-negative, distinct finite zeros of $q_f(\omega, -18)$ with odd multiplicities are

$$\omega_0 = 0, \omega_1 = 0.5195, \omega_2 = 0.6055, \omega_3 = 1.8804, \omega_4 = 3.6848.$$

Also define $\omega_5 = \infty$. Since $m + n = 10$ which is even, and $l(N(s)) = 2$ and $r(N(s)) = 1$,

$$l(N(s)) - r(N(s)) = 1$$

and

$$\{-1)^{l-1}\text{sgn}[q(\infty, -18\}] = -1,$$

it follows from Step 10 that every admissible string

$$\mathcal{I} = \{i_0, i_1, i_2, i_3, i_4, i_5\}$$

must satisfy

$$(i_0 - 2i_1 + 2i_2 - 2i_3 + 2i_4 - i_5).(-1) = 6.$$

Hence the admissible strings are

$$\begin{aligned}
\mathcal{I}_1 &= \{-1,-1,-1,1,-1,1\} \\
\mathcal{I}_2 &= \{-1,1,1,1,-1,1\} \\
\mathcal{I}_3 &= \{-1,1,-1,-1,-1,1\} \\
\mathcal{I}_4 &= \{-1,1,-1,1,1,1\} \\
\mathcal{I}_5 &= \{1,1,-1,1,-1,-1\}.
\end{aligned}$$

From Step 11, for I_1 it follows that the stabilizing (k_i, k_d) values corresponding to $k_p = -18$ must satisfy the string of inequalities:

$$\begin{cases}
p_1(\omega_0) + (k_i - k_d\omega_0^2)p_2(\omega_0) < 0 \\
p_1(\omega_1) + (k_i - k_d\omega_1^2)p_2(\omega_1) < 0 \\
p_1(\omega_2) + (k_i - k_d\omega_2^2)p_2(\omega_2) < 0 \\
p_1(\omega_3) + (k_i - k_d\omega_3^2)p_2(\omega_3) > 0 \\
p_1(\omega_4) + (k_i - k_d\omega_4^2)p_2(\omega_4) < 0 \\
p_1(\omega_5) + (k_i - k_d\omega_5^2)p_2(\omega_5) > 0.
\end{cases}$$

Substituting for $\omega_0, \omega_1, \omega_2, \omega_3, \omega_4$ and ω_5 in the above expressions, we obtain

$$\begin{cases}
k_i < 0 \\
k_i - 0.2699k_d < -4.6836 \\
k_i - 0.3666k_d < -10.0797 \\
k_i - 3.5358k_d > 3.912 \\
k_i - 13.5777k_d < 140.2055.
\end{cases}$$

The set of values of (k_i, k_d) for which last condition holds can be solved by linear programming and is denoted by S_1.

For I_2, we have

$$\begin{cases}
k_i < 0 \\
k_i - 0.2699k_d > -4.6836 \\
k_i - 0.3666k_d > -10.0797 \\
k_i - 3.5358k_d > 3.912 \\
k_i - 13.5777k_d < 140.2055.
\end{cases}$$

The set of values of (k_i, k_d) for which the last condition holds can be solved by linear programming and is denoted by S_2.

Similarly, we obtain

$$\begin{cases}
S_3 = \emptyset & \text{for } \mathcal{I}_3 \\
S_4 = \emptyset & \text{for } \mathcal{I}_4 \\
S_5 = \emptyset & \text{for } \mathcal{I}_5.
\end{cases}$$

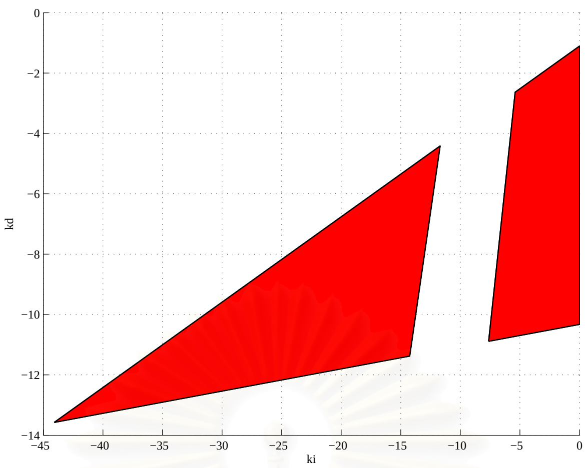

Figure 4.2: The stabilizing set of (k_i, k_d) values when $k_p = -18$.

Then, the stabilizing set of (k_i, k_d) values when $k_p = -18$ is given by

$$\begin{aligned} S_{(-18)} &= \cup_{x=1}^{5} S_x \\ &= S_1 \cup S_2. \end{aligned}$$

The set $S_{(-18)}$ and the corresponding S_1 and S_2 are shown in Fig. 4.2.

By sweeping over different k_p values and repeating the above algorithm at each stage, we can generate the set stabilizing (k_p, k_i, k_d) values. This set is shown in Fig. 4.3.

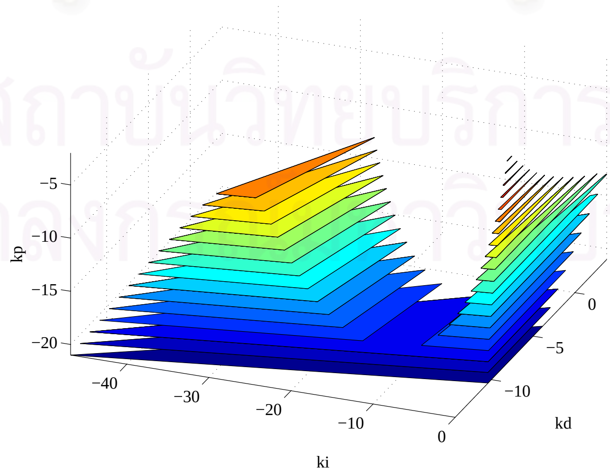

Figure 4.3: The set of stabilizing (k_p, k_i, k_d).

4.1.2 Comparison with Ziegler-Nichols Method

Consider the problem of choosing stabilizing PID gains for the plant $G(s) = \frac{N(s)}{D(s)}$ where

$$\begin{aligned} N(s) &= -4s + 1 \\ D(s) &= 0.8s^2 - 4.2s + 1 \end{aligned}$$

and $C(s)$ transfer function of PID controller, in the of

$$C(s) = k_p + \frac{ki}{s} + k_d s.$$

The closed-loop characteristic polynomial is

$$\delta(s, k_p, k_i, k_d) = sD(s) + (k_i + k_d s^2)N(s) + k_p s N(s).$$

The task is to determine those values of k_p, k_i and k_d, if any, for which $\delta(s, k_p, k_i, k_d)$ is Hurwitz. To do so, first using the root locus technique presented in section 3.3, the sweeping range of k_p values is narrowed down to $k_p \in (-1,\ 1.1)$. Then by sweeping over $k_p \in (-1,\ 1.1)$ and using the procedure in section 3.1, we obtain the stabilizing set of $(k_p,\ k_i,\ k_d)$ values in Fig. 4.4.

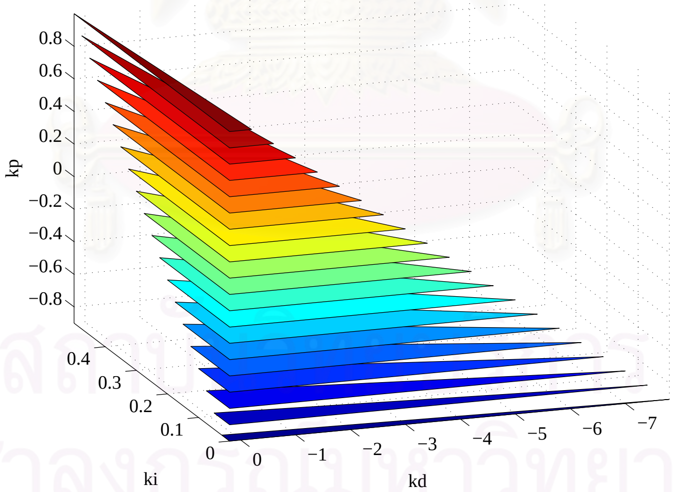

Figure 4.4: The stabilizing region of $(k_p,\ k_i,\ k_d)$.

Now let us examine where in this plot, the parameters obtained from the Ziegler-Nichols frequency response design would be located. For the plant of this example the ultimate gain $K_u = 1.05$ and the ultimate period $T_u = 3.925$. Hence, using the Ziegler-Nichols frequency response formulas, we obtain $k_p = 0.63$, $k_i = 0.321$, and $k_d = 0.3091$. Now for k_p fixed at 0.63, the set of stabilizing $(k_i,\ k_d)$ values can be obtained from Fig. 4.4.

This set is shown in Fig. 4.5. From Fig. 4.5, it is clear that for this example, the PID controller parameters which is obtained by the Ziegler-Nichols frequency response method are outside of the stabilizing region. It leads to closed-loop instability.

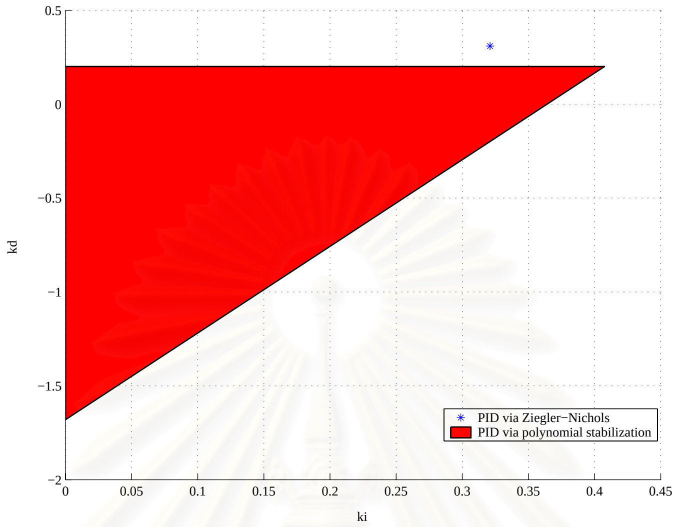

Figure 4.5: The stabilizing set of $(k_i,\ k_d)$ values when $k_p = 0.63$.

Again, we consider PID settings using the Ziegler-Nichols frequency response method for the plant $G(s)$ where

$$G(s) = \frac{1}{s^4 + 9s^3 + 19s^2 + 7s + 6}.$$

For this plant, the ultimate gain $K_u = 8.1728$ and the ultimate period $T_u = 7.1246$. Hence, using the Ziegler-Nichols frequency response formulas, we obtain $k_p = 4.9037$, $k_i = 1.3765$, and $k_d = 4.3671$. For $k_p = 4.9037$, using idea in section 3.1, we are able to obtain the set of all stabilizing $(k_i,\ k_d)$ values. This stabilizing set is shown in Fig. 4.6. From Fig. 4.6, it shows that the PID controller parameters obtained by the Ziegler-Nichols frequency response method are closed to the stability boundary. In this case, the resulting PID controller can be destabilized by small perturbations in the controller coefficient model.

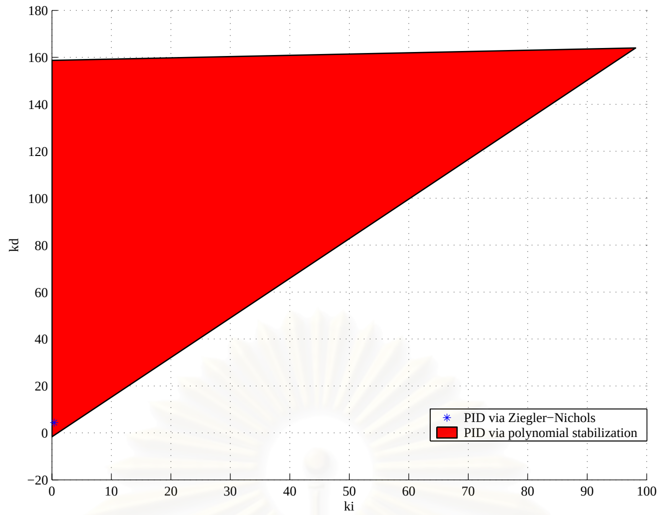

Figure 4.6: The stabilizing set of $(k_i,\ k_d)$ values when $k_p = 4.9037$.

4.1.3 PID Controller Synthesis Satisfying a Performance Specification

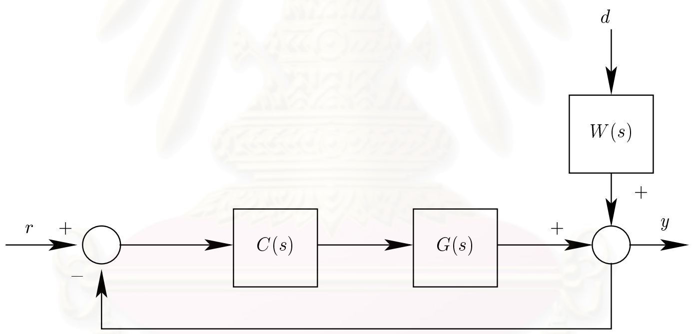

Figure 4.7: Feedback control system with a given performance.

Consider the plant $G(s) = \frac{N(s)}{D(s)}$ where

$$\begin{aligned} N(s) &= s - 1 \\ D(s) &= s^2 + 0.8s - 0.2. \end{aligned}$$

In this example, we consider the problem of determining the admissible PID controller gain values for which $||W(s)T(s)||_\infty < 1$, where $T(s)$ is the complementary sensitivity function:

$$T(s) = \frac{(k_d s^2 + k_p s + k_i)(s-1)}{s(s^2 + 0.8s - 0.2) + (k_d s^2 + k_p s + k_i)(s-1)}$$

and the weight $W(s)$ is chosen as a high pass transfer function:

$$W(s) = \frac{s + 0.1}{s + 1}.$$

The difference from section 3.2 problem setup is that, in this case we only have one weighting function, so that we only deal with one complex polynomial. Before we state condition for solving our problem, let state the following lemma.

Lemma 4.1 *Let* $F(s) = \frac{N_F(s)}{D_F(s)}$ *be a stable and proper rational function, where* $N_F(s)$ *and* $D_F(s)$ *are polynomials with* $\deg[d_F(s)] = \alpha$*. Then* $||F(s)||_\infty < 1$ *if and only if*

a) $|n_\alpha| < |d_\alpha|$*;*

b) $D_F(s) + e^{j\theta}N_F(s)$ *is Hurwitz for all* $\theta \in [0, 2\pi)$*.*

where n_α *and* d_α *are the leading coefficients of* $N_F(s)$ *and* $D_F(s)$*, respectively.*

To represent various robustness specifications, define $T_{cl}(s, k_p, k_i, k_d)$ as:

$$T_{cl}(s, k_i, k_d, k_p) = \frac{A(s) + (k_d s^2 + k_p s + k_i)B(s)}{sD(s) + (k_d s^2 + k_p s + k_i)N(s)}$$

for some real polynomial $A(s)$ and $B(s)$. Define the H_∞ optimization criteria to be

$$||W(s)T_{cl}(s, k_d, k_p, k_i)||_\infty < 1$$

where $W(s)$ is stable weighting function. Let the weighting function $W(s) = \frac{W_n(s)}{W_d(s)}$, where $W_n(s)$ and $W_d(s)$ are coprime polynomials; moreover, $W_d(s)$ is Hurwitz. For notational simplicity, we write the closed-loop characteristic polynomial

$$\rho(s, k_p, k_i, k_d) = sD(s) + (k_i + kp_s + k_d s^2)N(s)$$

and

$$\psi(s, k_p, k_i, k_d, \theta, \emptyset) \triangleq [sW_d(s)D(s) + e^{j\theta}W_n(s)A(s)] + (k_d s^2 + k_p s + k_i)[W_d(s)N(s) + e^{j\theta}W_n(s)B(s)].$$

Let

$$L(s) = sW_d(s)D(s) + e^{j\theta}W_n(s)A(s)$$

and

$$M(s) = W_d(s)N(s) + e^{j\theta}W_n(s)B(s).$$

Based on Lemma 4.1, we are already in the position to recast synthesis of H_∞ PID controller into the simultaneous polynomial stabilization. Given a weighted closed-loop transfer function of the form

$$W(s)T_{cl}(s, k_p, k_i, k_d) = \frac{W_n(s)}{W_d(s)} \frac{A(s) + (k_d s^2 + k_p s + k_i)B(s)}{sD(s) + (k_d s^2 + k_p s + k_i)N(s)}.$$

Using the Lemma 4.1 and the same idea which already present in section 3.2, we know that the admissible $(k_p,\ k_i,\ k_d)$ values exist if and only if the following conditions hold

a) $\rho(s, k_p, k_i, k_d)$ is Hurwitz;

b) $\psi(s, k_p, k_i, k_d, \theta)$ is Hurwitz for all $\theta \in [0, 2\pi)$;

c) $\mid W(\infty)T_{cl}(\omega, k_p, k_i, k_d) \mid < 1$.

Substitute the values with our numerical example equations, we get

a) $\rho(s, k_p, k_i, k_d) = s(s^2 + 0.8s - 0.2) + (k_d s^2 + k_p s + k_i)(s - 1)$ is Hurwitz;

b) $\psi(s, k_p, k_i, k_d, \theta) = s(s + 1)(s^2 + 0.8s - 0.2) + (k_d s^2 + k_p s + k_i)[(s + 1)(s - 1) + e^{j\theta}(s + 0.1)(s - 1)]$ is Hurwitz for all $\theta \in [0, 2\pi)$;

c) $|W(\infty)T(\infty)| = \left|\frac{k_d}{k_d+1}\right| < 1$.

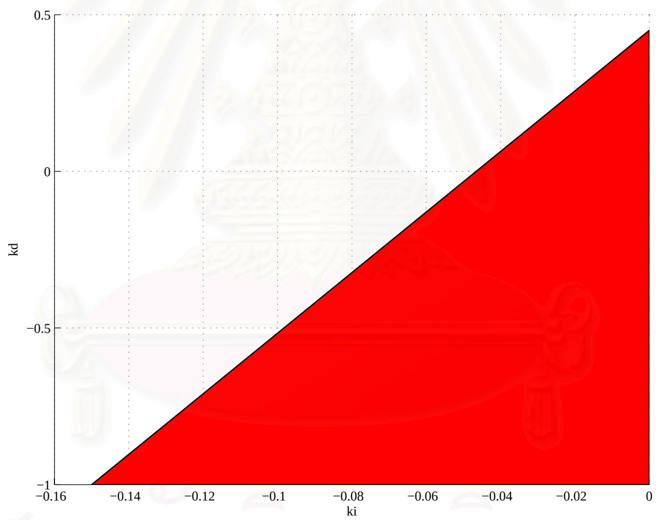

Figure 4.8: The admissible $(k_i,\ k_d)$ of $\mathcal{X}_{(1,\ -0.35)}$.

For condition (a), with a fixed k_p, for instance $k_p = -0.35$, and by setting $L(s) = s(s^2 + 0.8s - 0.2)$ and $M(s) = s - 1$, and using algorithm in section 3.2, we obtain the admissible set $\mathcal{X}_{(1,\ -0.35)}$ for which the closed-loop system is stable. $\mathcal{X}_{(1,\ -0.35)}$ is shown in Fig. 4.8. Now fixing $k_p = -0.35$, setting $L(s) = s(s + 1)(s - 1)$ and $M(s, \theta) = (s + 1)(s - 1) + e^{j\theta}(s + 0.1)(s - 1)$, sweeping over $\theta \in [0, 2\pi)$ and using the algorithm is section 3.2 at each stage, we obtain the admissible set $\mathcal{X}_{(2,\ 0.35)}$ is shown in Fig. 4.9. The condition (3) gives the constraint that $k_d > -0.5$. Hence the admissible set $\mathcal{X}_{(3,\ 0.35)}$ is given by

$$\mathcal{X}_{(3,\ 0.35)} = \{(k_i,\ k_d) \mid k_i \in \mathcal{R},\ k_d > -0.5\}.$$

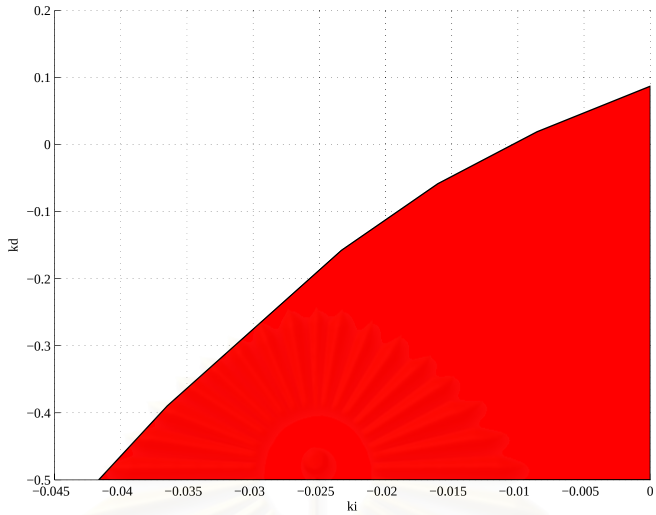

Figure 4.9: The admissible $(k_i,\ k_d)$ of $\mathcal{X}_{(2,\ -0.35)}$.

For $k_p = -0.35$, the admissible set of $(k_i,\ k_d)$ values for which $||W(s)T(s)||_\infty < 1$ is the intersection of $\mathcal{X}_{(1,\ 0.35)}$, $\mathcal{X}_{(1,\ 0.35)}$, and $\mathcal{X}_{(3,\ 0.35)}$. The solution region for this intersection is the Fig. 4.9 itself. By using root locus technique in section 3.3, it determine that a necessary condition for existance of stabilizing $(k_i,\ k_d)$ values is that $k_p \in (-1.8,\ -0.2)$. Thus, by sweeping over $k_p \in (-1.8,\ -0.2)$, and repeating the above procedure, we obtain the admissible $(k_p,\ k_i,\ k_d)$ values for which $||W(s)T(s)||_\infty < 1$. The entire admissible set is shown in Fig. 4.10.

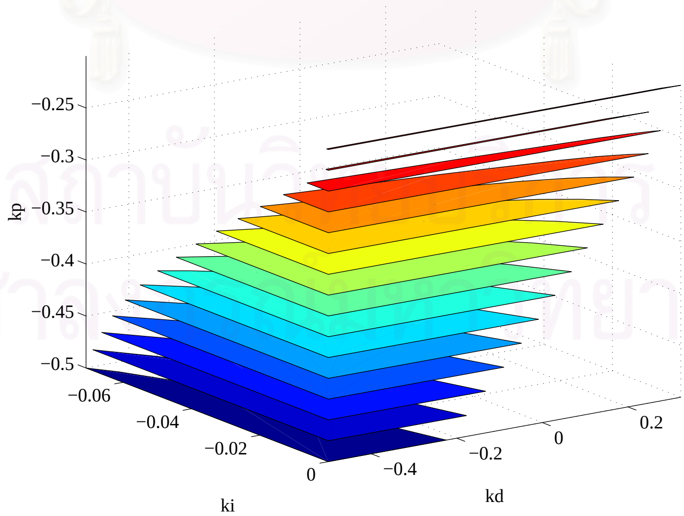

Figure 4.10: The set of admissible $(k_i,\ k_d,\ k_p)$.

4.2 Robust PID Tuning for Uncertain System

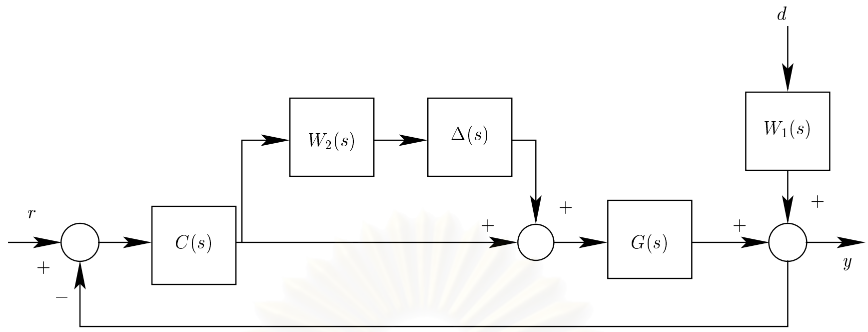

Figure 4.11: Feedback control system for a given uncertain model and robust performance.

Consider the plant $G(s) = \frac{N(s)}{D(s)}$ where

$$\begin{aligned} N(s) &= s - 15 \\ D(s) &= s^2 + s - 1. \end{aligned}$$

The weighting functions are chosen as:

$$\begin{aligned} W_1(s) &= \frac{0.2}{s+0.2} \\ W_2(s) &= \frac{s+0.1}{s+1}. \end{aligned}$$

Then, the sensitivity function and complementary sensitivity function are

$$\begin{aligned} S(s,k_p,k_i,k_d) &= \frac{s(s^2+s-1)}{s(s^2+s-1)+(k_ds^2+k_ps+k_i)(s-15)} \\ T(s,k_p,k_i,k_d) &= \frac{(k_ds^2+k_ps+k_i)(s-15)}{s(s^2+s-1)+(k_ds^2+k_ps+k_i)(s-a5)} \end{aligned}$$

The admissible $(k_ds^2 + k_ps + k_i)$ values exist if and oly if the following conditions hold

a) $\rho(s,k_p,k_i,k_d) = s(s^2+0.8s-0.2)+(k_ds^2+k_ps+k_i)(s-15)$ is Hurwitz;

b) $\psi(s,k_p,k_i,k_d,\theta,\o) = s(s+0.2)(s+1)(s^2+s-1)+e^{j\theta}s(0.2)(s+1)(s^2+s-1)+(k_ds^2+k_ps+k_i)[(s+0.2)(s+1)(s-15)+e^{j\o}(s+0.2)(s+0.1)(s-15)]$ is Hurwitz for all θ and $\o \in [0,2\pi)$;

c) $\mid W_1(\infty)S(\infty,k_p,k_i,k_d) \mid + \mid W_2(\infty)T(\infty,k_p,k_i,k_d) \mid < 1.$

With a fixed k_p, for instance $k_p = -0.3$, and using procedure in 3.2, we obtain the admissible set $\mathcal{X}_{(-0.3)}$ sketch in Fig. 4.12. By using root locus technique, we can de-

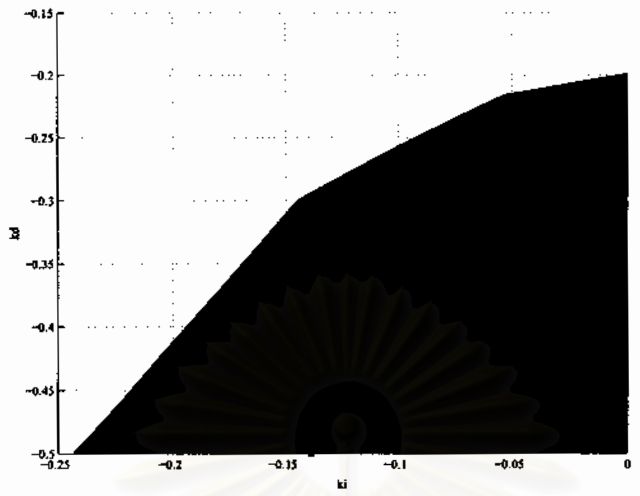

Figure 4.12: Robust PID controller gains when $k_p = -0.3$.

termine a necessary condition for the existence of admissible (k_i, k_d) values is that $k_p \in (-0.5079, -0.1155)$. Thus, by sweeping over $k_p \in (-0.5079, -0.1155)$, and using the design procedure in 3.2, we obtain the admissible set of (k_p, k_i, k_d) values for which

$$\| \, |W_1(\infty)S(\infty, k_p, k_i, k_d)| + |W_2(\infty)T(\infty, k_p, k_i, k_d)| \, \|_\infty < 1.$$

The robust PID controller for different value of k_p gains is sketched in Fig. 4.13.

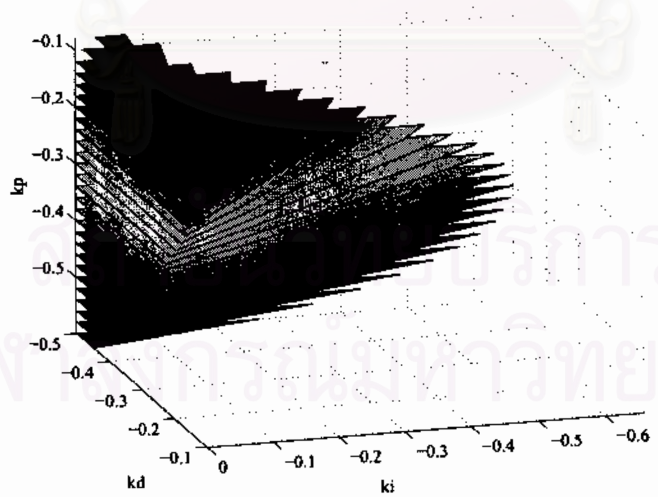

Figure 4.13: The set of robust PID controller gains.

4.3 Summary

In this chapter, for the sake of clarity, we have shown the step-by-step implementation of real polynomial stabilization algorithm. We also compared the result with PID tuning based on Ziegler-Nichols method. On this comparison we showed that, the Ziegler-Nichols PID sometimes gives fragile controller, and even cause the system to be unstable. However, for PID tuning via polynomial stabilization, we will have the all admissible region that stabilze the system, so that we have freedom of choose PID gain which is not fragile controller.

We also presented numerical examples on sample problems. These numerical examples could be solved by the polynomial stabilization method after some modification on problem formulation. First, we faced the problem for finding admissible PID gains satisfying the nominal performance in the form of H_∞ norm. Second, we dealt with the problem for finding admissible PID gains satisfying the robust performance. By using polynomial stabilization approach, we can characterize the all admissible PID gains satisfying nominal and robust performance.

Chapter 5

Robust PID Controller Synthesis for Belt Conveyor System

In this chapter, we will discuss about mathematical model of servo-driven belt conveyor system. The physical plant is the laboratory-scale device for experiment activity in Control System Research Laboratory of Chulalongkorn University. Our discussion in this chapter will begin with the nominal model of servo-driven belt conveyor system in §5.1. After that in §5.2, we will come out with result of weighting function represent uncertainty of the model. Then, we will apply our PID tuning procedure for belt conveyor system. The objective of our design is to make the system stable and satisfy a given performance, in spite of there is exist the uncertainty of the model. First in §5.3 We will carry explicitly the procedure to find PID gains that can stabilize the system. In section §5.4 we would like to find the robust PID tuning satisfy a given robust performance and plant uncertainty. The solution of this procedure needs us fulfill three given condition.

5.1 Nominal Model of Servo-Driven Belt Conveyor System

Figure 5.1: Physical components of belt conveyor system.

Consider the belt conveyor system depicted in Fig. 5.1, which consists of the belt lying on the iron plate with two shafts at the end. A DC motor is used to drive the belt through the shaft to transfer a load mass to the desired position. The system dynamics of belt conveyor systems can be divided into two mutual parts which are the motor and belt dynamic. Therefore, in this work, we consider the belt dynamic the feedback loop from the motor angle to the motor torque itself as shown in Fig. 5.2, where the input u is the voltage and the output y is the motor angle. The uncertainty is considered either of the following.

- Parametric uncertainty of motor
- Zero-pole uncertainty of load
- Unmodelled dynamics uncertainty of load
- The existence of nonlinearities such as friction.

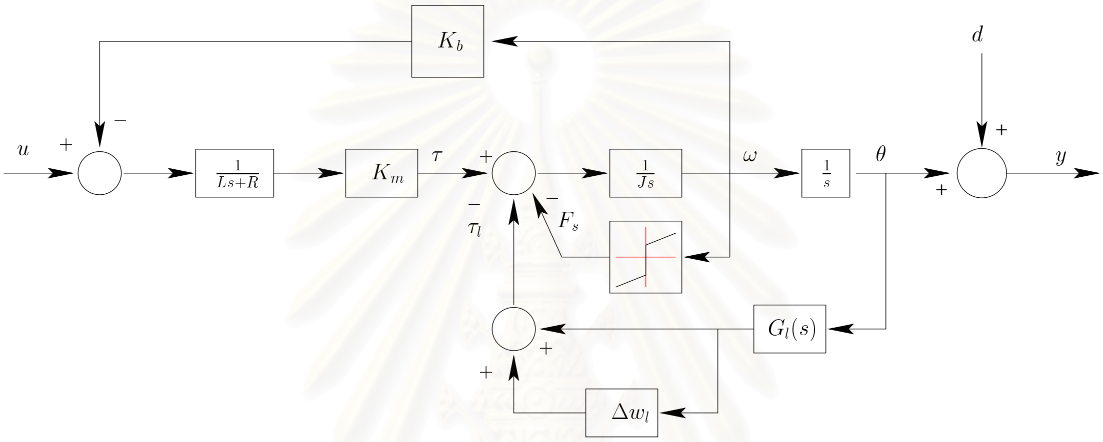

Figure 5.2: The belt conveyor system diagram including model uncertainty and friction.

We can simplify the model of belt conveyor systems on Fig. 5.2 into Fig. 5.3, where G_0 is nominal model, W_2 is the uncertainty weighting function. The complete works on this section can be found in [25]. The motor parameters are the data from a manufacturer [26]

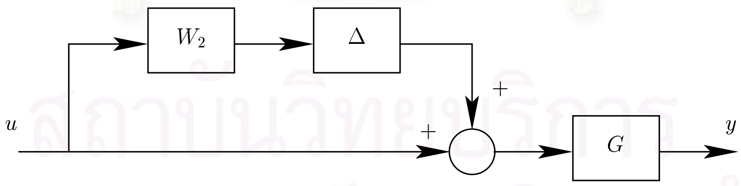

Figure 5.3: Plant model with multiplicative uncertainty.

$$\begin{aligned} R_0 &= 0.6\,\Omega, \quad L_0 = 1.24 \times 10^{-3} \text{ mH}, \\ J_0 &= 1.2 \times 10^{-5} \text{ kg}\cdot\text{m}^2, \quad B_0 = 2.2 \times 10^{-5} \text{ N}\cdot\text{m}\cdot\text{s}, \\ K_m^0 &= 3.42 \times 10^{-2} \text{ N}\cdot\text{m/A}, \quad K_b^0 = 3.42 \times 10^{-2} \text{ V/rad/s}. \end{aligned} \tag{5.1}$$

According to the parameter values in (5.1), the closed-loop transfer function from u to y without load is

$$G_{nl} = \frac{2.298 \times 10^6}{s^3 + 485.7s^2 + 7.949 \times 10^4 s}.$$

Suppose that, the load dynamic transfer function is

$$G_l = \frac{2(s+3)}{s^2 + 20s + 125}.$$

Therefore, the transfer function including the load dynamic is

$$G = \frac{(2.298 \times 10^6)s^2 + (4.597 \times 10^7)s + (2.873 \times 10^8)}{s^5 + 505.7s^4 + (8.933 \times 10^4)s^3 + (1.817\times)10^6 s^2 + (9.108 \times 10^7)s + (2.419 \times 10^8)}. \tag{5.2}$$

Many various identification methods are applied to belt conveyor system where the last transfer function is assumed as the plant transfer function [25]. After system identification, we get model of belt conveyor with load in form of the state space expression.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2.7702 \times 10^8 & -1.0407 \times 10^8 & -2.0292 \times 10^6 & -1.0134 \times 10^5 & -531.8121 \end{bmatrix} \tag{5.3}$$

$$B = \begin{bmatrix} 6.800 & -4.9459 \times 10^3 & 4.5452 \times 10^6 & -1.8773 \times 10^9 & 5.4743 \times 10^{11} \end{bmatrix}^T$$

$$C = [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0], \quad D = 0.$$

5.2 Model Uncertainty of Belt Conveyor System

In this section, we will consider only the uncertain model of the belt conveyor system. The model uncertainty is determined under the various conditions as follow.

- Parametric uncertainty of motor: all parameters of the motor are varied by $\pm 20\%$.
- Zero-pole uncertainty of load: zeroes and poles of transfer function of the load dynamic are varied by $\pm 10\%$.
- Unmodelled dynamics uncertainty of load: The load transfer function is multiplied by the first-order unmodelled dynamic.
- The existence of the nonlinearity such as friction.

We calculate the parametric uncertainty of the system by varying motor parameters, R, L, J, K_m, K_b $\pm 20\%$. Therefore, the parameters are in the form

$$R = R_0(1+\Delta_1), \quad L = L_0(1+\Delta_2),$$
$$J = J_0(1+\Delta_3), \quad B = B_0(1+\Delta_4),$$
$$K_m = K_{m0}(1+\Delta_5), \quad K_{b0}(1+\Delta_6).$$

where Δ_i is a real uncertainty such that $|\Delta_i| < 0.2$. After we proceed to find weighting function for uncertainty of model, we get

$$W_2 = \frac{1.979s^5 + 147.9s^4 + (1.768 \times 10^4)s^3 + (2.84 \times 10^5)s^2 + (1.78 \times 10^7)s + (4.457 \times 10^7)}{s^5 + 192.6s^4 + 6304s^3 + (3.675 \times 10^5)s^2 + (4.856 \times 10^6)s + 9.465 \times 10^7}.$$

5.3 PID Stabilization for Nominal Model of Belt Conveyor System

Consider our problem setup

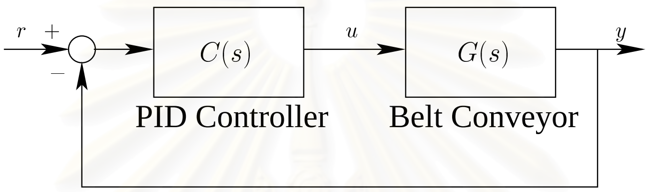

Figure 5.4: Control configuration of stabilizing belt conveyor system.

The transfer function of belt conveyor system (including the load dynamic), denoted as P is given as (5.3). If we change the state space representation in (5.3) into polynomial transfer function

$$\begin{aligned} G &= \frac{N(s)}{D(s)} \\ &= \frac{6.8s^4 - 1330s^3 + (2.604 \times 10^6)s^2 + (5.247 \times 10^7)s + 3.412 \times 10^8}{s^5 + 531.8s^4 + (1.013 \times 10^5)s^3 + (2.029 \times 10^6)s^2 + (1.041 \times 10^8)s + 2.77 \times 10^8}. \end{aligned}$$

The closed-loop characteristic polynomial is

$$\delta(s, k_p, k_i, k_d) = sD(s) + (k_i + k_d s^2)N(s) + k_p s N(s).$$

Thus $n = 6$ and $m = 4$. Also

$$\begin{aligned} N_e(s^2) &= 6.8s^4 + (2.604 \times 10^6)s^2 + 3.412 \times 10^8, \\ N_o(s^2) &= -1330s^2 + (5.247 \times 10^7), \\ D_e(s^2) &= 531.8s^4 + (2.029 \times 10^6)s^2 + 2.77 \times 10^8, \\ D_o(s^2) &= s^4 + (1.013 \times 10^5)s^2 + (1.041 \times 10^8), \end{aligned}$$

and

$$N^*(s) = (6.8s^4 + (2.604 \times 10^6)s^2 + 3.412 \times 10^8) - s(-1330s^2 + (5.247 \times 10^7)).$$

Multiply $\delta(s, k_p, k_i, k_d)$ with $N^*(s)$ so that

$$\delta(j\omega, k_p, k_i, k_d)N^*s(j\omega) = [p_1(\omega) + (k_i - k_d\omega^2)p_2(\omega)] + j[q_1(\omega) + k_p q_2(\omega)]$$

where

$$\begin{aligned} p_1(\omega) = &\ -6.8\omega^{10} + (4.002 \times 10^6)\omega^8 - (2.3973 \times 10^{11})\omega^6 + (1.9946 \times 10^{14})\omega^4 \\ &- (2.0969 \times 10^{16})\omega^2 \\ p_2(\omega) = &\ 46.24\omega^8 - (3.3647 \times 10^7)\omega^6 + (6.925 \times 10^{12})\omega^4 + (9.766 \times^{14})\omega^2 \\ &+ (1.164 \times^{17}) \\ q_1(\omega) = &\ (4.9459 \times 10^3\omega^9) - (1.4809 \times 10^9)\omega^7 - (2.8812 \times 10^{11})\omega^5 + (4.047 \times 10^{15})\omega^3 \\ &+ (9.4511 \times^{16})\omega \\ q_2(\omega) = &\ 46.24\omega^9 - (3.3647 \times^7)\omega^7 + (6.025 \times 10^{12})\omega^5 + (9.766 \times 10^{14})\omega^3 \\ &+ (1.164 \times 10^{17})\omega. \end{aligned}$$

The range of allowable k_p is determined by using root loci technique to be $(-1.7682, 48.8212)$. For instance, $k_p = 0.2$, we have

$$\begin{aligned} q(\omega, 0.2) = &\ q_1(\omega) + 0.2q_2(\omega) \\ = &\ (4.9552 \times 10^3)\omega^9 - (1.4876 \times 10^9)\omega^7 + (1.6731 \times 10^{12})\omega^5 + (4.2425 \times 10^{15})\omega^3 \\ &+ (1.1779 \times 10^{17})\omega. \end{aligned}$$

Then the real, non-negative, distinct finite zeros of $q_f(\omega, 0.2)$ with odd multiplicities are

$$\omega_0 = 0, \omega_1 = 48.620, \omega_2 = 546.88.$$

Also define $\omega_3 = \infty$. Since $m + n = 10$ which is even, and $l(N(s)) = 2$ and $r(N(s)) = 2$,

$$l(N(s)) - r(N(s)) = 0$$

and

$$\{-1)^{l-1}\text{sgn}[q(\infty, 0.2\}] = 1,$$

it follows from Step 10 that every admissible string

$$\mathcal{I} = \{i_0, i_1, i_2, i_3\}$$

must satisfy

$$(i_0 - 2i_1 + 2i_2 - i_3).(1) = 6.$$

Hence the admissible string is

$$\{1, -1, 1, -1\}$$

From Step 11, for $\mathcal{I}_1$ it follows that the stabilizing (k_i, k_d) values corresponding to $k_p = 40$ must satisfy the string of inequalities:

$$\begin{cases} p_1(\omega_0) + (k_i - k_d\omega_0^2)p_2(\omega_0) < 0 \\ p_1(\omega_1) + (k_i - k_d\omega_1^2)p_2(\omega_1) < 0 \\ p_1(\omega_2) + (k_i - k_d\omega_2^2)p_2(\omega_2) < 0 \\ p_1(\omega_3) + (k_i - k_d\omega_3^2)p_2(\omega_3) > 0 \end{cases}$$

Substituting for $\omega_0, \omega_1, \omega_2, \omega_3$ in the above expressions, we obtain

$$\begin{cases} k_i > 0 \\ k_i - (2.3639 \times 10^3)k_d < 48.607 \\ k_i - (2.9908 \times 10^5)k_d > -1.0426 \times 10^5 \\ -k_d < 0.14706. \end{cases}$$

The set of values of (k_i, k_d) for which last condition holds can be solved by linear programming and is denoted by S_1. The set stabilizing (k_p, k_i, k_d) and the corresponding S_1 are shown in Fig. 5.5.

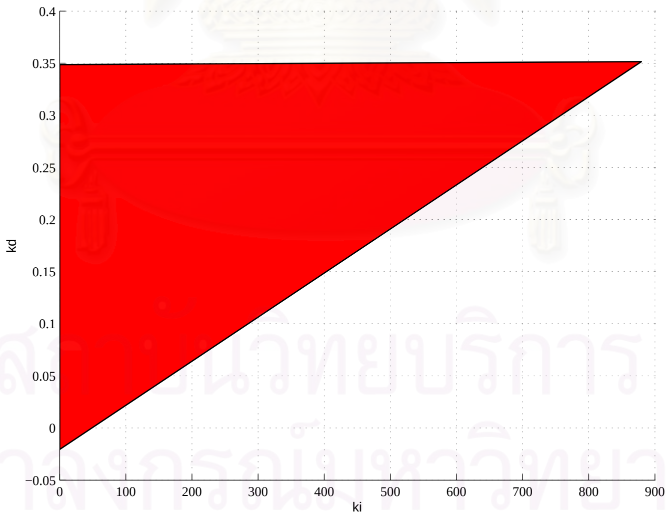

Figure 5.5: The set of (k_i, k_d) gains of conveyor system satisfy condition (a), when $k_p = 0.2$.

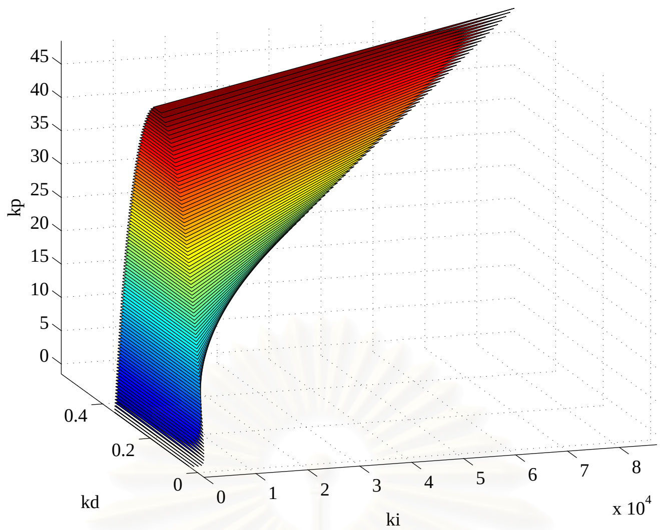

Figure 5.6: The stabilizing (k_p, k_i, k_d) gains of belt conveyor system.

By sweeping over different k_p values and repeating the algorithm at each stage, we can generate the set stabilizing (k_p, k_i, k_d) values. This set is shown in Fig. 5.6. To prove our controller gives us the right answer, we will pick the one point inside and ouside the stabilizing region of PID. The chosen point inside the region is $(k_p = 0.2, k_i = 10, k_d = 0.1)$. The closed-loop poles of this stable system is shown in Fig. 5.7. The chosen point outside the region is $(k_p = 0.2, k_i = 200, k_d = 0.05)$. The closed-loop poles of this unstable system is shown in Fig. 5.8.

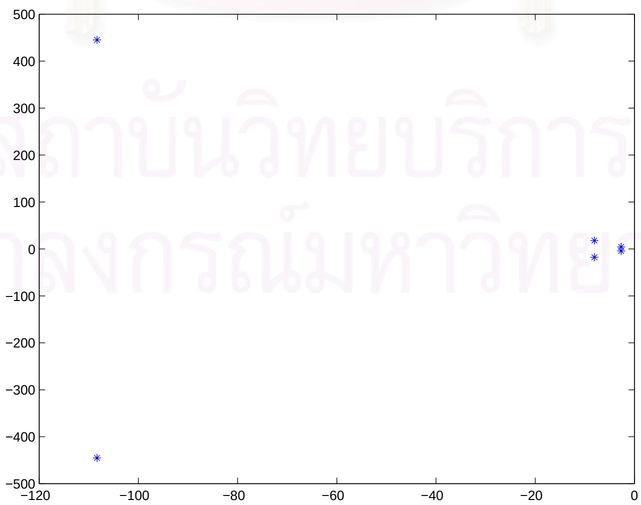

Figure 5.7: The closed-loop poles of belt nominal conveyor system when $k_p = 0.2, k_i = 10, k_d = 0.1$.

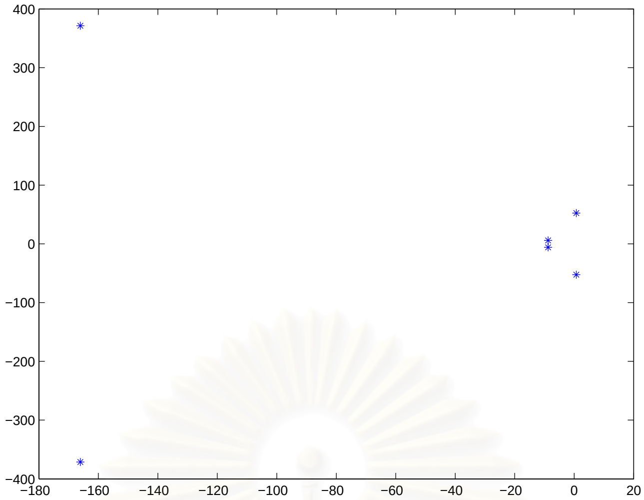

Figure 5.8: The closed-loop poles of nominal belt conveyor system when $k_p = 0.2, k_i = 200, k_d = 0.05$.

Our belt conveyor system has ultimate gain $K_u = 15.668$, and ultimate period $T_u = 0.024$. Hence, by using Zieler-Nichols formula, we obtain $k_p = 9.4008, k_i = 772.04$, and $k_d = 0.0286$. We compare the Ziegler-Nichols PID gain with the PID obtained by polynomial stabilization at the same value of k_p, shown in the Fig. 5.9. We can see that PID gain from the Zieler-Nichols method closed to unstable region, so that the controller is fragile when there is uncertainty of the plant model.

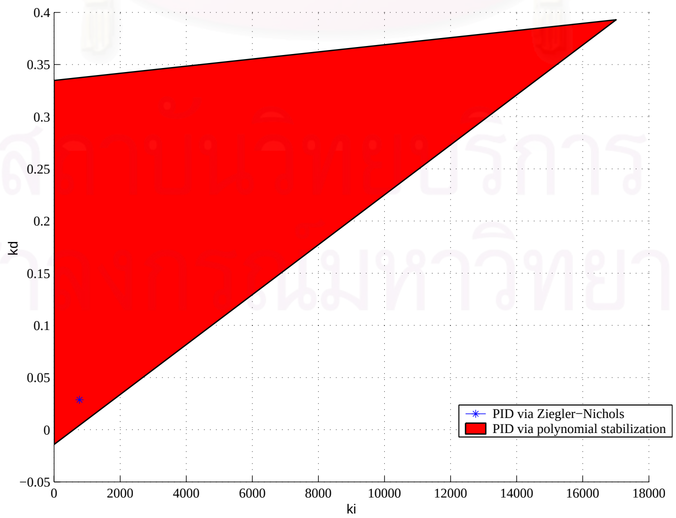

Figure 5.9: Comparison PID controllers via polynomial stabilization and via Ziegler-Nichols for belt conveyor system.

5.4 Robust PID Tuning for Belt Conveyor System

Consider our problem setup in Fig. 5.10.

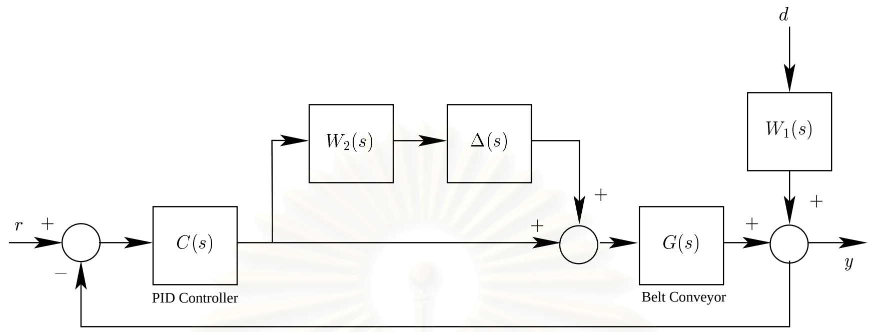

Figure 5.10: Control configuration of robust performance for belt conveyor system.

We raise again transfer function of belt conveyor system:

$$G = \frac{N(s)}{D(s)} = \frac{6.8s^4 - 1330s^3 + (2.604\times 10^6)s^2 + (5.247\times 10^7)s + 3.412\times 10^8}{s^5 + 531.8s^4 + (1.013\times 10^5)s^3 + (2.029\times 10^6)s^2 + (1.041\times 10^8)s + 2.77\times 10^8},$$

and the weighting function represent uncertainty of the system (W_2):

$$W_2 = \frac{N_2}{D_2} = \frac{1.979s^5 + 147.9s^4 + (1.768\times 10^4)s^3 + (2.84\times 10^5)s^2 + (1.78\times 10^7)s + (4.457\times 10^7)}{s^5 + 192.6s^4 + 6304s^3 + (3.675\times 10^5)s^2 + (4.856\times 10^6)s + 9.465\times 10^7}.$$

We set our performance specification as follows:

- First-order system.
- DC gain is 1.
- Time constant is 1 sec.

Therefore, our weighting function W_1 has the form of:

$$W_1 = \frac{N_1}{D_1} = \frac{1}{s+1}.$$

The procedure for determining the set of (k_p, k_i, k_d) values satisfy condition (a), (b), and (c) is already presented in section 3.2. By using root loci technique, it is determined that a necessary condition for existence of stabilizing (k_i, k_d) values is that $k_p \in (-1.7682, 48.8212)$.

For any fixed $k_p \in (-1.7682, 48.8212)$, we use algorithm in section 3.1 to solve condition (a) and algorithm in section 3.2 to solve condition (b). Then for a fixed k_p, we obtain the set of all (k_i, k_d) values for which $|| \, |W_1(\infty)S(\infty, k_p, k_i, k_d)| + |W_2(\infty)T(\infty, k_p, k_i, k_d)| \, ||_\infty < 1$ by taking intersection of the set (k_i, k_d) values satisfying condition (a), (b), and (c). If fixing $k_p = 0.2$, for condition (a) we get the solution which is the same with solution for stabilizing (k_p, k_i, k_d) when $k_p = 0.2$, see Fig. 5.5. For condition (b) we can see the solution in Fig. 5.11.

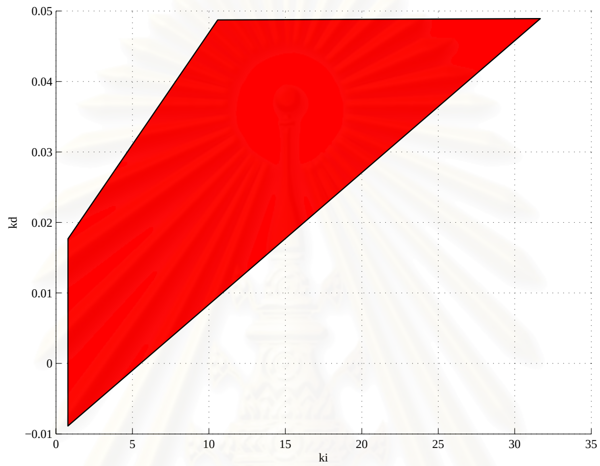

Figure 5.11: The set of (k_i, k_d) gains satisfy condition (b), when $k_p = 0.2$.

The intersection condition (a) with (b) is sketched in Fig.5.12, and we can conclude that the solution of of our PID gains when $k_p = 0.2$ is actually the solution for condition (b) alone. Remark on this result is the computation time for obtaining the solution region via

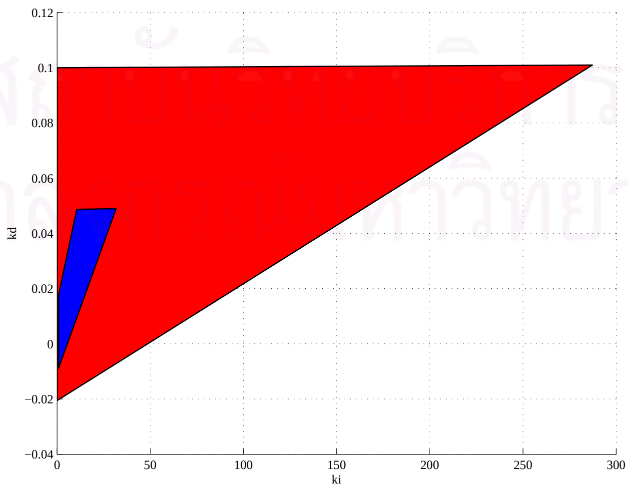

Figure 5.12: Intersection of condition (a) and (b), when $k_p = 0.2$.

polynomial stabilization increases exponentially with the increasing of the order of polynomial and resolution of θ and $\o$ (i.e., how many points between 0 and 2π that are sampled). Ideally, condition (b) indicates every θ and $\o \in [0, 2\pi)$ is Hurwitz, we should take resolution small enough for this purpose. The smaller the resolution is, the finer the solution region that we will have. To test whether our resolution point small enough, we can test by choosing some points within the solution region, and checking whether they satisfy our robust performance. Here, the test of robust performance condition is that the magnitude of frequency response must be less than 1 for all frequencies. If these controller gains satisfy our robust performance test, we can guarantee that our chosen resolution gives us the actual solution region; if not, we should take finer resolution.

In this calculation, the resolution of θ is chosen as $0 : 2\pi/10 : 2\pi$ and that of $\o$ with the same amount, then computation time was approximately four hours. We obtain the admissible PID region plotted in Fig. 5.11. . The post design test in frequency domain shows that not all the points satisfy robust performance condition. In particular, most gains satisfy robust performance condition, for example, a PID controller having $(k_p = 0.2, k_d = 0.01, k_i = 6)$ passes the frequency response test as shown in Fig. 5.13. The step response of the closed-loop system based on the chosen PID controller is illustrated in Fig. 5.14. The plot shows that the settling time of the closed-loop system is faster than the design specification. Note that the time response is closely related to the fact that there are some poles which are far away from imanigary axis, and the others which are near the imaginary axis. The control signal at $(k_p = 0.2, k_i = 6, k_d = 0.01)$, is shown in Fig. 5.15. The control input signal is not exceed 1.2 Volt, so it is quite reasonable to implement this control input. Note that there are certain gains in the admissible region which do not satisfy the robust performance condition. Hence, it is recommended to have a finer resolution of θ and $\o$ for the complex polynomial stabilization.

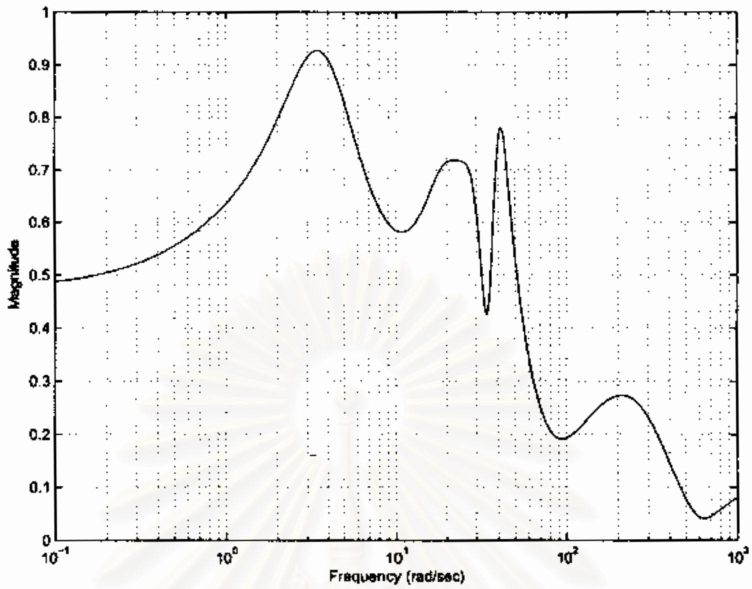

Figure 5.13: Frequency response of the belt conveyor system when $k_p = 0.2, k_i = 6, k_d = 0.01$.

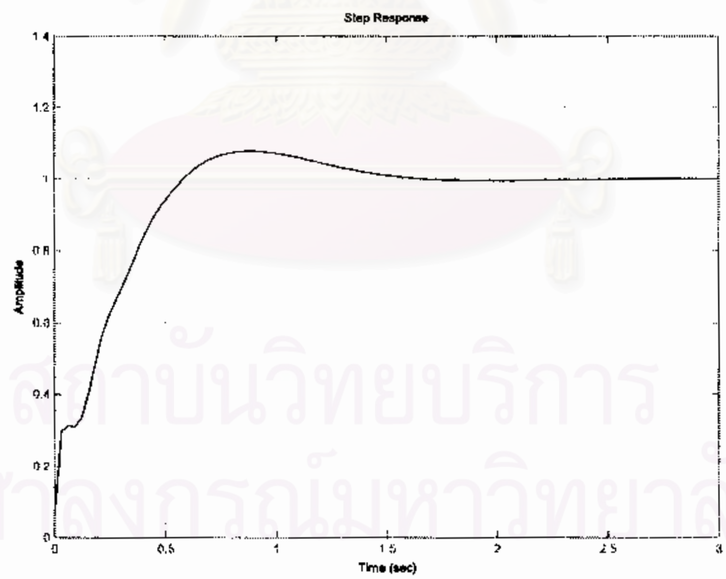

Figure 5.14: Step response of belt conveyor system when $k_p = 0.2, k_i = 6, k_d = 0.01$.

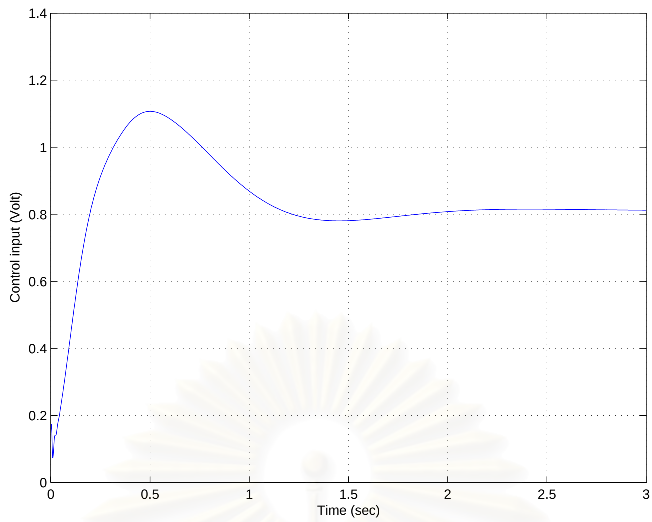

Figure 5.15: Control input from PID controller at $k_p = 0.2, k_i = 6, k_d = 0.01$.

To test the robustness our controller, we perturb the parameters in (5.2) to $\pm 10\%$, and $\pm 20\%$. and perform the frequency response test which is shown in Fig. 5.16. Here, we observe that the peak magnitue of frequency response is less than 1 for all frequencies. The step responses of the perturbed systems are displayed in Fig. 5.17. The perturbed systems have faster settling time which is close to the design specification. The control inputs from the perturbed system together with nominal model and model from system identification, are shown Fig. 5.18. Our control inputs are bounded within acceptable range.

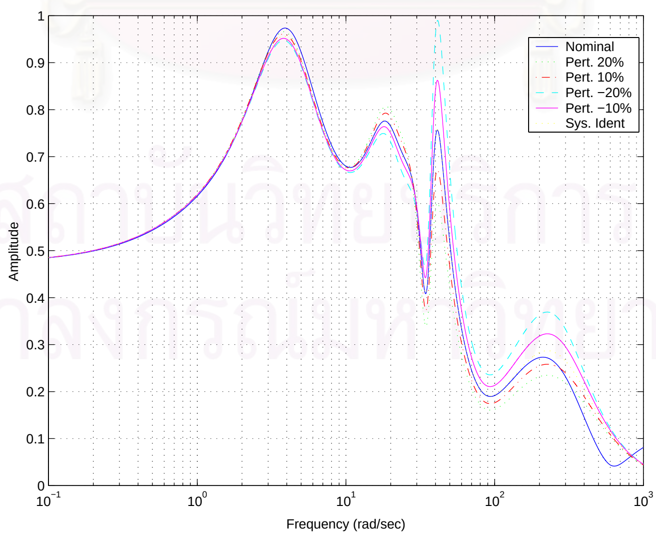

Figure 5.16: Frequency responses of the nominal and perturbed systems at $k_p = 0.2, k_i = 6, k_d = 0.01$.

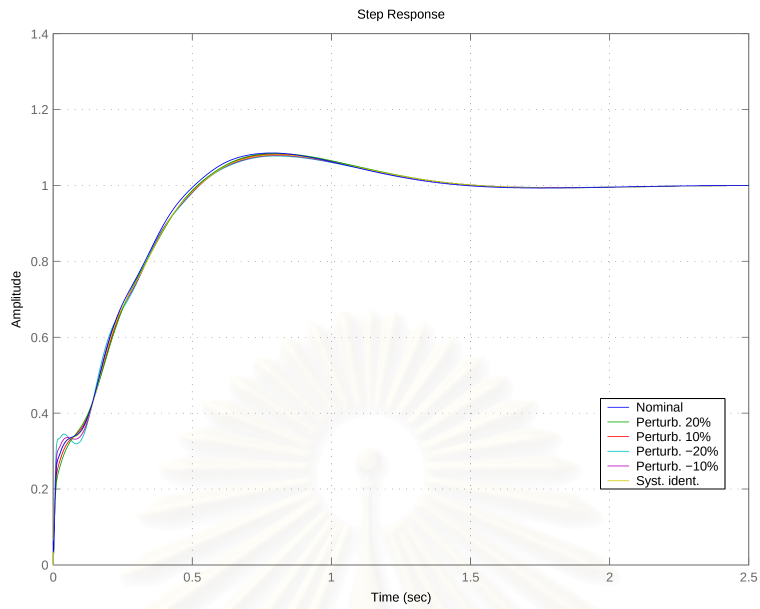

Figure 5.17: Step responses of the nominal and perturbed systems at $k_p = 0.2, k_i = 6, k_d = 0.01$.

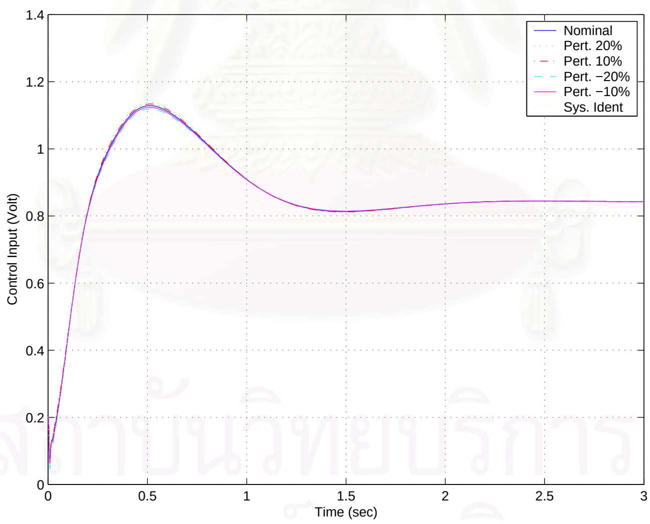

Figure 5.18: Control input from PID controller for the nominal and perturbed systems at $k_p = 0.2, k_i = 6, k_d = 0.01$.

5.5 Summary

In this chapter, we designed nominal and robust PID controllers for belt conveyor systems. For the nominal system, we characterized all stabilizing PID gains for the belt conveyor system. We plot the admissible region of PID controllers both in 2D and 3D. It is observed that the Ziegler-Nichols PID controller is quite near unstable region. Subsequently, we use the nominal model and uncertainty of the system to obtain the admissible PID gains satisfying the robust performance. A PID controller is chosen from the admissib le PID region and we conduct robustness tests in the frequency and time domain for the nominal and perturbed systems. The frequency responses indicate that robust performance condition is achieved, whereas the step responses show that the performance is reasonably following the design criteria. In addition, the PID controller gives acceptable control inputs. It is remarked that the computation time for sovling complex polynomial is much longer than solving real polynomial. This computation time will increase exponentially with respect to an increase in the plant's order and the finer resolution of θ and ϕ.

CHAPTER VI

CONCLUSIONS

6.1 Summary of Results

In this thesis, we deal with the problem to design PID controllers that stabilize the dynamical system subject to uncertainty. In Chapter 2, the system performance has been defined in terms of H_∞ norm and we have reviewed the conditions for nominal performance, robust stability and performance. In addition, we have provided their graphical interpretations based on the Nyquist plots. In Chapter 3, we have explained the polynomial stabilization approach and provided a systematic algorithm of polynomial stabilization. It hase been showed that finding the stabilizing PID gains for a nominal plant is equivalent to solving the real polynomial stabilization problem. Extending the result to the case of robust conntroller design, we need to solve three conditions simultaneously, involving real and complex polynomials. By employing the mathematical procedure, linear programming provides admissible PID controllers which not only stabilize the uncertain systems but also achieve specified robust performance specification. The important aspect of this approach is that the computational technique characterizes the set of admissible PID gain values.

Chapter 4 includes a number of numerical examples on polynomial stabilization. We have developed MATLAB programs and shown step-by-step procedure to determine stabilizing PID controllers for real and complex polynomials. Then, we compare the design results with the PID parameters obtained by the Zigler-Nichols method. It is clearly seen that the PID gains from the Ziegler-Nichols method are quite fragile and sometimes cannot stabilize the system under small perturbation. Because solving the polynomial stabilization yields the admissible region of stabilizing PID gains, we have freedom to choose appropriate PID gains so that to avoid fragile gains. Another example aims to obtain controllers satisfying nominal performance and involving three simultaneous conditions. These conditions consist of real polynomial (for obtaining stability of nominal system), and the complex polynomial and additional condition (for meeting nominal performance). The resulting PID gains were presented in both 2D and 3D plots. The final example dealt with performance for the uncertain system, so the design problem is to find the PID gains satisfying the robust performance. Like previous example, this robust performance problem can also be solved using three simultaneous equations. There is a small difference between the second and third problems. That is the forms of real and complex polynomials.

In Chapter 5, we have described the mathematical model of belt conveyor system as well as its uncertainty. Then, we applied the MATLAB programs to solve the real polynomial stabilization and obtained the stabilizing PID controller for nominal condition. It is observed that PID gains from the Zieger-Nichols method is rather close to boundary of the admissible PID region . Subsequently, we applied the design approach to search for robust PID controllers. Using the real and complex polynomial stabilization, robust PID gains are obtained at specified value of k_p. Note that a finer resolution of θ and $\o$ will ensure the satisfaction of robust performance.

6.2 Recommendations for Future Works

- Extension to Multi-Input-Multi-Output (MIMO) system:
  In this thesis, we have shown how to design nominal/robust PID controllers for a class of SISO systems. However, in many practical systems, there are many problems related to MIMO systems. The question how to generalize the procedure of MIMO polynomial stabilization is still open.

- Extensions to arbitrary fixed order controllers:
  We have shown a technique to design nominal and robust PID controllers. It is interesting to extend the research to design controllers having the form of arbitrary order of controllers. As a result, there will be many more candidates which satisfy the design specification.

- Extension of other polynomial stabilizations:
  We have the systematic ways to PID controller synthesis for polynomial stabilization. This fact can bring us to other applications related to polynomials, such as synthesis of PID controller for discrete time systems. We may also consider synthesis of PID controllers for continuous-time systems with time delay. Another example is minimum decay rate problem, i.e., that is to find PID controllers that make the closed-loop poles to the left half plane with a specified value of the negative axis. There are many problems that can be dealt within this approach, as long as we can convert the problems in form of polynomial stabilization. Some papers deal with this issue can be found in [27, 28, 29, 30, 31], and [1].

- Programming efficiency:
  In this work, the computational time was rather long when a high order of polynomial was considered. Therefore, there is a need to develop a more efficient programming, or an intermediate process which can reduce the order of system polynomial before coping with the stabilization problems.

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