เงื่อนไขเพียงพอสำหรับความต่อเนื่องอย่างอ่อนและเสถียรภาพเสมือนอย่างอ่อนในปริภูมิบานาค

นายสิทธิโชค ทรงสอาด

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SUFFICIENT CONDITIONS FOR WEAK CONTINUITY AND WEAK VIRTUAL STABILITY IN BANACH SPACES

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เราได้ให้เงื่อนไขบางประการที่รับประกันความต่อเนื่องอ่อนและเสถียรภาพเสมือนอ่อนสำหรับการส่งในปริภูมิบานาค โดยการแนะนำการส่งลิพชิทซ์เชิงฟังก์ชันนัล และ การส่งลิพชิทซ์เอกรูปเชิงฟังก์ชันนัล เรายังได้แสดงด้วยว่า สำหรับปริภูมิบานาคมิติ จำกัด ความเป็นลิพชิทซ์ (เอกรูป) เชิงฟังก์ชันนัลและ ความเป็นลิพชิทซ์ (เอกรูป) สมมูลกัน นอกจากนี้เรายังให้เกณฑ์ในการตรวจสอบความเป็นลิพชิทซ์ (เอกรูป) เชิงฟังก์ชันนัล ในปริภูมิบานาคมิติอนันต์อีกด้วย

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We give some conditions that guarantee the weak continuity and the weak virtual stability for mappings in Banach spaces by introducing the notions of functionally lipschitzian mappings and functionally uniformly lipschitzian mappings. We also show that, for finite-dimensional Banach spaces, functionally (uniformly) lipschitzian and (uniformly) lipschitzian conditions are equivalent. Some criteria for being functionally (uniformly) lipschitzian in infinite-dimensional Banach spaces are also given.

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CHAPTER I
INTRODUCTION AND PRELIMINARIES

1.1 Topological spaces

Definition 1.1. *A **topology** on a set X is a collection $\mathcal{T}$ of subset of X having the following properties:*

1. *$\varnothing$ and X are in $\mathcal{T}$,*
2. *the union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$,*
3. *the intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.*

A topological space is an ordered pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X, but we often omit specific mention of $\mathcal{T}$ if no confusion will arise.

Definition 1.2. *Let Y be a subset of topological space $(X, \mathcal{T}_X)$. We define the topology $\mathcal{T}_Y$ on Y by $\mathcal{T}_Y = \{A \cap Y \subseteq Y : A \in \mathcal{T}_X\}$ and call it the **subspace topology**.*

Definition 1.3. *Let $(X, \mathcal{T})$ be a topological space and O a subset of X. We say that*

1. *O is **open** if O belongs to the collection $\mathcal{T}$.*
2. *O is **closed** if $X - O$ is open.*
3. *For $x \in X$, O is a **neighborhood of** x if O is an open set containing x.*

Definition 1.4. *Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The **closure** of A is defined as the intersection of all closed sets (in X) containing A. The closure of A is denoted by $\overline{A}$.*

Note that : $\overline{A}$ is the smallest closed set containing A.

Definition 1.5. *If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) such that*

1. *for each $x \in X$, there is at least one basis element B containing x.*
2. *if x belongs to the intersection of two basis elements B_1 and B_2 , then there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.*

*If $\mathcal{B}$ satisfies these two conditions, then we define the **topology $\mathcal{T}$ generated by $\mathcal{B}$** as follows: A subset O of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in O$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Equivalentlly, $\mathcal{T}$ is the collection of all unions of basis elements.*

Definition 1.6. *A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subset of X whose union equals X. The **topology generated by the subbasis** $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$*

Note that : The collection of all finite intersection of elements of $\mathcal{S}$ is a basis for a topology $\mathcal{T}$.

Definition 1.7. *A **metric** on a nonempty set X is a mapping*

$$d : X \times X \to \mathbb{R}$$

having the following properties :

1. *$d(x, y) \geq 0$ for all $x, y \in X$; the equality holds if and only if $x = y$.*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$.*
3. *(Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.*

Given $x \in X$ and $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}.$$

*It is called the ϵ-**ball centered at** x. Sometimes we omit the metric d from the notation and write this ball simply as $B(x,\epsilon)$, when no confusion will arise.*

Example 1.8. *Let $d : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be defined by $d(x,y) = |x-y|$ for all $x.y \in \mathbb{R}$. Then d is a metric on $\mathbb{R}$. Now we say that d is a **euclidean metric** on $\mathbb{R}$, denoted by d_E.*

Definition 1.9. *If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x,\epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology no X, called the **metric topology** induced by d.*

Definition 1.10. *If X is a topological space, X is said to be **metrizable** if there exists a metric d on X that induces the topology of X. A **metric space** (X,d) is a metrizable space X together with a spacific metric d that gives the topology of X.*

Definition 1.11. *A relation $\leq$ on a set A is called a **partial order** relation if the following conditions hold for all $\alpha,\beta,\gamma \in A$*

1. *$\alpha \leq \alpha$.*
2. *If $\alpha \leq \beta$ and $\beta \leq \alpha$, then $\alpha = \beta$.*
3. *If $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$.*

*A **directed set** J is a set with a partial order $\leq$ such that for each pair α,β of elements of J, there exists an element γ of J having the property that $\alpha \leq \gamma$ and $\beta \leq \gamma$.*

Example 1.12. *$\mathbb{N}$ with a partial order $\leq$ is a directed set.*

Definition 1.13. *Let X be a topological space. A **net** in X is a function f from a directed set Λ to X. If $\alpha \in \Lambda$, we usually denote $f(\alpha)$ by x_α. We denote the net f itself by symbol $(x_\alpha)_{\alpha\in\Lambda}$, or merely by (x_α) if the index set is understood. Moreover if $\Lambda = \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}}$ is called a **sequence**.*

*The net (x_α) is said to **converge** to the point x of X (written $x_\alpha \to x$) if for each neighborhood U of x, there exists $\beta \in \Lambda$ such that for all $\gamma \geq \beta$, then $x_\gamma \in U$.*

Definition 1.14. *Let (X, d) be a metric space. A sequence (x_n) of points of X is said to be a* ***Cauchy sequence*** *in (X, d) if it has the property that given $\epsilon > 0$, there is an integer N such that*

$$d(x_n, x_m) < \epsilon \text{ whenever } n, m \geq N.$$

The metric space (X, d) is said to be ***complete*** *if every Cauchy sequence in X converges to some point in X.*

Example 1.15. *Let X be a metric space.*
Every convergent sequence in X is necessarily a Cauchy sequence.

Definition 1.16. *Let X and Y be topological spaces.*
A mapping $T : X \to Y$ is said to be ***continuous*** *if for each open subset V of Y, the set $T^{-1}(V) = \{x \in X : T(x) \in V\}$ is an open subset of X.*

Theorem 1.17 ([4], p.104)**.** *Let X and Y be topological spaces; let $T : X \to Y$. Then the followings are equivalent:*

1. *T is continuous;*
2. *For every open set C of Y, the set $T^{-1}(C)$ is open in X;*
3. *For every subset A of X, one has $T(\overline{A}) \subseteq \overline{T(A)}$;*
4. *For every closed set B of Y, the set $T^{-1}(B)$ is closed in X;*
5. *For each $x \in X$ and each neighborhood V of $T(x)$, there is a neighborhood U of x such that $T(U) \subseteq V$.*

If the condition in 5 holds for the point x of X, we say that T is ***continuous at the point*** *x.*

Definition 1.18. *A topological space X is said to be* ***Hausdorff*** *if each pair x, y of distinct points of X, there exist disjoint open sets containing x and y, respectively.*

Definition 1.19. *Suppose that one-point sets are closed in X. Then X is said to be* ***regular*** *if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.*

Definition 1.20. *Suppose that one-point sets are closed in X. Then X is said to be **completely regular** if for each pair consistiong of a point x and a closed set B disjoint from x, there exists a continuous mapping $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for each $y \in B$.*

Remark 1.21. *Every regular space is Hausdorff, and that a completely regular space is regular.*

Example 1.22. *Every metric space is regular.*

1.2 Banach spaces

Definition 1.23. *A set X is called a **vector space** (or a **linear space**) over $\mathbb{R}$ if we have a mapping $+$ from $X \times X$ to X and a mapping $\cdot$ from $\mathbb{R} \times X$ to X that satisfy the following conditions :*

1. $x + y = y + x$ *for all* $x, y \in X$.
2. $(x + y) + z = x + (y + z)$ *for all* $x, y, z \in X$.
3. *There is a vector* $0 \in X$ *such that* $x + 0 = x$ *for all* $x \in X$.
4. $\lambda(x + y) = \lambda x + \lambda y$ *for all* $\lambda \in \mathbb{R}$ *and* $x, y \in X$.
5. $(\lambda + \mu)x = \lambda x + \mu x$ *for all* $\lambda, \mu \in \mathbb{R}$ *and* $x \in X$.
6. $\lambda(\mu x) = (\lambda\mu)x$ *for all* $\lambda, \mu \in \mathbb{R}$ *and* $x \in X$.
7. $0 \cdot x = 0$ *and* $1 \cdot x = x$ *for all* $x \in X$.

*We call $+$ addition and $\cdot$ multiplication by scalars. Suppose that Y is a nonempty subset of X. We say that Y is a **subspace of** X if for each $x, y \in Y$ and $\alpha, \beta \in \mathbb{R}$, $\alpha x + \beta y$ belongs to Y.*

Definition 1.24. *A mapping $\|\cdot\|$ from a vector space X to $\mathbb{R}$ is called a **norm** if*

1. $\|x\| \geq 0$ *for all* $x \in X$*; the equality holds if and only if* $x = 0$.

2. $\|x+y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$.

3. $\|\alpha x\| = |\alpha| \, \|x\|$ *for all* $\alpha \in \mathbb{R}$ *and* $x \in X$.

We say that X *with a norm* $\|\cdot\|$ *is a* ***normed space*** *(or* ***normed vector space****), denoted by* $(X, \|\cdot\|)$.

Proposition 1.25. *Let* $d_{\|\cdot\|} : X \times X \to \mathbb{R}$ *be defined by* $d_{\|\cdot\|}(x,y) = \|x-y\|$. *Then* $d_{\|\cdot\|}$ *is a metric on* X*, so* $(X, d_{\|\cdot\|})$ *is a metric space. Therefore every normed space is a metric space.*

Example 1.26. *For* $0 < p < \infty$, $\ell_p = \{(x_n) \subseteq \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^p < \infty\}$ *and* $\|\cdot\|_p : \ell_p \to \mathbb{R}$ *be defined by* $\|(x_n)\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$. *Thus* ℓ_p *is a vector space (over* $\mathbb{R}$*) and* $\|\cdot\|_p$ *is a norm on* ℓ_p. *So,* ℓ_p *with this norm is a normed space.*

Definition 1.27. *A normed vector space* $(X, \|\cdot\|)$ *is a* ***Banach space*** *if* $(X, d_{\|\cdot\|})$ *is a complete metric space. If* (x_n) *is a sequence in* X*, the series* $\sum_{i=1}^{\infty} x_i$ *(or* $\sum x_n$*) is said to be* ***summable*** *if a sequence of partial sum* $(\sum_{i=1}^{n} x_i)$ *converges to some point in* X*, and it is called* ***absolutely summable*** *if* $\sum \|x_n\| < \infty$.

Theorem 1.28 ([2], p. 152)**.** *A normed vector space* X *is complete if and only if every absolutely summable series in* X *is summable.*

Example 1.29. *For any* $1 \leq p < \infty$, ℓ_p *is complete and hence* ℓ_p *is a Banach space.*

Definition 1.30. *Let* X *and* Y *be normed vector spaces and* $T : X \to Y$. *We say that* T *is a* ***linear mapping or linear operator*** *if for each* $x, y \in X$ *and* $\alpha, \beta \in \mathbb{R}$, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$. *In particular, if* $Y = \mathbb{R}$*, we call* T *a* ***linear functional****.*

A linear mapping T *is called* ***bounded*** *if there exists* $C > 0$ *such that for all* $x \in X$, $\|T(x)\| \leq C \|x\|$.

Proposition 1.31 ([2], p. 153)**.** *If* X *and* Y *are normed vector spaces and* $T : X \to Y$ *a linear mapping, the following are equivalent :*

1. T *is continuous;*

2. T *is continuous at 0;*

3. T *is bounded.*

Definition 1.32. *If X and Y are normed vector spaces, we denote the space of all bounded linear mappings from X to Y by $L(X,Y)$. Thus $L(X,Y)$ is a vector space. Let $\|\cdot\| : L(X,Y) \to \mathbb{R}$ be defined by*

$$\begin{aligned}\|T\| &= \sup\{\|T(x)\| : x \in X \text{ and } \|x\| = 1\} \\ &= \sup\{\frac{\|T(x)\|}{\|x\|} : x \in X \text{ and } x \neq 0\},\end{aligned}$$

*for all $T \in L(X,Y)$. Then $\|\cdot\|$ is a norm on $L(X,Y)$ and called the **operator norm**. Hence, $L(X,Y)$ with the operator norm is a normed vector space. In particular, The space $L(X,\mathbb{R})$ of bounded linear functional on X is called the **dual space** of X and denoted by X^*.*

Remark 1.33. *Every dual space of a normed vector space with the operator norm is a Banach space.*

Example 1.34. *Let $1 \leq p < \infty$ and $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$.*
Then the dual space of ℓ_p is isometrically isomorphic to ℓ_q; i.e., for each $f \in (\ell_p)^*$ there exists $(x_n) \in \ell_q$ such that $f(y_n) = \sum x_n y_n$ for all $(y_n) \in \ell_p$ and $\|f\| = \|(x_n)\|_q$.

Definition 1.35. *Let X be a Hausdorff space and $T : X \to X$ a continuous mapping. We say that*

1. $F(T) = \{x \in X : Tx = x\}$ *is the **fixed point set of** T.*

2. $C(T) = \{x \in X : \text{the sequence } (T^n x) \text{ converges}\}$ *is the **convergence set of** T.*

When $F(T) \neq \varnothing$, let $T^\infty : C(T) \to F(T)$ be defined by $T^\infty x = lim_{n\to\infty} T^n x$ for all $x \in C(T)$.

Definition 1.36. *Let (X, d) and (Y, ρ) be metric spaces and $T : X \to Y$. T is said to be*

1. ***nonexpansive*** *if $\rho(Tx, Ty) \leq d(x, y)$ for any $x, y \in X$;*
2. ***quasi-nonexpansive*** *if $T(X) \subseteq X \subseteq Y$, $F(T) \neq \varnothing$ and $\rho(Tx, p) \leq d(x, p)$ for any $x \in X$ and $p \in F(T)$;*
3. ***lipschitzian*** *if there is $k \geq 0$ such that $\rho(Tx, Ty) \leq kd(x, y)$ for any $x, y \in X$;*
4. ***uniformly lipschitzian*** *if $T(X) \subseteq X \subseteq Y$ and there is $k \geq 0$ such that $\rho(T^n x, T^n y) \leq kd(x, y)$ for any $x, y \in X$ and $n \in \mathbb{N}$.*

Definition 1.37. [1] *Let X be a nonempty Hausdorff space and $T : X \to X$ a continuous mapping. A fixed point x of T is said to be **virtually T-stable** if for each neighborhood U of x, there exist a neighborhood V of x and an increasing sequence (k_n) of positive integers such that $T^{k_n}(V) \subseteq U$ for all $n \in \mathbb{N}$. We simply call T **virtually stable** if every fixed point of T is virtually T-stable.*

Theorem 1.38. [1] *Suppose that X is a regular space and $T : X \to X$ a selfmapping with $F(T) \neq \varnothing$. If T is a virtually stable, then T^∞ is continuous.*

1.3 Weak topology

Proposition 1.39 ([3], p. 203)**.** *Let X be a set and let $\mathcal{F}$ be a family of mappings and $\{(Y_f, \mathcal{T}_f) : f \in \mathcal{F}\}$ a family of topological spaces such that for each $f \in \mathcal{F}$, $f(X) \subseteq Y_f$. Then there is the smallest topology for X with respect to which each member of $\mathcal{F}$ is continuous. That is, there is a unique topology $\mathcal{T}_\mathcal{F}$ for X such that the followings hold :*

1. *For each $f \in \mathcal{F}$, f is a continuous mapping from $(X, \mathcal{T}_\mathcal{F})$ into $(Y_f, \mathcal{T}_f)$.*
2. *If $\mathcal{T}$ is any topology for X such that for each $f \in \mathcal{F}$, f is a continuous mapping from $(X, \mathcal{T})$ into $(Y_f, \mathcal{T}_f)$, then $\mathcal{T}_\mathcal{F} \subseteq \mathcal{T}$.*

The topology $\mathcal{T}_{\mathcal{F}}$ has $\{f^{-1}(U) : f \in \mathcal{F}, U \in \mathcal{T}_f\}$ as a subbasis, and it is called ***the weak topology on X induced by $\mathcal{F}$****.*

Definition 1.40. *Let X be a normed space. Then the topology for X induced by the dual space of X (X^*) is* ***the weak topology*** *on X and also the topology induced by its norm is called* ***strong topology****.*

Definition 1.41. *Let Y be a subset of a normed space X.* ***The weak topology on*** *Y is a subspace topology of the weak topology on X*

Theorem 1.42 ([3], p. 215)**.** *Let X be a normed space. If X has finite dimension, the weak topology and strong topology are the same.*

Remark 1.43 ([3], p. 212)**.** *The weak topology on a normed space is completely regular.*

Definition 1.44. *Let X be a normed space and O a subset of X. We say that*

1. *O is* ***weakly open*** *if O belongs to the weak topology.*
2. *O is* ***weakly closed*** *if $X - O$ is weakly open.*
3. *For $x \in X$, O is a* ***weak neighborhood of*** *x if O is a weak open set containing x.*

Definition 1.45. *Let X and Y be normed spaces. A mapping $T : X \to Y$ is said to be* ***weakly continuous*** *(or* ***weak-to-weak continuous****) if for each weak open subset V of Y, the set $T^{-1}(V)$ is a weak open subset of X.*

Theorem 1.46. *Let X and Y be normed spaces. A mapping $T : X \to Y$ is weakly continuous if and only if for any $f \in Y^*$, $f \circ T$ is a weakly continuous functional.*

Definition 1.47. *Let X be a normed space and $T : X \to X$ a weakly continuous mapping. A fixed point x of T is said to be* ***weakly virtually T-stable*** *if for each weak neighborhood U of x, there exist a weak neighborhood V of x and an increasing sequence (k_n) of positive integers such that $T^{k_n}(V) \subseteq U$ for all $n \in \mathbb{N}$.*

*We simply call T **weakly virtually stable** if every fixed point of T is weakly virtually T-stable.*

Proposition 1.48 ([3], p. 212)**.** *A linear functional on a normed space is continuous with respect to the weak topology if and only if it is continuous with respect to the metric induced by its norm.*

Definition 1.49. [3] *A sequence (x_n) in an infinite-dimensional Banach space X is a **Schauder basis** for X if for each x in X there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n x_n$.*

Example 1.50. *Let $i \in \mathbb{N}$ and $e_i = (0, 0, \dots, 0, \overbrace{1}^{i^{th}}, 0, \dots)$. Then $e_i \in \ell_p$ for all $1 < p < \infty$ and a sequence (e_n) in ℓ_p is a Schauder basis for ℓ_p for all $1 < p < \infty$.*

Theorem 1.51. *([3], p. 351) If (x_n) is a Schauder basis for an infinite-dimensional Banach space, then $\|x_n\|^{-1} x_n$ is a Schauder basis for the space, so there is a Schauder basis (e_n) such that $\|e_i\| = 1$ for any $i \in \mathbb{N}$.*

Definition 1.52. *Let X be a Banach space. We say that X has a **normalized Schauder basis** if there is a Schauder basis (e_n) for X such that $\|e_i\| = 1$ for any $i \in \mathbb{N}$.*

Definition 1.53. *Let X be an infinite-dimensional Banach space with a Schauder basis (x_n). For each positive integer m, the m^{th} **coordinate functional** x_m^* for (x_n) is the mapping $\sum_n \alpha_n x_n \longmapsto \alpha_m$ from X into $\mathbb{R}$.*

Theorem 1.54. [3] *Each coordinate functional associated with a basis for Banach space is a continuous linear functional.*

CHAPTER II
WEAK CONTINUITY

It is well-known that every lipschitzian mapping is (uniformly) continuous. However, if $T : \ell_2 \to \ell_2$ defined by $T(x_1, x_2, \dots) = (\|(x_1, x_2, \dots)\|_2, 0, 0, \dots)$ for $(x_1, x_2, \dots) \in \ell_2$, then T is lipschitzian but T is not weakly continuous. Therefore, being lipschitzian does not imply the weak continuity. In this chapter, we will present some conditions that guarantee the weak continuity.

Thrughout the chapter let E be a Banach space, $X \subseteq E$ and $T : X \to E$.

Definition 2.1. *We say that T is* ***functionally lipschitzian*** *if for each f in E^* there exist $N \in \mathbb{N}$ and $g_1, g_2, \dots, g_N \in E^*$ such that*

$$|f(Tx - Ty)| \leq \sum_{i=1}^{N} |g_i(x - y)|$$

for any $x, y \in X$.

Proposition 2.2. *If T_1, T_2 are functionally lipschitzian and $a \in \mathbb{R}$, then $T_1 + T_2$ and $a \cdot T_1$ are functionally lipchitzian.*

Proof. It is easy to see that they are functionally lipschitzian because for each $f \in E^*$, $f((T_1 + T_2)x) = f(T_1x) + f(T_2x)$ and $f(a \cdot T_1x) = a \cdot f(T_1x)$. □

Theorem 2.3. *If T is functionally lipschitzian, then T is weakly continuous.*

Proof. Let $z \in X$ and U a weak neighborhood of Tz. Without loss of generality, we may assume that

$$U = \bigcap_{i=1}^{n} f_i^{-1}(f_i(Tz) - \epsilon, f_i(Tz) + \epsilon)$$

for some $\epsilon > 0$ and $f_1, f_2, .., f_n \in E^*$. Since T is functionally lipschitzian, for each $i = 1, \ldots, n$ there exist $N_i \in \mathbb{N}$ and $g_{(i,1)}, \ldots, g_{(i,N_i)} \in E^*$ such that

$$|f_i(Tx - Ty)| \leq \sum_{j=1}^{N_i} |g_{(i,j)}(x - y)|$$

for any $x, y \in X$. Then

$$V = \Big(\bigcap_{i=1}^{n} \bigcap_{j=1}^{N_i} g_{(i,j)}^{-1} \Big(g_{(i,j)}(z) - \frac{\epsilon}{N_i}, g_{(i,j)}(z) + \frac{\epsilon}{N_i} \Big) \Big) \cap X$$

is a weak neighborhood of z. To see that $T(V) \subseteq U$, let $y \in V$. Then

$$|f_i(Ty - Tz)| \leq \sum_{j=1}^{N_i} |g_{(i,j)}(y - z)| < \epsilon;$$

i.e., $Ty \in U$. Hence T is weakly continuous. $\square$

In the next theorem, we will show that when E has finite dimension, functionally lipschitzian and lipschitzian conditions are equivalent.

Theorem 2.4. *Suppose that E is finite dimensional. Then T is functionally lipschitzian if and only if T is lipschitzian.*

Proof. Let $\{e_1, \ldots, e_N\}$ be a normalized basis for E.
$(\Rightarrow)$ Assume that T is functionally lipschitzian and let $x, y \in X$. Then

$$\begin{aligned} \|Tx - Ty\| &\leq \sum_{i=1}^{N} |e_i^*(Tx - Ty)| \, \|e_i\| \\ &= \sum_{i=1}^{N} |e_i^*(Tx - Ty)|. \end{aligned}$$

Since T is functionally lipschitzian, for each $i = 1, 2, \ldots, N$, there exist $K_i \in \mathbb{N}$ and $g_{(i,1)}, g_{(i,2)}, \ldots, g_{(i,K_i)} \in E^*$ such that

$$|e_i^*(Tx - Ty)| \leq \sum_{m=1}^{K_i} |g_{(i,m)}(x - y)| \leq (\sum_{m=1}^{K_i} \left\| g_{(i,m)} \right\|) \left\| x - y \right\|.$$

Then

$$\|Tx - Ty\| \leq (\sum_{i=1}^{N} \sum_{m=1}^{K_i} \left\| g_{(i,m)} \right\|) \left\| x - y \right\|.$$

Thus T is lipschitzian.

$(\Leftarrow)$ Assume that T is lipschitzian and let $f \in E^*$ and $x, y \in X$.

Since T is lipschitzian, there is $L > 0$ such that

$$\|Tx - Ty\| \leq L \|x - y\| .$$

Then

$$\begin{aligned} |f(Tx - Ty)| &\leq \|f\| \|Tx - Ty\| \\ &\leq \|f\| L \|x - y\| \\ &\leq \|f\| L \sum_{i=1}^{N} |e_i^*(x - y)| \|e_i\| \\ &= \sum_{i=1}^{N} |(\|f\| L) e_i^*(x - y)|. \end{aligned}$$

Hence T is functionally lipschitzian. □

When E is infinite dimensional, the following theorem gives some criteria for being functionally lipschitzian.

Theorem 2.5. *Suppose that E is infinite dimensional with a normalized Schauder basis (e_n). Then T is functionally lipschitzian, if one of the followings holds :*

1. *There exist $N \in \mathbb{N}$, $g_1, g_2, \ldots, g_N \in E^*$ and (c_n) a sequence of non-negative real numbers such that $\sum_n c_n < \infty$ and for each $i \in \mathbb{N}$,*

$$|e_i^*(Tx - Ty)| \leq c_i \sum_{j=1}^{N} |g_j(x - y)|$$

for any $x, y \in X$.

2. *There exists $k \in \mathbb{N}$ such that for each $i \leq k$, there exist $N_i \in \mathbb{N}$ and $g_{(i,1)}, \ldots, g_{(i,N_i)} \in E^*$ such that for any $x, y \in X$*

$$|e_i^*(Tx - Ty)| \leq \sum_{j=1}^{N_i} |g_{(i,j)}(x - y)|$$

and for each $i > k$, $e_i^* \circ T = a \cdot e_i^* + b_i$ for some $a, b_i \in \mathbb{R}$.

Proof. (1) For each $f \in E^*$ and $x, y \in X$, we have

$$\begin{aligned}
|f(Tx - Ty)| &\leq \|f\| \, \|Tx - Ty\| \\
&\leq \|f\| \sum_{i=1}^{\infty} |e_i^*(Tx - Ty)| \, \|e_i\| \\
&\leq \|f\| \sum_{i=1}^{\infty} (c_i \sum_{j=1}^{N} |g_j(x - y)|) \\
&\leq \|f\| \, (\sum_{i=1}^{\infty} c_i) \sum_{j=1}^{N} |g_j(x - y)| \\
&= \sum_{j=1}^{N} | \, \|f\| \, (\sum_{i=1}^{\infty} c_i) \cdot g_j(x - y)|.
\end{aligned}$$

Hence T is functionally lipschitzian.

(2) For each $f \in E^*$ and $x, y \in X$, we have

$$\begin{aligned}
|f(\sum_{i=k+1}^{\infty} a \cdot e_i^*(x - y)e_i)| &= |a||f(\sum_{i=1}^{\infty} e_i^*(x - y)e_i) - f(\sum_{i=1}^{k} e_i^*(x - y)e_i)| \\
&\leq |a||f(\sum_{i=1}^{\infty} e_i^*(x - y)e_i)| + |a||f(\sum_{i=1}^{k} e_i^*(x - y)e_i)| \\
&= |a \cdot f(x - y)| + |a| \, \|f\| \sum_{i=1}^{k} |e_i^*(x - y)|
\end{aligned}$$

and

$$\begin{aligned}
|f(Tx - Ty)| &= |f(\sum_{i=1}^{\infty} e_i^*(Tx - Ty)e_i)| \\
&\leq |f(\sum_{i=1}^{k} e_i^*(Tx - Ty)e_i)| + |f(\sum_{i=k+1}^{\infty} e_i^*(Tx - Ty)e_i)| \\
&\leq \sum_{i=1}^{k} |f(e_i)||e_i^*(Tx - Ty)| + |f(\sum_{i=k+1}^{\infty} a \cdot e_i^*(x - y)e_i)| \\
&\leq \|f\| \sum_{i=1}^{k} \sum_{j=1}^{N_i} |g_{(i,j)}(x - y)| + |a| \, \|f\| \sum_{i=1}^{k} |e_i^*(x - y)| \\
&\quad + |a \cdot f(x - y)|.
\end{aligned}$$

Hence T is functionally lipschitzian. $\square$

Corollary 2.6. *Suppose that E is infinite dimensional with a normalized Schauder basis (e_n). If there exist $N \in \mathbb{N}$, $g_1, \dots, g_N \in E^*$ and for each $i \in \mathbb{N}$, there are lipschitzian self-mappings $h_{(i,1)}, \dots, h_{(i,N)}$ of $\mathbb{R}$ whose Lipschitz constants are $L_{(i,1)}, \dots, L_{(i,N)}$, respectively, such that*

$$e_i^* \circ T = \sum_{j=1}^{N} h_{(i,j)} \circ g_j|_X$$

and $\displaystyle\sum_{i=1}^{\infty} \max\{L_{(i,j)} : j = 1, \dots, N\} < \infty,$ *then T is functionally lipschitzian.*

Proof. Let $i \in \mathbb{N}$, $x, y \in X$. Then

$$\begin{aligned}
|e_i^*(Tx - Ty)| &\le |h_{(i,1)}(g_1(x)) - h_{(i,1)}(g_1(y))| + \cdots \\
&\quad + |h_{(i,N)}(g_N(x)) - h_{(i,N)}(g_N(y))| \\
&\le L_{(i,1)}|g_1(x-y)| + \cdots + L_{(i,N)}|g_N(x-y)| \\
&\le \max\{L_{(i,j)} : j = 1, \dots, N\} \sum_{k=1}^{N} |g_k(x-y)|.
\end{aligned}$$

By Theorem 2.5(1), T is functionally lipschitzian. □

The followings are some explicit examples of functionally lipschitzian (and hence, weakly continuous) mappings.

Example 2.7. *Let $1 < p < \infty$, $i \in \mathbb{N}$ and $T : \ell_p \to \mathbb{R}$ be defined by*

$$T_i(x_1, x_2, \dots) = \begin{cases} \frac{1}{2^i} \sin(x_1 + x_2) & \text{, if } i \text{ is odd ;} \\ \frac{1}{2^i} |x_3 - x_4| & \text{, if } i \text{ is even} \end{cases}$$

for any $(x_1, x_2, \dots) \in \ell_p$. Let $T : \ell_p \to \ell_p$ be defined by

$$T(x_1, x_2, \dots) = (T_1(x_1, x_2, \dots), T_2(x_1, x_2, \dots), \dots)$$

for any $(x_1, x_2, \dots) \in \ell_p$. Let $N = 2$,

$$g_1 = e_1^* + e_2^*, \quad g_2 = e_3^* - e_4^* \quad \text{and} \quad c_n = \frac{1}{2^n}.$$

Notice that for each $x \in \ell_p$,

$$Tx = \sum_{i=1}^{\infty} \Big(\frac{1}{2^{2i-1}} \sin(g_1(x))e_{2i-1} + \frac{1}{2^{2i}}|g_2(x)|e_{2i}\Big).$$

Since

$$\sum_{i=1}^{\infty} \Big\|\frac{1}{2^{2i-1}} \sin(g_1(x))e_{2i-1} + \frac{1}{2^{2i}}|g_2(x)|e_{2i}\Big\|_p \leq \sum_{i=1}^{\infty} \Big(\frac{1}{2^{2i-1}}|\sin(g_1(x))| + \frac{1}{2^{2i}}|g_2(x)|\Big),$$

$\displaystyle\sum_{i=1}^{\infty} \Big(\frac{1}{2^{2i-1}} \sin(g_1(x))e_{2i-1}+\frac{1}{2^{2i}}|g_2(x)|e_{2i}\Big)$ *is an absolutely summable series of* ℓ_p.

Then $Tx \in \ell_p$. *By Theorem* 2.5(1), T *is functionally lipschitzian.*

Example 2.8. *Let* $1 < p < \infty$ *and* $T : \ell_p \to \ell_p$ *be defined by*

$$T(x_1, x_2, \dots) = \Big(\frac{|x_1 + x_3|}{3}, \frac{|x_2 + x_4|}{3}, x_3, x_4, x_5, \dots\Big)$$

for any $(x_1, x_2, \dots) \in \ell_p$. *By Theorem* 2.5(2), T *is functionally lipschitzian by letting* $k = 2$, $N_1 = N_2 = 1$,

$$g_1 = \frac{1}{3}(e_1^* + e_3^*) \quad \text{and} \quad g_2 = \frac{1}{3}(e_2^* + e_4^*).$$

Example 2.9. *Let* $1 < p < \infty$ *and* $T : \ell_p \to \ell_p$ *be defined by*

$$T(x_1, x_2, \dots) = \Big(\frac{|x_1 + x_3|}{3}, \frac{|x_2 + x_4|}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\Big)$$

for any $(x_1, x_2, \dots) \in \ell_p$. *By Theorem* 2.5(2), T *is functionally lipschitzian by letting* $k = 2$, $N_1 = N_2 = 1$, $a = 0$,

$$b_{i+2} = \frac{1}{i} \quad \text{for any } i \in \mathbb{N}, \quad g_1 = \frac{1}{3}(e_1^* + e_3^*) \quad \text{and} \quad g_2 = \frac{1}{3}(e_2^* + e_4^*).$$

Example 2.10. *Let* $1 < p < \infty$, $i \in \mathbb{N}$ *and* $T_i : \ell_p \to \mathbb{R}$ *be defined by*

$$T_i(x_1, x_2, \dots) = \begin{cases} \frac{\sin(x_1)}{2^{i-1}} + \frac{|x_3+x_4-x_5|}{3^{i-1}} & , \text{ if } i \text{ is odd ;} \\ \frac{|x_1|}{2^{i-1}} + \frac{\sin(x_3+x_4-x_5)}{3^{i-1}} & , \text{ if } i \text{ is even} \end{cases}$$

for any $(x_1, x_2, \dots) \in \ell_p$. *Let* $T : \ell_p \to \ell_p$ *be defined by*

$$T(x_1, x_2, \dots) = (T_1(x_1, x_2, \dots), T_2(x_1, x_2, \dots), \dots)$$

for any $(x_1, x_2, \dots) \in \ell_p$. *Let* $N = 2$, $i \in \mathbb{N}$, $g_1 = e_1^*$, $g_2 = e_3^* + e_4^* - e_5^*$,

$$h_{(2i-1,1)} = \frac{1}{2^{2i-2}} \sin, \quad h_{(2i-1,2)} = \frac{1}{3^{2i-2}} |\cdot|, \quad h_{(2i,1)} = \frac{1}{2^{2i-1}} |\cdot|, \quad h_{(2i,2)} = \frac{1}{3^{2i-1}} \sin.$$

Then $Tx \in \ell_p$ *because the series*

$$\sum_{i=1}^{\infty} \Big(h_{(i,1)}(g(x))e_i + h_{(i,2)}(g_2(x))e_i \Big) \quad \textit{is absolutely summable series of} \quad \ell_p.$$

Let $L_{(i,j)}$ *be a Lipschitz constant of* $h_{(i,j)}$. *It is easy to see that*

$$L_{(i,1)} \leq \frac{1}{2^{i-1}} \quad \textit{and} \quad L_{(i,2)} \leq \frac{1}{3^{i-1}}.$$

Thus

$$\max\{L_{(i,1)}, L_{(i,2)}\} \leq \frac{1}{2^{i-1}}, \quad \textit{which implies} \quad \sum_{i=1}^{\infty} \max\{L_{(i,1)}, L_{(i,2)}\} < \infty.$$

By Corollary 2.6, T *is functionally lipschitzian.*

CHAPTER III
WEAK VIRTUAL STABILITY

Let E be a (real) Banach space , $X \subseteq E$ and $T : X \to X$ a self-mapping with $F(T) \neq \varnothing$. It is proved in [1] that nonexpansiveness is a condition that guarantees the strong virtual stability. In this chapter, we will present some conditions on T that guarantee the weak virtual stability.

Definition 3.1. *We say that*

1. *T is* ***functionally uniformly lipschitzian*** *if for each $f \in E^*$ there exist $N \in \mathbb{N}$ and $g_1, g_2, \ldots, g_N \in E^*$ such that for any $n \in \mathbb{N}$,*

$$|f(T^n x - T^n y)| \leq \sum_{i=1}^{N} |g_i(x-y)|$$

   for any $x, y \in X$.

2. *T is* ***functionally uniformly quasi-lipschitzian*** *if for each $f \in E^*$ there exist $N \in \mathbb{N}$ and $g_1, g_2, \ldots, g_N \in E^*$ such that for any $n \in \mathbb{N}$,*

$$|f(T^n x - T^n y)| \leq \sum_{i=1}^{N} |g_i(x-y)|$$

   for any $y \in X$ and x in $F(T)$.

Notice that every functionally uniformly lipschitzian mapping is functionally lipschitzian and functionally uniformly quasi-lipschitzian.

Theorem 3.2. *If T is functionally uniformly quasi-lipchitzian and weakly continuous, then T is weakly virtually stable.*

Proof. By the assumption, for each $n \in \mathbb{N}$, T^n is weakly continuous. Let $z \in F(T)$ and U a weak neighborhood of z. Without loss of generality, we may assume that

$$U = \bigcap_{i=1}^{n} f_i^{-1}(f_i(z) - \epsilon, f_i(z) + \epsilon)$$

for some $\epsilon > 0$ and $f_1, f_2, \ldots, f_n \in E^*$. Since T is functionally uniformly quasi-lipschitzian, for each i there exist $N_i \in \mathbb{N}$ and $g_{(i,1)}, \ldots, g_{(i,N_i)} \in E^*$ such that

$$|f_i(T^n x - T^n y)| \leq \sum_{j=1}^{N_i} |g_{(i,j)}(x-y)|$$

for any $n \in \mathbb{N}$, $y \in X$ and $x \in F(T)$. Then

$$V = \Big(\bigcap_{i=1}^{n} \bigcap_{j=1}^{N_i} g^{-1}_{(i,j)} \Big(g_{(i,j)}(z) - \frac{\epsilon}{N_i}, g_{(i,j)}(z) + \frac{\epsilon}{N_i} \Big) \Big) \cap X$$

is a weak neighborhood of z. To see that for each $n \in \mathbb{N}$, $T^n(V) \subseteq U$, let $n \in \mathbb{N}$ and $y \in V$. Then

$$|f_i(T^n y - z)| = |f_i(T^n y - T^n z)| \leq \sum_{j=1}^{N_i} |g_{(i,j)}(y - z)| < \epsilon;$$

i.e., $T^n y \in U$. Hence, T is weakly virtually stable with respect to the sequence of all natural numbers. □

Corollary 3.3. *If T is functionally uniformly lipchitzian, then T is weakly virtually stable.*

Proof. We have T is functionally uniformly quasi-lipchitzian and functionally lipchitzian. Then T is functionally uniformly quasi-lipchitzian and weakly continuous, by Theorem 3.2, T is weakly virtually stable. □

In the next theorem, we will show that when E has finite dimension, functionally uniformly lipschitzian and uniformly lipschitzian conditions are equivalent.

Theorem 3.4. *Suppose that E is a finite-dimensional Banach space with a normalized basis $(e_1, e_2, \ldots, e_N)$. Then T is functionally uniformly lipschitzian if and only if T is uniformly lipschitzian.*

Proof. ($\Rightarrow$) Assume that T is functionally uniformly lipschitzian.
Let $n \in \mathbb{N}$ and $x, y \in X$. Then

$$\begin{aligned} \|T^n x - T^n y\| &\leq \sum_{i=1}^{N} |e_i^*(T^n x - T^n y)| \, \|e_i\| \\ &= \sum_{i=1}^{N} |e_i^*(T^n x - T^n y)|. \end{aligned}$$

Since T is functionally uniformly lipschitzian, for each $i = 1, 2, \ldots, N$, there exist $K_i \in \mathbb{N}$ and $g_{(i,1)}, g_{(i,2)}, \ldots, g_{(i,K_i)} \in E^*$ such that for all $n \in \mathbb{N}$

$$|e_i^*(T^n x - T^n y)| \leq \sum_{m=1}^{K_i} |g_{(i,m)}(x-y)| \leq (\sum_{m=1}^{K_i} \|g_{(i,m)}\|) \|x - y\| .$$

Then

$$\|T^n x - T^n y\| \leq (\sum_{i=1}^{N} \sum_{m=1}^{K_i} \|g_{(i,m)}\|) \|x - y\| ; \quad \text{i.e., } T \text{ is uniformly lipschitzian.}$$

($\Leftarrow$) Assume that T is uniformly lipschitzian.
Let $f \in E^*$. Since T is uniformly lipschitzian, there is $L > 0$ such that for each $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \leq L \|x - y\| \quad \text{for all } x, y \in X.$$

Then for each $n \in \mathbb{N}$ and $x, y \in X$

$$\begin{aligned} |f(T^n x - T^n y)| &\leq \|f\| \|T^n x - T^n y\| \\ &\leq \|f\| L \|x - y\| \\ &\leq \|f\| L \sum_{i=1}^{N} |e_i^*(x-y)| \|e_i\| \\ &= \sum_{i=1}^{N} |(\|f\| L) e_i^*(x-y)| \end{aligned}$$

Hence T is functionally uniformly lipschitzian. □

When E is infinite dimensional, the following theorem gives some criteria for being functionally uniformly lipschitzian.

Theorem 3.5. *Suppose that E is infinite dimensional with a normalized Schauder basis (e_n). Then T is functionally uniformly lipschitzian if one of the followings holds :*

1. *There exist $N \in \mathbb{N}$, $g_1, g_2, \ldots, g_N \in E^*$ and (c_n) a sequence of nonnegative real numbers such that $\sum_n c_n < \infty$ and for each $i, n \in \mathbb{N}$,*

$$|e_i^*(T^n x - T^n y)| \leq c_i \sum_{j=1}^{N} |g_j(x-y)|$$

for any $x, y \in X$.

2. There exists $k \in \mathbb{N}$ such that for each $i \leq k$, there exist N_i and $g_{(i,1)}, \ldots, g_{(i,N_i)} \in E^*$ such that for any $n \in \mathbb{N}$ and $x, y \in X$

$$|e_i^*(T^n x - T^n y)| \leq \sum_{j=1}^{N_i} |g_{(i,j)}(x-y)|$$

and for each $i > k$, $e_i^* \circ T = ae_i^* + b_i$ for some $|a| \leq 1$ and $b_i \in \mathbb{R}$.

Proof. (1) For each $f \in E^*$, $n \in \mathbb{N}$ and $x, y \in X$, we have

$$\begin{aligned}
|f(T^n x - T^n y)| &\leq \|f\| \, \|T^n x - T^n y\| \\
&\leq \|f\| \sum_{i=1}^{\infty} |e_i^*(T^n x - T^n y)| \, \|e_i\| \\
&\leq \|f\| \sum_{i=1}^{\infty} (c_i \sum_{j=1}^{N} |g_j(x-y)|) \\
&\leq \|f\| \, (\sum_{i=1}^{\infty} c_i) \sum_{j=1}^{N} |g_j(x-y)| \\
&= \sum_{j=1}^{N} | \, \|f\| \, (\sum_{i=1}^{\infty} c_i) \cdot g_j(x-y)|.
\end{aligned}$$

Hence T is functionally uniformly lipschitzian.

(2) For each $f \in E^*$, $n \in \mathbb{N}$ and $x, y \in X$, we have

$$\begin{aligned}
|f(\sum_{i=k+1}^{\infty} a^n \cdot e_i^*(x-y)e_i)| &= |a^n||f(\sum_{i=1}^{\infty} e_i^*(x-y)e_i) - f(\sum_{i=1}^{k} e_i^*(x-y)e_i)| \\
&\leq |a^n|(|f(\sum_{i=1}^{\infty} e_i^*(x-y)e_i)| + |f(\sum_{i=1}^{k} e_i^*(x-y)e_i)|) \\
&\leq |f(x-y)| + \|f\| \sum_{i=1}^{k} |e_i^*(x-y)|
\end{aligned}$$

and

$$\begin{aligned}
|f(T^n x - T^n y)| &= |f(\sum_{i=1}^{\infty} e_i^*(T^n x - T^n y)e_i)| \\
&\le |f(\sum_{i=1}^{k} e_i^*(T^n x - T^n y)e_i)| + |f(\sum_{i=k+1}^{\infty} e_i^*(T^n x - T^n y)e_i)| \\
&\le \sum_{i=1}^{k} |f(e_i)||e_i^*(T^n x - T^n y)| + |f(\sum_{i=k+1}^{\infty} a^n \cdot e_i^*(x-y)e_i)| \\
&\le \|f\| \sum_{i=1}^{k} \sum_{j=1}^{N_i} |g_{(i,j)}(x-y)| + \|f\| \sum_{i=1}^{k} |e_i^*(x-y)| + |f(x-y)|
\end{aligned}$$

Hence T is functionally uniformly lipschitzian. □

Corollary 3.6. *Suppose that E is infinite dimensional with a normalized Schauder basis (e_n). If there exist $N \in \mathbb{N}$, $g_1, \ldots, g_N \in E^*$ and for each $i \in \mathbb{N}$, there are lipschitzian self-mappings $h_{(i,1)}, \ldots, h_{(i,N)}$ of $\mathbb{R}$ whose Lipschitz constants are $L_{(i,1)}, \ldots, L_{(i,N)}$, respectively, such that*

$$e_i^* \circ T = \sum_{j=1}^{N} h_{(i,j)} \circ g_j|_X \quad \text{and} \quad (\sum_{i=1}^{\infty} \max\{L_{(i,j)} : j = 1, \ldots, N\}) \sum_{k=1}^{N} \|g_k\| \le 1,$$

then T is functionally uniformly lipschitzian.

Proof. Let $A_i = \max\{L_{(i,j)} : j = 1, \ldots, N\}$ and $B = \sum_{k=1}^{N} \|g_k\|$. Claim that for each $i, n \in \mathbb{N}$,

$$|e_i^*(T^n x - T^n y)| \le A_i(\sum_{i=1}^{\infty} A_i)^{n-1} B^{n-1} \sum_{k=1}^{N} |g_k(x-y)| \quad \text{for any } x, y \in X.$$

Let $i \in \mathbb{N}$ and $x, y \in X$. By Corollary 2.6,

$$|e_i^*(Tx - Ty)| \le A_i \sum_{k=1}^{N} |g_k(x-y)|.$$

Assume that for some $m \in \mathbb{N}$,

$$|e_i^*(T^m x - T^m y)| \le A_i(\sum_{i=1}^{\infty} A_i)^{m-1} B^{m-1} \sum_{k=1}^{N} |g_k(x-y)|.$$

Then

$$
\begin{aligned}
|e_i^*(T^{m+1}x - T^{m+1}y)| &\leq A_i(\sum_{i=1}^{\infty} A_i)^{m-1} B^{m-1} \sum_{k=1}^{N} |g_k(Tx - Ty)| \\
&\leq A_i(\sum_{i=1}^{\infty} A_i)^{m-1} B^{m-1} B \, \|Tx - Ty\| \\
&\leq A_i(\sum_{i=1}^{\infty} A_i)^{m-1} B^{m} \sum_{i=1}^{\infty} |e_i^*(Tx - Ty)| \, \|e_i\| \\
&\leq A_i(\sum_{i=1}^{\infty} A_i)^{m-1} B^{m} \sum_{i=1}^{\infty} \Big(A_i \sum_{k=1}^{N} |g_k(x - y)| \Big) \\
&= A_i(\sum_{i=1}^{\infty} A_i)^{m-1} B^{m} (\sum_{i=1}^{\infty} A_i) \sum_{k=1}^{N} |g_k(x - y)| \\
&= A_i(\sum_{i=1}^{\infty} A_i)^{m} B^{m} \sum_{k=1}^{N} |g_k(x - y)|.
\end{aligned}
$$

By the claim and the assumption, we have for each $i, n \in \mathbb{N}$

$$
|e_i^*(T^n x - T^n y)| \leq A_i \sum_{k=1}^{N} |g_k(x - y)| \qquad \text{for any } x, y \in X.
$$

Hence T is functionally uniformly lipschitzian, by Theorem 3.5(1). □

Corollary 3.7. *Suppose that E is infinite dimensional with a normalized Schauder basis (e_n). If there exists $k \in \mathbb{N}$ such that for each $i \leq k$, there are $g_i \in E^*$ and a lipschitzian self-mapping h_i of $\mathbb{R}$ (whose Lipschitz constant is L_i) such that*

$$
e_i^* \circ T = h_i \circ g_i|_X
$$

and for each $i > k$, $e_i^* \circ T = a e_i^* + b_i$ for some $|a| \leq 1$, $b_i \in \mathbb{R}$ and $\sum_{j=1}^{k} L_j \|g_j\| < 1$, then T is functionally uniformly lipschitzian.

Proof. Let $i \leq k$ and $B = \sum_{j=1}^{k} L_j \|g_j\|$. Claim that for each $n \geq 2$ and $x, y \in X$,

$$\begin{aligned}|e_i^*(T^n x - T^n y)| \leq L_i \|g_i\| \Big(& B^{n-2} \sum_{j=1}^{k} L_j |g_j(x-y)| \\ & + (\sum_{r=1}^{n-2} |a|^{n-1-r} B^{r-1}) \sum_{j=1}^{k} L_j |g_j(\sum_{l=k+1}^{\infty} e_l^*(x-y) e_l)| \Big) \\ & + |a|^{n-1} L_i |g_i(\sum_{l=k+1}^{\infty} e_l^*(x-y) e_l)|.\end{aligned}$$

Since

$$\begin{aligned}|g_i(Tx - Ty)| &= |g_i(\sum_{j=1}^{\infty} |e^* \circ T(x-y))e_j|)| \\ &\leq \|g_i\| \sum_{j=1}^{k} |e_j^* \circ T(x-y)| + |a||g_i(\sum_{l=k+1}^{\infty} e_l^*(x-y) e_l)| \\ &\leq \|g_i\| \sum_{j=1}^{k} L_j |g_j(x-y)| + |a||g_i(\sum_{l=k+1}^{\infty} e_l^*(x-y) e_l)|, \qquad (*)\end{aligned}$$

then

$$\begin{aligned}|e_i^*(T^2 x - T^2 y)| &\leq L_i |g_i(Tx - Ty)| \\ &\leq L_i \|g_i\| \sum_{j=1}^{k} L_j |g_j(x-y)| + |a| L_i |g_i(\sum_{l=k+1}^{\infty} e_l^*(x-y) e_l)|.\end{aligned}$$

Assume that

$$\begin{aligned}|e_i^*(T^m x - T^m y)| \leq L_i \|g_i\| \Big(& B^{m-2} \sum_{j=1}^{k} L_j |g_j(x-y)| \\ & + (\sum_{r=1}^{m-2} |a|^{m-1-r} B^{r-1}) \sum_{j=1}^{k} L_j |g_j(\sum_{l=k+1}^{\infty} e_l^*(x-y) e_l)| \Big) \\ & + |a|^{m-1} L_i |g_i(\sum_{l=k+1}^{\infty} e_l^*(x-y) e_l)|\end{aligned}$$

for some $m \geq 2$.

Then

$$
\begin{aligned}
|e_i^*(T^{m+1}x - T^{m+1}y)| &\le L_i \,\|g_i\| \Big(B^{m-2} \sum_{j=1}^{k} L_j |g_j(Tx - Ty)| \\
&\quad + (\sum_{r=1}^{m-2} |a|^{m-1-r} B^{r-1}) \sum_{j=1}^{k} L_j |g_j(\sum_{l=k+1}^{\infty} e_l^*(Tx - Ty) e_l)| \Big) \\
&\quad + |a|^{m-1} L_i |g_i(\sum_{l=k+1}^{\infty} e_l^*(Tx - Ty) e_l)| \\
&\le L_i \,\|g_i\| \Big(B^{m-2} \sum_{j=1}^{k} L_j \Big[\|g_i\| \sum_{j=1}^{k} L_j |g_j(x - y)| \\
&\quad + |a| |g_i(\sum_{l=k+1}^{\infty} e_l^*(x - y) e_l)| \Big] \\
&\quad + (\sum_{r=1}^{m-2} |a|^{m-1-r} B^{r-1}) \sum_{j=1}^{k} L_j |g_j(\sum_{l=k+1}^{\infty} e_l^*(x - y) e_l)| \Big) \\
&\quad + |a|^{m-1} L_i |g_i(\sum_{l=k+1}^{\infty} e_l^*(x - y) e_l)| \qquad \text{(by(*))} \\
&\le L_i \,\|g_i\| \Big(B^{m-1} \sum_{j=1}^{k} L_j |g_j(x - y)| \\
&\quad + |a| B^{m-2} \sum_{j=1}^{k} L_j |g_j(\sum_{l=k+1}^{\infty} e_l^*(x - y) e_l)| \\
&\quad + (\sum_{r=1}^{m-2} |a|^{n-r} B^{r-1}) \sum_{j=1}^{k} L_j |g_j(\sum_{l=k+1}^{\infty} e_l^*(x - y) e_l)| \Big) \\
&\quad + |a|^{m} L_i |g_i(\sum_{l=k+1}^{\infty} e_l^*(x - y) e_l)| \\
&\le L_i \,\|g_i\| \Big(B^{m-1} \sum_{j=1}^{k} L_j |g_j(x - y)| \\
&\quad + (\sum_{r=1}^{m-1} |a|^{m-r} B^{r-1}) \sum_{j=1}^{k} L_j |g_j(\sum_{l=k+1}^{\infty} e_l^*(x - y) e_l)| \Big) \\
&\quad + |a|^{m} L_i |g_i(\sum_{l=k+1}^{\infty} e_l^*(x - y) e_l)|.
\end{aligned}
$$

By the induction hypothesis, we have the claim.

Also, we have

$$|e_i^*(Tx - Ty)| \leq L_i |g_i(x-y)| \qquad \text{for any } x, y \in X.$$

Since

$$|g_i(\sum_{l=k+1}^{\infty} e_l^*(x-y)e_l)| \leq \sum_{l=1}^{k} ||g_i(e_l)|e_l^*(x-y)| + |g_i(x-y)|,$$

$B < 1$ and $|a| \leq 1$, for each $n \in \mathbb{N}$ and $x, y \in X$,

$$\begin{aligned}
|e_i^*(T^n x - T^n y)| \leq{}& L_i \, \|g_i\| \Big(B^{n-2} \sum_{j=1}^{k} L_j |g_j(x-y)| \\
&+ (\sum_{r=1}^{n-2} |a|^{n-1-r} B^{r-1}) \sum_{j=1}^{k} L_j |g_j(\sum_{l=k+1}^{\infty} e_l^*(x-y)e_l)| \Big) \\
&+ |a|^{n-1} L_i |g_i(\sum_{l=k+1}^{\infty} e_l^*(x-y)e_l)| \\
\leq{}& L_i \, \|g_i\| \Big(B^{n-2} \sum_{j=1}^{k} L_j |g_j(x-y)| \\
&+ (\sum_{r=1}^{n-2} |a|^{n-1-r} B^{r-1}) \sum_{j=1}^{k} L_j \Big(\sum_{l=1}^{k} ||g_j(e_l)|e_l^*(x-y)| + |g_j(x-y)| \Big) \Big) \\
&+ |a|^{n-1} L_i \Big(\sum_{l=1}^{k} ||g_i(e_l)|e_l^*(x-y)| + |g_i(x-y)| \Big) \\
\leq{}& L_i \, \|g_i\| \Big(\sum_{j=1}^{k} L_j |g_j(x-y)| \\
&+ (\sum_{r=1}^{\infty} B^{r-1}) \sum_{j=1}^{k} L_j \Big(\sum_{l=1}^{k} ||g_j(e_l)|e_l^*(x-y)| + |g_j(x-y)| \Big) \Big) \\
&+ L_i \sum_{l=1}^{k} ||g_i(e_l)|e_l^*(x-y)| + L_i |g_i(x-y)|.
\end{aligned}$$

By Theorem 3.5(2), T is functionally uniformly lipschitzian. $\square$

The followings are some explicit examples of functionally uniformly lipschitzian (and hence, are weakly virtually stable by Corollary 3.3) mappings.

Notice that such mappings T may not be nonexpansive.

Example 3.8. *Let $T : \ell_2 \to \ell_2$ be defined by*

$$T(x_1, x_2, \dots) = \Big(\frac{|x_1 + x_3|}{3}, \frac{|x_2 + x_4|}{3}, x_3, x_4, x_5, \dots\Big)$$

for any $(x_1, x_2, \dots) \in \ell_2$. By Example 2.8, T is functionally lipschitzian. Let $k = 2$, $a = 1$, $b_{i+2} = 1$ for any $i \in \mathbb{N}$,

$$g_1 = \frac{1}{3}(e_1^* + e_3^*), \quad g_2 = \frac{1}{3}(e_2^* + e_4^*), \quad \textit{and} \quad h_1(x) = h_2(x) = |x|.$$

Since

$$g_1(x_1, x_2, \dots) = \frac{1}{3}(x_1 + x_3) \quad \textit{and} \quad g_2(x_1, x_2, \dots) = \frac{1}{3}(x_2 + x_4),$$

we have $\|g_1\| = \|g_2\| = \frac{\sqrt{2}}{3}$. Let L_1, L_2 be Lipschitz constants of h_1, h_2, respectively. Then $L_1 = L_2 \leq 1$, so $L_1 \|g_1\| + L_2 \|g_2\| \leq \frac{2\sqrt{2}}{3} < 1$. By Corollary 3.7, T is functionally uniformly lipschitzian.
Suppose $T(x_1, x_2, \dots) = (x_1, x_2, \dots)$ for finding the fixed point set of T.

Then

$$\begin{aligned} x_1 &= \frac{1}{3}|x_1 + x_3| \geq 0 \\ 9x_1^2 &= (x_1 + x_3)^2 \\ 0 &= (4x_1 + x_3)(2x_1 - x_3) \end{aligned} \qquad \textit{and} \qquad \begin{aligned} x_2 &= \frac{1}{3}|x_2 + x_4| \geq 0 \\ 9x_2^2 &= (x_2 + x_4)^2 \\ 0 &= (4x_2 + x_4)(2x_2 - x_4). \end{aligned}$$

Therefore, we have $(x_1, x_2, \dots) \in F(T)$ if and only if ($x_3 = -4x_1$ or $x_3 = 2x_1$) and ($x_4 = -4x_2$ or $x_4 = 2x_2$) for any $x_1, x_2 \geq 0$. Thus

$$(1, 1, 2, 2, 0, \dots), (1, 1, -4, -4, 0, \dots) \in F(T),$$

but

$$\frac{1}{2}(1, 1, 2, 2, 0, \dots) + \frac{1}{2}(1, 1, -4, -4, 0, \dots) = (1, 1, -1, -1, 0, \dots) \notin F(T).$$

Therefore, $F(T)$ is not convex. Moreover, since

$$\begin{aligned}\|(1,1,8,8,0,\dots)-(0,0,\dots)\|_2 &= \|(1,1,8,8,0,\dots)\|_2 \\ &= \sqrt{2+2\cdot 8^2} \\ &< \sqrt{2\cdot 3^2+2\cdot 8^2} \\ &= \|(3,3,8,8,0,\dots)\|_2 \\ &= \|T(1,1,8,8,0,\dots)\|_2 \\ &= \|T(1,1,8,8,0,\dots)-T(0,0,\dots)\|_2\,,\end{aligned}$$

we have T is not nonexpansive.

Example 3.9. *Let $X = \{(x_1, x_2, \dots) \in \ell_2 : x_i \geq 0$ for any $i \in \mathbb{N}\}$ and $T : X \to X$ be defined by*

$$T(x_1,x_2,\dots) = \Big(\frac{|x_1+x_3|}{3}, \frac{|x_2+x_4|}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\Big)$$

for any $(x_1, x_2, \dots) \in \ell_2$. Let $k = 2$, $a = 0$,

$$b_{i+2} = \frac{1}{i} \quad \textit{for any } i \in \mathbb{N} \quad g_1 = \frac{1}{3}(e_1^* + e_3^*), \quad g_2 = \frac{1}{3}(e_2^* + e_4^*) \quad \textit{and} \quad h_1(x) = h_2(x) = |x|.$$

Since

$$g_1(x_1,x_2,\dots) = \frac{1}{3}(x_1+x_3) \quad \textit{and} \quad g_2(x_1,x_2,\dots) = \frac{1}{3}(x_2+x_4),$$

we have $\|g_1\| = \|g_2\| = \frac{\sqrt{2}}{3}$. Let L_1, L_2 be Lipschitz constants of h_1, h_2, respectively. Then $L_1 = L_2 \leq 1$, so $L_1\|g_1\| + L_2\|g_2\| \leq \frac{2\sqrt{2}}{3} < 1$. By Corollary 3.7, T is functionally uniformly lipschitzian.
Suppose $T(x_1, x_2, \dots) = (x_1, x_2, \dots)$ for finding the fixed point set of T . Then

$$x_1 = \frac{1}{3}|x_1+1| \geq 0, \quad x_2 = \frac{1}{3}|x_2+\frac{1}{2}| \geq 0 \quad \textit{and for any} \quad i = 3,4,\dots, \quad x_i = \frac{1}{i-2}.$$

It follows that $F(T) = \{(\frac{1}{2}, \frac{1}{4}, 1, \frac{1}{2}, \dots)\}$ and hence $F(T)$ is convex.

Let $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in X$.
Then

$$\begin{aligned}
\|Tx - Ty\|_2 &= \Big(\frac{1}{4^2}\,\big||x_1+x_3| - |y_1+y_3|\big|^2 + \frac{1}{4^2}\,\big||x_2+x_4| - |y_2+y_4|\big|^2\Big)^{\frac{1}{2}} \\
&= \frac{1}{4}\Big[|x_1+x_3|^2 + |y_1+y_3|^2 - 2|(x_1+x_3)(y_1+y_3)| \\
&\quad + |x_2+x_4|^2 + |y_2+y_4|^2 - 2|(x_2+x_4)(y_2+y_4)|\Big]^{\frac{1}{2}} \\
&= \frac{1}{4}\Big[|x_1+x_3|^2 + |y_1+y_3|^2 - 2(x_1+x_3)(y_1+y_3) \\
&\quad + |x_2+x_4|^2 + |y_2+y_4|^2 - 2(x_2+x_4)(y_2+y_4)\Big]^{\frac{1}{2}} \\
&= \frac{1}{4}\Big[|x_1+x_3|^2 + |y_1+y_3|^2 - 2(x_1y_1 + x_1y_3 + x_3y_1 + x_3y_3) \\
&\quad + |x_2+x_4|^2 + |y_2+y_4|^2 - 2(x_2y_2 + x_2y_4 + x_4y_2 + x_4y_4)\Big]^{\frac{1}{2}} \\
&= \frac{1}{4}\Big[|x_1-y_1|^2 + |x_3-y_3|^2 - 2x_1y_3 - 2x_3y_1 + 2x_1x_3 + 2y_1y_3 \\
&\quad + |x_2-y_2|^2 + |x_4-y_4|^2 - 2x_2y_4 - 2x_4y_2 + 2x_2x_4 + 2y_2y_4\Big]^{\frac{1}{2}} \\
&= \frac{1}{4}\Big[|x_1-y_1|^2 + |x_3+y_3|^2 + 2(x_3-y_3)(x_1-y_1) \\
&\quad + |x_2+y_2|^2 + |x_4+y_4|^2 + 2(x_4-y_4)(x_2-y_2)\Big]^{\frac{1}{2}} \\
&\le \frac{1}{4}\Big[2|x_1-y_1|^2 + 2|x_3+y_3|^2 + 2|x_2+y_2|^2 + 2|x_4+y_4|^2\Big]^{\frac{1}{2}} \\
&\le \|x-y\|_2\,.
\end{aligned}$$

Therefore, T is nonexpansive.

Example 3.10. *Let $1 < p < \infty$ and $h : \mathbb{R} \to \mathbb{R}$ be defined by*

$$h(x) = \begin{cases} 1 & , \textit{ if } x > 1; \\ x & , \textit{ if } 0 \le x \le 1; \\ 0 & , \textit{ if } x < 0 \end{cases} \qquad \textit{for any } x \in \mathbb{R}.$$

Let $T : \ell_p \to \ell_p$ be defined by

$$T(x_1, x_2, \dots) = \Big(h(\frac{x_1+x_3}{4}), h(\frac{x_2+x_4}{4}), x_3, x_4, x_5, \dots\Big)$$

for any $(x_1, x_2, \dots) \in \ell_p$. *Let* $q = \frac{p}{p-1} > 1$ *and* $k = 2$, $a = 1$, $b_{i+2} = 1$ *for any* $i \in \mathbb{N}$,

$$h_1 = h_2 = h, \quad g_1 = \frac{e_1^* + e_3^*}{4} \quad \textit{and} \quad g_2 = \frac{e_2^* + e_4^*}{4}.$$

Since

$$g_1(x_1, x_2, \dots) = \frac{x_1 + x_3}{4} \quad \textit{and} \quad g_2(x_1, x_2, \dots) = \frac{x_2 + x_4}{4},$$

we have $\|g_1\| = \|g_2\| = \frac{\sqrt[q]{2}}{4}$. *Let* L_1, L_2 *be Lipschitz constants of* h_1, h_2, *respectively. Then* $L_1 = L_2 \leq 1$ *and* $q > 1$, *so* $L_1 \|g_1\| + L_2 \|g_2\| \leq \frac{2\sqrt[q]{2}}{4} < 1$. *By Corollary* 3.7, T *is functionally uniformly lipschitzian. It is easy to see that*

$$(1, 1, 8, 8, 0, 0, \dots), \quad (0, 0, 0, 0, \dots) \in F(T),$$

but

$$(\frac{1}{4}, \frac{1}{4}, 2, 2, 0, 0, \dots) \notin F(T).$$

Hence, F(T) is not convex. Since

$$\begin{aligned} \left\| (\frac{1}{4}, \frac{1}{4}, 2, 2, 0, \dots) - (0, 0, \dots) \right\|_p &= \left\| (\frac{1}{4}, \frac{1}{4}, 2, 2, 0, \dots) \right\|_p \\ &= \sqrt[p]{\frac{2}{4^p} + 2^{p+1}} \\ &< \sqrt[p]{2(\frac{9}{16})^p + 2^{p+1}} \\ &= \left\| (\frac{9}{16}, \frac{9}{16}, 2, 2, 0, \dots) - (0, 0, \dots) \right\|_p \\ &= \left\| T(\frac{1}{4}, \frac{1}{4}, 2, 2, 0, \dots) - T(0, 0, \dots) \right\|_p, \end{aligned}$$

we have that T *is not nonexpansive.*

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