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MULTI-VALUED HOMOMORPHISMS BETWEEN
SOME GROUPS AND HYPERGROUPS

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ไฮเปอร์กรุป คือ ระบบ $(H,\circ)$ โดยที่ H เป็นเซตไม่ว่าง และ $\circ$ เป็นการดำเนินการไฮเปอร์ ซึ่งสอดคล้องเงื่อนไข

$$x\circ(y\circ z)=(x\circ y)\circ z \text{ และ } x\circ H=H\circ x=H \text{ สำหรับทุก } x,y,z\in H$$

สาทิสสัณฐานหลายค่า จากไฮเปอร์กรุป $(H,\circ)$ ไปยัง $(H',\circ')$ หมายถึง ฟังก์ชันหลายค่าจาก H ไปยัง H' ซึ่งสอดคล้องสมบัติว่า

$$f(x\circ y)=f(x)\circ' f(y) \text{ สำหรับทุก } x,y\in H$$

และ เรากล่าวว่า f *ทั่วถึง* เมื่อ $f(H)\left(=\bigcup_{x\in H} f(x)\right)=H'$ สำหรับจำนวนเต็มบวก k ให้ $(\mathbb{Z},\circ_k)$ เป็นไฮเปอร์กรุปที่นิยามการดำเนินการไฮเปอร์ $\circ_k$ บน $\mathbb{Z}$ โดย

$$x\circ_k y=x+y+k\mathbb{Z} \text{ สำหรับทุก } x,y\in\mathbb{Z}$$

และเรานิยามไฮเปอร์กรุป $(\mathbb{Z}_n,\circ_k)$ ในทำนองเดียวกัน นั่นคือ

$$[x]_n\circ_k[y]_n=[x]_n+[y]_n+k\mathbb{Z}_n \text{ สำหรับทุก } x,y\in\mathbb{Z}$$

ในการวิจัยนี้ เราให้ลักษณะของสาทิสสัณฐานหลายค่าและสาทิสสัณฐานหลายค่าแบบทั่วถึง ระหว่างกรุป $(\mathbb{Z},+)$, $(\mathbb{Z}_n,+)$ และไฮเปอร์กรุปที่อยู่ในรูปแบบ $(\mathbb{Z},\circ_k)$, $(\mathbb{Z}_n,\circ_k)$ ยิ่งไปกว่านั้น เรายังพิจารณาจำนวนเชิงการนับของเซตของฟังก์ชันหลายค่าเช่นนั้นด้วย

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A *hypergroup* is a system $(H, \circ)$ where H is a nonempty set and $\circ$ is a hyperoperation such that

$$x \circ (y \circ z) = (x \circ y) \circ z \text{ and } x \circ H = H \circ x = H \text{ for all } x, y, z \in H.$$

By a *multi-valued homomorphism* from a hypergroup $(H, \circ)$ into a hypergroup $(H', \circ')$ we mean a multi-valued function f from H into H' such that

$$f(x \circ y) = f(x) \circ' f(y) \text{ for all } x, y \in H$$

and we say that f is *surjective* if $f(H)(= \bigcup_{x \in H} f(x)) = H'$. For a positive integer k, let $(\mathbb{Z}, \circ_k)$ be the hypergroup with the hyperoperation $\circ_k$ on $\mathbb{Z}$ defined by

$$x \circ_k y = x + y + k\mathbb{Z} \text{ for all } x, y \in \mathbb{Z}.$$

The hypergroup $(\mathbb{Z}_n, \circ_k)$ is defined analogously, that is,

$$[x]_n \circ_k [y]_n = [x]_n + [y]_n + k\mathbb{Z}_n \text{ for all } x, y \in \mathbb{Z}.$$

In this research, we characterize the multi-valued homomorphisms and the surjective multi-valued homomorphisms between the groups $(\mathbb{Z}, +)$, $(\mathbb{Z}_n, +)$ and the hypergroups of the forms $(\mathbb{Z}, \circ_k)$, $(\mathbb{Z}_n, \circ_k)$. In addition, the cardinalities of the sets of such multi-valued functions are determined.

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CONTENTS

CHAPTER I
INTRODUCTION

The concept of homomorphism has been introduced and studied in any algebraic structure. As we know, the concept of continuity plays a crucial role in topological structures. To generalize the concept of continuous function, the lower semi-continuity and the upper semi-continuity of multi-valued functions have been considered and studied. See for example in [11], [7] and [3]. This motivated Triphop, Harnchoowong and Kemprasit [10] to consider multi-valued functions in an algebraic sense. They defined *multi-valued homomorphisms* or *multihomomorphisms* between groups. Then characterizations of multihomomorphisms among all the cyclic groups (up to isomorphism) were provided in [10]. In addition, the numbers of such multihomomorphisms were determined. In fact, multi-valued endomorphisms of hypergroups in a more general sense have been introduced to obtain an example of feeble hyperrings ([2], page 176). Nenthein and Lertwichitsilp [5] studied extensively by making use of the results in [10]. They defined a *surjective multihomomorphism* in a natural way and characterized and counted the surjective multihomomorphisms between cyclic groups. Some interesting necessary conditions of the multihomomorphisms from any group into a subgroup of the additive group of real numbers and a subgroup of the multiplicative group of nonzero real numbers were provided by Youngkhong and Savettaseranee in [12].

In this research, *multi-valued homomorphisms* and *surjective multi-valued homomorphisms* between hypergroups are defined exactly the same as those were given in [10] and [5] for groups, respectively. Such multi-valued functions between the cyclic groups $(\mathbb{Z}, +)$, $(\mathbb{Z}_n, +)$ and the hypergroups of the forms $(\mathbb{Z}, \circ_k)$ and $(\mathbb{Z}_n, \circ_k)$ are characterized where k is a positive integer, $x \circ_k y = x + y + k\mathbb{Z}$ and $[x]_n \circ_k [y]_n = [x]_n + [y]_n + k\mathbb{Z}_n$ for all $x, y \in \mathbb{Z}$. In addition, most of them are counted.

As given in [10] and [5], let $\mathrm{MHom}((H,\circ),(H',\circ'))$ and $\mathrm{SMHom}((H,\circ),(H',\circ'))$ be the sets of all multi-valued homomorphisms and all surjective multi-valued homomorphisms from the hypergroup $(H,\circ)$ into the hypergroup $(H',\circ')$.

The preliminaries and notations used in this research are given in Chapter II. Multi-valued homomorphisms and surjective multi-valued homomorphisms from our target groups into the hypergroups of our interest are characterized in Chapter III. That is, the elements of $\mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k))$, $\mathrm{SMHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k))$, $\mathrm{MHom}((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k))$, $\mathrm{SMHom}((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k))$, $\mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z}_n,\circ_k))$, $\mathrm{SMHom}((\mathbb{Z},+),(\mathbb{Z}_n,\circ_k))$, $\mathrm{MHom}((\mathbb{Z}_m,+),(\mathbb{Z}_n,\circ_k))$ and $\mathrm{SMHom}((\mathbb{Z}_m,+),(\mathbb{Z}_n,\circ_k))$ are determined. Also, the cardinalities of these sets are all provided in this chapter.

Chapter IV provides characterizations determining when multi-valued functions from the hypergroups of the above forms into the cyclic groups $(\mathbb{Z},+)$ and $(\mathbb{Z}_n,+)$ are multi-valued homomorphisms and surjective multi-valued homomorphisms. That is, the elements of $\mathrm{MHom}((\mathbb{Z},\circ_k),(\mathbb{Z},+))$, $\mathrm{SMHom}((\mathbb{Z},\circ_k),(\mathbb{Z},+))$, $\mathrm{MHom}((\mathbb{Z},\circ_k),(\mathbb{Z}_n,+))$, $\mathrm{SMHom}((\mathbb{Z},\circ_k),(\mathbb{Z}_n,+))$, $\mathrm{MHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z},+))$, $\mathrm{SMHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z},+))$, $\mathrm{MHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z}_m,+))$ and $\mathrm{SMHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z}_m,+))$ are characterized. Moreover, we show that the cardinalities of all the sets $\mathrm{MHom}((\mathbb{Z},\circ_k),(\mathbb{Z},+))$, $\mathrm{SMHom}((\mathbb{Z},\circ_k),(\mathbb{Z},+))$, $\mathrm{MHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z},+))$ and $\mathrm{SMHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z},+))$ with $n \nmid k$ are $2^{\aleph_0}$.

CHAPTER II

PRELIMINARIES

The cardinality of a set X is denoted by $|X|$.

A *multi-valued function* from a nonempty set X into a nonempty set Y is a function $f : X \to \mathcal{P}^*(Y)$ where $\mathcal{P}(Y)$ is the power set of Y and $\mathcal{P}^*(Y) = \mathcal{P}(Y) \smallsetminus \{\varnothing\}$ and for $A \subseteq X$, let

$$f(A) = \bigcup_{a \in A} f(a).$$

A *hyperoperation* on a nonempty set H is a multi-valued function $\circ$ from $H \times H$ into H, that is, $\circ : H \times H \to \mathcal{P}^*(H)$, and for $x, y \in H$, $x \circ y$ denotes the value of $(x, y) \in H \times H$ under $\circ$. In this case, $(H, \circ)$ is called a *hypergroupoid*. For nonempty subsets A, B of H, let

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b.$$

If $\varnothing \neq A \subseteq H$ and $x \in H$, let $A \circ x$ and $x \circ A$ stand for $A \circ \{x\}$ and $\{x\} \circ A$, respectively. We say that a hypergroupoid $(H, \circ)$ is *commutative* if $x \circ y = y \circ x$ for all $x, y \in H$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if

$$(x \circ y) \circ z = x \circ (y \circ z) \text{ for all } x, y, z \in H.$$

A *hypergroup* is a semihypergroup $(H, \circ)$ such that

$$H \circ x = x \circ H = H \text{ for all } x \in H.$$

Notice that every group is a hypergroup. In fact, hypergroupoids, semihypergroups and hypergroups are generalizations of groupoids, semigroups and groups, respectively.

Let G be a group and N a normal subgroup of G. If $\circ_N$ is the hyperoperation on G defined by

$$x \circ_N y = xyN \text{ for all } x, y \in G,$$

then $(G, \circ_N)$ is a hypergroup ([2], page 11). It is clearly seen that

$$x_1 \circ_N x_2 \circ_N \cdots \circ_N x_l = x_1 x_2 \cdots x_l N \text{ for all } x_1, x_2, \ldots, x_l \in G \text{ with } l > 1.$$

Notice that if G is abelian, then $(G, \circ_N)$ is a commutative hypergroup. Also, if $N = \{e\}$, then $(G, \circ_N) = G$.

The set of integers is denoted by $\mathbb{Z}$ and let $\mathbb{Z}^+ = \{x \in \mathbb{Z} \mid x > 0\}$ and $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$. Let $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +)$(where $n \in \mathbb{Z}^+$) denote respectively the additive group of integers and the additive group of integers modulo n and for $x \in \mathbb{Z}$, let $[x]_n$ be the congruence class modulo n of x. For $a, b \in \mathbb{Z}$ with $a \neq 0$, $a \mid b$ means that b is divisible by a in $\mathbb{Z}$. Also, if b is not divisible by a, we write $a \nmid b$. Recall that every infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$,

$$\mathbb{Z}_n = \{[x]_n \mid x \in \mathbb{Z}\} = \{[0]_n, [1]_n, \ldots, [n-1]_n\},\ |\,\mathbb{Z}_n| = n$$

and every finite cyclic group of order n is isomorphic to $(\mathbb{Z}_n, +)$. For $a_1, \ldots, a_l \in \mathbb{Z}$, not all 0, let $(a_1, \ldots, a_l)$ be the g.c.d. of $a_1, \ldots, a_l$. Then $(a_1, \ldots, a_l) = x_1 a_1 + x_2 a_2 + \cdots + x_l a_l$ for some $x_1, \ldots, x_l \in \mathbb{Z}$. It is easily shown that

$$\left(\frac{l}{(l,m)}, \frac{m}{(l,m)}\right) = 1 \text{ for all } l, m \in \mathbb{Z}, \text{ not both } 0,$$

$$l\mathbb{Z} = m\mathbb{Z} \Leftrightarrow l = \pm m \text{ for all } l, m \in \mathbb{Z},$$

$$l\mathbb{Z} + m\mathbb{Z} = (l,m)\mathbb{Z}, \quad l\mathbb{Z}_n + m\mathbb{Z}_n = (l,m)\mathbb{Z}_n \text{ for all } l, m \in \mathbb{Z}, \text{ not both } 0,$$

$$l\mathbb{Z}_n = (l,n)\mathbb{Z}_n = (|\,l|, n)\mathbb{Z}_n = |\,l|\mathbb{Z}_n = \{[0]_n, [(l,n)]_n, \ldots, \left(\frac{n}{(l,n)} - 1\right)[(l,n)]_n\},$$

$$|\,l\mathbb{Z}_n| = \frac{n}{(l,n)} \text{ for all } l \in \mathbb{Z}.$$

Hence

$$l\mathbb{Z}_n = m\mathbb{Z}_n \Leftrightarrow (l,n) = (m,n) \quad \text{for all } l,m \in \mathbb{Z}.$$

Moreover, every subgroup of $(\mathbb{Z},+)$ is of the form $l\mathbb{Z}$. Also, every subgroup of $(\mathbb{Z}_n,+)$ is of the form $l\mathbb{Z}_n$. Recall that the Euler φ-function is defined by $\varphi(1) = 1$ and for $k \in \mathbb{Z}$ with $k > 1$, $\varphi(k)$ is the number of positive integers less than k and relatively prime to k. Then

$$\varphi(k) = |\{a \in \{1,2,\ldots,k\} \mid (a,k) = 1\}| \ \text{ for all } k \in \mathbb{Z}^+.$$

It is known that for $k \in \mathbb{Z}^+$, $\sum\limits_{l|k} \varphi(l) = k$ ([6], page 191).

For $k \in \mathbb{Z}^+$, let $(\mathbb{Z},\circ_k)$ and $(\mathbb{Z}_n,\circ_k)$ be the hypergroups with

$$\begin{aligned} x \circ_k y &= x + y + k\mathbb{Z}, \\ [x]_n \circ_k [y]_n &= [x]_n + [y]_n + k\mathbb{Z}_n (= [x+y]_n + k\mathbb{Z}_n) \\ &\text{for all } x,y \in \mathbb{Z}. \end{aligned}$$

By a *muli-valued homomorphism* or a *multihomomorphism* from a hypergroup $(H,\circ)$ into a hypergroup $(H',\circ')$ we mean a multi-valued function from H into H' satisfying the condition

$$f(x \circ y) = f(x) \circ' f(y) \ \text{ for all } x,y \in H.$$

Denote by $\mathrm{MHom}((H,\circ),(H',\circ'))$ the set of all multi-valued homomorphisms from $(H,\circ)$ into $(H',\circ')$ and set $\mathrm{MHom}(H,\circ) :=\mathrm{MHom}((H,\circ),(H,\circ))$. We say that $f \in \mathrm{MHom}((H,\circ),(H',\circ'))$ is *surjective* if

$$f(H) \left(= \bigcup_{h \in H} f(h) \right) = H'.$$

Let $\mathrm{SMHom}((H,\circ),(H',\circ'))$ be the set of all surjective multi-valued homomorphisms from $(H,\circ)$ into $(H',\circ')$ and also set $\mathrm{SMHom}(H,\circ) :=\mathrm{SMHom}((H,\circ),(H,\circ))$.

Characterizations of multi-valued homomorphisms and surjective multi-valued homomorphisms between cyclic groups were provided in [10] and [5], respectively. Also, such elements were counted.

In the remainder of this research, let m,n be positive integers.

Theorem 2.1 ([10]). *For a multi-valued function f from $\mathbb{Z}$ into itself, $f \in$ MHom$(\mathbb{Z},+)$ if and only if there exist a subsemigroup H of $(\mathbb{Z},+)$ containing 0 and an element $a \in \mathbb{Z}$ such that*

$$f(x) = ax + H \text{ for all } x \in \mathbb{Z}.$$

Theorem 2.2 ([10]). $|MHom(\mathbb{Z},+)| = \aleph_0$.

We note here that Theorem 2.2 was proved in [10] by exploiting the fact that every subsemigroup of $(\mathbb{Z}_0^+,+)$ containing 0 is finitely generated, that is, if S is a subsemigroup of $(\mathbb{Z}_0^+,+)$ containing 0, then there are $a_1, a_2, \ldots, a_l \in S$ such that

$$S = a_1\mathbb{Z}_0^+ + a_2\mathbb{Z}_0^+ + \ldots + a_l\mathbb{Z}_0^+.$$

This fact was mentioned in [1].

Theorem 2.3 ([5]). *For a multi-valued function f from $\mathbb{Z}$ into itself, $f \in$ SMHom $(\mathbb{Z},+)$ if and only if there exist a subsemigroup H of $(\mathbb{Z},+)$ containing 0 and $a \in \mathbb{Z}$ such that*

$$f(x) = ax + H \text{ for all } x \in \mathbb{Z},$$
$$(a,h) = 1 \text{ for some } h \in H \text{ and}$$
$$H = \mathbb{Z} \text{ whenever } a = 0.$$

Theorem 2.4 ([5]). $|SMHom(\mathbb{Z},+)| = \aleph_0$.

Theorem 2.5 ([10]). *For a multi-valued function f from $\mathbb{Z}_n$ into $\mathbb{Z}$, $f \in$ MHom $((\mathbb{Z}_n,+),(\mathbb{Z},+))$ if and only if either*

(i) *there exists a subsemigroup H of $(\mathbb{Z},+)$ containing 0 such that*

$$f([x]_n) = H \quad \text{for all } x \in \mathbb{Z}$$

or

(ii) *there exist $l, a \in \mathbb{Z}$ such that $l \neq 0$, $\frac{l}{(l,n)} \mid a$ and*

$$f([x]_n) = ax + l\mathbb{Z} \quad \text{for all } x \in \mathbb{Z}.$$

Theorem 2.6 ([10]). $|MHom((\mathbb{Z}_n, +), (\mathbb{Z}, +))| = \aleph_0$.

Theorem 2.7 ([5]). *For a multi-valued function f from $\mathbb{Z}_n$ into $\mathbb{Z}$, $f \in SMHom((\mathbb{Z}_n, +), (\mathbb{Z}, +))$ if and only if there exist $l, a \in \mathbb{Z}$ such that $l \neq 0$, $l \mid n$, $(a, l) = 1$ and*

$$f([x]_n) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

Theorem 2.8 ([5]). $|SMHom((\mathbb{Z}_n, +), (\mathbb{Z}, +))| = n$.

Theorem 2.9 ([10]). *For a multi-valued function f from $\mathbb{Z}$ into $\mathbb{Z}_n$, $f \in MHom((\mathbb{Z}, +), (\mathbb{Z}_n, +))$ if and only if there exist $l, a \in \mathbb{Z}$ such that*

$$f(x) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

Theorem 2.10 ([10]). $|MHom((\mathbb{Z}, +)), (\mathbb{Z}_n, +)| = \sum\limits_{\substack{l \in \mathbb{Z}^+ \\ l \mid n}} l$.

Theorem 2.11 ([5]). *For a multi-valued function f from $\mathbb{Z}$ into $\mathbb{Z}_n$, $f \in SMHom((\mathbb{Z}, +), (\mathbb{Z}_n, +))$ if and only if there are $l, a \in \mathbb{Z}$ such that $(a, l, n) = 1$ and*

$$f(x) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

Theorem 2.12 ([5]). $|SMHom((\mathbb{Z}, +)), (\mathbb{Z}_n, +)| = n$.

Theorem 2.13 ([10]). *For a multi-valued function f from $\mathbb{Z}_m$ into $\mathbb{Z}_n$, $f \in MHom((\mathbb{Z}_m, +), (\mathbb{Z}_n, +))$ if and only if there are $l, a \in \mathbb{Z}$ such that $\frac{(l, n)}{(l, m, n)} \mid a$ and*

$$f([x]_m) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

Theorem 2.14 ([10]). $|MHom((\mathbb{Z}_m, +)), (\mathbb{Z}_n, +)| = \sum\limits_{\substack{l \in \mathbb{Z}^+ \\ l \mid n}} (l, m)$. *In particular,* $|MHom(\mathbb{Z}_n, +)| = \sum\limits_{\substack{l \in \mathbb{Z}^+ \\ l \mid n}} l$.

Theorem 2.15 ([5]). *For a multi-valued function f from $\mathbb{Z}_m$ into $\mathbb{Z}_n$, $f \in SMHom((\mathbb{Z}_m, +), (\mathbb{Z}_n, +))$ if and only if there exist $l, a \in \mathbb{Z}$ such that $(l, n) \mid m$, $(a, l, n) = 1$ and*

$$f([x]_m) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

Theorem 2.16 ([5]). $|\,SMHom((\mathbb{Z}_m,+)),(\mathbb{Z}_n,+)\,| = (m,n)$. *In particular,* $|SMHom(\mathbb{Z}_n,+)| = n$.

The following basic facts of sets and cardinal numbers will be used.

(1) For any set X, $|\mathcal{P}(X)| = 2^{|X|}$.

(2) For nonempty sets X and Y, $|\{f \mid f : X \to Y\}| = |Y|^{|X|}$.

(3) $(2^{\aleph_0})^n = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ for all $n \in \mathbb{Z}^+$ ([4], page 98).

Let k be a positive integer in the remainder of this research. Also, let $(\mathbb{Z}, \circ_k)$ and $(\mathbb{Z}_n, \circ_k)$ be the hypergroups defined previously.

CHAPTER III

MULTI-VALUED HOMOMORPHISMS FROM GROUPS INTO HYPERGROUPS

This chapter gives characterizations of multi-valued homomorphisms and surjective multi-valued homomorphisms from the groups $(\mathbb{Z},+)$, $(\mathbb{Z}_n,+)$ into the hypergroups of the forms $(\mathbb{Z},\circ_k)$ and $(\mathbb{Z}_n,\circ_k)$. The cardinalities of the sets of such multi-valued functions of all pairs of those groups and hypergroups are also provided.

3.1 Multi-valued Homomorphisms from the Group $(\mathbb{Z},+)$ into the Hypergroup $(\mathbb{Z},\circ_k)$

We begin this section by recalling the following fact given in [10].

Lemma 3.1.1 ([10]). *If H is a subsemigroup of $(\mathbb{Z},+)$ such that $H\cap\mathbb{Z}^+\neq\varnothing$ and $H\cap\mathbb{Z}^-\neq\varnothing$, then $H=l\mathbb{Z}$ for some $l\in\mathbb{Z}\smallsetminus\{0\}$.*

The following lemmas are also needed.

Lemma 3.1.2. *Let G be a group with identity e. If $f\in MHom(G,(\mathbb{Z},\circ_k))$, then $f(e)=l\mathbb{Z}$ for some $l\in\mathbb{Z}\smallsetminus\{0\}$ with $l\mid k$.*

Proof. Let $f\in\mathrm{MHom}(G,(\mathbb{Z},\circ_k))$. Then

$$\begin{aligned} f(e)=f(ee)=f(e)\circ_k f(e)&=f(e)+f(e)+k\mathbb{Z}\\ &\supseteq f(e)+f(e) \end{aligned} \tag{1}$$

since $0\in k\mathbb{Z}$. This implies that $f(e)$ is a subsemigroup of $(\mathbb{Z},+)$. Let $a\in f(e)$. Then from (1),

$$2a+k\mathbb{Z}=a+a+k\mathbb{Z}\subseteq f(e)+f(e)+k\mathbb{Z}=f(e). \tag{2}$$

Let $b, c \in \mathbb{Z}$ be such that $b > \dfrac{-2a}{k} > c$. Then $kb > -2a > kc$ which implies from (2) that

$$0 < 2a + kb \in f(e) \text{ and } 0 > 2a + kc \in f(e).$$

It follows that $f(e) \cap \mathbb{Z}^{+} \neq \varnothing$ and $f(e) \cap \mathbb{Z}^{-} \neq \varnothing$. By Lemma 3.1.1, $f(e) = l\mathbb{Z}$ for some $l \in \mathbb{Z} \smallsetminus \{0\}$. Hence by (1),

$$l\mathbb{Z} = l\mathbb{Z} + l\mathbb{Z} + k\mathbb{Z} = l\mathbb{Z} + k\mathbb{Z} = (l, k)\mathbb{Z}.$$

Consequently, $l = \pm(l, k)$, so $l \mid k$. □

Lemma 3.1.3. *Let G be a group with identity e and $f \in MHom(G, (\mathbb{Z}, \circ_k))$. Then for every $x \in G$, there exists an element $a \in f(x)$ such that*

$$f(x^t) = at + f(e) \text{ for all } t \in \mathbb{Z}.$$

Proof. By Lemma 3.1.2, $f(e) = l\mathbb{Z}$ for some $l \in \mathbb{Z} \smallsetminus \{0\}$ with $l \mid k$. Then $l\mathbb{Z} + k\mathbb{Z} = (l, k)\mathbb{Z} = l\mathbb{Z}$. Let $x \in G$ be given. Then

$$f(x) = f(xe) = f(x) \circ_k f(e) = f(x) + l\mathbb{Z} + k\mathbb{Z} = f(x) + l\mathbb{Z}, \tag{1}$$

and similarly,

$$f(x^{-1}) = f(x^{-1}) + l\mathbb{Z}. \tag{2}$$

Since $l\mathbb{Z} + k\mathbb{Z} = l\mathbb{Z}$, we obtain respectively from (1) and (2) that

$$f(x) + k\mathbb{Z} \quad = f(x) + l\mathbb{Z} = f(x), \tag{3}$$

$$f(x^{-1}) + k\mathbb{Z} = f(x^{-1}) + l\mathbb{Z} = f(x^{-1}). \tag{4}$$

These imply that

$$\begin{aligned} l\mathbb{Z} &= f(e) \\ &= f(xx^{-1}) \\ &= f(x) \circ_k f(x^{-1}) \\ &= f(x) + f(x^{-1}) + k\mathbb{Z} \\ &= f(x) + (f(x^{-1}) + k\mathbb{Z}) \\ &= f(x) + f(x^{-1}) \quad \text{from (4).} \end{aligned} \tag{5}$$

Since $0 \in l\mathbb{Z}$, from (5), there are $a \in f(x)$ and $b \in f(x^{-1})$ such that $0 = a + b$. Then $-a = b \in f(x^{-1})$. Thus from (5), we have

$$\begin{aligned} &f(x) - a \subseteq f(x) + f(x^{-1}) = l\mathbb{Z}, \\ &a + f(x^{-1}) \subseteq f(x) + f(x^{-1}) = l\mathbb{Z} \end{aligned}$$

which imply that

$$f(x) \subseteq a + l\mathbb{Z} \text{ and } f(x^{-1}) \subseteq -a + l\mathbb{Z}. \tag{6}$$

By (1), (2) and (6),

$$\begin{aligned} &f(x) \subseteq a + l\mathbb{Z} \subseteq f(x) + l\mathbb{Z} = f(x), \\ &f(x^{-1}) \subseteq -a + l\mathbb{Z} \subseteq f(x^{-1}) + l\mathbb{Z} = f(x^{-1}). \end{aligned}$$

Consequently,

$$f(x) = a + l\mathbb{Z} = a + f(e) \text{ and } f(x^{-1}) = -a + l\mathbb{Z} = -a + f(e). \tag{7}$$

Note that $f(x^0) = f(e) = a0 + f(e)$. If $t \in \mathbb{Z}^+$ and $t > 1$, then

$$\begin{aligned} f(x^t) &= f(x) \circ_k f(x) \circ_k \cdots \circ_k f(x) \ \ \text{(t copies)} \\ &= \underbrace{f(x) + \cdots + f(x)}_{\text{t copies}} + k\mathbb{Z} \\ &= (f(x) + k\mathbb{Z}) + \cdots + (f(x) + k\mathbb{Z}) \ \ \text{(t brackets)} \\ &= f(x) + \cdots + f(x) \ \ \text{(t copies)} \quad \text{from (3)} \\ &= (a + l\mathbb{Z}) + \cdots + (a + l\mathbb{Z}) \ \ \text{(t brackets)} \quad \text{from (7)} \\ &= at + l\mathbb{Z} \\ &= at + f(e), \end{aligned}$$

$$
\begin{aligned}
f(x^{-t}) &= f((x^{-1})^t) \\
&= f(x^{-1}) \circ_k f(x^{-1}) \circ_k \cdots \circ_k f(x^{-1}) \ \ \text{(t copies)} \\
&= \underbrace{f(x^{-1}) + \cdots + f(x^{-1})}_{\text{t copies}} + k\mathbb{Z} \\
&= (f(x^{-1}) + k\mathbb{Z}) + \cdots + (f(x^{-1}) + k\mathbb{Z}) \ \ \text{(t brackets)} \\
&= f(x^{-1}) + \cdots + f(x^{-1}) \ \ \text{(t copies)} \quad \text{from (4)} \\
&= (-a + l\mathbb{Z}) + \cdots + (-a + l\mathbb{Z}) \ \ \text{(t brackets)} \quad \text{from (7)} \\
&= (-at) + l\mathbb{Z} \\
&= a(-t) + f(e).
\end{aligned}
$$

Hence the desired result follows. □

Theorem 3.1.4. *For a multi-valued function f from $\mathbb{Z}$ into itself, $f \in MHom((\mathbb{Z}, +), (\mathbb{Z}, \circ_k))$ if and only if there are $l, a \in \mathbb{Z}$ such that $l \neq 0$, $l \mid k$ and*

$$f(x) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

Proof. Let $f \in \text{MHom}((\mathbb{Z}, +), (\mathbb{Z}, \circ_k))$. By Lemma 3.1.3, there is an element $a \in f(1)$ such that

$$f(x1) = ax + f(0) \text{ for all } x \in \mathbb{Z},$$

that is,

$$f(x) = ax + f(0) \text{ for all } x \in \mathbb{Z}.$$

By Lemma 3.1.2, $f(0) = l\mathbb{Z}$ for some $l \in \mathbb{Z} \setminus \{0\}$ with $l \mid k$. Hence

$$f(x) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

Conversely, assume that there are $l, a \in \mathbb{Z}$ such that $l \neq 0$, $l \mid k$ and

$$f(x) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

Since $l \mid k$, we have $l\mathbb{Z} + k\mathbb{Z} = (l,k)\mathbb{Z} = l\mathbb{Z}$. Then for all x, $y \in \mathbb{Z}$,

$$\begin{aligned} f(x+y) &= a(x+y) + l\mathbb{Z} \\ &= ax + ay + l\mathbb{Z} \\ &= ax + ay + l\mathbb{Z} + k\mathbb{Z} \\ &= (ax + l\mathbb{Z}) + (ay + l\mathbb{Z}) + k\mathbb{Z} \\ &= f(x) + f(y) + k\mathbb{Z} \\ &= f(x) \circ_k f(y). \end{aligned}$$

Hence $f \in \mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k))$, as desired. □

Theorem 3.1.5. *For a multi-valued function f from $\mathbb{Z}$ into itself, $f \in$ SMHom $((\mathbb{Z},+),(\mathbb{Z},\circ_k))$ if and only if there exist $l, a \in \mathbb{Z}$ such that $l \neq 0$, $l \mid k$, $(a,l) = 1$ and*

$$f(x) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

Proof. Assume that $f \in \mathrm{SMHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k))$. Then $f \in \mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k))$ and $f(\mathbb{Z}) = \mathbb{Z}$. By Theorem 3.1.4, there are $l \in \mathbb{Z} \setminus \{0\}$ and $a \in \mathbb{Z}$ such that $l \mid k$,

$$f(x) = ax + l\mathbb{Z} \quad \text{for all } x \in \mathbb{Z}.$$

Consequently,

$$\begin{aligned} \mathbb{Z} = f(\mathbb{Z}) = \bigcup_{x\in\mathbb{Z}} f(x) &= \bigcup_{x\in\mathbb{Z}} (ax + l\mathbb{Z}) \\ &= a\mathbb{Z} + l\mathbb{Z} = (a,l)\mathbb{Z} \end{aligned}$$

which implies that $(a,l) = 1$.

Conversely, assume that $l \in \mathbb{Z} \setminus \{0\}$, $a \in \mathbb{Z}$, $l \mid k$, $(a,l) = 1$ and

$$f(x) = ax + l\mathbb{Z} \quad \text{for all } x \in \mathbb{Z}.$$

By Theorem 3.1.4, $f \in \mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k))$. Since $(a,l) = 1$, we have

$$f(\mathbb{Z}) = a\mathbb{Z} + l\mathbb{Z} = (a,l)\mathbb{Z} = \mathbb{Z}.$$

Therefore $f \in \mathrm{SMHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k))$. □

For $l \in \mathbb{Z} \setminus \{0\}$ with $l \mid k$ and $a \in \mathbb{Z}$, let $F_{l,a} \in \text{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k))$ be defined by

$$F_{l,a}(x) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

To determine $|\text{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k))|$ and $|\text{SMHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k))|$, the following lemma is needed.

Lemma 3.1.6. *Let $l, t \in \mathbb{Z} \setminus \{0\}$ with $l \mid k$ and $t \mid k$ and $a, b \in \mathbb{Z}$. Then $F_{l,a} = F_{t,b}$ if and only if $t = \pm l$ and $b \equiv a \mod |l|$.*

Proof. If $F_{l,a} = F_{t,b}$, then

$$ax + l\mathbb{Z} = F_{l,a}(x) = F_{t,b}(x) = bx + t\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

In particular,

$$l\mathbb{Z} = a0 + l\mathbb{Z} = b0 + t\mathbb{Z} = t\mathbb{Z}$$

which implies that $t = \pm l$. Hence

$$a + l\mathbb{Z} = a1 + l\mathbb{Z} = b1 + l\mathbb{Z} = b + l\mathbb{Z},$$

so $b - a \in l\mathbb{Z}$. Therefore $b \equiv a \mod |l|$.

Conversely, assume that $t = \pm l$ and $b \equiv a \mod |l|$. Then $b - a \in |l|\mathbb{Z} = l\mathbb{Z}$ and $l\mathbb{Z} = t\mathbb{Z}$. Since

$$\text{for all } x \in \mathbb{Z},\ \ bx - ax = (b-a)x \in l\mathbb{Z}x \subseteq l\mathbb{Z},$$

it follows that

$$ax + l\mathbb{Z} = bx + l\mathbb{Z} = bx + t\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

Hence

$$F_{l,a}(x) = ax + l\mathbb{Z} = ax + t\mathbb{Z} = F_{t,b}(x) \text{ for all } x \in \mathbb{Z},$$

so $F_{l,a} = F_{t,b}$. □

Theorem 3.1.7. $|MHom((\mathbb{Z},+),(\mathbb{Z},\circ_k))| = \sum_{\substack{l \in \mathbb{Z}^+ \\ l \mid k}} l$ *and*

$$|SMHom((\mathbb{Z},+),(\mathbb{Z},\circ_k))| = k.$$

Proof. From Theorem 3.1.4 and Theorem 3.1.5, we have

$$\mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k)) = \{F_{l,a} |\ l \in \mathbb{Z} \smallsetminus \{0\},\ a \in \mathbb{Z} \text{ and } l \mid k\}, \tag{1}$$

$$\begin{aligned}\mathrm{SMHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k)) = \{F_{l,a} |\ l \in \mathbb{Z} \smallsetminus \{0\},\ a \in \mathbb{Z},\ l \mid k \text{ and}\\ (a,l) = 1\}.\end{aligned} \tag{2}$$

Then (1), (2) and Lemma 3.1.6 yield the followings :

$$\mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k)) = \{F_{l,a} \mid l \in \mathbb{Z}^+,\ l \mid k \text{ and } a \in \{0,\ 1, \ldots,\ l-1\}\}, \tag{3}$$

$$\begin{aligned}\mathrm{SMHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k)) = \{F_{l,a} \mid l \in \mathbb{Z}^+,\ l \mid k,\ a \in \{0,\ 1, \ldots,\ l-1\}\\ \text{and } (a,l) = 1\}.\end{aligned} \tag{4}$$

Again, by (3), (4) and Lemma 3.1.6, we have

$$|\mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k))| = \sum_{\substack{l \in \mathbb{Z}^+ \\ l \mid k}} l,$$

$$|\mathrm{SMHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k))| = \sum_{\substack{l \in \mathbb{Z}^+ \\ l \mid k}} \varphi(l) = k.$$

□

Remark 3.1.8. Let us compare the results of this section with Theorem 2.1 - Theorem 2.4 where H is a subsemigroup of $(\mathbb{Z},+)$ containing 0 and $l,\ a \in \mathbb{Z}$.

	Characterization	**Cardinality**
$\text{MHom}((\mathbb{Z},+),(\mathbb{Z},+))$	$f(x) = ax + H$	$\aleph_0$
$\text{SMHom}((\mathbb{Z},+),(\mathbb{Z},+))$	$f(x) = ax + H$, $(a, h) = 1$ for some $h \in H$, $a = 0 \Rightarrow H = \mathbb{Z}$	$\aleph_0$
$\text{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k))$	$f(x) = ax + l\mathbb{Z}$, $l \neq 0$, $l \mid k$	$\sum\limits_{\substack{l\in\mathbb{Z}^+ \\ l\mid k}} l$
$\text{SMHom}((\mathbb{Z},+),(\mathbb{Z},\circ_k))$	$f(x) = ax + l\mathbb{Z}$, $l \neq 0$, $l \mid k$, $(a, l) = 1$	k

3.2 Multi-valued Homomorphisms from the Group $(\mathbb{Z}_n, +)$ into the Hypergroup $(\mathbb{Z}, \circ_k)$

In this section, the following result is needed. It was proved in [10].

Lemma 3.2.1 ([10]). *Let $l, a \in \mathbb{Z}$ and define*

$$f([x]_n) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

Then f is a well-defined multi-valued function from $\mathbb{Z}_n$ into $\mathbb{Z}$ if and only if either

(i) $l = a = 0$ *or*

(ii) $l \neq 0$ *and* $\dfrac{l}{(l,n)} \mid a$.

Theorem 3.2.2. *For a multi-valued function f from $\mathbb{Z}_n$ into $\mathbb{Z}$, $f \in MHom((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k))$ if and only if there exist $l, a \in \mathbb{Z}$ such that $l \neq 0$, $l \mid k$, $\dfrac{l}{(l,n)} \mid a$ and*

$$f([x]_n) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

Proof. Assume that $f \in \mathrm{MHom}((\mathbb{Z}_n, +), (\mathbb{Z}, \circ_k))$. By Lemma 3.1.2 and Lemma 3.1.3, there are $l \in \mathbb{Z} \smallsetminus \{0\}$ and $a \in f([1]_n)$ such that $l \mid k$ and

$$f([x]_n)(= f(x[1]_n)) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

By Lemma 3.2.1, we have $\dfrac{l}{(l,n)} \mid a$.

For the converse, assume that $l \in \mathbb{Z} \smallsetminus \{0\}$, $a \in \mathbb{Z}$, $l \mid k$, $\dfrac{l}{(l,n)} \mid a$ and

$$f([x]_n) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

Then by Lemma 3.2.1, f is well-defined. Since $l \mid k$, we have $l\mathbb{Z} + k\mathbb{Z} = l\mathbb{Z}$. If $x, y \in \mathbb{Z}$, then

$$\begin{aligned} f([x]_n + [y]_n) &= f([x+y]_n) \\ &= a(x+y) + l\mathbb{Z} \\ &= ax + ay + l\mathbb{Z} + k\mathbb{Z} \\ &= (ax + l\mathbb{Z}) + (ay + l\mathbb{Z}) + k\mathbb{Z} \\ &= f([x]_n) + f([y]_n) + k\mathbb{Z} \\ &= f([x]_n) \circ_k f([y]_n). \end{aligned}$$

Hence $f \in \mathrm{MHom}((\mathbb{Z}_n, +), (\mathbb{Z}, \circ_k))$. □

Lemma 3.2.3. *For $l \in \mathbb{Z} \smallsetminus \{0\}$ and $a \in \mathbb{Z}$, $l \mid k$, $\dfrac{l}{(l,n)} \mid a$ and $(a,l) = 1$ if and only if $l \mid (k,n)$ and $(a,l) = 1$.*

Proof. Assume that $l \mid k$, $\dfrac{l}{(l,n)} \mid a$ and $(a,l) = 1$. Since $(a,l) = 1$, $\left(a, \dfrac{l}{(l,n)}\right) = 1$. But $\dfrac{l}{(l,n)} \mid a$, so $\dfrac{|l|}{(l,n)} = \left(a, \dfrac{l}{(l,n)}\right) = 1$ which implies that $(l,n) = |l|$, so $l \mid n$. But since $l \mid k$, we have $l \mid (k,n)$.

Conversely, assume that $l \mid (k,n)$. Then $l \mid k$ and $l \mid n$. Thus $(l,n) = |l|$, so $\dfrac{l}{(l,n)} = \pm 1$. Hence $\dfrac{l}{(l,n)} \mid a$. □

Theorem 3.2.4. *For a multi-valued function f from $\mathbb{Z}_n$ into $\mathbb{Z}$, $f \in$ SMHom $((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k))$ if and only if there exist $l, a \in \mathbb{Z}$ such that $l \neq 0$, $l \mid (k,n)$, $(a,l)=1$ and*

$$f([x]_n) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

Proof. Assume that $f \in \mathrm{SMHom}((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k))$. Then $f \in \mathrm{MHom}((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k))$ and $f(\mathbb{Z}_n) = \mathbb{Z}$. By Theorem 3.2.2, there are $l \in \mathbb{Z} \setminus \{0\}$ and $a \in \mathbb{Z}$ such that $l \mid k$, $\dfrac{l}{(l,n)} \mid a$ and

$$f([x]_n) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

This implies that

$$\begin{aligned}\mathbb{Z} = f(\mathbb{Z}_n) &= \bigcup_{x\in\mathbb{Z}} f([x]_n)\\ &= \bigcup_{x\in\mathbb{Z}} (ax + l\mathbb{Z})\\ &= a\mathbb{Z} + l\mathbb{Z} = (a,l)\mathbb{Z},\end{aligned}$$

Thus $(a,l) = 1$. It follows from Lemma 3.2.3 that $l \mid (k,n)$.

For the converse, let l, a and f be as above. By Lemma 3.2.3, $l \mid k$, $\dfrac{l}{(l,n)} \mid a$ and $(a,l) = 1$. By Theorem 3.2.2, $f \in \mathrm{MHom}((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k))$. Since $(a,l) = 1$, it follows that

$$f(\mathbb{Z}_n) = a\mathbb{Z} + l\mathbb{Z} = (a,l)\mathbb{Z} = \mathbb{Z}.$$

□

For $l \in \mathbb{Z} \setminus \{0\}$, $a \in \mathbb{Z}$, $l \mid k$ and $\dfrac{l}{(l,n)} \mid a$, let $G_{l,a} \in \mathrm{MHom}((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k))$ be defined by

$$G_{l,a}([x]_n) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

Lemma 3.2.5. *Let $l, t \in \mathbb{Z} \setminus \{0\}$, $a, b \in \mathbb{Z}$, $l \mid k$, $t \mid k$, $\dfrac{l}{(l,n)} \mid a$ and $\dfrac{t}{(t,n)} \mid b$. Then $G_{l,a} = G_{t,b}$ if and only if $t = \pm l$ and $b \equiv a \mod |l|$.*

Proof. The proof is analogous to that of Lemma 3.1.6 □

Theorem 3.2.6. $|MHom((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k))| = \sum_{\substack{l\in\mathbb{Z}^+\\ l|k}} (l,n)$
and

$$|SMHom((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k))| = (k,n).$$

Proof. From Theorem 3.2.2 and Theorem 3.2.4, we have

$$\text{MHom}((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k)) = \{G_{l,a} |\ l \in \mathbb{Z}\setminus\{0\},\ a\in\mathbb{Z},\ l \mid k \text{ and } \frac{l}{(l,n)} \mid a\}, \quad (1)$$

$$\begin{aligned}\text{SMHom}((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k)) = \{G_{l,a} |\ l \in \mathbb{Z}\setminus\{0\},\ a\in\mathbb{Z},\ l \mid (k,n) \text{ and }\\ (a,l)=1\},\end{aligned} \quad (2)$$

respectively. We deduce from (1), (2) and Lemma 3.2.5 that

$$\begin{aligned}\text{MHom}((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k)) &= \{G_{l,a} \mid l\in\mathbb{Z}^+,\ l\mid k,\ a\in\{0,\,1,\dots,\,l-1\}\\ &\qquad \text{and } \frac{l}{(l,n)} \mid a\},\\ &= \{G_{l,a} \mid l\in\mathbb{Z}^+,\ l\mid k \text{ and}\\ &\qquad a\in\{0,\,\frac{l}{(l,n)},\dots,\,((l,n)-1)\frac{l}{(l,n)}\}\},\end{aligned} \quad (3)$$

$$\begin{aligned}\text{SMHom}((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k)) = \{G_{l,a} \mid l\in\mathbb{Z}^+,\ l\mid (k,n),\ a\in\{0,\,1,\dots,\,l-1\}\\ \text{and } (a,l)=1\}.\end{aligned} \quad (4)$$

Hence (3), (4) and Lemma 3.2.5 give

$$|\text{MHom}((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k))| = \sum_{\substack{l\in\mathbb{Z}^+\\ l|k}} (l,n),$$

$$|\text{SMHom}((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k))| = \sum_{\substack{l\in\mathbb{Z}^+\\ l|(k,n)}} \varphi(l) = (k,n).$$

□

Remark 3.2.7. The following diagram gives a comparison of the results of this section and Theorem 2.5 - Theorem 2.8 where H is a subsemigroup of $(\mathbb{Z},+)$ containing 0 and $l,\,a\in\mathbb{Z}$.

	Characterization	Cardinality
$\mathrm{MHom}((\mathbb{Z}_n,+),(\mathbb{Z},+))$	$f([x]_n)=H$ or $f([x]_n)=ax+l\mathbb{Z}$, $l\neq 0$, $\frac{l}{(l,n)}\mid a$	$\aleph_0$
$\mathrm{SMHom}((\mathbb{Z}_n,+),(\mathbb{Z},+))$	$f([x]_n)=ax+l\mathbb{Z}$, $l\neq 0$, $l\mid n$, $(a,l)=1$	n
$\mathrm{MHom}((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k))$	$f([x]_n)=ax+l\mathbb{Z}$, $l\neq 0$, $l\mid k$, $\frac{l}{(l,n)}\mid a$	$\sum_{\substack{l\in\mathbb{Z}^+\\ l\mid k}}(l,n)$
$\mathrm{SMHom}((\mathbb{Z}_n,+),(\mathbb{Z},\circ_k))$	$f([x]_n)=ax+l\mathbb{Z}$, $l\neq 0$, $l\mid(k,n)$, $(a,l)=1$	(k,n)

3.3 Multi-valued Homomorphisms from the Group $(\mathbb{Z},+)$ into the Hypergroup $(\mathbb{Z}_n,\circ_k)$

First, we recall that a subsemigroup of a finite group G must be a subgroup of G. Thus if S is a subsemigroup of $(\mathbb{Z}_n,+)$, then $S=l\mathbb{Z}_n$ for some $l\in\mathbb{Z}$.

The following two lemmas are similar to Lemma 3.1.2 and Lemma 3.1.3. They are needed to obtain our main results of this section.

Lemma 3.3.1. *Let G be a group with identity e. If $f\in MHom(G,(\mathbb{Z}_n,\circ_k))$, then $f(e)=l\mathbb{Z}_n$ for some $l\in\mathbb{Z}$ such that $(l,n)\mid k$.*

Proof. If $f\in\mathrm{MHom}(G,(\mathbb{Z}_n,\circ_k))$, then

$$\begin{aligned}f(e)=f(ee)&=f(e)\circ_k f(e)\\&=f(e)+f(e)+k\mathbb{Z}_n\\&\supseteq f(e)+f(e)\end{aligned}$$

since $[0]_n \in k\mathbb{Z}_n$. Thus $f(e)$ is a subsemigroup of $(\mathbb{Z}_n, +)$. Hence $f(e) = l\mathbb{Z}_n$ for some $l \in \mathbb{Z}$. Consequently,

$$\begin{aligned} l\mathbb{Z}_n = f(e) = f(e) + f(e) + k\mathbb{Z}_n &= l\mathbb{Z}_n + l\mathbb{Z}_n + k\mathbb{Z}_n \\ &= l\mathbb{Z}_n + k\mathbb{Z}_n = (l,k)\mathbb{Z}_n \end{aligned}$$

which implies that $(l,n) = (l,k,n) = ((l,n),k)$. Hence $(l,n) \mid k$. □

Lemma 3.3.2. *Let G be a group with identity e and $f \in MHom(G, (\mathbb{Z}_n, \circ_k))$. Then for every $x \in G$, there exists $a \in \mathbb{Z}$ such that $[a]_n \in f(x)$ and*

$$f(x^t) = [at]_n + f(e) \text{ for all } t \in \mathbb{Z}.$$

Proof. By Lemma 3.3.1, $f(e) = l\mathbb{Z}_n$ for some $l \in \mathbb{Z}$ with $(l,n) \mid k$. We also have that

$$\begin{aligned} l\mathbb{Z}_n + k\mathbb{Z}_n = (l,k)\mathbb{Z}_n &= (l,k,n)\mathbb{Z}_n \\ &= ((l,n),k)\mathbb{Z}_n = (l,n)\mathbb{Z}_n = l\mathbb{Z}_n. \end{aligned} \tag{1}$$

Let $x \in G$ be given. Then from (1),

$$f(x) = f(xe) = f(x) \circ_k f(e) = f(x) + l\mathbb{Z}_n + k\mathbb{Z}_n = f(x) + l\mathbb{Z}_n \tag{2}$$

and similarly

$$f(x^{-1}) = f(x^{-1}) + l\mathbb{Z}_n. \tag{3}$$

From (1), (2) and (3), we have

$$f(x) + k\mathbb{Z}_n = f(x) + l\mathbb{Z}_n = f(x), \tag{4}$$

$$f(x^{-1}) + k\mathbb{Z}_n = f(x^{-1}) + l\mathbb{Z}_n = f(x^{-1}). \tag{5}$$

It follows that

$$\begin{aligned} l\mathbb{Z}_n &= f(e) \\ &= f(xx^{-1}) \\ &= f(x) \circ_k f(x^{-1}) \\ &= f(x) + f(x^{-1}) + k\mathbb{Z}_n \\ &= f(x) + (f(x^{-1}) + k\mathbb{Z}_n) \\ &= f(x) + f(x^{-1}) \quad \text{ from (5).} \end{aligned} \tag{6}$$

Since $[0]_n \in l\mathbb{Z}_n$, from (6), there is an element $a \in \mathbb{Z}$ such that $[a]_n \in f(x)$, $-[a]_n \in f(x^{-1})$. It follows from (6) that

$$f(x) - [a]_n \subseteq f(x) + f(x^{-1}) = l\mathbb{Z}_n,$$
$$[a]_n + f(x^{-1}) \subseteq f(x) + f(x^{-1}) = l\mathbb{Z}_n$$

which imply that

$$f(x) \subseteq [a]_n + l\mathbb{Z}_n \text{ and } f(x^{-1}) \subseteq -[a]_n + l\mathbb{Z}_n. \tag{7}$$

We deduce from (2), (3) and (7) that

$$f(x) \subseteq [a]_n + l\mathbb{Z}_n \subseteq f(x) + l\mathbb{Z}_n = f(x),$$
$$f(x^{-1}) \subseteq -[a]_n + l\mathbb{Z}_n \subseteq f(x^{-1}) + l\mathbb{Z}_n = f(x^{-1}).$$

Consequently,

$$f(x) = [a]_n + l\mathbb{Z}_n = [a]_n + f(e) \text{ and } f(x^{-1}) = -[a]_n + l\mathbb{Z}_n = [-a]_n + f(e). \tag{8}$$

We have that $f(x^0) = f(e) = [a0]_n + f(e)$. If $t \in \mathbb{Z}^+$ and $t > 1$, then

$$\begin{aligned} f(x^t) &= \underbrace{f(x) + \cdots + f(x)}_{\text{t copies}} + k\mathbb{Z}_n \\ &= (f(x) + k\mathbb{Z}_n) + \cdots + (f(x) + k\mathbb{Z}_n) \quad \text{(t brackets)} \\ &= f(x) + \cdots + f(x) \quad \text{(t copies)} \qquad \text{from (4)} \\ &= ([a]_n + l\mathbb{Z}_n) + \cdots + ([a]_n + l\mathbb{Z}_n) \quad \text{(t brackets)} \qquad \text{from (8)} \\ &= [at]_n + l\mathbb{Z}_n \\ &= [at]_n + f(e), \end{aligned}$$

$$\begin{aligned} f(x^{-t}) &= f((x^{-1})^t) \\ &= \underbrace{f(x^{-1}) + \cdots + f(x^{-1})}_{\text{t copies}} + k\mathbb{Z}_n \\ &= (f(x^{-1}) + k\mathbb{Z}_n) + \cdots + (f(x^{-1}) + k\mathbb{Z}_n) \quad \text{(t brackets)} \\ &= f(x^{-1}) + \cdots + f(x^{-1}) \quad \text{(t copies)} \qquad \text{from (5)} \\ &= (-[a]_n + l\mathbb{Z}_n) + \cdots + (-[a]_n + l\mathbb{Z}_n) \quad \text{(t brackets)} \qquad \text{from (8)} \\ &= (-[at]_n) + l\mathbb{Z}_n \\ &= [a(-t)]_n + f(e). \end{aligned}$$

Therefore the proof is complete. □

Theorem 3.3.3. *For a multi-valued function f from $\mathbb{Z}$ into $\mathbb{Z}_n$, $f \in MHom$ $((\mathbb{Z}, +), (\mathbb{Z}_n, \circ_k))$ if and only if there are $l, a \in \mathbb{Z}$ such that $(l, n) \mid k$ and*

$$f(x) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

Proof. Assume $f \in \text{MHom}((\mathbb{Z}, +), (\mathbb{Z}, \circ_k))$. By Lemma 3.3.2, there is an element $a \in \mathbb{Z}$ such that $[a]_n \in f(1)$ and

$$f(x)\,(= f(x1)) = [ax]_n + f(0) \text{ for all } x \in \mathbb{Z}. \tag{1}$$

By Lemma 3.3.1, $f(0) = l\mathbb{Z}_n$ for some $l \in \mathbb{Z}$ with $(l, n) \mid k$. Hence from (1),

$$f(x) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

For the converse, let $l, a \in \mathbb{Z}$ be such that $(l, n) \mid k$ and

$$f(x) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

Since $(l, n) \mid k$, from (1) of the proof of Lemma 3.3.2, we have

$$l\mathbb{Z}_n + k\mathbb{Z}_n = l\mathbb{Z}_n. \tag{2}$$

If $x, y \in \mathbb{Z}$, then

$$\begin{aligned} f(x+y) &= [a(x+y)]_n + l\mathbb{Z}_n \\ &= [ax]_n + [ay]_n + l\mathbb{Z}_n + k\mathbb{Z}_n \quad \text{from (2)} \\ &= ([ax]_n + l\mathbb{Z}_n) + ([ay]_n + l\mathbb{Z}_n) + k\mathbb{Z}_n \\ &= f(x) + f(y) + k\mathbb{Z}_n \\ &= f(x) \circ_k f(y). \end{aligned}$$

Hence $f \in \text{MHom}((\mathbb{Z}, +), (\mathbb{Z}_n, \circ_k))$. □

Theorem 3.3.4. *For a multi-valued function f from $\mathbb{Z}$ into $\mathbb{Z}_n$, $f \in$ SMHom $((\mathbb{Z}, +), (\mathbb{Z}_n, \circ_k))$ if and only if there exist $l, a \in \mathbb{Z}$ such that $(l, n) \mid k$, $(a, l, n) = 1$ and*

$$f(x) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

Proof. Assume that $f \in \text{SMHom}((\mathbb{Z}, +), (\mathbb{Z}_n, \circ_k))$. Then $f \in \text{MHom}((\mathbb{Z}, +), (\mathbb{Z}_n, \circ_k))$ and $f(\mathbb{Z}) = \mathbb{Z}_n$. From Theorem 3.3.3, there are $l, a \in \mathbb{Z}$ such that $(l, n) \mid k$ and

$$f(x) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z},$$

and hence

$$\begin{aligned} \mathbb{Z}_n = f(\mathbb{Z}) &= \bigcup_{x \in \mathbb{Z}} ([ax]_n + l\mathbb{Z}_n) \\ &= \bigcup_{x \in \mathbb{Z}} a[x]_n + l\mathbb{Z}_n \\ &= a\mathbb{Z}_n + l\mathbb{Z}_n \\ &= (a, l)\mathbb{Z}_n. \end{aligned}$$

This implies that $(1,n) = (a,l,n)$, so $(a,l,n) = 1$.

Conversely, assume that l, a, f are given as above. By Theorem 3.3.3, $f \in \mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z}_n,\circ_k))$. Since $(a,l,n) = 1$, it follows that

$$\begin{aligned} f(\mathbb{Z}) &= a\mathbb{Z}_n + l\mathbb{Z}_n \\ &= (a,l)\mathbb{Z}_n \\ &= (a,l,n)\mathbb{Z}_n \\ &= \mathbb{Z}_n. \end{aligned}$$

Hence $f \in \mathrm{SMHom}((\mathbb{Z},+),(\mathbb{Z}_n,\circ_k))$. $\square$

For $l, a \in \mathbb{Z}$ with $(l,n) \mid k$, let $H_{l,a} \in \mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z}_n,\circ_k))$ be defined by

$$H_{l,a}(x) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

Note that if $l \mid (k,n)$, then $(l,n) \mid (k,n)$ and $(k,n) \mid k$, so $H_{l,a}$ is meaningful.

Lemma 3.3.5. (i) *$MHom((\mathbb{Z},+),(\mathbb{Z}_n,\circ_k)) = \{H_{l,a} \mid l \in \mathbb{Z}^+,\ l \mid (k,n)$ and $a \in \{0,1,\ldots,l-1\}\}$.*

(ii) *$SMHom((\mathbb{Z},+),(\mathbb{Z}_n,\circ_k)) = \{H_{l,a} \mid l \in \mathbb{Z}^+,\ l \mid (k,n),\ a \in \{0,1,\ldots,l-1\}$ and $(a,l) = 1\}$.*

(iii) *If $l, t \in \mathbb{Z}^+$, $l \mid (k,n)$, $t \mid (k,n)$, $a \in \{0,1,\ldots,l-1\}$ and $b \in \{0,1,\ldots,t-1\}$, then $H_{l,a} = H_{t,b}$ implies $l = t$ and $a = b$.*

Proof. (i) From Theorem 3.3.3 and the fact mentioned above, it suffices to show that for $l, a \in \mathbb{Z}$ with $(l,n) \mid k$, there are $t \in \mathbb{Z}^+$ and $b \in \{0,1,\ldots,t-1\}$ such that $t \mid (k,n)$ and $H_{l,a} = H_{t,b}$. Let $l, a \in \mathbb{Z}$ be such that $(l,n) \mid k$. Let $t = (l,n)$ and $b \in \{0,1,\ldots,t-1\}$ be such that $a = pt + b$ for some $p \in \mathbb{Z}$. Hence $t \in \mathbb{Z}^+$, $t \mid k$ and $t \mid n$, so $t \mid (k,n)$. Also, we have that

$$
\begin{aligned}
\text{for all } x \in \mathbb{Z},\ H_{t,b}(x) &= [bx]_n + t\mathbb{Z}_n \\
&= [(a-pt)x]_n + t\mathbb{Z}_n \\
&= [ax]_n - [tpx]_n + t\mathbb{Z}_n \\
&= [ax]_n - t[px]_n + t\mathbb{Z}_n \\
&= [ax]_n + t\mathbb{Z}_n \\
&= [ax]_n + (l,n)\mathbb{Z}_n \\
&= [ax]_n + l\mathbb{Z}_n \\
&= H_{l,a}(x).
\end{aligned}
$$

(ii) Let $l \in \mathbb{Z}^+$ and $a \in \{0, 1, \dots, l-1\}$ be such that $l \mid (k, n)$. Since

$$H_{l,a}(\mathbb{Z}) = a\mathbb{Z}_n + l\mathbb{Z}_n = (a, l)\mathbb{Z}_n,$$

it follows that $H_{l,a}(\mathbb{Z}) = \mathbb{Z}_n$ if and only if $1 = (1, n) = (a, l, n) = (a, l)$. From this fact and (i), (ii) follows.

(iii) If $l, t \in \mathbb{Z}^+$, $l \mid (k, n)$, $t \mid (k, n)$, $a \in \{0, 1, \dots, l-1\}$ and $b \in \{0, 1, \dots, t-1\}$ are such that $H_{l,a} = H_{t,b}$, then $l \mid n$, $t \mid n$ and

$$l\mathbb{Z}_n = H_{l,a}(0) = H_{t,b}(0) = t\mathbb{Z}_n,$$

so $l = (l, n) = (t, n) = t$. Thus $H_{l,a} = H_{l,b}$, so

$$[a]_n + l\mathbb{Z}_n = H_{l,a}(1) = H_{l,b}(1) = [b]_n + l\mathbb{Z}_n.$$

Then

$$[|a-b|]_n \in l\mathbb{Z}_n = \{[0]_n, [l]_n, \dots, (\frac{n}{l} - 1)[l]_n\}.$$

But $|a-b| \in \{0, 1, \dots, l-1\}$, so $|a-b| = 0$. Thus $a = b$. □

Theorem 3.3.6. $|MHom((\mathbb{Z}, +), (\mathbb{Z}_n, \circ_k))| = \sum_{\substack{l \in \mathbb{Z}^+ \\ l \mid (k,n)}} l$

and

$$|SMHom((\mathbb{Z}, +), (\mathbb{Z}_n, \circ_k))| = (k, n).$$

Proof. By Lemma 3.3.5(i) and (iii),

$$|\text{MHom}((\mathbb{Z},+),(\mathbb{Z}_n,\circ_k))| = \sum_{\substack{l\in\mathbb{Z}^+\\ l|(k,n)}} l.$$

We have from Lemma 3.3.5(ii) and (iii) that

$$|\text{SMHom}((\mathbb{Z},+),(\mathbb{Z}_n,\circ_k))| = \sum_{\substack{l\in\mathbb{Z}^+\\ l|(k,n)}} \varphi(l) = (k,n).$$

$\square$

Remark 3.3.7. We compare the results in this section and Theorem2.9 - Theorem 2.12 by the following diagram where $l, a \in \mathbb{Z}$.

	Characterization	**Cardinality**
$\text{MHom}((\mathbb{Z},+),(\mathbb{Z}_n,+))$	$f(x) = [ax]_n + l\mathbb{Z}_n$	$\sum_{\substack{l\in\mathbb{Z}^+\\ l\|n}} l$
$\text{SMHom}((\mathbb{Z},+),(\mathbb{Z}_n,+))$	$f(x) = [ax]_n + l\mathbb{Z}_n$, $(a,l,n) = 1$	n
$\text{MHom}((\mathbb{Z},+),(\mathbb{Z}_n,\circ_k))$	$f(x) = [ax]_n + l\mathbb{Z}_n$, $(l,n) \mid k$	$\sum_{\substack{l\in\mathbb{Z}^+\\ l\|(k,n)}} l$
$\text{SMHom}((\mathbb{Z},+),(\mathbb{Z}_n,\circ_k))$	$f(x) = [ax]_n + l\mathbb{Z}_n$, $(l,n) \mid k$, $(a,l,n) = 1$	(k,n)

3.4 Multi-valued Homomorphisms from the Group $(\mathbb{Z}_m,+)$ into the Hypergroup $(\mathbb{Z}_n,\circ_k)$

The following known fact is needed in this section.

Lemma 3.4.1 ([10]). *Let $l, a \in \mathbb{Z}$ and define*

$$f([x]_m) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

Then f is a well-defined multi-valued function from $\mathbb{Z}_m$ into $\mathbb{Z}_n$ if and only if $\frac{(l,n)}{(l,m,n)} \mid a$.

Theorem 3.4.2. *For a multi-valued function f from $\mathbb{Z}_m$ into $\mathbb{Z}_n$, $f \in MHom$ $((\mathbb{Z}_m, +),(\mathbb{Z}_n, \circ_k))$ if and only if there exist $l, a \in \mathbb{Z}$ such that $(l, n) \mid k$, $\frac{(l,n)}{(l,m,n)} \mid a$ and*

$$f([x]_m) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

Proof. Assume that $f \in \text{MHom}((\mathbb{Z}_m, +), (\mathbb{Z}_n, \circ_k))$. By Lemma 3.3.1 and Lemma 3.3.2, there are $l, a \in \mathbb{Z}$ such that $(l, n) \mid k$ and

$$f([x]_m) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

Lemma 3.4.1 yields the fact that $\frac{(l,n)}{(l,m,n)} \mid a$.

For the converse, let l, a, f be as above. By Lemma 3.4.1, f is well-defined. Since $(l, n) \mid k$, we have as before that

$$l\mathbb{Z}_n + k\mathbb{Z}_n = l\mathbb{Z}_n.$$

Hence for all $x, y \in \mathbb{Z}$,

$$\begin{aligned} f([x]_m + [y]_m) &= f([x + y]_m) \\ &= [a(x + y)]_n + l\mathbb{Z}_n \\ &= [ax]_n + [ay]_n + l\mathbb{Z}_n + k\mathbb{Z}_n \\ &= ([ax]_n + l\mathbb{Z}_n) + ([ay]_n + l\mathbb{Z}_n) + k\mathbb{Z}_n \\ &= f([x]_m) + f([y]_m) + k\mathbb{Z}_n \\ &= f([x]_m) \circ_k f([y]_m). \end{aligned}$$

Therefore $f \in \text{MHom}((\mathbb{Z}_m, +), (\mathbb{Z}_n, \circ_k))$, as desired. □

Lemma 3.4.3. *For $l, a \in \mathbb{Z}$, $(l,n) \mid k$, $\dfrac{(l,n)}{(l,m,n)} \mid a$ and $(a,l,n) = 1$ if and only if $(l,n) \mid (k,m)$ and $(a,l,n) = 1$.*

Proof. Assume that $(l,n) \mid k$, $\dfrac{(l,n)}{(l,m,n)} \mid a$ and $(a,l,n) = 1$. Then $\left(a, \dfrac{(l,n)}{(l,m,n)}\right) = 1$ since $(a,(l,n)) = (a,l,n) = 1$. But $\dfrac{(l,n)}{(l,m,n)} \mid a$, thus $\dfrac{(l,n)}{(l,m,n)} = 1$. That is, $(l,n) = (l,m,n)$. This implies that $(l,n) \mid m$. Hence $(l,n) \mid (k,m)$.

For the converse, assume that $(l,n) \mid (k,m)$ and $(a,l,n) = 1$. Then $(l,n) \mid k$ and $(l,n) \mid m$. Thus $(l,m,n) = (l,n)$, so $\dfrac{(l,n)}{(l,m,n)} = 1$. Hence $\dfrac{(l,n)}{(l,m,n)} \mid a$. $\square$

Theorem 3.4.4. *For a multi-valued function f from $\mathbb{Z}_m$ into $\mathbb{Z}_n$, $f \in$ SMHom $((\mathbb{Z}_m,+),(\mathbb{Z}_n,\circ_k))$ if and only if there exist $l, a \in \mathbb{Z}$ such that $(l,n) \mid (k,m)$, $(a,l,n) = 1$ and*

$$f([x]_m) = [ax]_n + l\mathbb{Z}_n \textit{ for all } x \in \mathbb{Z}.$$

Proof. Assume that $f \in \mathrm{SMHom}((\mathbb{Z}_m,+),(\mathbb{Z}_n,\circ_k))$. Then $f \in \mathrm{MHom}((\mathbb{Z}_m,+),(\mathbb{Z}_n,\circ_k))$ and $f(\mathbb{Z}_m) = \mathbb{Z}_n$. By Theorem 3.4.2, there are $l, a \in \mathbb{Z}$ such that $(l,n) \mid k$, $\dfrac{(l,n)}{(l,m,n)} \mid a$ and

$$f([x]_m) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

It follows that

$$\mathbb{Z}_n = f(\mathbb{Z}_m) = a\mathbb{Z}_n + l\mathbb{Z}_n = (a,l)\mathbb{Z}_n$$

and hence $1 = (1,n) = (a,l,n)$. By Lemma 3.4.3, we have $(l,n) \mid (k,m)$.

Conversely, let l, a, f be as above. By Theorem 3.4.2 and Lemma 3.4.3, we deduce that $f \in \mathrm{MHom}((\mathbb{Z}_m,+),(\mathbb{Z}_n,\circ_k))$ and $(a,l,n) = 1$. Since $(a,l,n) = 1$, we have

$$f(\mathbb{Z}_m) = a\mathbb{Z}_n + l\mathbb{Z}_n = (a,l)\mathbb{Z}_n = (a,l,n)\mathbb{Z}_n = \mathbb{Z}_n.$$

Therefore $f \in \mathrm{SMHom}((\mathbb{Z}_m,+),(\mathbb{Z}_n,\circ_k))$. $\square$

For $l, a \in \mathbb{Z}$ with $(l,n) \mid k$ and $\dfrac{(l,n)}{(l,m,n)} \mid a$, let $I_{l,a}$ be the element of MHom $((\mathbb{Z}_m,+),(\mathbb{Z}_n,\circ_k))$ defined by

$$I_{l,a}([x]_m) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}.$$

If $l \neq 0, l \mid (k,n)$ and $\dfrac{l}{(l,m)} \mid a$, then $(l,n) \mid l$, $l \mid k$, so $(l,n) \mid k$. Since $(l,n) \mid l$, $\dfrac{l}{(l,m)} \mid a$ and $(l,m) \mid m$, it follows that $(l,n) \mid am$, so $\dfrac{(l,n)}{(l,m,n)} \mid a\dfrac{m}{(l,m,n)}$. Since $\left(\dfrac{(l,n)}{(l,m,n)}, \dfrac{m}{(l,m,n)}\right) = 1$, we have $\dfrac{(l,n)}{(l,m,n)} \mid a$. Hence $I_{l,a}$ is defined.

Also, if $l \neq 0$ and $l \mid (k,m,n)$, then $l \mid (k,n)$ and $\dfrac{l}{(l,m)} = \dfrac{l}{|l|} = \pm 1$ which divides a, so by the above proof, $I_{l,a}$ is also defined.

Lemma 3.4.5. (i) *MHom*$((\mathbb{Z}_m,+),(\mathbb{Z}_n,\circ_k)) = \{I_{l,a} \mid l \in \mathbb{Z}^+,\ l \mid (k,n),$ $a \in \{0,1,\ldots,l-1\}$ *and* $\dfrac{l}{(l,m)} \mid a\}$.

(ii) *SMHom*$((\mathbb{Z}_m,+),(\mathbb{Z}_n,\circ_k)) = \{I_{l,a} \mid l \in \mathbb{Z}^+,\ l \mid (k,m,n),\ a \in \{0,1,\ldots,l-1\}$ *and* $(a,l) = 1\}$.

(iii) *If* $l, t \in \mathbb{Z}^+$, $l \mid (k,n)$, $t \mid (k,n)$, $a \in \{0,1,\ldots,l-1\}$, $b \in \{0,1,\ldots,t-1\}$, $\dfrac{l}{(l,m)} \mid a$ *and* $\dfrac{t}{(t,m)} \mid b$, *then* $I_{l,a} = I_{t,b}$ *implies* $l = t$ *and* $a = b$.

Proof. (i) As mentioned above, to prove (i), by Theorem 3.4.2, it suffices to prove that for $l, a \in \mathbb{Z}$ with $(l,n) \mid k$ and $\dfrac{(l,n)}{(l,m,n)} \mid a$, there are $t \in \mathbb{Z}^+$ and $b \in \{0,1,\ldots,t-1\}$ such that $\dfrac{t}{(t,m)} \mid b$ and $I_{l,a} = I_{t,b}$. Let $l, a \in \mathbb{Z}$ be such that $(l,n) \mid k$ and $\dfrac{(l,n)}{(l,m,n)} \mid a$. Let $t = (l,n)$ and $b \in \{0,1,\ldots,t-1\}$ be such that $a = pt + b$ for some $p \in \mathbb{Z}$. Then $t \in \mathbb{Z}^+$, $t \mid k$, $t \mid n$. Thus $t \mid (k,n)$. Since $\dfrac{t}{(t,m)} = \dfrac{(l,n)}{(l,m,n)}$, $\dfrac{(l,n)}{(l,m,n)} \mid a$, $\dfrac{t}{(t,m)} \mid t$ and $b = a - pt$, we deduce that $\dfrac{t}{(t,m)} \mid b$.

We also have that

$$\begin{aligned}\text{for every } x \in \mathbb{Z},\ I_{t,b}([x_m]) &= [bx]_n + t\mathbb{Z}_n \\ &= [(a-pt)x]_n + t\mathbb{Z}_n \\ &= [ax]_n - t[px]_n + t\mathbb{Z}_n \\ &= [ax]_n + t\mathbb{Z}_n \\ &= [ax]_n + (l,n)\mathbb{Z}_n \\ &= [ax]_n + l\mathbb{Z}_n \\ &= I_{l,a}([x]_m).\end{aligned}$$

(ii) If $l \in \mathbb{Z} \setminus \{0\}$ and $a \in \mathbb{Z}$ are such that $l \mid (k,m,n)$ and $(a,l) = 1$, then $(l,n) \mid l$, $l \mid (k,m)$ and $(a,l,n) = ((a,l),n) = (1,n) = 1$. It follows from Lemma 3.4.3 and Theorem 3.4.4 that

$$\begin{aligned}\{I_{l,a} \mid l \in \mathbb{Z}^+,\ l \mid (k,m,n), a \in \{0,1,\dots,l-1\} \text{ and } (a,l) = 1\}& \\ \subseteq \mathrm{SMHom}((\mathbb{Z}_m,+),(\mathbb{Z},\circ_k)).&\end{aligned}$$

To prove the reverse inclusion by Lemma 3.4.3 and Theorem 3.4.4, let $l, a \in \mathbb{Z}$ be such that $(l,n) \mid (k,m)$ and $(a,l,n) = 1$. Let $t = (l,n)$ and $b \in \{0,1,\dots,t-1\}$ be such that $a = pt + b$ for some $p \in \mathbb{Z}$. Then $t \mid n$, $t \mid (k,m)$ and $(a,t) = (a,(l,n)) = 1$, that is, $t \in \mathbb{Z}^+$, $t \mid (k,m,n)$ and $(a,t) = 1$. We have $(t,n) = t = (l,n)$, $(l,n) \mid (k,m)$ and

$$\begin{aligned}(b,t,n) &= (a-pt,t,n) \\ &= (a-p(l,n),(l,n),n).\end{aligned}$$

If $c \in \mathbb{Z}^+$ is such that $c \mid a - p(l,n)$, $c \mid (l,n)$ and $c \mid n$, then $c \mid a$, so $c \mid (a,(l,n))$. But $(a,l,n) = 1$, so $c = 1$. This shows that $(b,t,n) = 1$. Since $t \mid n$, $(b,t) = (b,t,n) = 1$. From Lemma 3.4.3 and Theorem 3.4.4, $I_{t,b} \in \mathrm{SMHom}((\mathbb{Z}_m,+),(\mathbb{Z}_n,\circ_k))$. The proof in (i) shows that $I_{t,b} = I_{l,a}$.

(iii) Let $l, t \in \mathbb{Z}^+$, $a \in \{0,1,\dots,l-1\}$ and $b \in \{0,1,\dots,t-1\}$ such that $l \mid (k,n)$, $t \mid (k,n)$, $\dfrac{l}{(l,m)} \mid a$ and $\dfrac{t}{(t,m)} \mid b$. Assume that $I_{l,a} = I_{t,b}$. Then

$$l\mathbb{Z}_n = I_{l,a}([0]_m) = I_{t,b}([0]_m) = t\mathbb{Z}_n,$$

so $(l, n) = (t, n)$. Since $l \mid n$ and $t \mid n$, we have $l = t$. Then $I_{l,a} = I_{l,b}$. Thus

$$[a]_n + l\mathbb{Z}_n = I_{l,a}([1]_m) = I_{l,b}([1]_m) = [b]_n + l\mathbb{Z}_n,$$

so $[|a-b|]_n \in l\mathbb{Z}_n = \{[0]_n, [l]_n, \ldots, \left(\frac{n}{l} - 1\right)[l]_n\}$. Since $|a - b| \in \{0, 1, \ldots, l-1\}$, it follows that $|a - b| = 0$. Thus $a = b$.

Hence the lemma is proved. □

Theorem 3.4.6. $|MHom((\mathbb{Z}_m, +), (\mathbb{Z}_n, \circ_k))| = \sum_{\substack{l \in \mathbb{Z}^+ \\ l \mid (k,n)}} (l, m)$ *and*

$$|SMHom((\mathbb{Z}_m, +), (\mathbb{Z}_n, \circ_k))| = (k, m, n).$$

Proof. From Lemma 3.4.5(i), we have

$$\text{MHom}((\mathbb{Z}_m, +), (\mathbb{Z}_n, \circ_k)) = \{I_{l,a} \mid l \in \mathbb{Z}^+,\ l \mid (k, n) \text{ and } a \in \{0, \frac{l}{(l,m)}, \ldots, ((l, m) - 1)\frac{l}{(l,m)}\}\}.$$

Hence by Lemma 3.4.5(iii),

$$|\text{MHom}((\mathbb{Z}_m, +), (\mathbb{Z}_n, \circ_k))| = \sum_{\substack{l \in \mathbb{Z}^+ \\ l \mid (k,n)}} (l, m).$$

It follows directly from Lemma 3.4.5(ii) and (iii) that

$$|\text{SMHom}((\mathbb{Z}_m, +), (\mathbb{Z}_n, \circ_k))| = \sum_{\substack{l \in \mathbb{Z}^+ \\ l \mid (k,m,n)}} \varphi(l) = (k, m, n).$$

□

Remark 3.4.7. The following diagram shows a comparison of Theorem 2.13 - Theorem 2.16 and the results obtained in this section where $l, a \in \mathbb{Z}$.

	Characterization	**Cardinality**
$\mathrm{MHom}((\mathbb{Z}_m,+),(\mathbb{Z}_n,+))$	$f([x]_m)=[ax]_n+l\mathbb{Z}_n$, $\dfrac{(l,n)}{(l,m,n)} \mid a$	$\sum\limits_{\substack{l\in\mathbb{Z}^+\\ l\mid n}}(l,m)$
$\mathrm{SMHom}((\mathbb{Z}_m,+),(\mathbb{Z}_n,+))$	$f([x]_m)=[ax]_n+l\mathbb{Z}_n$, $(l,n)\mid m$, $(a,l,n)=1$	(m,n)
$\mathrm{MHom}((\mathbb{Z}_m,+),(\mathbb{Z}_n,\circ_k))$	$f([x]_m)=[ax]_n+l\mathbb{Z}_n$, $(l,n)\mid k$, $\dfrac{(l,n)}{(l,m,n)} \mid a$	$\sum\limits_{\substack{l\in\mathbb{Z}^+\\ l\mid (k,n)}}(l,m)$
$\mathrm{SMHom}((\mathbb{Z}_m,+),(\mathbb{Z}_n,\circ_k))$	$f([x]_m)=[ax]_n+l\mathbb{Z}_n$, $(l,n)\mid (k,m)$, $(a,l,n)=1$	(k,m,n)

CHAPTER IV
MULTI-VALUED HOMOMORPHISMS FROM HYPERGROUPS INTO GROUPS

Multi-valued homomorphisms and surjective multi-valued homomorphisms from the hypergroups $(\mathbb{Z}, \circ_k)$ and $(\mathbb{Z}_n, \circ_k)$ into the group $(\mathbb{Z}, +)$ and $(\mathbb{Z}_m, +)$ are determined in this chapter. We give characterizations of such multi-valued functions. It is also shown that the cardinalities of the sets of such multi-valued homomorphisms and surjective multi-valued homomorphisms into $(\mathbb{Z},+)$ where $n \nmid k$ are $2^{\aleph_0}$.

4.1 Multi-valued Homomorphisms from the Hypergroup $(\mathbb{Z}, \circ_k)$ into the Group $(\mathbb{Z}, +)$

Lemma 3.1.1 and the following three lemmas are needed.

Lemma 4.1.1. *If H is a subsemigroup of $(\mathbb{Z}, +)$ such that $H + H = H$, then $0 \in H$.*

Proof. If $H \subseteq \mathbb{Z}^+$, then $H + H \subseteq \mathbb{Z}^+$ and $\min(H + H) = 2\min H > \min H$ which is a contradiction since $H + H = H$. Hence $H \nsubseteq \mathbb{Z}^+$. Also, if $H \subseteq \mathbb{Z}^-$, then $\max(H + H) = 2\max H < \max H$ which is contrary to that $H + H = H$. Then either $0 \in H$ or $H \cap \mathbb{Z}^+ \neq \varnothing$ and $H \cap \mathbb{Z}^- \neq \varnothing$, so by Lemma 3.1.1, $0 \in H$. □

Lemma 4.1.2. *If $f \in MHom((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))$, then the following statements hold.*

(i) $f(k\mathbb{Z}) = f(x_1) + \cdots + f(x_t)$ *for all* $x_1, \ldots, x_t \in \mathbb{Z}$ *with* $x_1 + \cdots + x_t \in k\mathbb{Z}$ *and* $t > 1$

(ii) $f(k\mathbb{Z}) = f(x) + f(y)$ *for all* $x, y \in k\mathbb{Z}$.

(iii) $f(k\mathbb{Z})$ *is a subsemigroup of* $(\mathbb{Z}, +)$ *containing 0.*

(iv) $f(x) + f(y) = f(x) + f(y) + f(k\mathbb{Z})$ *for all* $x, y \in \mathbb{Z}$.

(v) $f(x + k\mathbb{Z}) = f(x) + f(k\mathbb{Z})$ *for all* $x \in \mathbb{Z}$.

Proof. (i) If $x_1, \ldots, x_t \in \mathbb{Z}$ are such that $t > 1$ and $x_1 + \cdots + x_t \in k\mathbb{Z}$, then $x_1 + \cdots + x_t + k\mathbb{Z} = k\mathbb{Z}$, so

$$\begin{aligned} f(k\mathbb{Z}) &= f(x_1 + \cdots + x_t + k\mathbb{Z}) \\ &= f(x_1 \circ_k \cdots \circ_k x_t) \\ &= f(x_1) + \cdots + f(x_t). \end{aligned}$$

(ii) follows directly from (i).

(iii) If $x, y \in f(k\mathbb{Z})$, then $x \in f(s)$ and $y \in f(t)$ for some $s, t \in k\mathbb{Z}$, so by (ii), $x + y \in f(s) + f(t) = f(k\mathbb{Z})$. This shows that $f(k\mathbb{Z})$ is a subsemigroup of $(\mathbb{Z}, +)$. Also, by (i),

$$f(k\mathbb{Z}) + f(k\mathbb{Z}) = f(0) + f(0) + f(0) + f(0) = f(k\mathbb{Z}).$$

Hence by Lemma 4.1.1, $0 \in f(k\mathbb{Z})$.

(iv) If $x, y \in \mathbb{Z}$, then

$$\begin{aligned} f(x) + f(y) &= f(x \circ_k y) \\ &= f(x + y + k\mathbb{Z}) \\ &= f(x + y + 0 + 0 + k\mathbb{Z}) \\ &= f(x \circ_k y \circ_k 0 \circ_k 0) \\ &= f(x) + f(y) + f(0) + f(0) \\ &= f(x) + f(y) + f(k\mathbb{Z}) \qquad \text{by (ii)}. \end{aligned}$$

(v) For every $x \in \mathbb{Z}$,

$$\begin{aligned} f(x + k\mathbb{Z}) &= f(x + 0 + 0 + k\mathbb{Z}) \\ &= f(x \circ_k 0 \circ_k 0) \\ &= f(x) + f(0) + f(0) \\ &= f(x) + f(k\mathbb{Z}) \qquad \text{by (ii)}. \end{aligned}$$

□

Lemma 4.1.3. *If* $f \in MHom((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))$, *then there exists an element* $a \in f(1)$ *such that*

$$f(x + k\mathbb{Z}) = ax + f(k\mathbb{Z}) \;\; \textit{for all}\, x \in \mathbb{Z}.$$

Proof. By Lemma 4.1.2(i) and (iii), $0 \in f(k\mathbb{Z}) = f(1) + f(-1)$, so there is an element $a \in \mathbb{Z}$ such that $a \in f(1)$ and $-a \in f(-1)$. Since $f(1) + f(-1) = f(k\mathbb{Z})$, $f(k\mathbb{Z}) = f(k\mathbb{Z}) + f(k\mathbb{Z})$, $f(1 + k\mathbb{Z}) = f(1) + f(k\mathbb{Z})$ and $f(-1 + k\mathbb{Z}) = f(-1) + f(k\mathbb{Z})$ by Lemma 4.1.2(i),(iii) and (v), respectively, it follows that

$$\begin{aligned} f(1 + k\mathbb{Z}) &= f(1) + f(k\mathbb{Z}) \\ &\supseteq a + f(k\mathbb{Z}) \\ &= a + f(k\mathbb{Z}) + f(k\mathbb{Z}) \\ &= a + f(-1) + f(1) + f(k\mathbb{Z}) \\ &\supseteq a - a + f(1) + f(k\mathbb{Z}) \\ &= f(1) + f(k\mathbb{Z}) \\ &= f(1 + k\mathbb{Z}) \end{aligned}$$

and

$$\begin{aligned} f(-1 + k\mathbb{Z}) &= f(-1) + f(k\mathbb{Z}) \\ &\supseteq -a + f(k\mathbb{Z}) \\ &= -a + f(k\mathbb{Z}) + f(k\mathbb{Z}) \\ &= -a + f(1) + f(-1) + f(k\mathbb{Z}) \\ &\supseteq -a + a + f(-1) + f(k\mathbb{Z}) \\ &= f(-1) + f(k\mathbb{Z}) \\ &= f(-1 + k\mathbb{Z}) \end{aligned}$$

which imply that $f(1 + k\mathbb{Z}) = a + f(k\mathbb{Z})$ and $f(-1 + k\mathbb{Z}) = -a + f(k\mathbb{Z})$.

If $l \in \mathbb{Z}^+$ with $l > 1$, then

$$\begin{aligned}
f(l + k\mathbb{Z}) &= f(\underbrace{1 + \cdots + 1}_{l \text{ copies}} + k\mathbb{Z}) \\
&= f(\underbrace{1 \circ_k \cdots \circ_k 1}_{l \text{ copies}}) \\
&= f(1) + \cdots + f(1) \\
&= f(1) + \cdots + f(1) + f(k\mathbb{Z}) && \text{by Lemma 4.1.2(iv)} \\
&= (f(1) + f(k\mathbb{Z})) + \cdots + (f(1) + f(k\mathbb{Z})) && \text{by Lemma 4.1.2(iii)} \\
&= f(1 + k\mathbb{Z}) + \cdots + f(1 + k\mathbb{Z}) && \text{by Lemma 4.1.2(v)} \\
&= (a + f(k\mathbb{Z})) + \cdots + (a + f(k\mathbb{Z})) \\
&= al + f(k\mathbb{Z})
\end{aligned}$$

and

$$\begin{aligned}
f(-l + k\mathbb{Z}) &= f(\underbrace{-1 + \cdots - 1}_{l \text{ copies}} + k\mathbb{Z}) \\
&= f(\underbrace{(-1) \circ_k \cdots \circ_k (-1)}_{l \text{ copies}}) \\
&= f(-1) + \cdots + f(-1) \\
&= f(-1 + k\mathbb{Z}) + \cdots + f(-1 + k\mathbb{Z}) && \text{by Lemma 4.1.2(iii), (iv) and (v)} \\
&= (-a + f(k\mathbb{Z})) + \cdots + (-a + f(k\mathbb{Z})) \\
&= a(-l) + f(k\mathbb{Z}).
\end{aligned}$$

Hence the lemma is proved. □

Theorem 4.1.4. *For a multi-valued function f from $\mathbb{Z}$ into itself, $f \in MHom((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))$ if and only if one of the following two conditions holds.*

(i) *There exists a subsemigroup H of $(\mathbb{Z}, +)$ containing 0 such that*

$$\begin{aligned}
f(x + k\mathbb{Z}) &= H \quad \text{for all } x \in \mathbb{Z} \text{ and} \\
f(x) + f(y) &= H \quad \text{for all } x, y \in \mathbb{Z}.
\end{aligned}$$

(ii) *There exist* $l, a \in \mathbb{Z}$ *such that* $l \neq 0$, $\dfrac{l}{(l,k)} \mid a$,

$$f(x + k\mathbb{Z}) = ax + l\mathbb{Z} \quad \textit{for all } x \in \mathbb{Z} \textit{ and}$$

$$f(x) + f(y) = f(x) + f(y) + l\mathbb{Z} \quad \textit{for all } x, y \in \mathbb{Z}.$$

Proof. Assume that $f \in \mathrm{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))$. By Lemma 4.1.2(iii), $f(k\mathbb{Z})$ is a subsemigroup of $(\mathbb{Z}, +)$ containing 0. By Lemma 4.1.2(iv),

$$f(x) + f(y) = f(x) + f(y) + f(k\mathbb{Z}) \quad \text{for all } x, y \in \mathbb{Z}. \tag{1}$$

By Lemma 4.1.3, there exists $a \in f(1)$ such that

$$f(x + k\mathbb{Z}) = ax + f(k\mathbb{Z}) \quad \text{for all } x \in \mathbb{Z}. \tag{2}$$

Case 1: $a = 0$. From (2), we have

$$f(x + k\mathbb{Z}) = f(k\mathbb{Z}) \quad \text{for all } x \in \mathbb{Z} \tag{3}$$

and for all $x, y \in \mathbb{Z}$,

$$\begin{aligned} f(x) + f(y) &= f(x \circ_k y) \\ &= f(x + y + k\mathbb{Z}) \\ &= f(k\mathbb{Z}) \qquad \text{from (3).} \end{aligned}$$

Thus f satisfies (i).

Case 2: $a \neq 0$. It follows from (2) that

$$f(k\mathbb{Z}) = f(k + k\mathbb{Z}) = ak + f(k\mathbb{Z}) \quad \text{and}$$

$$f(k\mathbb{Z}) = f(-k + k\mathbb{Z}) = -ak + f(k\mathbb{Z}).$$

Since $0 \in f(k\mathbb{Z})$, $ak, -ak \in f(k\mathbb{Z})$, so $f(k\mathbb{Z}) \cap \mathbb{Z}^+ \neq \varnothing$ and $f(k\mathbb{Z}) \cap \mathbb{Z}^- \neq \varnothing$. Then by Lemma 3.1.1, $f(k\mathbb{Z}) = l\mathbb{Z}$ for some $l \in \mathbb{Z}$ and $l \neq 0$. But $ak \in l\mathbb{Z}$, so $l \mid ak$. Thus $\dfrac{l}{(l,k)} \mid a$. From (1) and (2), we have

$$f(x) + f(y) = f(x) + f(y) + l\mathbb{Z} \quad \text{for all } x, y \in \mathbb{Z}$$

and

$$f(x + k\mathbb{Z}) = ax + l\mathbb{Z} \quad \text{for all } x \in \mathbb{Z},$$

respectively. Hence (ii) holds.

Conversely, assume that f satisfies (i) or (ii). If f satisfies (i), then for all $x, y \in \mathbb{Z}$,

$$f(x \circ_k y) = f(x + y + k\mathbb{Z}) = H = f(x) + f(y),$$

which implies that $f \in \mathrm{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))$.

Next, assume that f satisfies (ii). First we show that f defined on each coset is independent on its representatives. If $x, y \in \mathbb{Z}$ are such that $x + k\mathbb{Z} = y + k\mathbb{Z}$, then $x - y \in k\mathbb{Z}$. Since $\frac{l}{(l,k)} \mid a$, we have $l \mid ak$, so

$$ax - ay \in ak\mathbb{Z} \subseteq l\mathbb{Z}$$

which implies that $ax + l\mathbb{Z} = ay + l\mathbb{Z}$. To show that $f \in \mathrm{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))$, let $x, y \in \mathbb{Z}$. Since $f(x) \subseteq f(x + k\mathbb{Z}) = ax + l\mathbb{Z}$ and $f(y) \subseteq f(y + k\mathbb{Z}) = ay + l\mathbb{Z}$, we deduce that $f(x) = ax + A$ and $f(y) = ay + B$ for some nonempty subsets A, B of $l\mathbb{Z}$. Therefore $A + B + l\mathbb{Z} = l\mathbb{Z}$ and hence

$$\begin{aligned}
f(x \circ_k y) &= f(x + y + k\mathbb{Z}) \\
&= a(x + y) + l\mathbb{Z} \\
&= ax + ay + l\mathbb{Z} \\
&= ax + A + ay + B + l\mathbb{Z} \\
&= f(x) + f(y) + l\mathbb{Z} \\
&= f(x) + f(y) \quad \text{by assumption.}
\end{aligned}$$

Therefore the proof is complete. □

We give a note here that if a multi-valued function f from $\mathbb{Z}$ into itself satisfies (ii) of Theorem 4.1.4 with $a = 0$, then f satisfies (i). To show this, assume that

$$f(x + k\mathbb{Z}) = l\mathbb{Z} \qquad \text{for all } x \in \mathbb{Z} \text{ and}$$
$$f(x) + f(y) = f(x) + f(y) + l\mathbb{Z} \qquad \text{for all } x, y \in \mathbb{Z}.$$

Then for $x, y \in \mathbb{Z}$, $f(x) \subseteq f(x + k\mathbb{Z}) = l\mathbb{Z}$ and $f(y) \subseteq f(y + k\mathbb{Z}) = l\mathbb{Z}$ which implies that $f(x) + f(y) + l\mathbb{Z} = l\mathbb{Z}$. Hence $f(x) + f(y) = l\mathbb{Z}$.

Lemma 4.1.5. *For $l \in \mathbb{Z} \smallsetminus \{0\}$ and $a \in \mathbb{Z}$, $\frac{l}{(l,k)} \mid a$ and $(a, l) = 1$ if and only if $l \mid k$ and $(a, l) = 1$.*

Proof. Assume that $\frac{l}{(l,k)} \mid a$ and $(a, l) = 1$. These imply that $\left(a, \frac{l}{(l,k)}\right) = \frac{|l|}{(l,k)}$ and $\left(a, \frac{l}{(l,k)}\right) = 1$, respectively. Thus $\frac{|l|}{(l,k)} = 1$, so $|l| = (l, k)$. Hence $l \mid k$.

If $l \mid k$, then $\frac{l}{(l,k)} = \frac{l}{|l|} = \pm 1$ which divides a.

Therefore the lemma is proved. $\square$

Theorem 4.1.6. *For a multi-valued function f from $\mathbb{Z}$ into itself, $f \in$ SMHom $((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))$ if and only if one of the following two conditions holds.*

(i) $f(x + k\mathbb{Z}) = \mathbb{Z}$ *for all* $x \in \mathbb{Z}$ *and*

$f(x) + f(y) = \mathbb{Z}$ *for all* $x, y \in \mathbb{Z}$.

(ii) *There exist $l, a \in \mathbb{Z}$ such that $l \neq 0$, $l \mid k$, $(a, l) = 1$,*

$$f(x + k\mathbb{Z}) = ax + l\mathbb{Z} \qquad \textit{for all } x \in \mathbb{Z} \textit{ and}$$
$$f(x) + f(y) = f(x) + f(y) + l\mathbb{Z} \qquad \textit{for all } x, y \in \mathbb{Z}.$$

Proof. Assume that $f \in \text{SMHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))$. Then f satisfies (i) or (ii) of Theorem 4.1.4. If f satisfies (i) of Theorem 4.1.4, then (i) holds since $\mathbb{Z} = f(\mathbb{Z}) = \bigcup_{x \in \mathbb{Z}} f(x + k\mathbb{Z}) = H$.

Next, assume that f satisfies (ii) of Theorem 4.1.4. Then there are $l, a \in \mathbb{Z}$ such that $l \neq 0$, $\frac{l}{(l,k)} \mid a$,

$$f(x+k\mathbb{Z}) = ax + l\mathbb{Z} \qquad \text{for all } x \in \mathbb{Z} \text{ and}$$
$$f(x)+f(y) = f(x)+f(y)+l\mathbb{Z} \quad \text{for all } x, y \in \mathbb{Z}.$$

Since $f(\mathbb{Z}) = \mathbb{Z}$, it follows that

$$\mathbb{Z} = f(\mathbb{Z}) = \bigcup_{x\in\mathbb{Z}} f(x+k\mathbb{Z}) = a\mathbb{Z} + l\mathbb{Z} = (a,l)\mathbb{Z}$$

which implies that $(a,l) = 1$. By Lemma 4.1.5, $l \mid k$ and $(a,l) = 1$.

For the converse, assume that f satisfies (i) or (ii). If f satisfies (i), then by Theorem 4.1.4, $f \in \mathrm{MHom}((\mathbb{Z},\circ_k),(\mathbb{Z},+))$ and $\mathbb{Z} = f(1+k\mathbb{Z}) \subseteq f(\mathbb{Z})$, so $f(\mathbb{Z}) = \mathbb{Z}$.

If f satisfies (ii), then by Theorem 4.1.4 and Lemma 4.1.5, $f \in \mathrm{MHom}((\mathbb{Z},\circ_k),(\mathbb{Z},+))$. Since $(a,l) = 1$ and

$$f(\mathbb{Z}) \;= f(\bigcup_{x\in\mathbb{Z}}(x+k\mathbb{Z})) = \bigcup_{x\in\mathbb{Z}}(ax+l\mathbb{Z}) = a\mathbb{Z}+l\mathbb{Z} = (a,l)\mathbb{Z},$$

it follows that $f(\mathbb{Z}) = \mathbb{Z}$.

Therefore the theorem is proved. □

Theorem 4.1.7. $|MHom((\mathbb{Z},\circ_k),(\mathbb{Z},+))| = |SMHom((\mathbb{Z},\circ_k),(\mathbb{Z},+))| = 2^{\aleph_0}$.

Proof. Note that

$$\begin{aligned}((2\mathbb{Z}+1)\cup\{0\})+((2\mathbb{Z}+1)\cup\{0\}) &= (2\mathbb{Z}+1+2\mathbb{Z}+1)\cup(2\mathbb{Z}+1)\cup\{0\}\\ &= (2\mathbb{Z}+2)\cup(2\mathbb{Z}+1)\cup\{0\}\\ &= 2\mathbb{Z}\cup(2\mathbb{Z}+1) = \mathbb{Z}. \qquad (1)\end{aligned}$$

Let $X \subseteq 2\mathbb{Z}\smallsetminus\{0\}$ and define $f_X : \mathbb{Z} \to \mathcal{P}^*(\mathbb{Z})$ by

$$\begin{aligned} f_X(0) &= ((2\mathbb{Z}+1)\cup\{0\})\cup X \text{ and}\\ f_X(x) &= \mathbb{Z} \qquad \text{for all } x \in \mathbb{Z}\smallsetminus\{0\}. \end{aligned} \qquad (2)$$

Then $f_X(x) \supseteq (2\mathbb{Z}+1)\cup\{0\}$ for all $x \in \mathbb{Z}$, so by (1), we have

$$f_X(x) + f_X(y) = \mathbb{Z} \qquad \text{for all } x, y \in \mathbb{Z}. \tag{3}$$

Moreover, if $x \in \mathbb{Z}$, then $x + k\mathbb{Z}$ is infinite since $k > 0$. Therefore we have

$$f_X(x + k\mathbb{Z}) = \bigcup_{t \in \mathbb{Z}} f_X(x + kt) = \mathbb{Z}. \tag{4}$$

By (3) and (4), f_X satisfies (i) of Theorem 4.1.6, so $f_X \in \text{SMHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))$.

If X and X' are distinct subsets of $2\mathbb{Z} \smallsetminus \{0\}$, then $((2\mathbb{Z}+1) \cup \{0\}) \cup X \neq ((2\mathbb{Z}+1) \cup \{0\}) \cup X'$, so from (2), we have $f_X(0) \neq f_{X'}(0)$. Consequently, $f_X \neq f_{X'}$ for all distinct subsets X and X' of $2\mathbb{Z} \smallsetminus \{0\}$. Hence we have

$$\begin{aligned} |\text{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))| &\geq |\text{SMHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))| \\ &\geq |\{f_X \mid X \subseteq 2\mathbb{Z} \smallsetminus \{0\}| \\ &= |\{X \mid X \subseteq 2\mathbb{Z} \smallsetminus \{0\}| \\ &= 2^{\aleph_0}. \end{aligned}$$

But

$$\begin{aligned} |\text{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))| &\leq |\{f \mid f : \mathbb{Z} \to \mathcal{P}^*(\mathbb{Z})\}| \\ &= (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}, \end{aligned}$$

so we have $|\text{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))| = |\text{SMHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))| = 2^{\aleph_0}$, as desired.

□

Remark 4.1.8. Note that for $l \in \mathbb{Z}^+$, $l\mathbb{Z} = l(2\mathbb{Z} \cup (2\mathbb{Z}+1)) = 2l\mathbb{Z} \cup (2l\mathbb{Z} + l)$. We can see from the proof of Theorem 4.1.7 that if $l \in \mathbb{Z}^+$ and $X \subseteq 2l\mathbb{Z} \smallsetminus \{0\}$, then $g_X : \mathbb{Z} \to \mathcal{P}^*(l\mathbb{Z}) \subseteq \mathcal{P}^*(\mathbb{Z})$ defined by

$$\begin{aligned} &g_X(0) = ((2l\mathbb{Z} + l) \cup \{0\}) \cup X \text{ and} \\ &g_X(x) = l\mathbb{Z} \quad \text{for all } x \in \mathbb{Z} \smallsetminus \{0\} \end{aligned}$$

belongs to $\text{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))$ with $g_X(\mathbb{Z}) = l\mathbb{Z}$. Also, $g_X \neq g_{X'}$ for all distinct nonempty subsets X and X' of $2l\mathbb{Z} \setminus \{0\}$. If $l > 1$, then $l\mathbb{Z} \subsetneq \mathbb{Z}$ and thus

$$\begin{aligned} 2^{\aleph_0} &\geq |\text{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +)) \setminus \text{SMHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))| \\ &\geq |\{g_X \mid X \subseteq 2l\mathbb{Z} \setminus \{0\}\}| \\ &= |\{X \mid X \subseteq 2l\mathbb{Z} \setminus \{0\}\}| \\ &= 2^{\aleph_0}. \end{aligned}$$

This implies that $|\text{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +)) \setminus \text{SMHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))| = 2^{\aleph_0}$.

Remark 4.1.9. It can be seen from Theorem 2.1 that each pair H, a determines a unique $f \in \text{MHom}(\mathbb{Z}, +)$ with

$$f(x) = ax + H \qquad \text{for all } x \in \mathbb{Z}$$

where H is a subsemigroup of $(\mathbb{Z}, +)$ containing 0 and $a \in \mathbb{Z}$. If $a \neq 0$ and $(a, h) = 1$ for some $h \in H$, then the pair H, a also determines a unique $f \in \text{SMHom}(\mathbb{Z}, +)$ which satisfies the above equality.

In contrast, we can see from the proof of Theorem 4.1.7 that for every subset X of $2\mathbb{Z} \setminus \{0\}$,

$$\begin{aligned} f_X(x + k\mathbb{Z}) &= \mathbb{Z} \qquad \text{for all } x \in \mathbb{Z}, \\ f_X(x) + f_X(y) &= \mathbb{Z} \qquad \text{for all } x, y \in \mathbb{Z} \end{aligned}$$

which imply that

$$\begin{aligned} f_X(x + k\mathbb{Z}) &= ax + \mathbb{Z} \qquad \text{for all } a, x \in \mathbb{Z}, \\ f_X(x) + f_X(y) &= f_X(x) + f_X(y) + \mathbb{Z} \qquad \text{for all } x, y \in \mathbb{Z}. \end{aligned}$$

Note that $\dfrac{1}{(1,k)} \mid a$, $1 \mid k$ and $(a, 1) = 1$. Therefore we deduce that a subsemigroup H of $(\mathbb{Z}, +)$ containing 0 does not necessarily determine a unique $f \in \text{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))$ $[\text{SMHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))]$ satisfying (i) of Theorem 4.1.4 [Theorem 4.1.6]. Also, each pair l, a with $l \neq 0$ and $\dfrac{l}{(l,k)} \mid a$ $[l \neq 0, l \mid k$ and $(a, l) = 1]$ does not necessarily determine a unique $f \in \text{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))$ $[\text{SMHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}, +))]$ satisfying (ii) of Theorem 4.1.4 [Theorem 4.1.6].

Remark 4.1.10. Theorem 2.1 - Theorem 2.4 are compared with the results of this section as follows: where H is a subsemigroup of $(\mathbb{Z},+)$ containing 0 and $l, a \in \mathbb{Z}$.

	Characterization	**Cardinality**
$\text{MHom}((\mathbb{Z},+),(\mathbb{Z},+))$	$f(x) = ax + H$	$\aleph_0$
$\text{SMHom}((\mathbb{Z},+),(\mathbb{Z},+))$	$f(x) = ax + H$, $(a,h) = 1$ for some $h \in H$, $a = 0 \Rightarrow H = \mathbb{Z}$	$\aleph_0$
$\text{MHom}((\mathbb{Z},\circ_k),(\mathbb{Z},+))$	(i) $f(x+k\mathbb{Z}) = H$, $f(x)+f(y) = H$ or (ii) $f(x+k\mathbb{Z}) = ax + l\mathbb{Z}$, $f(x)+f(y) = f(x)+f(y)+l\mathbb{Z}$, $l \neq 0$, $\frac{l}{(l,k)} \mid a$	$2^{\aleph_0}$
$\text{SMHom}((\mathbb{Z},\circ_k),(\mathbb{Z},+))$	(i) $f(x+k\mathbb{Z}) = \mathbb{Z}$, $f(x)+f(y) = \mathbb{Z}$ or (ii) $f(x+k\mathbb{Z}) = ax + l\mathbb{Z}$, $f(x)+f(y) = f(x)+f(y)+l\mathbb{Z}$, $l \neq 0$, $l \mid k$, $(a,l) = 1$	$2^{\aleph_0}$

4.2 Multi-valued Homomorphisms from the Hypergroup $(\mathbb{Z},\circ_k)$ into the Group $(\mathbb{Z}_n,+)$

Recall that if H is a subsemigroup of the group $(\mathbb{Z}_n,+)$, then $H = l\mathbb{Z}_n$ for some $l \in \mathbb{Z}$. With this fact, the following two lemmas can be proved analogously to those of Lemma 4.1.2 and Lemma 4.1.3.

Lemma 4.2.1. *If $f \in MHom((\mathbb{Z}, \circ_k), (\mathbb{Z}_n, +))$, then the following statements hold.*

(i) $f(k\mathbb{Z}) = f(x_1) + \cdots + f(x_t)$ *for all* $x_1, \ldots, x_t \in \mathbb{Z}$ *with* $x_1 + \cdots + x_t \in k\mathbb{Z}$ *and* $t > 1$.

(ii) $f(k\mathbb{Z}) = f(x) + f(y)$ *for all* $x, y \in k\mathbb{Z}$.

(iii) $f(k\mathbb{Z}) = l\mathbb{Z}_n$ *for some* $l \in \mathbb{Z}$.

(iv) $f(x) + f(y) = f(x) + f(y) + f(k\mathbb{Z})$ *for all* $x, y \in \mathbb{Z}$.

(v) $f(x + k\mathbb{Z}) = f(x) + f(k\mathbb{Z})$ *for all* $x \in \mathbb{Z}$.

Lemma 4.2.2. *If $f \in MHom((\mathbb{Z}, \circ_k), (\mathbb{Z}_n, +))$, then there exists an element $a \in \mathbb{Z}$ such that $[a]_n \in f(1)$ and*

$$f(x + k\mathbb{Z}) = [ax]_n + f(k\mathbb{Z}) \text{ for all } x \in \mathbb{Z}.$$

Theorem 4.2.3. *For a multi-valued function f from $\mathbb{Z}$ into $\mathbb{Z}_n$, $f \in MHom((\mathbb{Z}, \circ_k), (\mathbb{Z}_n, +))$ if and only if there exist $l, a \in \mathbb{Z}$ such that $\dfrac{(l,n)}{(l,k,n)} \mid a$ and*

$$f(x + k\mathbb{Z}) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z},$$
$$f(x) + f(y) = f(x) + f(y) + l\mathbb{Z}_n \text{ for all } x, y \in \mathbb{Z}.$$

Proof. Assume that $f \in \mathrm{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}_n, +))$. By Lemma 4.2.2, there is an element $a \in \mathbb{Z}$ such that $[a]_n \in f(1)$ and

$$f(x + k\mathbb{Z}) = [ax]_n + f(k\mathbb{Z}) \quad \text{for all } x \in \mathbb{Z}.$$

By Lemma 4.2.1(iii), $f(k\mathbb{Z}) = l\mathbb{Z}_n$ for some $l \in \mathbb{Z}$. Hence

$$f(x + k\mathbb{Z}) = [ax]_n + l\mathbb{Z}_n \quad \text{for all } x \in \mathbb{Z}. \tag{1}$$

Also, from Lemma 4.2.1(iii) and (iv),

$$f(x) + f(y) = f(x) + f(y) + l\mathbb{Z}_n \quad \text{for all } x, y \in \mathbb{Z}.$$

From $f(k\mathbb{Z}) = l\mathbb{Z}_n$, (1) implies that

$$l\mathbb{Z}_n = f(k\mathbb{Z}) = f(k + k\mathbb{Z}) = [ak]_n + l\mathbb{Z}_n,$$

so $[ak]_n \in l\mathbb{Z}_n$. Hence

$$ak\mathbb{Z}_n = [ak]_n\mathbb{Z}_n \subseteq l\mathbb{Z}_n\mathbb{Z}_n = l\mathbb{Z}_n$$

which implies that $|ak\mathbb{Z}_n| \mid |l\mathbb{Z}_n|$. Thus $\dfrac{n}{(ak,n)} \mid \dfrac{n}{(l,n)}$, so $(l,n) \mid (ak,n)$. Hence $(l,n) \mid ak$. It follows that $\dfrac{(l,n)}{(l,k,n)} \mid a$.

For the converse, assume that there are $l, a \in \mathbb{Z}$ such that $\dfrac{(l,n)}{(l,k,n)} \mid a$ and

$$f(x + k\mathbb{Z}) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}, \tag{2}$$

$$f(x) + f(y) = f(x) + f(y) + l\mathbb{Z}_n \text{ for all } x, y \in \mathbb{Z}. \tag{3}$$

Then $(l,n) \mid ak$. To show that f is defined independently to the representatives of cosets, let $x, y \in \mathbb{Z}$ be such that $x + k\mathbb{Z} = y + k\mathbb{Z}$. Then $x - y \in k\mathbb{Z}$, so

$$\begin{aligned}
[ax]_n - [ay]_n + l\mathbb{Z}_n &= [a(x-y)]_n + l\mathbb{Z}_n \\
&\subseteq ak\mathbb{Z}_n + l\mathbb{Z}_n \\
&= (ak, l)\mathbb{Z}_n \\
&= (ak, l, n)\mathbb{Z}_n \\
&= (l, n)\mathbb{Z}_n \ \text{ since } (l,n) \mid ak \\
&= l\mathbb{Z}_n.
\end{aligned}$$

This implies that $[ax]_n - [ay]_n + l\mathbb{Z}_n = l\mathbb{Z}_n$ and thus $[ax]_n + l\mathbb{Z}_n = [ay]_n + l\mathbb{Z}_n$.

Let $x, y \in \mathbb{Z}$. From (2),

$$f(x) \subseteq f(x + k\mathbb{Z}) \subseteq [ax]_n + l\mathbb{Z}_n, \ f(y) \subseteq f(y + k\mathbb{Z}) \subseteq [ay]_n + l\mathbb{Z}_n,$$

so there are nonempty subsets A, B of $l\mathbb{Z}_n$ such that $f(x) = [ax]_n + A$ and $f(y) = [ay]_n + B$. Thus $A + B + l\mathbb{Z}_n = l\mathbb{Z}_n$. Hence

$$
\begin{aligned}
f(x \circ_k y) &= f(x + y + k\mathbb{Z}) \\
&= [a(x+y)]_n + l\mathbb{Z}_n \\
&= [ax]_n + [ay]_n + l\mathbb{Z}_n \\
&= [ax]_n + A + [ay]_n + B + l\mathbb{Z}_n \\
&= f(x) + f(y) + l\mathbb{Z}_n \\
&= f(x) + f(y) \quad \text{from (3).}
\end{aligned}
$$

Therefore the proof of the theorem is complete. □

Lemma 4.2.4. *For $l, a \in \mathbb{Z}$, $\dfrac{(l,n)}{(l,k,n)} \mid a$ and $(a,l,n) = 1$ if and only if $(l,n) \mid k$ and $(a,l,n) = 1$.*

Proof. Note that $(l,n) \neq 0$. By Lemma 4.1.5, we have $\dfrac{(l,n)}{((l,n),k)} \mid a$ and $(a,(l,n)) = 1$ if and only if $(l,n) \mid k$ and $(a,(l,n)) = 1$. Therefore the desired result follows. □

Theorem 4.2.5. *For a multi-valued function f from $\mathbb{Z}$ into $\mathbb{Z}_n$, $f \in SMHom$ $((\mathbb{Z}, \circ_k), (\mathbb{Z}_n, +))$ if and only if there exist $l, a \in \mathbb{Z}$ such that $(l,n) \mid k$, $(a,l,n) = 1$ and*

$$f(x + k\mathbb{Z}) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z},$$

$$f(x) + f(y) = f(x) + f(y) + l\mathbb{Z}_n \text{ for all } x, y \in \mathbb{Z}.$$

Proof. Assume that $f \in \text{SMHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}_n, +))$. Then $f(\mathbb{Z}) = \mathbb{Z}_n$ and by Theorem 4.2.3, there are $l, a \in \mathbb{Z}$ such that $\dfrac{(l,n)}{(l,k,n)} \mid a$ and

$$f(x + k\mathbb{Z}) = [ax]_n + l\mathbb{Z}_n \text{ for all } x \in \mathbb{Z}, \tag{1}$$

$$f(x) + f(y) = f(x) + f(y) + l\mathbb{Z}_n \text{ for all } x, y \in \mathbb{Z}. \tag{2}$$

Since $f(\mathbb{Z}) = \mathbb{Z}_n$, by (1)

$$\mathbb{Z}_n = f(\mathbb{Z}) = f(\mathbb{Z} + k\mathbb{Z}) = a\mathbb{Z}_n + l\mathbb{Z}_n = (a,l)\mathbb{Z}_n$$

which implies that $1 = (1, n) = (a, l, n)$. By Lemma 4.2.4, $(l, n) \mid k$.

Conversely, assume that $l, a \in \mathbb{Z}$ such that $(l, n) \mid k$, $(a, l, n) = 1$ and f satisfies (1) and (2). It follows from Theorem 4.2.3 and Lemma 4.2.4 that $f \in \mathrm{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}_n, +))$. Since $(a, l, n) = 1$, by (1), we have

$$f(\mathbb{Z}) = a\mathbb{Z}_n + l\mathbb{Z}_n = (a, l)\mathbb{Z}_n = (a, l, n)\mathbb{Z}_n = 1\mathbb{Z}_n = \mathbb{Z}_n.$$

Hence $f \in \mathrm{SMHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}_n, +))$. □

Remark 4.2.6. Let us compare Theorem 2.9 - Theorem 2.12 with the results of this section where $l, a \in \mathbb{Z}$.

	Characterization	**Cardinality**
$\mathrm{MHom}((\mathbb{Z}, +), (\mathbb{Z}_n, +))$	$f(x) = [ax]_n + l\mathbb{Z}_n$	$\sum_{\substack{l \in \mathbb{Z}^+ \\ l \mid n}} l$
$\mathrm{SMHom}((\mathbb{Z}, +), (\mathbb{Z}_n, +))$	$f(x) = [ax]_n + l\mathbb{Z}_n$, $(a, l, n) = 1$	n
$\mathrm{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}_n, +))$	$f(x + k\mathbb{Z}) = [ax]_n + l\mathbb{Z}_n$, $f(x) + f(y) = f(x) + f(y) + l\mathbb{Z}_n$, $\frac{(l, n)}{(l, k, n)} \mid a$	–
$\mathrm{SMHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}_n, +))$	$f(x + k\mathbb{Z}) = [ax]_n + l\mathbb{Z}_n$, $f(x) + f(y) = f(x) + f(y) + l\mathbb{Z}_n$, $(l, n) \mid k$, $(a, l, n) = 1$	–

We give a remark that the cardinalities of $\mathrm{MHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}_n, +))$ and SMHom $((\mathbb{Z},\circ_k),(\mathbb{Z}_n,+))$ are not known in this research. It is easily seen that $|\mathrm{MHom}((\mathbb{Z},\circ_k), (\mathbb{Z}_1, +))| = |\mathrm{SMHom}((\mathbb{Z}, \circ_k), (\mathbb{Z}_1, +))| = 1$.

4.3 Multi-valued Homomorphisms from the Hypergroup $(\mathbb{Z}_n, \circ_k)$ into the Group $(\mathbb{Z}, +)$

First, we provide the lemmas analogous to Lemma 4.1.2 and Lemma 4.1.3. The proofs can be given analogously.

Lemma 4.3.1. *If $f \in MHom((\mathbb{Z}_n, \circ_k), (\mathbb{Z}, +))$, then the following statements hold.*

(i) $f(k\mathbb{Z}_n) = f([x_1]_n) + \cdots + f([x_t]_n)$ *for all* $x_1, \ldots, x_t \in \mathbb{Z}$ *with* $[x_1]_n + \cdots + [x_t]_n \in k\mathbb{Z}_n$.

(ii) $f(k\mathbb{Z}_n) = f([x]_n) + f([y]_n)$ *for all* $x, y \in \mathbb{Z}$ *with* $[x]_n, [y]_n \in k\mathbb{Z}_n$.

(iii) $f(k\mathbb{Z}_n)$ *is a subsemigroup of* $(\mathbb{Z}, +)$ *containing 0.*

(iv) $f([x]_n) + f([y]_n) = f([x]_n) + f([y]_n) + f(k\mathbb{Z}_n)$ *for all* $x, y \in \mathbb{Z}$.

(v) $f([x]_n + k\mathbb{Z}_n) = f([x]_n) + f(k\mathbb{Z}_n)$ *for all* $x \in \mathbb{Z}$.

Lemma 4.3.2. *If $f \in MHom((\mathbb{Z}_n, \circ_k), (\mathbb{Z}, +))$, then there exists an element $a \in f([1]_n)$ such that*

$$f([x]_n + k\mathbb{Z}_n) = [ax]_n + f(k\mathbb{Z}_n) \text{ for all } x \in \mathbb{Z}.$$

Theorem 4.3.3. *For a multi-valued function f from $\mathbb{Z}_n$ into $\mathbb{Z}$, $f \in MHom((\mathbb{Z}_n,\circ_k), (\mathbb{Z}, +))$ if and only if one of the following two conditions holds.*

(i) *There exists a subsemigroup H of $(\mathbb{Z}, +)$ containing 0 such that*

$$f([x]_n + k\mathbb{Z}_n) = H \quad \text{for all } x \in \mathbb{Z} \text{ and}$$
$$f([x]_n) + f([y]_n) = H \quad \text{for all } x, y \in \mathbb{Z}.$$

(ii) *There exist $l, a \in \mathbb{Z}$ such that $l \neq 0$, $\dfrac{l}{(l,k,n)} \mid a$ and*

$$f([x]_n + k\mathbb{Z}_n) = ax + l\mathbb{Z} \quad \text{for all } x \in \mathbb{Z} \text{ and}$$
$$f([x]_n) + f([y]_n) = f([x]_n) + f([y]_n) + l\mathbb{Z} \quad \text{for all } x, y \in \mathbb{Z}.$$

Proof. Assume that $f \in \mathrm{MHom}((\mathbb{Z}_n, \circ_k), (\mathbb{Z}, +))$. By Lemma 4.3.1(iii), $f(k\mathbb{Z}_n)$

is a subsemigroup of $(\mathbb{Z}, +)$ containing 0. By Lemma 4.3.1(iv),

$$f([x]_n) + f([y]_n) = f([x]_n) + f([y]_n) + f(k\mathbb{Z}_n) \text{ for all } x, y \in \mathbb{Z}. \tag{1}$$

By Lemma 4.3.2, there exists $a \in f([1]_n)$ such that

$$f([x]_n + k\mathbb{Z}_n) = ax + f(k\mathbb{Z}_n) \text{ for all } x \in \mathbb{Z}. \tag{2}$$

Case 1: a=0. From (2), we have

$$f([x]_n + k\mathbb{Z}_n) = f(k\mathbb{Z}_n) \text{ for all } x \in \mathbb{Z} \tag{3}$$

and for all $x, y \in \mathbb{Z}$,

$$\begin{aligned} f([x]_n) + f([y]_n) &= f([x]_n \circ_k [y]_n) \\ &= f([x]_n + [y]_n + k\mathbb{Z}_n) \\ &= f(k\mathbb{Z}_n) \quad \text{ from (3).} \end{aligned}$$

Thus f satisfies (i).

Case 2: $a \neq 0$. It follows from (2) that

$$\begin{aligned} f(k\mathbb{Z}_n) &= f([k]_n + k\mathbb{Z}_n) = ak + f(k\mathbb{Z}_n) \text{ and} \\ f(k\mathbb{Z}_n) &= f([-k]_n + k\mathbb{Z}_n) = -ak + f(k\mathbb{Z}_n). \end{aligned}$$

Since $0 \in f(k\mathbb{Z}_n)$, ak, $-ak \in f(k\mathbb{Z}_n)$, so $f(k\mathbb{Z}_n) \cap \mathbb{Z}^+ \neq \varnothing$ and $f(k\mathbb{Z}_n) \cap \mathbb{Z}^- \neq \varnothing$. Then by Lemma 3.1.1, $f(k\mathbb{Z}_n) = l\mathbb{Z}$ for some $l \in \mathbb{Z} \smallsetminus \{0\}$. Since $ak \in l\mathbb{Z}$, $l \mid ak$. Also, we have

$$l\mathbb{Z} = f(k\mathbb{Z}_n) = f([n]_n + k\mathbb{Z}_n) = an + l\mathbb{Z},$$

so $an \in l\mathbb{Z}$. Thus $l \mid an$. But $(l, k, n) = xl + yk + zn$ for some $x, y, z \in \mathbb{Z}$, so $a(l, k, n) = axl + y(ak) + z(an)$ which implies that $l \mid a(l, k, n)$. Hence $\dfrac{l}{(l, k, n)} \mid a$. From (1) and (2), we have

$$f([x]_n) + f([y]_n) = f([x]_n) + f([y]_n) + l\mathbb{Z} \text{ for all } x, y \in \mathbb{Z}$$

and

$$f([x]_n + k\mathbb{Z}_n) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z},$$

respectively. Hence (ii) holds.

Conversely, assume that f satisfies (i) or (ii). If f satisfies (i), then for all $x, y \in \mathbb{Z}$,

$$\begin{aligned} f([x]_n \circ_k [y]_n) &= f([x]_n + [y]_n + k\mathbb{Z}_n) \\ &= f([x+y]_n + k\mathbb{Z}_n) \\ &= H \\ &= f([x]_n) + f([y]_n), \end{aligned}$$

which implies that $f \in \mathrm{MHom}((\mathbb{Z}_n, \circ_k), (\mathbb{Z}, +))$.

Next, assume that f satisfies (ii). First we show that f defined on each coset is independent on its representatives, let $x, y \in \mathbb{Z}$ be such that $[x]_n + k\mathbb{Z}_n = [y]_n + k\mathbb{Z}_n$. Then $[x - y]_n \in k\mathbb{Z}_n$. Thus $x - y = ks + nt$ for some $s, t \in \mathbb{Z}$. Since $\dfrac{l}{(l,k,n)} \mid a$, we have $l \mid a(l, k, n)$, so $l \mid ak$ and $l \mid an$. Thus $ak, an \in l\mathbb{Z}$. It follows that

$$ax - ay = a(x - y) = a(ks + nt) = aks + ant \in l\mathbb{Z}$$

which implies that $ax + l\mathbb{Z} = ay + l\mathbb{Z}$.

To show that $f \in \mathrm{MHom}((\mathbb{Z}_n, \circ_k), (\mathbb{Z}, +))$, let $x, y \in \mathbb{Z}$. Since

$$f([x]_n) \subseteq f([x]_n + k\mathbb{Z}_n) = ax + l\mathbb{Z},\ f([y]_n) \subseteq f([y]_n + k\mathbb{Z}_n) = ay + l\mathbb{Z},$$

we deduce that $f([x]_n) = ax + A$ and $f([y]_n) = ay + B$ for some nonempty subsets A, B of $l\mathbb{Z}$. Therefore $A + B + l\mathbb{Z} = l\mathbb{Z}$ and hence

$$\begin{aligned} f([x]_n \circ_k [y]_n) &= f([x]_n + [y]_n + k\mathbb{Z}_n) \\ &= a(x + y) + l\mathbb{Z} \\ &= ax + ay + l\mathbb{Z} \\ &= ax + A + ay + B + l\mathbb{Z} \\ &= f([x]_n) + f([y]_n) + l\mathbb{Z} \\ &= f([x]_n) + f([y]_n). \end{aligned}$$

Hence $f \in \mathrm{MHom}((\mathbb{Z}_n, \circ_k), (\mathbb{Z}, +))$. $\square$

Lemma 4.3.4. *For $l \in \mathbb{Z} \in \{0\}$ and $a \in \mathbb{Z}$, $\dfrac{l}{(l,k,n)} \mid a$ and $(a,l) = 1$ if and only if $l \mid (k,n)$ and $(a,l) = 1$.*

Proof. Since $(l,k,n) = (l,(k,n))$, the desired result follows directly from Lemma 4.1.5. □

Theorem 4.3.5. *For a multi-valued function f from $\mathbb{Z}_n$ into $\mathbb{Z}$, $f \in SMHom$ $((\mathbb{Z}_n, \circ_k), (\mathbb{Z}, +))$ if and only if one of the following two conditions holds.*

(i) $$f([x]_n + k\mathbb{Z}_n) = \mathbb{Z} \quad \textit{for all } x \in \mathbb{Z} \textit{ and}$$
$$f([x]_n) + f([y]_n) = \mathbb{Z} \quad \textit{for all } x, y \in \mathbb{Z}.$$

(ii) *There exist $l, a \in \mathbb{Z}$ such that $l \neq 0, l \mid (k,n), (a,l) = 1$,*

$$f([x]_n + k\mathbb{Z}_n) = ax + l\mathbb{Z} \quad \textit{for all } x \in \mathbb{Z} \textit{ and}$$
$$f([x]_n) + f([y]_n) = f([x]_n) + f([y]_n) + l\mathbb{Z} \quad \textit{for all } x, y \in \mathbb{Z}.$$

Proof. Assume that $f \in \text{SMHom}((\mathbb{Z}_n, \circ_k), (\mathbb{Z}, +))$. Then f satisfies (i) or (ii) of Theorem 4.3.3 and $f(\mathbb{Z}_n) = \mathbb{Z}$. If f satisfies (i) of Theorem 4.3.3, then (i) holds.

Next, assume that f satisfies (ii) of Theorem 4.3.3. Then there are $l, a \in \mathbb{Z}$ such that $l \neq 0$, $\dfrac{l}{(l,k,n)} \mid a$ and

$$f([x]_n + k\mathbb{Z}_n) = ax + l\mathbb{Z} \text{ for all } x \in \mathbb{Z} \text{ and}$$
$$f([x]_n) + f([y]_n) = f([x]_n) + f([y]_n) + l\mathbb{Z} \text{ for all } x, y \in \mathbb{Z}.$$

Since $f(\mathbb{Z}_n) = \mathbb{Z}$, it follows that

$$\begin{aligned} \mathbb{Z} = f(\mathbb{Z}_n) &= f(\bigcup_{x \in \mathbb{Z}} ([x]_n + k\mathbb{Z}_n)) \\ &= \bigcup_{x \in \mathbb{Z}} f([x]_n + k\mathbb{Z}_n) \\ &= \bigcup_{x \in \mathbb{Z}} ax + l\mathbb{Z} = a\mathbb{Z} + l\mathbb{Z} = (a,l)\mathbb{Z} \end{aligned}$$

which implies that $(a,l) = 1$. By Lemma 4.3.4, $l \mid (k,n)$.

For the converse, assume that f satisfies (i) or (ii). By Theorem 4.3.3 and

Lemma 4.3.4, $f \in \text{MHom}((\mathbb{Z}_n, \circ_k), (\mathbb{Z},+))$. If f satisfies (i), then $\mathbb{Z} = f([1]_n + k\mathbb{Z}_n) \subseteq f(\mathbb{Z}_n)$, so $f(\mathbb{Z}_n) = \mathbb{Z}$. If f satisfies (ii), then

$$f(\mathbb{Z}_n) = f(\bigcup_{x \in \mathbb{Z}}([x]_n + k\mathbb{Z}_n)) = a\mathbb{Z} + l\mathbb{Z} = (a,l)\mathbb{Z} = \mathbb{Z}.$$

Hence $f \in \text{SMHom}((\mathbb{Z}_n, \circ_k), (\mathbb{Z}, +))$. □

Theorem 4.3.6. *The cardinalities of MHom*$((\mathbb{Z}_n,\circ_k),(\mathbb{Z},+))$ *and SMHom*$((\mathbb{Z}_n,\circ_k),(\mathbb{Z},+))$ *are the followings.*

(i) *If* $n \mid k$, *then*

$$|MHom((\mathbb{Z}_n, \circ_k), (\mathbb{Z}, +))| = \aleph_0,\ |SMHom((\mathbb{Z}_n, \circ_k), (\mathbb{Z}, +))| = n.$$

(ii) *If* $n \nmid k$, *then*

$$|MHom((\mathbb{Z}_n, \circ_k), (\mathbb{Z}, +))| = |SMHom((\mathbb{Z}_n, \circ_k), (\mathbb{Z}, +))| = 2^{\aleph_0}.$$

Proof. If $n \mid k$, then $k\mathbb{Z}_n = \{[0]_n\}$, so $(\mathbb{Z}_n, \circ_k) = (\mathbb{Z}_n, +)$, hence (i) holds by Theorem 2.6 and Theorem 2.8.

Next, assume that $n \nmid k$. Then $n > 1$, so $|\mathbb{Z}_n| = n > 1$. We have that

$$((2\mathbb{Z}+1) \cup \{0\}) + ((2\mathbb{Z}+1) \cup \{0\}) = 2\mathbb{Z} \cup (2\mathbb{Z}+1) = \mathbb{Z}. \tag{1}$$

Let $X \subseteq 2\mathbb{Z} \smallsetminus \{0\}$ and define $f_X : \mathbb{Z}_n \to \mathcal{P}^*(\mathbb{Z})$ by

$$\begin{aligned} f_X([0]_n) &= ((2\mathbb{Z}+1) \cup \{0\}) \cup X \text{ and} \\ f_X([x]_n) &= \mathbb{Z} \quad \text{for all } x \in \mathbb{Z} \smallsetminus n\mathbb{Z}. \end{aligned} \tag{2}$$

Then $f_X([x]_n) \supseteq (2\mathbb{Z}+1) \cup \{0\}$ for all $x \in \mathbb{Z}$, so by (1), we have

$$f_X([x]_n) + f_X([y]_n) = \mathbb{Z} \text{ for all } x, y \in \mathbb{Z}. \tag{3}$$

Since $n \nmid k$, $|k\mathbb{Z}_n| = |(k,n)\mathbb{Z}_n| = \dfrac{n}{(k,n)} > 1$. Thus for any $x \in \mathbb{Z}$, $|[x]_n + k\mathbb{Z}_n| = |k\mathbb{Z}_n| > 1$. It follows that

$$f_X([x]_n + k\mathbb{Z}_n) = \bigcup_{t \in \mathbb{Z}} f_X([x]_n + k[t]_n) = \mathbb{Z}. \tag{4}$$

By (3) and (4), f_X satisfies (i) of Theorem 4.3.5, so $f_X \in \text{SMHom}((\mathbb{Z}_n, \circ_k), (\mathbb{Z}, +))$.

If X and X' are distinct subsets of $2\mathbb{Z} \smallsetminus \{0\}$, then $((2\mathbb{Z}+1) \cup \{0\}) \cup X \neq$

$((2\mathbb{Z}+1)\cup\{0\})\cup X'$, so from (2), we have $f_X([0]_n) \neq f_{X'}([0]_n)$. Consequently, $f_X \neq f_{X'}$ for all distinct subsets X and X' of $2\mathbb{Z}\setminus\{0\}$. Hence we have

$$\begin{aligned} |\mathrm{MHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z},+))| &\geq |\mathrm{SMHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z},+))| \\ &\geq |\{f_X \mid X \subseteq 2\mathbb{Z}\setminus\{0\}\}| \\ &= |\{X \mid X \subseteq 2\mathbb{Z}\setminus\{0\}\}| \\ &= 2^{\aleph_0}. \end{aligned}$$

But

$$\begin{aligned} |\mathrm{MHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z},+))| &\leq |\{f \mid f : \mathbb{Z}_n \to \mathcal{P}^*(\mathbb{Z})\}| \\ &= (2^{\aleph_0})^n \\ &= 2^{\aleph_0}, \end{aligned}$$

so $\quad |\mathrm{MHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z},+))| = |\mathrm{SMHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z},+))| = 2^{\aleph_0}$ if $n \nmid k$. □

Remark 4.3.7. The following diagram gives a comparison between Theorem 2.5 - Theorem 2.8 and the theorems in this section where $l, a \in \mathbb{Z}$ and H is a subsemigroup of $(\mathbb{Z},+)$ containing 0.

	Characterization	Cardinality
$\mathrm{MHom}((\mathbb{Z}_n,+),(\mathbb{Z},+))$	(i) $f([x]_n) = H$ or (ii) $f([x]_n) = ax + l\mathbb{Z}$, $l \neq 0$, $\dfrac{l}{(l,n)} \mid a$	$\aleph_0$
$\mathrm{SMHom}((\mathbb{Z}_n,+),(\mathbb{Z},+))$	$f([x]_n) = ax + l\mathbb{Z}$, $l \neq 0$, $l \mid n$, $(a,l) = 1$	n
$\mathrm{MHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z},+))$	(i) $f([x]_n + k\mathbb{Z}_n) = H$, $f([x]_n) + f([y]_n) = H$ or (ii) $f([x]_n + k\mathbb{Z}_n) = ax + l\mathbb{Z}$, $f([x]_n) + f([y]_n)$ $= f([x]_n) + f([y]_n) + l\mathbb{Z}$, $l \neq 0$, $\dfrac{l}{(l,k,n)} \mid a$	$\aleph_0$ if $n \mid k$, $2^{\aleph_0}$ if $n \nmid k$.
$\mathrm{SMHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z},+))$	(i) $f([x]_n + k\mathbb{Z}_n) = \mathbb{Z}$, $f([x]_n) + f([y]_n) = \mathbb{Z}$ or (ii) $f([x]_n + k\mathbb{Z}_n) = ax + l\mathbb{Z}$, $f([x]_n) + f([y]_n)$ $= f([x]_n) + f([y]_n) + l\mathbb{Z}$, $l \neq 0, l \mid (k,n), (a,l) = 1$	n if $n \mid k$, $2^{\aleph_0}$ if $n \nmid k$.

4.4 Multi-valued Homomorphisms from the Hypergroup $(\mathbb{Z}_n,\circ_k)$ into the Group $(\mathbb{Z}_m,+)$

Lemma 4.4.1 and Lemma 4.4.2 given below can be proved analogously to the proofs of Lemma 4.1.2 and Lemma 4.1.3, respectively. Note that a subsemigroup of $(\mathbb{Z}_m,+)$ must be of the form $l\mathbb{Z}_m$, $l \in \mathbb{Z}$.

Lemma 4.4.1. *If* $f \in MHom((\mathbb{Z}_n, \circ_k), (\mathbb{Z}_m, +))$*, then the following statements hold.*

(i) $f(k\mathbb{Z}_n) = f([x_1]_n) + \cdots + f([x_t]_n)$ *for all* $x_1, \ldots, x_t \in \mathbb{Z}$ *with* $[x_1]_n + \cdots + [x_t]_n \in k\mathbb{Z}_n$.

(ii) $f(k\mathbb{Z}_n) = f([x]_n) + f([y]_n)$ *for all* $x, y \in \mathbb{Z}$ *with* $[x]_n, [y]_n \in k\mathbb{Z}_n$.

(iii) $f(k\mathbb{Z}_n) = l\mathbb{Z}_m$ *for some* $l \in \mathbb{Z}$.

(iv) $f([x]_n) + f([y]_n) = f([x]_n) + f([y]_n) + f(k\mathbb{Z}_n)$ *for all* $x, y \in \mathbb{Z}$.

(v) $f([x]_n + k\mathbb{Z}_n) = f([x]_n) + f(k\mathbb{Z}_n)$ *for all* $x \in \mathbb{Z}$.

Lemma 4.4.2. *If* $f \in MHom((\mathbb{Z}_n, \circ_k), (\mathbb{Z}_m, +))$*, then there exists an element* $a \in \mathbb{Z}$ *such that* $[a]_m \in f([1]_n)$ *and*

$$f([x]_n + k\mathbb{Z}_n) = [ax]_m + f(k\mathbb{Z}_n) \text{ for all } x \in \mathbb{Z}.$$

Theorem 4.4.3. *For a multi-valued function* f *from* $\mathbb{Z}_n$ *into* $\mathbb{Z}_m$, $f \in MHom((\mathbb{Z}_n, \circ_k), (\mathbb{Z}_m, +))$ *if and only if there exist* $l, a \in \mathbb{Z}$ *such that* $\frac{(l,m)}{(l,k,m,n)} \mid a$ *and*

$$f([x]_n + k\mathbb{Z}_n) = [ax]_m + l\mathbb{Z}_m \text{ for all } x \in \mathbb{Z},$$

$$f([x]_n) + f([y]_n) = f([x]_n) + f([y]_n) + l\mathbb{Z}_m \text{ for all } x, y \in \mathbb{Z}.$$

Proof. Assume that $f \in \mathrm{MHom}((\mathbb{Z}_n, \circ_k), (\mathbb{Z}_m, +))$. By Lemma 4.4.2, there is an element $a \in \mathbb{Z}$ such that $[a]_m \in f([1]_n)$ and

$$f([x]_n + k\mathbb{Z}_n) = [ax]_m + f(k\mathbb{Z}_n) \text{ for all } x \in \mathbb{Z}.$$

By Lemma 4.4.1(iii), $f(k\mathbb{Z}_n) = l\mathbb{Z}_m$ for some $l \in \mathbb{Z}$. Hence

$$f([x]_n + k\mathbb{Z}_n) = [ax]_m + l\mathbb{Z}_m \text{ for all } x \in \mathbb{Z}.$$

Also, from Lemma 4.4.1(iii) and (iv),

$$f([x]_n) + f([y]_n) = f([x]_n) + f([y]_n) + l\mathbb{Z}_m \text{ for all } x, y \in \mathbb{Z}.$$

From $f(k\mathbb{Z}_n) = l\mathbb{Z}_m$, we have

$$l\mathbb{Z}_m = f(k\mathbb{Z}_n) = f([k]_n + k\mathbb{Z}_n) = [ak]_m + l\mathbb{Z}_m,$$

$$l\mathbb{Z}_m = f(k\mathbb{Z}_n) = f([n]_n + k\mathbb{Z}_n) = [an]_m + l\mathbb{Z}_m,$$

so $[ak]_m, [an]_m \in l\mathbb{Z}_m$. From the proof of Theorem 4.2.3, we have that $(l,m) \mid ak$ and $(l,m) \mid an$. Since $(l,k,m,n) = ((l,m),k,n) = x(l,m) + yk + zn$ for some $x,y,z \in \mathbb{Z}$, it follows that $(l,m) \mid a(l,k,m,n)$. Hence $\frac{(l,m)}{(l,k,m,n)} \mid a$.

For the converse, let l,a,f be as above. To show that f is defined independently to the representatives of cosets, let $x,y \in \mathbb{Z}$ be such that $[x]_n + k\mathbb{Z}_n = [y]_n + k\mathbb{Z}_n$. Then $[x-y]_n \in k\mathbb{Z}_n$. Thus $x - y = ks + nt$ for some $s,t \in \mathbb{Z}$. Since $\frac{(l,m)}{(l,k,m,n)} \mid a$, we have $(l,m) \mid a(l,k,m,n)$, so $(l,m) \mid ak$ and $(l,m) \mid an$. Thus $[ak]_m, [an]_m \in (l,m)\mathbb{Z}_m = l\mathbb{Z}_m$. It follows that

$$\begin{aligned}[ax]_m - [ay]_m &= a[x-y]_m \\ &= a[ks+nt]_m \\ &= [ak]_m[s]_m + [an]_m[t]_m \in l\mathbb{Z}_m\end{aligned}$$

which implies that $[ax]_m + l\mathbb{Z}_m = [ay]_m + l\mathbb{Z}_m$.

To show that $f \in \mathrm{MHom}((\mathbb{Z}_n, \circ_k), (\mathbb{Z}_m, +))$, let $x,y \in \mathbb{Z}$. Then

$$f([x]_n) \subseteq f([x]_n + k\mathbb{Z}_n) = [ax]_m + l\mathbb{Z}_m, f([y]_n) \subseteq f([y]_n + k\mathbb{Z}_n) = [ay]_m + l\mathbb{Z}_m.$$

Then there are nonempty subsets A, B of $l\mathbb{Z}_m$ such that $f([x]_n) = [ax]_m + A$ and $f([y]_n) = [ay]_m + B$. Therefore $A + B + l\mathbb{Z}_m = l\mathbb{Z}_m$ and hence

$$\begin{aligned}f([x]_n \circ_k [y]_n) &= f([x]_n + [y]_n + k\mathbb{Z}_n) \\ &= [a(x+y)]_m + l\mathbb{Z}_m \\ &= [ax]_m + [ay]_m + l\mathbb{Z}_m \\ &= [ax]_m + A + [ay]_m + B + l\mathbb{Z}_m \\ &= f([x]_n) + f([y]_n) + l\mathbb{Z}_m \\ &= f([x]_n) + f([y]_n).\end{aligned}$$

Therefore the proof is complete. □

Lemma 4.4.4. *For $l,a \in \mathbb{Z}$, $\frac{(l,m)}{(l,k,m,n)} \mid a$ and $(a,l,m) = 1$ if and only if $(l,m) \mid (k,n)$ and $(a,l,m) = 1$.*

Proof. By Lemma 4.1.5, we have that $\dfrac{(l,m)}{((l,m),(k,n))} \mid a$ and $(a,(l,m))=1$ if and only if $(l,m)\mid(k,n)$ and $(a,(l,m))=1$. Hence the result follows, as desired. □

Theorem 4.4.5. *For a multi-valued function f from $\mathbb{Z}_n$ into $\mathbb{Z}_m$, $f \in SMHom$ $((\mathbb{Z}_n,\circ_k),(\mathbb{Z}_m,+))$ if and only if there exist l, $a\in\mathbb{Z}$ such that $(l,m)\mid(k,n)$, $(a,l,m)=1$ and*

$$f([x]_n + k\mathbb{Z}_n) = [ax]_m + l\mathbb{Z}_m \text{ for all } x \in \mathbb{Z},$$
$$f([x]_n) + f([y]_n) = f([x]_n) + f([y]_n) + l\mathbb{Z}_m \text{ for all } x, y \in \mathbb{Z}.$$

Proof. Assume that $f \in \mathrm{SMHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z}_m,+))$. Then $f\in \mathrm{MHom}((\mathbb{Z}_n,\circ_k),$ $(\mathbb{Z}_m,+))$ and $f(\mathbb{Z}_n)=\mathbb{Z}_m$. By Theorem 4.4.3, there exist $l,a\in\mathbb{Z}$ such that $\dfrac{(l,m)}{(l,k,m,n)}\mid a$ and

$$f([x]_n + k\mathbb{Z}_n) = [ax]_m + l\mathbb{Z}_m \text{ for all } x \in \mathbb{Z},$$
$$f([x]_n) + f([y]_n) = f([x]_n) + f([y]_n) + l\mathbb{Z}_m \text{ for all } x, y \in \mathbb{Z}.$$

Since $f(\mathbb{Z}_n)=\mathbb{Z}_m$, we have

$$\mathbb{Z}_m = f(\mathbb{Z}_n) = a\mathbb{Z}_m + l\mathbb{Z}_m = (a,l)\mathbb{Z}_m.$$

Thus $1=(1,m)=(a,l,m)$. By Lemma 4.4.4, $(l,m)\mid(k,n)$.

Conversely, assume that l, a, f are as before. By Theorem 4.4.3 and Lemma 4.4.4, we have $f\in\mathrm{MHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z}_m,+))$. Since $(a,l,m)=1$,

$$f(\mathbb{Z}_n) = a\mathbb{Z}_m + l\mathbb{Z}_m = (a,l)\mathbb{Z}_m = (a,l,m)\mathbb{Z}_m = \mathbb{Z}_m.$$

Therefore $f\in\mathrm{SMHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z}_m,+))$. □

Remark 4.4.6. We also compare the results of this section with Theorem 2.13 - Theorem 2.16 by the following diagram where $l,a\in\mathbb{Z}$.

	Characterization	**Cardinality**
$\mathrm{MHom}((\mathbb{Z}_n,+),(\mathbb{Z}_m,+))$	$f([x]_n)=[ax]_m+l\mathbb{Z}_m$, $\frac{(l,m)}{(l,m,n)} \mid a$	$\sum_{\substack{l\in\mathbb{Z}^+\\ l\mid m}}(l,n)$
$\mathrm{SMHom}((\mathbb{Z}_n,+),(\mathbb{Z}_m,+))$	$f([x]_n)=[ax]_m+l\mathbb{Z}_m$, $(l,m)\mid n,(a,l,m)=1$	(m,n)
$\mathrm{MHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z}_m,+))$	$f([x]_n+k\mathbb{Z}_n)=[ax]_m+l\mathbb{Z}_m$, $f([x]_n)+f([y]_n)$ $=f([x]_n)+f([y]_n)+l\mathbb{Z}_m$, $\frac{(l,m)}{(l,k,m,n)} \mid a$	–
$\mathrm{SMHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z}_m,+))$	$f([x]_n+k\mathbb{Z}_n)=[ax]_m+l\mathbb{Z}_m$, $f([x]_n)+f([y]_n)$ $=f([x]_n)+f([y]_n)+l\mathbb{Z}_m$, $(l,m)\mid(k,n),(a,l,m)=1$	–

We note here that counting the elements of $\mathrm{MHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z}_m,+))$ and $\mathrm{SMHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z}_m,+))$ is still open in this research. However, these two sets are finite. It is clear $|\mathrm{MHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z}_1,+))|=|\mathrm{SMHom}((\mathbb{Z}_n,\circ_k),(\mathbb{Z}_1,+))|=1$. It is not difficult to see that $|\mathrm{MHom}((\mathbb{Z}_1,\circ_k),(\mathbb{Z}_m,+))|=\sum_{\substack{l\in\mathbb{Z}^+\\ l\mid m}}1$ and $|\mathrm{SMHom}((\mathbb{Z}_1,\circ_k),(\mathbb{Z}_m,+))|=1$.

REFERENCES

[1] Brown, W. and Curtis, F. Numerical semigroups of maximal and almost maximal length. **Semigroup Forum** 42(1991): 218 – 231.

[2] Corsini, P. **Prolegomena of hypergroup theory**. Udine: Aviani Editore, 1993.

[3] Feichtinger, O. More on lower semi-continuity. **Amer. Math. Monthly** 83(1978): 39.

[4] Hrbacek, K. and Jech, T. **Introduction to set theory**. New York: Marchel Dekker, 1999.

[5] Nenthein, S. and Lertwichitsilp, P. Surjective multihomomorphisms between cyclic groups. **Thai J. Math.** 4(2006): 35 – 42.

[6] Niven, I., Zuckerman, H. S. and Montgomery, H. L. **An introduction to theory of numbers**. New York: John Wiley & Sons, 1991.

[7] Smithson, R. E. A characterization of lower semi-continuity. **Amer. Math. Monthly** 75(1968): 500.

[8] Teparos, W. **Multi-valued homomorphisms of semigroups and regularity of semigroups of multi-valued functions**. Master Thesis, Chulalongkorn University, 2006.

[9] Teparos, W. and Kemprasit, Y. Regularity of semigroups of multihomomor phisms of $(\mathbb{Z}_n, +)$. **Thai J. Math.**, Spec. Issue for Annual Meeting in Math. 2006: 25 – 30.

[10] Triphop, N., Harnchoowong, A. and Kemprasit, Y. Multihomomorphisms between cyclic groups. **Set-valued Math. and Appl.** 1(2007): 39 – 48.

[11] Whyburn, G. T. Continuity of multifunctions. **Proc. Nat. Acad. Sciences** 54(1965): 1494 – 1501.

[12] Youngkhong, P. and Savettaseranee, K. Multihomomorphisms from groups into groups of real numbers. **Thai J. Math.** 4(2006): 43 – 48.

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