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NEARMEDIA AND THEIR REPRESENTATIONS

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เอกสิทธิ์ สังข์วิสุทธิ์ : เนียร์มีเดียและตัวแทนของเนียร์มีเดีย. (NEARMEDIA AND THEIR REPRESENTATIONS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: ผศ.ดร.ยศนันต์ มีมาก, 39 หน้า.

เนียร์มีเดียประกอบด้วยเซต $\mathscr{S}$ เป็นเซตของสถานะแปลงโดยการกระทำของเซต $\mathscr{T}$ ของการแปลงเรียกว่าโทเค็น ซึ่งสอดคล้องกับเงื่อนไข สำหรับทุก ๆ สถานะ S และ T ที่ต่างกันใน $\mathscr{S}$ มีข้อความกระชับแปลง S ไป T วิทยานิพนธ์ฉบับนี้นำเสนอเนียร์มีเดียบางพวกและสมบัติพื้นฐานของเนียร์มีเดีย ตลอดจนตัวแทนของเนียร์มีเดียในรูปของกราฟ และ พวกเรียงขนาด ยิ่งกว่านั้นเราประยุกต์เนียร์มีเดียไปยังทฤษฎีการประเมินผลอีกด้วย

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A nearmedium consists of a set $\mathscr{S}$ of states transformed by the actions of a set $\mathscr{T}$ of transformations, called tokens, satisfying the condition: For any two distinct states S and T in $\mathscr{S}$, there is a concise message transforming S into T. We present some families and elementary properties of nearmedia and their representations as graphs and graded families. Moreover, we give some application to assessment theory using our nearmedia.

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CHAPTER I

PRELIMINARIES

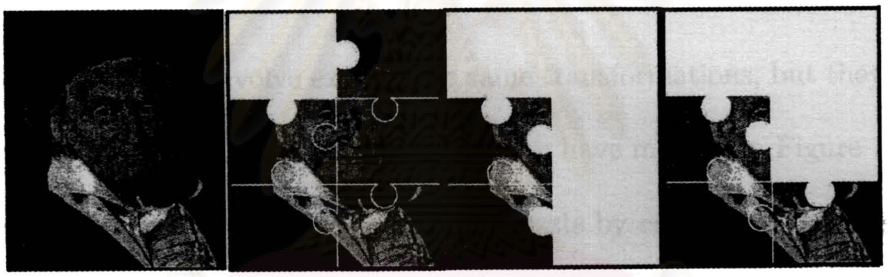
Media were introduced by Falmagne in 1997 in order to generalize a class of models of preference changes. A medium consists of a set of states transformed by the actions of a set of transformations, called tokens, satisfying two axioms. We shall formally give them in Section 1.2. We begin with an example from everyday life, which will serve as an informal introduction to the main concepts of media theory.

1.1 A Jigsaw Puzzle[1]

Consider a jigsaw puzzle of size 3×2. We call a *state* of this puzzle any partial solution, formed by a linked subset of the puzzle pieces in their correct positions. Four such states are displayed in Figure 1.1.1 (a), (b), (c) and (d). Thus, the completed puzzle is a state. We also regard as states the initial situation (the empty board), and any single piece appropriately placed on the board. A careful count gives us 41 states (see Figure 1.1.2). To each of the six pieces of the puzzle correspond exactly two transformations which consist of placing or removing a piece. In the first case, a piece is placed either on an empty board, or so that it

[1] We closely follow Section 1.1 of [5] for this first foundation example of media theory.

can be linked to some pieces already on the board. In the second case, the piece is already on the board and removing it either leaves the board empty or does not disconnect the remaining pieces. By convention, these two types of transformations apply artificially to all the states in the sense that placing a piece already on the board or removing a piece that is not on the board leaves the state unchanged. This provides the first example of a *medium*, a concept based on a pair $(\mathscr{S}, \mathscr{T})$ of sets: a set $\mathscr{S}$ states, and a collection $\mathscr{T}$ of transformations capable, in some cases, of converting a state into a different one. The formal definition given in the next section of such a structure relies on two constraining axioms.



(a) (b) (c) (d)

Figure 1.1.1: The figures illustrate some states of a jigsaw puzzle.

By the above definition, none of these transformations is one-to-one. For instance, applying the transformation adding the upper left piece of the puzzle to either of the states pictured in Figure 1.1.1 (a) or (b) results in the same state, namely (a). In the first case, we have thus a loop. Accordingly, the two transformations associated with each piece are not mutual inverses. However, each of the transformations in a pair can undo the action of the other. We shall say that these transformations are reverses of one another. When the number of states is finite, it may be convenient to represent a medium by its graph and we shall often do so. The medium of a 3×2 jigsaw puzzle has its graph displayed in Figure 1.1.2 below. As usual, we omit loops.

An examination of this graph leads to further insight. For any two states S and T, it is possible to find a sequence of transformations whose successive applications from S results in forming T. This path from S to T never strays from the allowed set of states, and can be made minimally short, that is: its length is equal to the number of pieces which are not common to both states. Moreover, any two such paths from S to T will involve exactly the same transformations, but they may be applied in different orders. As an illustration, we have marked in Figure 1.1.2 two such paths from state $\boxed{34}$ to the completed puzzle by coloring their edges in red and blue, respectively. These two paths are

$$\boxed{34} \xmapsto{1} \boxed{134} \xmapsto{5} \boxed{1345} \xmapsto{2} \boxed{12345} \xmapsto{6} \boxed{123456} \tag{1.1.1}$$

$$\boxed{34} \xmapsto{6} \boxed{346} \xmapsto{2} \boxed{2346} \xmapsto{5} \boxed{23456} \xmapsto{1} \boxed{123456} \tag{1.1.2}$$

Observe also from the graph that for any two states S and T, we can always find a minimally short path from S and T that does not contain transformations in exact opposite orders.

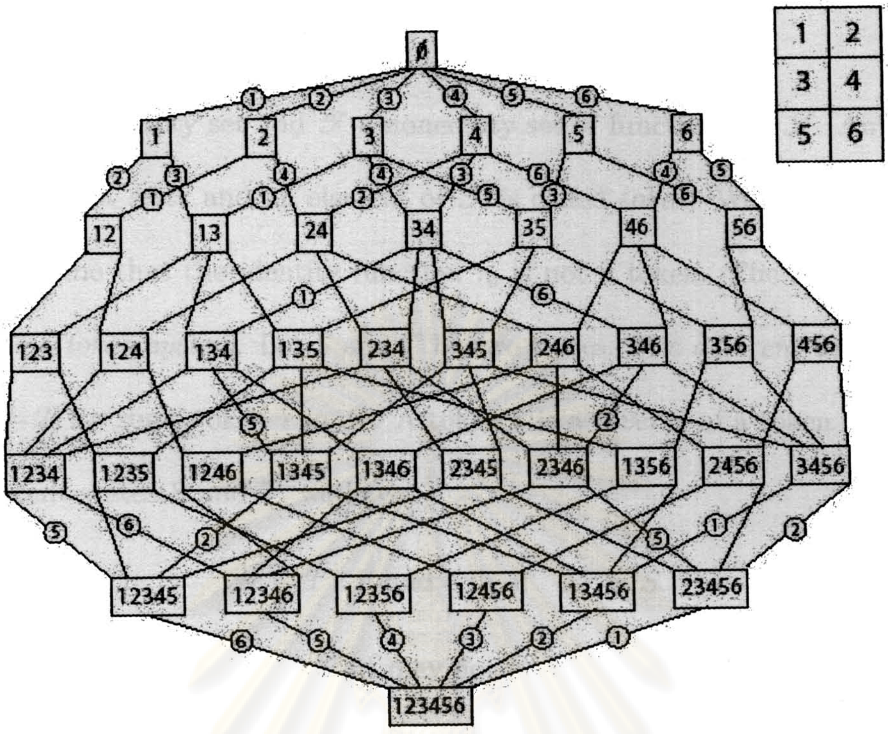


Figure 1.1.2: Graph of a 3 × 2 jigsaw puzzle medium. A schematic of the puzzle is the upper right of the graph, with the six pieces numbered 1, . . . , 6. Each of the 41 vertices of the graph represent one state of the medium, that is, one partial solution of the puzzle symbolized by a rectangle containing the list of its pieces. Each edge represents a pair of mutually reverse transformations, one adding a piece, and the other removing it. To avoid cluttering the figure, only some of the edges are labeled (by a circle).

1.2 Media and Nearmedia

Let $\mathscr{S}$ be a nonempty set and $\mathscr{T}$ a nonempty set of functions on $\mathscr{S}$. An element of $\mathscr{S}$ is called a *state* and an element of $\mathscr{T}$ is called *token*. We write $\tau(S) = S\tau$ and we assume that the identity function τ_0 is not a token. The pair $(\mathscr{S}, \mathscr{T})$ is said to be a *token system*. Let S and T be two states. T is *adjacent* to S if $S \neq T$ and $S\tau = T$ for some token τ in $\mathscr{T}$. A token $\tilde{\tau}$ is a *reverse* of a token τ if for any two adjacent states S and T, we have

$$S\tau = T \quad \text{if and only if} \quad T\tilde{\tau} = S.$$

Note that $\tilde{\tau}$ (if exists) is unique but may not be in $\mathscr{T}$. If the reverse $\tilde{\tau}$ of τ exists, then τ and $\tilde{\tau}$ are mutual reverses and $\tilde{\tilde{\tau}} = \tau$.

A finite composition of tokens $m = \tau_1 \ldots \tau_n$ is called a *message*. Its *content* is the set $\mathscr{C}(m) = \{\tau_1, \ldots, \tau_n\}$. For two distinct states S and T, if $Sm = T$ for some message m, then we say that m *produces* T *from* S. A message m is said to be *effective* (resp. *ineffective*) for a state S if $Sm \neq S$ (resp. $Sm = S$). It is called *stepwise effective* if $S\tau_1 \ldots \tau_i \neq S\tau_1 \ldots \tau_{i+1}$ for all $i = 1, \ldots, n-1$. A message is *inconsistent* if $\tau, \tilde{\tau} \in \mathscr{C}(m)$ for some $\tau \in \mathscr{C}(m)$ and *consistent* otherwise. A *concise message for a state* S is a message which is stepwise effective for S, consistent, and any token occurs at most once in the message. A message is *closed* for a state S if it is stepwise effective and ineffective for S. A message $m = \tau_1 \ldots \tau_n$ is *vacuous* if the set of indices $\{1, \ldots, n\}$ can be partitioned into pairs $\{i, j\}$, such τ_i and τ_j are mutual reverses.

Let $(\mathscr{S}, \mathscr{T})$ be a token system. The digraph $G = (V, E)$ is said to *represent* $(\mathscr{S}, \mathscr{T})$ if V is the set of states $\mathscr{S}$ and the set E of ordered pairs of states (S, T) is considered to be directed from S to T when $S\tau = T$ for some $\tau \in \mathscr{T}$.

We illustrate the above terminologies in the following example.

Example 1.2.1. Given a token system $(\mathscr{S}, \mathscr{T})$ with $\mathscr{S} = \{S, T, V, W, X\}$ and $\mathscr{T} = \{\tau_1, \tau_2, \tau_3, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3\}$ as shown in Figure 1.2.1.

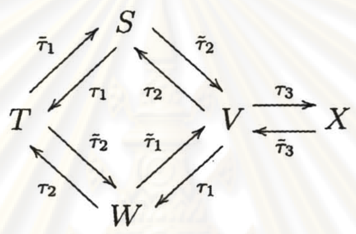


Figure 1.2.1: A token system.

Let $m_1 = \tau_1\tau_2\tau_1$, $m_2 = \tau_1\tilde{\tau}_2\tilde{\tau}_1$ and $m_3 = \tau_1\tilde{\tau}_2\tilde{\tau}_1\tau_2$. Then m_1, m_2, m_3 are messages. Their contents are $\mathscr{C}(m_1) = \{\tau_1, \tau_2\}, \mathscr{C}(m_2) = \{\tau_1, \tau_2, \tilde{\tau}_1\}$ and $\mathscr{C}(m_3) = \{\tau_1, \tau_2, \tilde{\tau}_1, \tilde{\tau}_2\}$, respectively. Since $Sm_2 = V$, we have m_2 produces V from S. Notice that $Sm_3 = S\tau_1\tilde{\tau}_2\tilde{\tau}_1\tau_2 = T\tilde{\tau}_2\tilde{\tau}_1\tau_2 = W\tilde{\tau}_1\tau_2 = V\tau_2 = S$. Thus m_2 and m_3 are effective and ineffective, respectively and both m_2 and m_3 are stepwise effective. However, m_1 is not stepwise effective since $Sm_1 = S\tau_1\tau_2\tau_1 = T\tau_2\tau_1 = T\tau_1 = T$. Furthermore, the messages m_1, m_2 and m_3 are inconsistent. Clearly, m_3 is vacuous. Some examples of consistent and concise messages are τ_1, $\tau_2\tau_1$ and $\tilde{\tau}_3\tau_2\tau_1$.

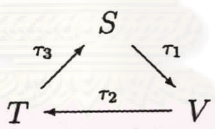
A token system $(\mathscr{S}, \mathscr{T})$ is called a *medium* if the following two axioms are fulfilled.

$[M_1]$ For any two distinct states S and T in $\mathscr{S}$, there is a concise message transforming S into T.

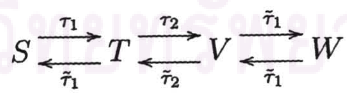
$[M_2]$ A message, which is closed for some state, is vacuous.

The token system given in Example 1.2.1 satisfies axioms $[M_1]$ and $[M_2]$, so it is a medium.

Example 1.2.2. The token system $(\mathscr{S}, \mathscr{T})$ with $\mathscr{S} = \{S, T, V\}$ and $\mathscr{T} = \{\tau_1, \tau_2, \tau_3\}$ as shown satisfies $[M_1]$ but does not fulfill $[M_2]$. Note that the token system which satisfies only $[M_1]$ may not have a reverse in $\mathscr{T}$.



Example 1.2.3. The token system $(\mathscr{S}, \mathscr{T})$ with $\mathscr{S} = \{S, T, V, W\}$ and $\mathscr{T} = \{\tau_1, \tau_2, \tilde{\tau}_1, \tilde{\tau}_2\}$ as shown satisfies $[M_2]$ but does not fulfill $[M_1]$.



Hence we obtain

Remark. The axioms $[M_1]$ and $[M_2]$ are independent.

Example 1.2.4. The 3×2 jigsaw in Section 1.1 is a medium where the set $\mathscr{S}$ of states consists any partial solution obtained by a linked subset of the puzzle pieces in their correct positions and the set $\mathscr{T}$ of tokens forms from adding and removing one piece of puzzle and the remain puzzle pieces must be connected.

Example 1.2.5. Let $P(X)$ be the power set of X, i.e., the family of all subsets of X, and let $\mathscr{G} = \bigcup_{x\in X}\{\gamma_x, \tilde{\gamma}_x\}$ be the family of functions from $P(X)$ into $P(X)$ where γ_x and $\tilde{\gamma}_x$ are defined by

$$\gamma_x : S \mapsto S\gamma_x = \begin{cases} S \cup \{x\}, & \text{if } \{x\} \subseteq S^c; \\ S, & \text{otherwise,} \end{cases}$$

and

$$\tilde{\gamma}_x : S \mapsto S\tilde{\gamma}_x = \begin{cases} S \setminus \{x\}, & \text{if } \{x\} \subseteq S; \\ S, & \text{otherwise} \end{cases}$$

for all $x \in X$. Corollary 2.1.3 in Chapter II says that $(P(X), \mathscr{G})$ is a medium. Its graph is called a "cube" which we shall define later in this section.

If $X = \{a\}$ or $X = \{a, b\}$ or $X = \{a, b, c\}$, then their graphs are respectively shown in Figure 1.2.2.

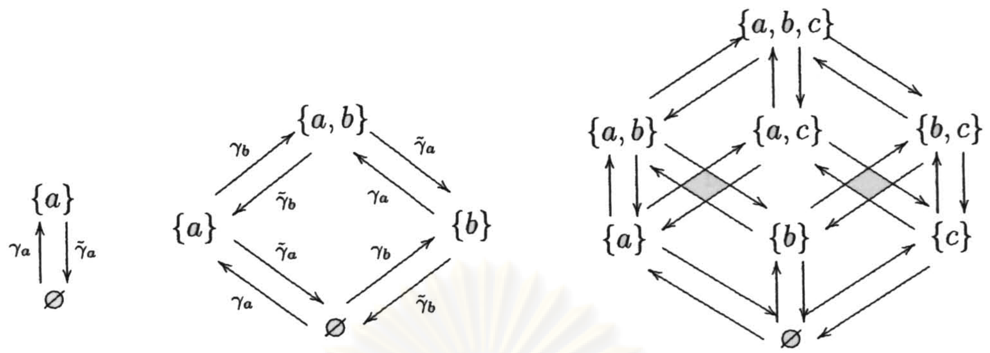


Figure 1.2.2: The graphs of $(P(X), \mathscr{G})$ when $X = \{a\}$, $X = \{a, b\}$ and $X = \{a, b, c\}$, respectively. For simplicity, we omit the token labels in the last graph.

Example 1.2.6. Let $\mathbb{Z}^2$ denote the set of all integral lattice points with the set of transformations $\mathscr{T}$ consisting of all pairs of tokens $\tau_i, \tilde{\tau}_i$ and $\sigma_i, \tilde{\sigma}_i$ on $\mathbb{Z}^2$ defined by

$$\tau_i : (k, q) \mapsto (k, q)\tau_i = \begin{cases} (k+1, q), & \text{if } i = k; \\ (k, q), & \text{if } i \neq k; \end{cases}$$

and

$$\tilde{\tau}_i : (k, q) \mapsto (k, q)\tilde{\tau}_i = \begin{cases} (k-1, q), & \text{if } i = k-1; \\ (k, q), & \text{if } i \neq k-1; \end{cases}$$

and

$$\sigma_i : (k,q) \mapsto (k,q)\sigma_i = \begin{cases} (k,q+1), & \text{if } i = q; \\ (k,q), & \text{if } i \neq q; \end{cases}$$

and

$$\tilde{\sigma}_i : (k,q) \mapsto (k,q)\tilde{\sigma}_i = \begin{cases} (k,q-1), & \text{if } i = q-1; \\ (k,q), & \text{if } i \neq q-1. \end{cases}$$

for all $i \in \mathbb{Z}$. Then $(\mathbb{Z}^2, \mathscr{T})$ is a medium and it is an example of infinite media. This medium is displayed in Figure 1.2.3.

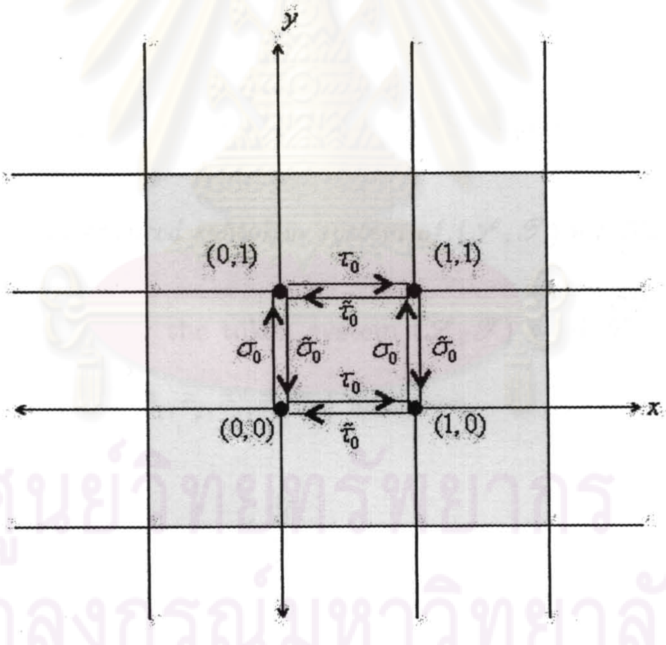


Figure 1.2.3: An example of infinite media.

To see this, let (a,b) and (c,d) be two distinct states in $\mathscr{S}$. Then there exist integers k and l such that $c = a + k$ and $d = b + l$. Clearly,

$$m = \tau_a \tau_{a+1} \cdots \tau_{a+k-1} \sigma_b \sigma_{b+1} \cdots \sigma_{b+l-1}$$

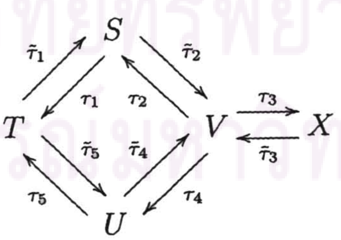
is a concise message sending (a,b) to (c,d). Thus we have $[M_1]$. To see that our token system fulfills $[M_2]$, we may assume without loss of generality that a closed message m' sends $(0,0)$ to $(0,0)$. That is, $(0,0)m' = (0,0)$. It follows that the number of going up tokens σ_i is the same as the number of going down tokens $\tilde{\sigma}_i$ and the number of going forward tokens τ_j is the same as the number of going backward tokens, $\tilde{\tau}_j$. Hence m is vacuous.

Let $(\mathscr{S}, \mathscr{T})$ be a token system and $\varnothing \neq \mathscr{S}' \subseteq \mathscr{S}$. Write

$$\mathscr{T}_{\mathscr{S}'} = \{\tau \in \mathscr{T} \mid S\tau \neq S,\ S\tau \in \mathscr{S}' \text{ for some } S \text{ in } \mathscr{S}\}.$$

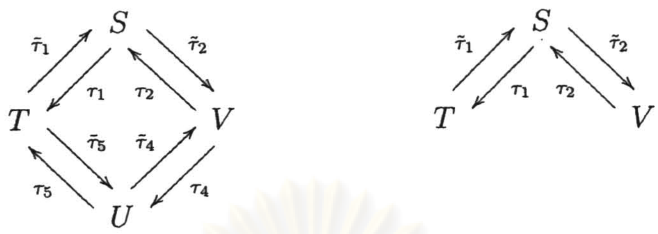
We call $(\mathscr{S}', \mathscr{T}_{\mathscr{S}'})$ an *induced subtoken system of* $(\mathscr{S}, \mathscr{T})$ for $\mathscr{T}_{\mathscr{S}'} \neq \varnothing$.

Example 1.2.7. Consider the token system $(\mathscr{S}, \mathscr{T})$ with $\mathscr{S} = \{S, T, U, V, X\}$ and $\mathscr{T} = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4, \tilde{\tau}_5\}$ as shown.

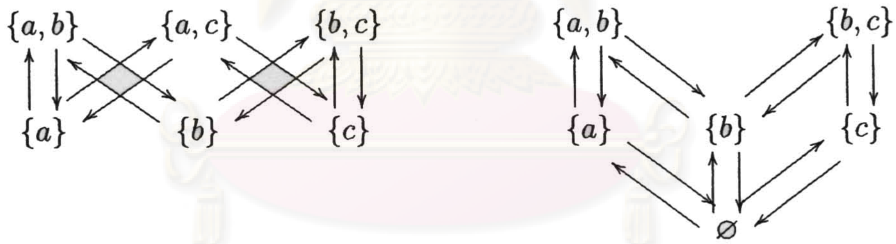


For $\mathscr{S}' = \{S, T, U, V\}$, we have $\mathscr{T}_{\mathscr{S}'} = \{\tau_1, \tau_2, \tau_4, \tau_5, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_4, \tilde{\tau}_5\}$ and for $\mathscr{S}'' = \{S, T, V\}$, we have $\mathscr{T}_{\mathscr{S}''} = \{\tau_1, \tau_2, \tilde{\tau}_1, \tilde{\tau}_2\}$. The induced subtoken systems $(\mathscr{S}', \mathscr{T}')$

and $(\mathscr{S}'', \mathscr{T}'')$ are respectively displayed below.



Example 1.2.8. Consider the medium $(P(X), \mathscr{G})$ when $X = \{a, b, c\}$ where $P(X) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $\mathscr{G} = \{\gamma_a, \gamma_b, \gamma_c, \tilde{\gamma}_a, \tilde{\gamma}_b, \tilde{\gamma}_c\}$ defined in Example 1.2.5. The graph of this medium is shown in the right-most of Figure 1.2.2. For $\mathscr{F} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $\mathscr{F}' = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$, the induced token systems $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ and $(\mathscr{F}', \mathscr{G}_{\mathscr{F}'})$ are respectively displayed below.



The *cube* on a set X is the graph $\mathscr{H}$ with the vertex set $P(X)$ and subsets P and Q of X are adjacent if their symmetric difference $P \Delta Q = (P \setminus Q) \cup (Q \setminus P)$ is a singleton. A graph G is called a *partial cube* if it is isometrically embeddable into the cube $\mathscr{H}(X)$ for some set X. The cubes $\mathscr{H}(X)$ for $|X| = 1, 2$ and 3 are respectively shown in Figure 1.2.4 and some partial cubes of $\mathscr{H}(X)$ when $|X| = 3$ are displayed in Figure 1.2.5.

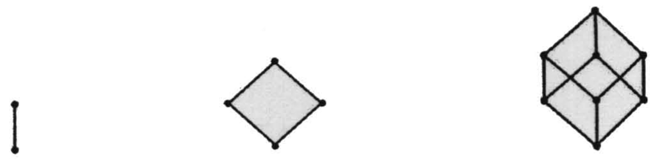


Figure 1.2.4: The cubes $\mathscr{H}(X)$ when $|X| = 1, 2$ and 3, respectively.

Figure 1.2.5: Some partial cubes of $\mathscr{H}(X)$ when $|X| = 3$.

The following representation is due to Ovchinnikov.

Theorem 1.2.9. *[7] Let $\mathscr{F} \subseteq P(X)$. The graph G represents the medium $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ if and only if G is a partial cube on X.*

In this work, we soften the definition of a medium to a *nearmedium* defined as a token system $(\mathscr{S}, \mathscr{T})$ which satisfies the condition $[M_1]$. Therefore, a medium is a nearmedium and $(\mathscr{S}, \mathscr{T})$ in Example 1.2.2 is a nearmedium which is not a medium. This thesis is organized as follows. In chapter II, we construct two families of token systems which are nearmedia based on the medium given in Example 1.2.5 and derive elementary properties and their representations as graded families and graphs. These families $P_n(X)$ and $Q_n(X)$ are presented in Sections 2.1 and 2.2, respectively. Section 2.3 gives an example of infinite nearmedia. The final chapter demonstrates some applications and real-world examples for assessment theory of nearmedia described in Chapter II.

CHAPTER II

FAMILIES OF NEARMEDIA

This chapter presents two families of nearmedia and provides examples of infinite token system which respectively, does not fulfill $[M_2]$ and does not fullfil $[M_1]$, respectively, in the final section.

2.1 $P_n(X)$-family

Let X be a set. For $n \geq 1$, we define

$$P_n(X) = \{S \subseteq X : S \text{ is finite and } n \text{ divides } |S|\}.$$

Let $\mathscr{G} = \{\gamma_{a_1 \ldots a_n}, \tilde{\gamma}_{a_1 \ldots a_n} : a_1, \ldots, a_n$ are distinct elements in $X\}$ be the family of functions on $P_n(X)$ given by

$$\gamma_{a_1 \ldots a_n} : S \mapsto S\gamma_{a_1 \ldots a_n} = \begin{cases} S \cup \{a_1, \ldots, a_n\}, & \text{if } \{a_1, \ldots, a_n\} \subseteq S^c; \\ S, & \text{otherwise,} \end{cases}$$

and

$$\tilde{\gamma}_{a_1 \ldots a_n} : S \mapsto S\tilde{\gamma}_{a_1 \ldots a_n} = \begin{cases} S \smallsetminus \{a_1, \ldots, a_n\}, & \text{if } \{a_1, \ldots, a_n\} \subseteq S; \\ S, & \text{otherwise} \end{cases}$$

for any distinct elements $a_1, \ldots, a_n$ in X. It is clear that $(P_n(X), \mathscr{G})$ is a token system with $\gamma_{a_1 \ldots a_n}$ and $\tilde{\gamma}_{a_1 \ldots a_n}$ are mutual reverses. We give some examples with their graphs in order to demonstrate the above definition and induced subtoken systems.

Example 2.1.1. 1. Let $X = \{1, 2, 3\}$. Then $P_2(X) = \{\varnothing, \{1,2\}, \{1,3\}, \{2,3\}\}$ and $\mathscr{G} = \{\gamma_{12}, \gamma_{13}, \gamma_{23}, \tilde{\gamma}_{12}, \tilde{\gamma}_{13}, \tilde{\gamma}_{23}\}$.

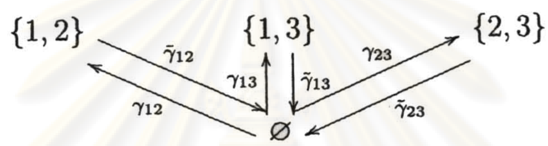


Figure 2.1.1: The graph of $(P_2(X), \mathscr{G})$.

2. Let $X = \{1, 2, 3, 4\}$. Consider $\mathscr{F} = \{\varnothing, \{1,2\}, \{1,3\}, \{1,2,3,4\}\} \subseteq P_2(X)$. Then the induced subtoken system $\mathscr{G}_{\mathscr{F}} = \{\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{24}, \tilde{\gamma}_{12}, \tilde{\gamma}_{13}, \tilde{\gamma}_{14}, \tilde{\gamma}_{24}\}$.

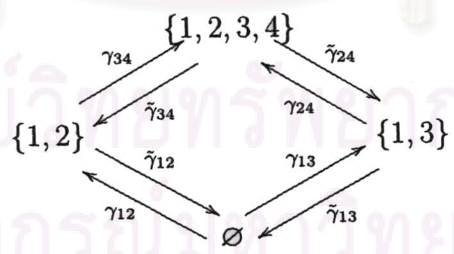


Figure 2.1.2: The graph of $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$.

3. Let $X = \{1,2,3,4,5,6\}$. Consider $\mathscr{F}' = \{\varnothing, \{1,2\}, \{1,3\}, \{1,2,3,4,5,6\}\} \subseteq P_2(X)$. Then the induced subtoken system $\mathscr{G}_{\mathscr{F}'} = \{\gamma_{12}, \gamma_{13}, \tilde{\gamma}_{12}, \tilde{\gamma}_{13}\}$.

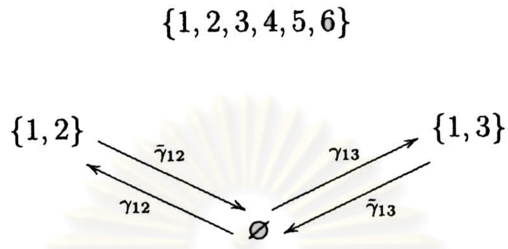


Figure 2.1.3: The graph of $(\mathscr{F}', \mathscr{G}_{\mathscr{F}'})$.

Lemma 2.1.2. *Let X be a set. Then*

1. *$(P_n(X), \mathscr{G})$ satisfies $[M_1]$ for all $n \geq 1$;*

2. *$(P_1(X), \mathscr{G})$ satisfies $[M_2]$;*

3. *For $n \geq 2$, $(P_n(X), \mathscr{G})$ satisfies $[M_2]$ if and only if $|X| < 2n$.*

Proof. 1. Let $P, Q \in P_n(X)$. Assume that $|P \cap Q| \equiv k \mod n$ for some $k = 0, \ldots, n-1$. Then $|P \cap Q| = nh + k$, $|P \setminus Q| = nj - k$ and $|Q \setminus P| = nl - k$ for some positive integers h, j, k, l and $k < n$. Write $P \setminus Q = \{p_1, \ldots, p_{nj-k}\}, Q \setminus P = \{q_1, \ldots, q_{nl-k}\}$ and $P \cap Q = \{a_1, \ldots, a_{nh+k}\}$. Let

$$m = \tilde{\gamma}_{p_1 \cdots p_n} \cdots \tilde{\gamma}_{p_{n(j-2)+1} \cdots p_{n(j-1)}} \tilde{\gamma}_{p_{n(j-1)+1} \cdots p_{nj-k} a_1 \cdots a_k}$$

$$\gamma_{q_1 \cdots q_n} \cdots \gamma_{q_{n(l-2)+1} \cdots q_{n(l-1)}} \gamma_{q_{n(l-1)+1} \cdots q_{(nl-k)} a_1 \cdots a_k}.$$

It is clear from the construction that m is a concise message and $Pm = Q$.

2. Let m be a closed message for some state P. That is, m is stepwise effective and ineffective for P and $Pm = P$. Suppose that $\gamma_x \in \mathscr{C}(m)$ and $\tilde{\gamma}_x \notin \mathscr{C}(m)$. Then $x \in Pm$ and $x \notin P$. We have a contradiction since $Pm = P$. Thus, for each token γ_x in $\mathscr{C}(m)$, there is an appearance of the reverse token $\tilde{\gamma}_x$ in m. Because m is stepwise effective, the appearances of token γ_x and $\tilde{\gamma}_x$ in m must alternate. Suppose that the sequence of appearances of γ_x and $\tilde{\gamma}_x$ begins and ends with γ_x. Since the message m is stepwise effective for state P and ineffective for this state, we must have $x \notin P$ and $x \in Pm = P$, a contradiction. Hence m is vacuous.

3. Let $n \geq 2$. Assume that $|X| \geq 2n$ and let $a_1, \ldots, a_{2n}$ be $2n$ distinct elements in X. Notice that $\varnothing$, $\{a_1, \ldots, a_{2n}\}$, $\{a_1, \ldots, a_n\}$ and $\{a_n, \ldots, a_{2n-1}\}$ are in $P_n(X)$. Then we may choose the message $m = \gamma_{a_1 \ldots a_n} \gamma_{a_{n+1} \ldots a_{2n}} \tilde{\gamma}_{a_1 \ldots a_{n-1} a_{2n}} \tilde{\gamma}_{a_n \ldots a_{2n-1}}$, so that $\varnothing m = \varnothing$. It is clearly that m is stepwise effective and ineffective but m is not vacuous. Thus $(P_n(X), \mathscr{G})$ does not satisfy $[M_2]$. Conversely, suppose that $|X| < 2n$. Let $S \in P_n(X)$. It forces that $S = \varnothing$ or $|S| = n$. Therefore a message which is stepwise effective for some state and ineffective for this state must be vacuous. □

Corollary 2.1.3. *The token system $(P_n(X), \mathscr{G})$ is a nearmedium for all $n \geq 1$. In particular, $(P_1(X), \mathscr{G})$ is a medium and $(P_n(X), \mathscr{G})$ is a medium whenever $n \geq 2$ and $|X| < 2n$.*

Let X be a set. For subsets P and Q of X, the *distance between P and Q* is defined by

$$d(P,Q) = \begin{cases} |P\Delta Q| = |(P \setminus Q) \cup (Q \setminus P)|, & \text{if } P\Delta Q \text{ is a finite set;} \\ \infty, & \text{otherwise.} \end{cases}$$

A family $\mathscr{F}$ of subsets of X is called *well-graded* if for any two distinct sets P and Q in $\mathscr{F}$, there is a sequence of sets $P = R_0, R_1, \ldots, R_k = Q$ in $\mathscr{F}$ such that $d(R_{i-1}, R_i) = 1$ for $i = 1, \ldots, k$ and $d(P,Q) = k$. Ovchinnikov established that

Theorem 2.1.4. *[7] Let $\mathscr{F} \subseteq P_1(X)$. Then a token system $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ is a medium if and only if $\mathscr{F}$ is a well-graded family of subsets of X.*

We say that a family $\mathscr{F} \subseteq P_n(X)$ is *graded* if for any two distinct sets P and Q in $\mathscr{F}$ where $|P \cap Q| = nh + k, |P \setminus Q| = nj - k$ and $|Q \setminus P| = nl - k$ for some $h, j, l \in \mathbb{Z}^+, k \in \mathbb{Z}_0^+$ and $k < n$, there is a sequence of sets $P = R_0, R_1, \ldots, R_{j+l} = Q$ in $\mathscr{F}$ with $R_{i-1} \subseteq R_i$ or $R_i \subseteq R_{i-1}$ such that $d(R_{i-1}, R_i) = n$ for $i = 1, \ldots, j+l$ and $d(P,Q) = n(j+l) - 2k$.

Remark. For $n = 1$, the terms graded and well-graded are coincide.

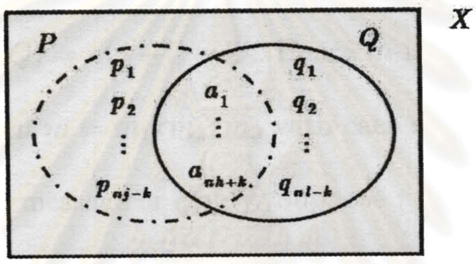
Example 2.1.5. Let $X = \{1,2,3\}$. Then $P_1(X) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. Consider $\mathscr{F} = \{\varnothing, \{1\}, \{3\}\}$ and $\mathscr{F}' = \{\varnothing, \{1\}, \{3\}, \{1,2\}, \{2,3\}\}$. We have that $\mathscr{F}$ is a graded family but $\mathscr{F}'$ is not a graded family because there is no sequence of the length 2 from $\{1,2\}$ to $\{2,3\}$.

Theorem 2.1.6. *Let $\mathscr{F} \subseteq P_n(X)$. Then a token system $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ is a nearmedium if and only if $\mathscr{F}$ is a graded family of X.*

Proof. Assume that $(\mathscr{F}, \mathscr{G}_{\mathscr{F}})$ satisfies $[M_1]$. Let P and Q be in $\mathscr{F}$. Then $Pm = Q$ for some concise message m. By the proof of Theorem 2.1.2 1., we have $|P \cap Q| = nh + k$, $|P \smallsetminus Q| = nj - k$, $|Q \smallsetminus P| = nl - k$ for some $n, j, k, l \in \mathbb{Z}^+$ and $k < n$, $P \smallsetminus Q = \{p_1, \ldots, p_{nj-k}\}, Q \smallsetminus P = \{q_1, \ldots, q_{nl-k}\}, P \cap Q = \{a_1, \ldots, a_{nh+k}\}$ and

$$m = \tilde{\gamma}_{p_1 \ldots p_n} \cdots \tilde{\gamma}_{p_{n(j-2)+1} \cdots p_{n(j-1)}} \tilde{\gamma}_{p_{n(j-1)+1} \cdots p_{nj-k} a_1 \ldots a_k}$$

$$\gamma_{q_1 \ldots q_n} \cdots \gamma_{q_{n(l-2)+1} \cdots q_{n(l-1)}} \gamma_{q_{n(l-1)+1} \cdots q_{(nl-k)} a_1 \ldots a_k}.$$



Define a sequence of sets in $\mathscr{F}$ as follows:

$$R_0 = P, R_1 = R_0 \tilde{\gamma}_{p_1 \ldots p_n}, \ldots,$$

$$R_j = R_{j-1} \tilde{\gamma}_{p_{n(j-1)+1} \cdots p_{nj-k} a_1 \ldots a_k} \gamma_{q_1 \ldots q_n}, \ldots,$$

$$\text{and } R_{j+l} = R_{j+l-1} \gamma_{q_{n(l-1)+1} \cdots q_{(nl-k)} a_1 \ldots a_k}.$$

It is clear that $d(R_{i-1}, R_i) = n$ for $i = 1, \ldots, j + l$ and $d(P, Q) = n(j + l) - 2k$. Finally, $R_{i-1} \subseteq R_i$ or $R_i \subseteq R_{i-1}$ since m is stepwise effective.

Conversely, we assume that $\mathscr{F}$ is a graded family. Let $P, Q \in \mathscr{F} \subseteq P_n(X)$. Then $|P \smallsetminus Q| = nj - k$, $|Q \smallsetminus P| = nl - k$, there is a sequence of sets $P = R_0, R_1, \ldots, R_{j+l} = Q$, $R_{i-1} \subseteq R_i$ or $R_i \subseteq R_{i-1}$ such that $d(R_{i-1}, R_i) = n$ for

$i = 1, \ldots, j+l$ and $d(P,Q) = n(j+l) - 2k$. Define a message m by

$$m = \tau_1 \ldots \tau_{j+l} \quad \text{where } \tau_i = \begin{cases} \gamma_{R_i \smallsetminus R_{i-1}}, & R_{i-1} \subseteq R_i; \\ \tilde{\gamma}_{R_{i-1} \smallsetminus R_i}, & R_i \subseteq R_{i-1} \end{cases} \quad \text{for all } i = 1, \ldots, j+l.$$

Then $Pm = Q$. Since $R_{i-1} \subseteq R_i$ or $R_i \subseteq R_{i-1}$ for $i = 1, \ldots, j+l$, m is stepwise effective. Suppose that τ and $\tilde{\tau}$ are contents of m. Thus

$$d(P,Q) \leq n(j+l-2) = n(j+l) - 2n < n(j+l) - 2k,$$

which contradicts $d(P,Q) = n(j+l) - 2k$. Hence m is consistent. Suppose that τ occurs twice in m. Then $m = m_1 \tau m_2 \tau m_3$ with each m_1, m_2 and m_3 is a stepwise effective and consistent message or empty. Suppose $\tau = \gamma_{a_1 \ldots a_n}$. Thus $\{a_1, \ldots, a_n\}$ is removed by the message m_2. Therefore

$$d(P,Q) \leq n(j+l-2) = n(j+l) - 2n < n(j+l) - 2k.$$

On the other hand, $d(P,Q) = n(j+l) - 2k$, so we obtain a contradiction. The proof for $\tau = \tilde{\gamma}_{a_1 \ldots a_n}$ can be done in a similar way. Hence any token occurs at most once in the message m. □

2.2 $Q_n(X)$-family

Let X be a set. For $n \geq 1$, we define

$$Q_n(X) = \{S \subseteq X : |S| = n\}.$$

Let $\mathscr{G} = \{\gamma_{a_i}^{a_j} : a_i$ and a_j are distinct elements in $X\}$ be the family of functions on $Q_n(X)$ given by

$$\gamma_{a_i}^{a_j} : S \mapsto S\gamma_{a_i}^{a_j} = \begin{cases} (S \cup \{a_i\}) \setminus \{a_j\}, & \text{if } a_i \notin S \text{ and } a_j \in S; \\ S, & \text{otherwise} \end{cases}$$

for any a_i and a_j in X with $a_i \neq a_j$. It is immediate that $(Q_n(X), \mathscr{G})$ is a token system and $\gamma_{a_i}^{a_j}$ and $\gamma_{a_j}^{a_i}$ are mutual reverses. Hence the graph represented this nearmedium is undirected.

Example 2.2.1. Let $X = \{1,2,3\}$. Then $Q_1(X) = \{\{1\},\{2\},\{3\}\}$ and $\mathscr{G} = \{\gamma_1^2, \gamma_1^3, \gamma_2^3, \gamma_2^1, \gamma_3^1, \gamma_3^2\}$. The token system $(Q_1(X), \mathscr{G})$ is shown below.

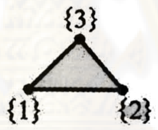


Figure 2.2.1: The graph of $(Q_1(X), \mathscr{G})$ when $X = \{1,2,3\}$.

Example 2.2.2. Let $X = \{1,2,3,4\}$. Then

1. $Q_1(X) = \{\{1\},\{2\},\{3\},\{4\}\}$ and $\mathscr{G} = \{\gamma_1^2, \gamma_1^3, \gamma_1^4, \gamma_2^3, \gamma_2^4, \gamma_3^4, \gamma_2^1, \gamma_3^1, \gamma_4^1, \gamma_3^2, \gamma_4^2, \gamma_4^3\}$.

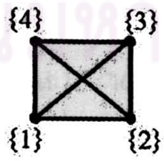


Figure 2.2.2: The graph of $(Q_1(X), \mathscr{G})$ when $X = \{1,2,3,4\}$.

2. $Q_2(X) = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$ and

$$\mathscr{G} = \{\gamma_1^2, \gamma_1^3, \gamma_1^4, \gamma_2^3, \gamma_2^4, \gamma_3^4, \gamma_2^1, \gamma_3^1, \gamma_4^1, \gamma_3^2, \gamma_4^2, \gamma_4^3\}.$$

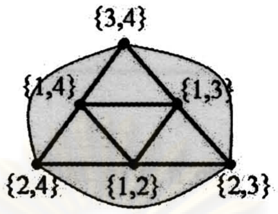


Figure 2.2.3: The graph of $(Q_2(X), \mathscr{G})$ when $X = \{1,2,3,4\}$.

3. $Q_3(X) = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$ and

$$\mathscr{G} = \{\gamma_1^2, \gamma_1^3, \gamma_1^4, \gamma_2^3, \gamma_2^4, \gamma_3^4, \gamma_2^1, \gamma_3^1, \gamma_4^1, \gamma_3^2, \gamma_4^2, \gamma_4^3\}.$$

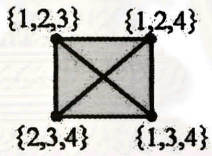
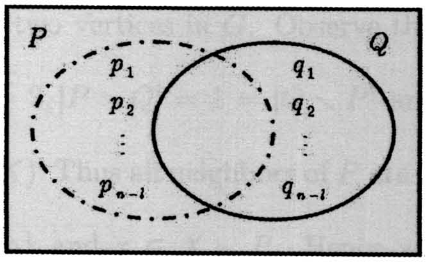


Figure 2.2.4: The graph of $(Q_3(X), \mathscr{G})$ when $X = \{1,2,3,4\}$.

Lemma 2.2.3. *Let X be a set and $n \geq 1$. Then the token system $(Q_n(X), \mathscr{G})$ satisfies $[M_1]$. In other words, $(Q_n(X), \mathscr{G})$ is a nearmedium.*

Proof. Let $P, Q \in Q_n(X)$. Assume that $|P \cap Q| = l$. Then $|P \setminus Q| = n - l = |Q \setminus P|$. Write $P \setminus Q = \{p_1, \ldots, p_{n-l}\}$ and $Q \setminus P = \{q_1, \ldots, q_{n-l}\}$, and let $m = \gamma_{q_1}^{p_1} \ldots \gamma_{q_{n-l}}^{p_{n-l}}$.



It is clear from the construction that m is a concise message and $Pm = Q$.

□

Corollary 2.2.4. *Let $P, Q \in Q_n(X)$. If $d(P, Q) = 2n$, then there is a sequence of sets $P = R_0, R_1, \ldots, R_n = Q$ in $Q_n(X)$ such that $d(R_{i-1}, R_i) = 2$ for $i = 1, \ldots, n$ and $d(P, Q) = 2n$.*

Proof. Let $P, Q \in Q_n(X)$ be such that $d(P, Q) = |P \Delta Q| = 2n$. Then $P \cap Q = \varnothing$. The proof of the above lemma gives the message $m = \gamma_{q_1}^{p_1} \ldots \gamma_{q_n}^{p_n}$ so that $Pm = Q$. Define a sequence of sets in $Q_n(X)$ as follows:

$$R_0 = P, R_1 = R_0\gamma_{q_1}^{p_1}, \ldots, \text{ and } R_n = R_{n-1}\gamma_{q_n}^{p_n} = Q.$$

It is obvious that $d(R_{i-1}, R_i) = 2$ for $i = 1, \ldots, n$ and $d(P, Q) = 2n$. □

A graph is said to be *complete* if all vertices are adjacent and it is said to be *r-regular* if all vertices have degree r

Theorem 2.2.5. *Let $|X| = l$. If G is the graph which represents $(Q_n(X), \mathscr{G})$, then G is $n(l-n)$-regular.*

Proof. Let P and Q be two vertices in G. Observe that if vertices P and Q are adjacent, then $|P\Delta Q| = 2, |P \setminus Q| = 1 = |Q \setminus P|$ and $|P \cap Q| = n-1$. Write $P = \{p_1, \ldots, p_n\} \in Q_n(X)$. Thus all neighbors of P are of the form $(P \setminus \{p_i\}) \cup \{x\}$ for some $i \in \{1, 2, \ldots, n\}$ and $x \in X \setminus P$. Hence we have $n(l-n)$ choices of neighbours of P, and so G is $n(l-n)$-regular. □

Remark. Every state in $(Q_1(X), \mathscr{G})$ is adjacent and so its graph is a complete graph as shown in Figures 2.2.1 and 2.2.2.

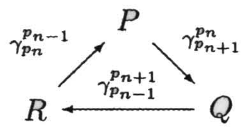
Corollary 2.2.6. *Let $|X| = l$. If $1 \le r < l$, then the graph which represents $Q_r(X)$ and $Q_{l-r}(X)$ are isomorphic.*

Proof. Since $\binom{l}{r} = \binom{l}{l-r}$, $|Q_r(X)| = |Q_{l-r}(X)|$. We define a function $f : Q_r(X) \to Q_{l-r}(X)$ by $S \mapsto X \setminus S$ for all $S \in Q_r(X)$. Let S, T be vertices of $Q_r(X)$ such that S is adjacent to T. Then there are $a, b \in X$ such that $S\gamma_a^b = T$, so $a \notin S, b \in S, a \in T$ and $b \notin T$. Consequently, $a \in X \setminus S, b \notin X \setminus S, a \notin X \setminus T$ and $b \in X \setminus T$. Thus $(X \setminus S)\gamma_b^a = X \setminus T$, and hence $X \setminus S$ is adjacent to $X \setminus T$. Therefore $Q_r(X)$ and $Q_{l-r}(X)$ are isomorphic as required. □

Corollary 2.2.7. *Let $|X| = l$. If $l > n$, then $(Q_n(X), \mathscr{G})$ does not satisfy $[M_2]$. Furthermore, its graph always contains a triangle.*

Proof. Let $p_1, \ldots, p_n, p_{n+1}$ be $n+1$ distinct elements in X. Then $P = \{p_1, \ldots, p_n\}$, $Q = \{p_1, \ldots, p_{n-1}, p_{n+1}\}$ and $R = \{p_1, \ldots, p_{n-2}, p_n, p_{n+1}\}$ are three distinct states in $Q_n(X)$. Observe that the states P, Q and R form a triangle with tokens

$\gamma_{p_{n+1}}^{p_n}, \gamma_{p_n}^{p_{n-1}}$ and $\gamma_{p_{n-1}}^{p_{n+1}}$, respectively,



and the message $m = \gamma_{p_{n+1}}^{p_n}\gamma_{p_n}^{p_{n-1}}\gamma_{p_{n-1}}^{p_{n+1}}$ is a nonvacuous stepwise effective and ineffective message. Hence $Q_n(X)$ does not satisfy $[M_2]$. $\square$

2.3 Infinite Nearmedia

In this section, we give an example of infinite nearmedia and an example of an infinite token system which does not fulfill $[M_1]$.

Example 2.3.1. Let $\mathbb{Z}^2$ denote the set of all integral lattice points with the set of transformations $\mathscr{T}$ consisting of all pairs of tokens $\sigma_i, \tilde{\sigma}_i$ and $\nu_i, \tilde{\nu}_i$ on $\mathbb{Z}^2$ defined by

$$\sigma_i : (k,q) \mapsto (k,q)\sigma_i = \begin{cases} (k, q+1), & \text{if } i = q; \\ (k,q), & \text{if } i \neq q; \end{cases}$$

and

$$\tilde{\sigma}_i : (k,q) \mapsto (k,q)\tilde{\sigma}_i = \begin{cases} (k, q-1), & \text{if } i = q-1; \\ (k,q), & \text{if } i \neq q-1; \end{cases}$$

and

$$\nu_i : (k,q) \mapsto (k,q)\nu_i = \begin{cases} (k-1, q+1), & \text{if } i = k; \\ (k,q), & \text{if } i \neq k; \end{cases}$$

and

$$\tilde{\nu}_i : (k,q) \mapsto (k,q)\tilde{\nu}_i = \begin{cases} (k+1, q-1), & \text{if } i = k-1; \\ (k,q), & \text{if } i \neq k-1; \end{cases}$$

for all $i \in \mathbb{Z}$. Since $(0,0)\nu_0\sigma_1\tilde{\nu}_0\tilde{\sigma}_0 = (0,0)$ but $\nu_0\sigma_1\tilde{\nu}_0\tilde{\sigma}_0$ is not vacuous, this token system does not enjoy $[M_2]$. Next we will show that it satisfies $[M_1]$.

To show this, let (a,b) and (c,d) be two distinct states in $\mathscr{S}$. Then there exist integers k and l such that $c = a-k$ and $d = b+l$. Clearly $m_1 = \tilde{\nu}_a\tilde{\nu}_{a-1}\ldots\tilde{\nu}_{a-k+1}$, with $(a,b)m_1 = (a-k, b+k)$. We may assume without loss of generality that $l = k+r$ for some integer r. Thus we obtain the message $m_2 = \sigma_{b+k}\sigma_{b+k+1}\ldots\sigma_{b+k+r-1}$ with $(a-k, b+k)m_2 = (a-k, b+l) = (c,d)$. It is obvious that

$$m = m_1m_2 = \tilde{\nu}_a\tilde{\nu}_{a-1}\ldots\tilde{\nu}_{a-k+1}\sigma_{b+k}\sigma_{b+k+1}\ldots\sigma_{b+k+r-1}$$

is a concise message sending (a,b) to (c,d).

The graph of this nearmedia is displayed below.

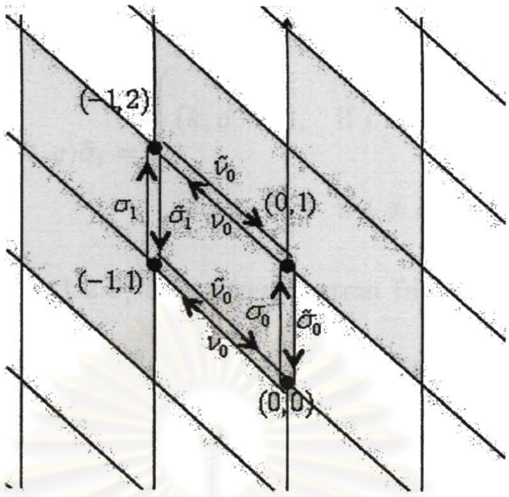


Figure 2.3.1: An infinite nearmedium.

Example 2.3.2. Let $\mathbb{Z}^2$ denote the set of all integral lattice points with the set of transformations $\mathscr{T}$ consisting of all pairs of tokens $\tau_i, \tilde{\tau}_i$ and $\sigma_i, \tilde{\sigma}_i$ on $\mathbb{Z}^2$ defined by

$$\tau_i : (k,q) \mapsto (k,q)\tau_i = \begin{cases} (k+1,q), & \text{if } i = k; \\ (k,q), & \text{if } i \neq k; \end{cases}$$

and

$$\tilde{\tau}_i : (k,q) \mapsto (k,q)\tilde{\tau}_i = \begin{cases} (k-1,q), & \text{if } i = k-1; \\ (k,q), & \text{if } i \neq k-1; \end{cases}$$

$$\sigma_i : (k,q) \mapsto (k,q)\sigma_i = \begin{cases} (k,q+1), & \text{if } i = q, k \neq 0, q \neq 0; \\ (k,q), & \text{if } i \neq q; \end{cases}$$

and

$$\tilde{\sigma}_i : (k,q) \mapsto (k,q)\tilde{\sigma}_i = \begin{cases} (k, q-1), & \text{if } i = q-1, k \neq 0, q \neq 1; \\ (k,q), & \text{if } i \neq q-1; \end{cases}$$

By the proof in Example 1.2.6, this token system fulfills $[M_2]$. Since there is no concise message from $(0,0)$ to $(0,1)$, this token system does not satisfy $[M_1]$. The graph of this token system is given in the following figure.

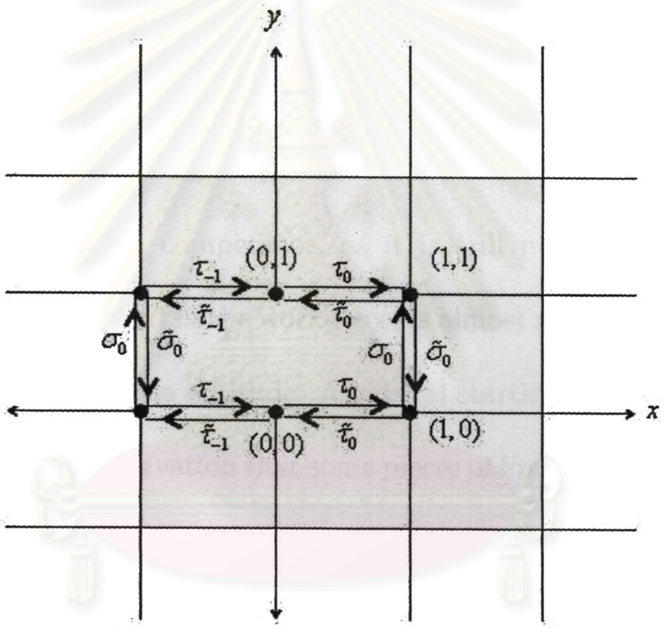


Figure 2.3.2: An infinite token system which does not satisfy $[M_1]$.

CHAPTER III

APPLICATIONS

We provide an application of nearmedia constructed in the previous chapter in this final chapter.

3.1 Knowledge Structures[2]

The assessment of human competence, as it is still performed today by many specialists in the schools and in the workplace, is almost systematically base on the numerical evaluation of some aptitude. A natural starting point for an assessment theory stems from the observation that some pieces of knowledge normally precede, in time, other pieces of knowledge. In our context, some algebra problem may be solvable by a student only if some other problems have already been mastered by that student. This may not only be because some prerequisites are required to master a problem, but may also be due to historical or other circumstances. For example, in a given environment, some concepts are always taught in a particular order, even though there is probably no logical or pedagogical reason to do so. Whatever its genesis may be, this precedence relation may be used to design an

[2]The detail of this section summerizes from [6].

efficient assessment mechanism.

The six types of algebra problems comprising Arithmetic, Beginning Abstract Algebra, Intermediate Algebra and Pre-Calculus are displayed with examples in Table 3.1.2.

A simple example of a precedence relation between problems is illustrated by Figure 3.1.1, which displays a plausible precedence diagram pertaining to the six types of algebra problems given in Table 3.1.2. Note in passing that we distinguish between a type of problem and an instance of that type. Thus, a type of problem is an abstract formulation subsuming a possibly large class of instances. For the rest of this section, problem is almost always intended to mean problem type. The exceptions will be apparent from the context.

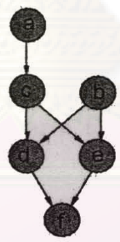


Figure 3.1.1: Precedence diagram for the six types of algebra problems illustrated in Table 3.1.2.

Name of problem type	Example of instance
a Word problem on proportions (Type 1)	A car on the freeway at an average speed of 52 miles per hour. How many miles does it travel in 5 hours and 30 minutes?
b Plotting a point in the coordinate plane	Using the pencil, mark the point at the coordinates $(1, 3)$.
c Multiplication of monomials	Perform the following multiplication: $4x^4y^4 \cdot 2x \cdot 5y^2$ and simplify your answer as much as possible.
d Greatest common factor of two monomials	Find the greatest common factor of the expression $14t^6y$ and $4tu^5y^8$. Simplify your answer as much as possible.
e Graphing the line through a given point with a given slope	Graph the line with slope -7 passing through the point $(-3, -2)$
f Writing the equation of the line through a given point and perpendicular to a given line	Write an equation for the line that passes through the point $(-5, 3)$ and is perpendicular to the line $8x + 5y = 11$.

Table 3.1.2: Six types of problems in Elementary Algebra.

The precedence relation between problems is symbolized by the downward arrows. For example, Problem **e** is preceded by Problems **b**, **c** and **a**. In other words, the mastery of Problem **e** implies that of **b**, **c** and **a**. In the case of these six problems, the precedence relation proposed by the diagram of Figure 3.1.1 is a credible one. For example, if a student responds correctly to an instance of Problem **f**, it is highly plausible that the same student has also mastered the other five problems. Note that this particular precedence relation is part of a much bigger one, representing a comprehensive coverage of all of Beginning Algebra, starting with the solution of simple linear equations and ending with problem types such as **f** in Table 3.1.2.

The knowledge states. The precedence diagram of Figure 3.1.1 completely specifies the feasible knowledge states. The respondent can certainly have mastered just Problem **a**: having mastered **a** does not imply knowing anything else. But if he or she knows **e**, for example, then $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ must also have been mastered, forming a knowledge state which we represent as the set of problems $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}\}$ or more compactly **abce**. Analyzing carefully the precedence diagram of Figure 3.1.1, we see that there are exactly 10 knowledge states consistent with it, forming the set

$$\mathcal{K} = \{\varnothing, \mathbf{a}, \mathbf{b}, \mathbf{ab}, \mathbf{ac}, \mathbf{abc}, \mathbf{abc}, \mathbf{abce}, \mathbf{abcde}, \mathbf{abcdef}\},$$

where $\varnothing$ symbolizes the empty state: the respondent is unable to solved any of the 6 problems. The set $\mathcal{K}$ is our basic concept, and is called a *knowledge structure*. Note that a useful knowledge structure is not necessarily representable by a precedence diagram and may simply be specified by the collection of knowledge states.

The learning paths. This knowledge structure allows several learning paths. Starting from the naive state $\varnothing$, the full mastery of state **abcdef** can be achieved by mastering first **a**, and then successively the other problems in the order $\mathbf{b} \mapsto \mathbf{c} \mapsto \mathbf{d} \mapsto \mathbf{e} \mapsto \mathbf{f}$. But there are other possible ways to learn. All in all, there are 6 possible learning paths consistent with the knowledge structure $\mathcal{K}$, which are displayed in Figure 3.1.3.

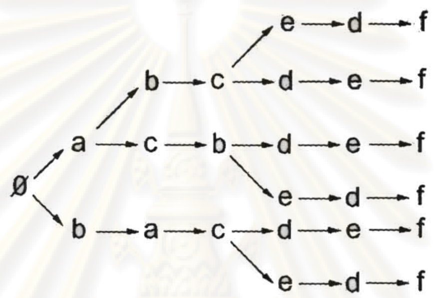


Figure 3.1.3: The 6 possible learning paths consistent with the precedence diagram of Figure 3.1.1.

3.2 n-learning Spaces

Doignon and Falmagne [3] formalized the concept of a knowledge structure (with respect to a topic) as a family $\mathcal{K}$ of subsets of a basic set Q of items[3] of knowledge. Each of the sets in $\mathcal{K}$ is a knowledge state, representing the competence of a particular individual in the population of reference. It is assumed that $\varnothing, Q \in \mathcal{K}$. Two compelling learning axioms are:

[3]In a scholarly context, an item might be a type of problem to be solved, such as long division in arithmetic.

[K_1] If $K \subseteq L$ are two states, with $|L \setminus K| = n$, then there is a chain of states

$$K_0 = K \subseteq K_1 \subset \ldots \subset K_n = L$$

such that $K_i = K_{i-1} + \{q_i\}$ with $q_i \in Q$ for $1 \leq i \leq n$. (We use $+$ to denote disjoint union.) In words, intuitively: If the state K of the learner is included in some other state L, then the learner can reach state L by learning one item at a time.

[K_2] If $K \subseteq L$ are two states, with $K \cup \{q\} \in \mathcal{K}$ and $q \notin L$, then $L \cup \{q\} \in \mathcal{K}$. In words: If item q is learnable from state K, then it is also learnable from any state L that can be reached from K by learning more items.

A knowledge structure $\mathcal{K}$ satisfying Axioms [K_1] and [K_2] is called a *learning space*. To cast a learning space as a medium, we take any knowledge state to be a state of the medium. The transformations consist in adding (or removing) an item $q \in Q$ to (from) a state; thus, they take the form of the two functions: $\tau_q : \mathcal{K} \to \mathcal{K} : K \mapsto K + \{q\}$ and $\tau_q : \mathcal{K} \to \mathcal{K} : K \mapsto K \setminus \{q\}$. The study of media is thus instrumental in our understanding of learning spaces as defined by [K_1] and [K_2]. Note that a learning space is known in the combinatorics literature as an "antimatroid", a structure introduced by Dilworth [2].

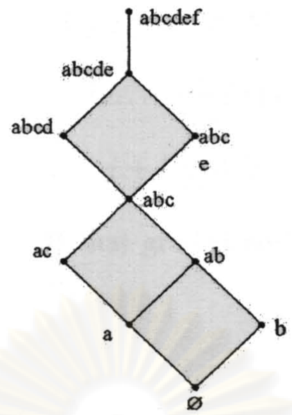


Figure 3.2.1: The graph of a learning space when we case as a medium.

The knowledge structure $\mathcal{K} \subseteq P_n(Q)$ is said to be an *n-learning space* if $\mathcal{K}$ fulfills

[K_1'] If $K \subseteq L$ are two states in $\mathcal{K}$, with $|L \setminus K| = rn$ for some positive integer r, then there is a chain of states

$$K_0 = K \subset K_1 \subset \ldots \subset K_n = L$$

such that $K_i = K_{i-1} + \{q_i, \ldots, q_{i+n-1}\}$ with $q_i \in Q$ for $1 \leq i \leq n$. Again, $+$ stands for disjoint union.

A family $\mathcal{K}$ of subset Q is called *well-graded* if for any $A, B \in \mathcal{K}$, there exists a finite sequence of sets $A = K_0, K_1, \ldots K_k = B$ in $\mathcal{K}$ such that $d(A, B) = k$, and $d(K_{i-1}, K_i) = 1$, $i = 1, \ldots, k$. Following this concept, we say that a family $\mathcal{K} \subseteq P_n(Q)$ is *graded* if, for any $A, B \in \mathcal{K}$ where $|A \cap B| = nh + k$, $|A \setminus B| = nj - k$ and $|B \setminus A| = nl - k$ for some $h, j, l \in \mathbb{Z}^+, k \in \mathbb{Z}_0^+$ and $k < n$, there exists a finite sequence of sets $A = K_0, K_1, \dots, K_{j+l} = B$ in $\mathcal{K}$ with $K_{i-1} \subsetneq K_i$ or $K_i \subsetneq K_{i-1}$ such that $d(K_{i-1}, K_i) = n$ for $i = 1, \dots, j+l$ and $d(A, B) = n(j+l) - 2k$ such that $d(K_{i-1}, K_i) = n$ for $i = 1, \dots, j+l$ and $d(A, B) = n(j+l) - 2k$. Finally, we remark similarly to Chapter II that graded and well-graded are coincide when $n = 1$.

Theorem 3.2.1. *Let $\mathcal{K}$ be a knowledge structure which is closed under union, that is, for any two states K and L in $\mathcal{K}$, we have $K \cup L \in \mathcal{K}$. Then $\mathcal{K}$ is a graded family of Q if and only if $\mathcal{K}$ satisfies $[K_1']$.*

Proof. Let $K, L \in \mathcal{K}$ and $K \subsetneq L$. Then $|L \setminus K| = rn$ for some $r \in \mathbb{Z}^+$. Consequently, $|K \setminus L| = 0$ and $d(K, L) = rn$. Since $\mathcal{K}$ is graded, there is a sequence $K = K_0, K_1, \dots, K_r = L$ in $\mathcal{K}$ such that $K_{i-1} \subsetneq K_i$ and $d(K_i, K_{i-1}) = n$. Hence $\mathcal{K}$ satisfies $[K_1']$.

Conversely, we assume that $\mathcal{K}$ satisfies $[K_1']$. Let $A, B \in \mathcal{K}$. Since $\mathcal{K}$ is closed under union, $A \cup B \in K$. Then $|A| = nh$, $|B| = ni$ and $|A \cap B| = nj$ for some $h, i, j \in \mathbb{Z}^+$. Similarly, there is a chain of states $K_0 = B \subsetneq K_1 \subsetneq \dots \subsetneq K_{i-j} = A \cup B$. Since $A \subsetneq A \cup B$ and $|A \setminus B| = n(h-j)$, there is a chain of state $K_0' = A \subsetneq K_1' \subsetneq \dots \subsetneq K_{h-j}' = A \cup B$. Then we obtain the inclusions

$$K_0 = A \subsetneq K_1 \subsetneq \dots \subsetneq K_{i-j} = A \cup B \supsetneq K_{h-j}' \supsetneq \dots \supsetneq \dots \supsetneq K_1' \supsetneq K_0' = B.$$

Thus $K_0 = A, K_1, \dots, K_{i-j} = A \cup B, K_{h-j}', \dots, K_1', K_0' = B$ is a desired sequence of set in $\mathcal{K}$ such that $d(A, B) = n(h + i - 2j)$. Hence $\mathcal{K}$ is graded as required. ☐

By Theorem 2.1.6, we have

Corollary 3.2.2. *Let $\mathcal{K} \subseteq P_n(Q)$ which is closed under union. Then a knowledge structure $\mathcal{K}$ is a nearmedium if and only if $\mathcal{K}$ satisfies $[K_1']$.*

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