# CHAPTER II

## TRANSFORMATION SEMIGROUPS

The main purpose of this chapter is to characterize the partial transformation semigroup, the full transformation semigroup, the semigroup of almost identical partial transformations and the semigroup of almost identical full transformations on a set X which are regular-* or *-regular in term of the cardinality of X.

Let X be a set. Let $T_X$ and $\mathcal{T}_X$ denote the partial transformation semigroup on the set X, and the full transformation semigroup on the set X, respectively. For $\alpha \in T_X$, $\alpha$ is an idempotent of $T_X$ if and only if $\nabla\alpha \subseteq \Delta\alpha$ and $x\alpha = x$ for all $x \in \nabla\alpha$.

An inverse semigroup is a regular-* semigroup and a *-regular semigroup. Then the symmetric inverse semigroup on a set X, $I_X$, is regular-* and *-regular.

Recall that for a regular-* semigroup S, the product of two projections of S is an idempotent of S [10, Theorem 2.5].



The two following theorems show that for any set X, $T_X$ is regular-* or *-regular if and only if $|X| \leq 1$.

2.1 <u>Theorem</u>. For any set X, the partial transformation semigroup on X, $T_X$, is regular-* if and only if $|X| \leq 1$.

<u>Proof</u> : Assume that $T_X$ is a regular-* semigroup with an involution *. Suppose $|X| \geq 2$. Let a, b be two distinct elements in X. For each $x \in X$, let $\alpha_x$ be the element of $T_X$ such that $\Delta\alpha_x = X$, $\nabla\alpha_x = \{x\}$, and for $x, y \in X$, let $\beta_{x,y}$ be the element of $T_X$ such that $\Delta\beta_{x,y} = \{x\}$, $\nabla\beta_{x,y} = \{y\}$, Observe that for any $x \in X$, $\alpha_x$ is an idempotent of $T_X$, and for $x, y \in X$, $\beta_{x,y}$ is an idempotent of $T_X$ if and only if $x = y$. Since $\alpha_a, \beta_{a,a} \in T_X$ and $T_X$ is a regular-* semigroup, $\alpha_a = \alpha_a\alpha^*_a\alpha_a$ and $\beta_{a,a} = \beta_{a,a}\beta^*_{a,a}\beta_{a,a}$. Because $a = a\alpha_a = a\alpha_a\alpha^*_a\alpha_a = a\alpha^*_a\alpha_a$, it follows that $a \in \Delta\alpha^*_a\alpha_a \subseteq \Delta\alpha^*_a$, so $a \in \Delta\alpha^*_a$. Hence $a\alpha^*_a = c$ for some $c \in X$. Because $\Delta\alpha_a\alpha^*_a = (\nabla\alpha_a \cap \Delta\alpha^*_a)\alpha_a^{-1} = (\{a\} \cap \Delta\alpha^*_a)\alpha_a^{-1} = \{a\}\alpha_a^{-1} = X = \Delta\alpha_c$ and $\nabla\alpha_a\alpha^*_a = (\nabla\alpha_a \cap \Delta\alpha^*_a)\alpha^*_a = (\{a\} \cap \Delta\alpha^*_a)\alpha^*_a = \{a\}\alpha^*_a = \{c\} = \nabla\alpha_c$, we have that $\alpha_a\alpha^*_a = \alpha_c$, so $\alpha^*_c = (\alpha_a\alpha^*_a)^* = \alpha_a\alpha^*_a = \alpha_c$. Thus $\alpha_c$ is a projection of $T_X$. Since $a = a\beta_{a,a} = a\beta_{a,a}\beta^*_{a,a}\beta_{a,a} = a\beta^*_{a,a}\beta_{a,a}$, $a \in \Delta\beta^*_{a,a}\beta_{a,a} \subseteq \Delta\beta^*_{a,a}$, so $a \in \Delta\beta^*_{a,a}$. Because $a\beta_{a,a} = a\beta_{a,a}\beta^*_{a,a}\beta_{a,a} = (a\beta^*_{a,a})\beta_{a,a}$, it follows that $a\beta^*_{a,a} = a$. From $\Delta\beta_{a,a}\beta^*_{a,a} = (\nabla\beta_{a,a} \cap \Delta\beta^*_{a,a})\beta_{a,a}^{-1} = (\{a\} \cap \Delta\beta^*_{a,a})\beta_{a,a}^{-1} = \{a\}\beta_{a,a}^{-1} = \{a\} = \Delta\beta_{a,a}$ and $\nabla\beta_{a,a}\beta^*_{a,a} = (\nabla\beta_{a,a} \cap \Delta\beta^*_{a,a})\beta^*_{a,a} = (\{a\} \cap \Delta\beta^*_{a,a})\beta^*_{a,a} = \{a\}\beta^*_{a,a} = \{a\} = \nabla\beta_{a,a}$, we have that $\beta_{a,a}\beta^*_{a,a} = \beta_{a,a}$. Hence $\beta^*_{a,a} = (\beta_{a,a}\beta^*_{a,a})^* = \beta_{a,a}\beta^*_{a,a} = \beta_{a,a}$. Similarly, we can show that $\beta^*_{b,b} = \beta_{b,b}$. Now we have that $\alpha_c$, $\beta_{a,a}$ and $\beta_{b,b}$ are projections of $T_X$. By [10, Theorem 2.5], $\beta_{a,a}\alpha_c$ and $\beta_{b,b}\alpha_c$ are idempotents of $T_X$. But $\beta_{a,a}\alpha_c = \beta_{a,c}$ and $\beta_{b,b}\alpha_c = \beta_{b,c}$. It then follows that $c = a$ and $c = b$. It is a contradiction since $a \neq b$. This proves that if $T_X$ is regular-*, then $|X| \leq 1$.

If $|X| \leq 1$, then $T_X = I_X$ which is an inverse semigroup, so it is *-regular. #

The next theorem characterizes full transformation semigroups which are *-semigroup.

2.3 <u>Theorem</u>. For any set X, the full transformation semigroup on X, $\mathcal{T}_X$, is a *-semigroup if and only if $|X| \leq 1$.

<u>Proof</u> : Let the full transformation semigroup, $\mathcal{T}_X$, be a *-semigroup with an involution *. Suppose $|X| \geq 2$. For each $x \in X$, Let $\alpha_x$ be an element of $T_X$ such that $\Delta\alpha_x = X$, $\nabla\alpha_x = \{x\}$. Let a, b be two distinct elements in X. Then $\alpha_a$ and $\alpha_b$ are different elements in $\mathcal{T}_X$. From the definition of $\alpha_x$, $x \in X$, it follows that for all $\beta \in \mathcal{T}_X$, $\beta\alpha_x = \alpha_x$. In particular, $\alpha_a^*\alpha_a = \alpha_a$, $\alpha_b^*\alpha_b = \alpha_b$ and $\alpha_a = \alpha_b\alpha_a$. Therefore, $\alpha_a = \alpha_a^*$ and $\alpha_b = \alpha_b^*$. Thus $\alpha_a = \alpha_b\alpha_a = \alpha_b^*\alpha_a^* = (\alpha_a\alpha_b)^* = \alpha_b^* = \alpha_b$, which is a contradiction. This proves that if $\mathcal{T}_X$ is a *-semigroup, then $|X| \leq 1$.

If $|X| \leq 1$, then $\mathcal{T}_X$ is a trivial semigroup, so it is a *-semigroup. #



2.4 <u>Corollary</u>. For any set X, the full transformation semigroup on X, $\mathcal{T}_X$, is a regular-* semigroup if and only if $|X| \leq 1$.

2.5 <u>Corollary</u>. For any set X, the full transformation semigroup on X, $\mathcal{T}_X$, is a *-regular semigroup if and only if $|X| \leq 1$.

A partial transformation $\alpha$ of a set X is <u>almost identical</u> if $x\alpha \neq x$ for a finite number of elements $x$ in the domain of $\alpha$.

For any set X, the semigroup of almost identical 1-1 partial transformations on X, $W_X$, is an inverse semigroup, so it is regular-* and *-regular.

Let X be a set. Since the semigroup of almost identical partial transformations on X, $U_X$, is a subsemigroup of $T_X$ and the semigroup of almost identical full transformations on X, $V_X$, is a subsemigroup of $\mathcal{T}_X$, it follows that $E(U_X) = E(T_X) \cap U_X$ and $E(V_X) = E(\mathcal{T}_X) \cap V_X$. But

$E(T_X) = \{\alpha \in T_X \mid \nabla\alpha \subseteq \Delta\alpha \text{ and } a\alpha = a \text{ for all } a \in \nabla\alpha\}$ and

$E(\mathcal{T}_X) = \{\alpha \in T_X \mid \Delta\alpha = X \text{ and } a\alpha = a \text{ for all } a \in \nabla\alpha\} =$

$\{\alpha \in \mathcal{T}_X \mid a\alpha = a \text{ for all } a \in \nabla\alpha\}$. Hence $E(U_X) =$

$\{\alpha \in T_X \mid \nabla\alpha \subseteq \Delta\alpha,\ a\alpha = a \text{ for all } a \in \nabla\alpha \text{ and } |\Delta\alpha \setminus \nabla\alpha| < \infty\}$,

$E(V_X) = \{\alpha \in \mathcal{T}_X \mid a\alpha = a \text{ for all } a \in \nabla\alpha \text{ and } |X \setminus \nabla\alpha| < \infty\}$, that is, for $\alpha \in T_X$, $\alpha$ is an idempotent of $U_X$ if and only if $\nabla\alpha \subseteq \Delta\alpha$, $a\alpha = a$ for all $a \in \nabla\alpha$ and $|\Delta\alpha \setminus \nabla\alpha| < \infty$, and for $\alpha \in \mathcal{T}_X$, $\alpha$ is an idempotent of $V_X$ if and only if $a\alpha = a$ for all $a \in \nabla\alpha$ and $|X \setminus \nabla\alpha| < \infty$.

The following theorems characterize the semigroup of almost identical partial transformations and the semigroup of almost identical full transformations on a set X which are regular-* or *-regular.

2.6 <u>Theorem</u>. For any set X, the semigroup of almost identical partial transformations on X, $U_X$, is regular-* if and only if $|X| \leq 1$.

<u>Proof</u> : If $|X| \leq 1$, then $U_X = T_X$, so by Theorem 2.1, $U_X$ is regular-*.

Conversely, assume that $U_X$ is regular-*. Suppose $|X| \geq 2$. For x, y $\in$ X, let $\alpha_{x,y}$ be the element of $T_X$ such that $\Delta\alpha_{x,y} = X$, $\nabla\alpha_{x,y} = X \setminus \{y\}$, $y\alpha_{x,y} = x$ and $z\alpha_{x,y} = z$ for all $z \in X \setminus \{y\}$, and let $\beta_{x,y}$ be the element of $T_X$ such that $\Delta\beta_{x,y} = \{x\}$, $\nabla\beta_{x,y} = \{y\}$. Then for all x, y $\in$ X, $\alpha_{x,y} \in E(U_X)$, and $\beta_{x,y} \in E(U_X)$ if and only if x = y. Let a, b be two distinct elements in X. Then $\alpha_{a,b}$, $\beta_{a,b}$, $\beta_{b,b} \in U_X$ and so $\alpha^*_{a,b}$, $\beta^*_{a,b}$, $\beta^*_{b,b}$ exist in $U_X$. From the proof of Theorem 2.1, we have that $\beta^*_{a,a} = \beta_{a,a}$ and $\beta^*_{b,b} = \beta_{b,b}$. Because $\beta_{a,a}$, $\beta_{b,b} \in E(U_X)$, it follows that $\beta_{a,a}$ and $\beta_{b,b}$ are projections in $U_X$. Claim that $X \setminus \{b\} \subseteq \Delta\alpha^*_{a,b}$, $a\alpha^*_{a,b} = a$ or b and $x\alpha^*_{a,b} = x$ for all $x \in X \setminus \{a, b\}$. If $y \in X \setminus \{b\}$, then $y = y\alpha_{a,b} = y\alpha_{a,b}\alpha^*_{a,b}\alpha_{a,b} = y\alpha^*_{a,b}\alpha_{a,b}$, so $y \in \Delta\alpha^*_{a,b}\alpha_{a,b} \subseteq \Delta\alpha^*_{a,b}$. Thus $X \setminus \{b\} \subseteq \Delta\alpha^*_{a,b}$. Since $a = a\alpha_{a,b} = a\alpha_{a,b}\alpha^*_{a,b}\alpha_{a,b} = (a\alpha^*_{a,b})\alpha_{a,b}$, we have that $a\alpha^*_{a,b} = a$ or b. If $x \in X \setminus \{a, b\}$, then $x = x\alpha_{a,b} = x\alpha_{a,b}\alpha^*_{a,b}\alpha_{a,b} = (x\alpha^*_{a,b})\alpha_{a,b}$, so by the definition of $\alpha_{a,b}$, it follows that $x\alpha^*_{a,b} = x$.

<u>Case $a\alpha^*_{a,b} = a$</u>. Then $\alpha_{a,b}\alpha^*_{a,b} = \alpha_{a,b}$, so $\alpha^*_{a,b} = \alpha_{a,b}$. But $\alpha_{a,b} \in E(U_X)$, it follows that $\alpha_{a,b}$ is a projection in $U_X$. By [10, Theorem 2.5], $\beta_{b,b}\alpha_{a,b}$ is an idempotent in $U_X$. But $\beta_{b,b}\alpha_{a,b} = \beta_{b,a} \notin E(U_X)$. It is a contradiction.

Case $a\alpha^*_{a,b} = b$. Then $\alpha_{a,b}\alpha^*_{a,b} = \alpha_{b,a}$, so $\alpha^*_{b,a} = \alpha_{b,a}$. Thus $\alpha_{b,a}$ is a projection of $U_X$. By [10, Theorem 2.5], $\beta_{a,a}\alpha_{b,a}$ is an idempotent in $U_X$. But $\beta_{a,a}\alpha_{b,a} = \beta_{a,b} \notin E(U_X)$. It is a contradiction.

This proves that if $U_X$ is a regular-* semigroup, then $|X| \leq 1$. #

2.7 Theorem. For any set X, the semigroup of almost identical partial transformations on X, $U_X$, is *-regular if and only if $|X| \leq 1$.

Proof : If $|X| \leq 1$, then $U_X = T_X$, so by Theorem 2.2, $U_X$ is *-regular.

Conversely, assume that $U_X$ is *-regular. Suppose $|X| \geq 2$. For x, y $\in$ X, let $\alpha_{x,y}$ and $\beta_{x,y}$ be elements of $T_X$ defined as in the proof of Theorem 2.6, that is, $\Delta\alpha_{x,y} = X$, $\nabla\alpha_{x,y} = X\setminus\{y\}$, $y\alpha_{x,y} = x$, $z\alpha_{x,y} = z$ for all $z \in X\setminus\{y\}$, $\Delta\beta_{x,y} = \{x\}$ and $\nabla\beta_{x,y} = \{y\}$. For x, y, z $\in$ X, let $\eta_{x,y,z}$ be the element of $T_X$ such that $\Delta\eta_{x,y,z} = \{x, y\}$, $\nabla\eta_{x,y,z} = \{z\}$. Let a, b be two distinct elements in X. Then $\alpha_{a,b}$, $\beta_{a,a}$, $\beta_{b,b} \in U_X$. By the same proof of the proof of Theorem 2.1, we have that $\beta^*_{a,a} = \beta_{a,a}$ and $\beta^*_{b,b} = \beta_{b,b}$. Since $U_X$ is *-regular, $\alpha^\dagger_{a,b}$, $\beta^\dagger_{a,a}$ exist in $U_X$. Claim that $X\setminus\{b\} \subseteq \Delta\alpha^\dagger_{a,b}$, $a\alpha^\dagger_{a,b} = a$ or $b$ and $x\alpha^\dagger_{a,b} = x$ for all $x \in X\setminus\{a, b\}$. If $y \in X\setminus\{b\}$, then $y = y\alpha_{a,b} = y\alpha_{a,b}\alpha^\dagger_{a,b}\alpha_{a,b} = y\alpha^\dagger_{a,b}\alpha_{a,b}$, so $y \in \Delta\alpha^\dagger_{a,b}\alpha_{a,b} \subseteq \Delta\alpha^\dagger_{a,b}$. Thus $X\setminus\{b\} \subseteq \Delta\alpha^\dagger_{a,b}$. Since $a = a\alpha_{a,b} = a\alpha_{a,b}\alpha^\dagger_{a,b}\alpha_{a,b} = (a\alpha^\dagger_{a,b})\alpha_{a,b}$, it implies that $a\alpha^\dagger_{a,b} = a$ or $b$. If $x \in X\setminus\{a, b\}$, then $x = x\alpha_{a,b} = x\alpha_{a,b}\alpha^\dagger_{a,b}\alpha_{a,b} = (x\alpha^\dagger_{a,b})\alpha_{a,b}$ which implies $x\alpha^\dagger_{a,b} = x$.

Case $a\alpha^{\dagger}_{a,b} = a$. Then $\alpha_{a,b}\alpha^{\dagger}_{a,b} = \alpha_{a,b}$. Hence $\alpha_{a,b} = \alpha^*_{a,b}$. It is easily seen that $\eta_{a,b,a} = \alpha_{a,b}\beta_{a,a}$. Thus $\eta^*_{a,b,a} = (\alpha_{a,b}\beta_{a,a})^* = \beta^*_{a,a}\alpha^*_{a,b} = \beta_{a,a}\alpha_{a,b} = \beta_{a,a} = \beta^*_{a,a}$ which implies $\eta_{a,b,a} = \beta_{a,a}$. It is a contradiction because $\Delta\eta_{a,b,a} = \{a, b\}$ but $\Delta\beta_{a,a} = \{a\}$.

Case $a\alpha^{\dagger}_{a,b} = b$. Then $\alpha_{a,b}\alpha^{\dagger}_{a,b} = \alpha_{b,a}$, so $\alpha_{b,a} = \alpha^*_{b,a}$. Since $\eta_{a,b,b} = \alpha_{b,a}\beta_{b,b}$, we have that $\eta^*_{a,b,b} = (\alpha_{b,a}\beta_{b,b})^* = \beta^*_{b,b}\alpha^*_{b,a} = \beta_{b,b}\alpha_{b,a} = \beta_{b,b} = \beta^*_{b,b}$. Hence $\eta_{a,b,b} = \beta_{b,b}$ which is a contradiction because $\Delta\eta_{a,b,b} = \{a, b\}$ and $\Delta\beta_{b,b} = \{b\}$.

This proves that if $U_X$ is a *-regular semigroup, then $|X| \leq 1$. #

2.8 Theorem. For any set X, the semigroup of almost identical full transformations on X, $V_X$, is regular-* if and only if $|X| \leq 1$.

Proof : If $|X| \leq 1$, then $V_X = \mathcal{T}_X$, so by Corollary 2.4, $V_X$ is regular-*.

Conversely, assume that $V_X$ is regular-*. Suppose $|X| \geq 2$. For x, y $\in$ X, let $\alpha_{x,y}$ be the element of $T_X$ defined as in the proof of Theorem 2.6, that is, $\Delta\alpha_{x,y} = X$, $\nabla\alpha_{x,y} = X\setminus\{y\}$, $y\alpha_{x,y} = x$ and $z\alpha_{x,y} = z$ for all $z \in X\setminus\{y\}$. For x, y $\in$ X, let $\lambda_{x,y}$ be the element of $T_X$ such that $\Delta\lambda_{x,y} = X = \nabla\lambda_{x,y}$, $x\lambda_{x,y} = y$, $y\lambda_{x,y} = x$ and $z\lambda_{x,y} = z$ for all $z \in X\setminus\{x, y\}$. Let a, b be two distinct elements in X. Then $\alpha_{a,b}$, $\lambda_{a,b} \in V_X$. Since $V_X$ is a regular-* semigroup, $\alpha^*_{a,b}$, $\lambda^*_{a,b}$ exist in $V_X$. Claim that $\lambda^*_{a,b} = \lambda_{a,b}$. Since $b\lambda_{a,b} = b\lambda_{a,b}\lambda^*_{a,b}\lambda_{a,b} = (a\lambda^*_{a,b})\lambda_{a,b}$, by the definition of $\lambda_{a,b}$, $a\lambda^*_{a,b} = b$. Similarly, we can show that $b\lambda^*_{a,b} = a$. If $x \in X\setminus\{a, b\}$, then

$x\lambda_{a,b} = x\lambda_{a,b}\lambda^*_{a,b}\lambda_{a,b} = x\lambda^*_{a,b}\lambda_{a,b} = (x\lambda^*_{a,b})\lambda_{a,b}$, so $x\lambda^*_{a,b} = x$. Therefore, $\lambda^*_{a,b} = \lambda_{a,b}$.

Next, we claim that $a\alpha^*_{a,b} = a$ or $b$, and $x\alpha^*_{a,b} = x$ for all $x \in X\setminus\{a, b\}$. Since $a = a\alpha_{a,b} = a\alpha_{a,b}\alpha^*_{a,b}\alpha_{a,b} = (a\alpha^*_{a,b})\alpha_{a,b}$, it implies $a\alpha^*_{a,b} = a$ or $b$. For $x \in X\setminus\{a, b\}$, $x = x\alpha_{a,b} = x\alpha_{a,b}\alpha^*_{a,b}\alpha_{a,b}$ $= x\alpha^*_{a,b}\alpha_{a,b} = (x\alpha^*_{a,b})\alpha_{a,b}$ which implies $x\alpha^*_{a,b} = x$.

<u>Case $a\alpha^*_{a,b} = a$.</u> Then $\alpha_{a,b}\alpha^*_{a,b} = \alpha_{a,b}$, so $\alpha^*_{a,b} = \alpha_{a,b}$. But $\lambda_{a,b}\alpha_{a,b}$ $= \alpha_{a,b}$, then $\alpha_{a,b} = \alpha^*_{a,b} = (\lambda_{a,b}\alpha_{a,b})^* = \alpha^*_{a,b}\lambda^*_{a,b} = \alpha_{a,b}\lambda_{a,b} = \alpha_{b,a}$, which is a contradiction since $b\alpha_{a,b} = a$ but $b\alpha_{b,a} = b$.

<u>Case $a\alpha^*_{a,b} = b$.</u> Then $\alpha_{a,b}\alpha^*_{a,b} = \alpha_{b,a}$, so $\alpha^*_{b,a} = \alpha_{b,a}$. Thus $\alpha_{b,a} =$ $\alpha^*_{b,a} = (\lambda_{a,b}\alpha_{b,a})^* = \alpha^*_{b,a}\lambda^*_{a,b} = \alpha_{b,a}\lambda_{a,b} = \alpha_{a,b}$ which is a contradiction because $a\alpha_{b,a} = b$ but $a\alpha_{a,b} = a$.

Hence, the theorem is completely proved. #

2.9 <u>Theorem.</u> For any set X, the semigroup of almost identical full transformations on X, $V_X$, is *-regular if and only if $|X| \leq 1$.

<u>Proof</u> : If $|X| \leq 1$, then $V_X = \mathcal{T}_X$, so by Corollary 2.5, $V_X$ is *-regular.

Conversely, assume that $V_X$ is *-regular. Suppose $|X| \geq 2$. For $x, y \in X$, let $\lambda_{x,y}$ be the element of $T_X$ defined as in the proof of Theorem 2.8; that is, $\Delta\lambda_{x,y} = X = \nabla\lambda_{x,y}$, $x\lambda_{x,y} = y$, $y\lambda_{x,y} = x$, $z\lambda_{x,y} = z$ for all $z \in X\setminus\{x, y\}$. For $x, y, z \in X$, let $\alpha_{x,y,z}$ be the element of $T_X$ such that $\Delta\alpha_{x,y,z} = X$,

$$t\alpha_{x,y,z} = \begin{cases} x & \text{if } t \in \{x, y, z\}, \\ t & \text{otherwise.} \end{cases}$$

Let a, b be two distinct elements in X. Then $\alpha_{a,b,b}$, $\alpha_{b,a,a}$, $\lambda_{a,b} \in V_X$; in particular, $\alpha_{a,b,b}$, $\alpha_{b,a,a} \in E(V_X)$. Since $V_X$ is a *-regular semigroup, $\alpha^\dagger_{a,b,b}$, $\alpha^\dagger_{b,a,a}$, $\lambda^\dagger_{a,b}$ exist in $V_X$. Claim that $\lambda^\dagger_{a,b} = \lambda_{a,b}$. Because $a\lambda_{a,b} = a\lambda_{a,b}\lambda^\dagger_{a,b}\lambda_{a,b} = (b\lambda^\dagger_{a,b})\lambda_{a,b}$ and $\lambda_{a,b}$ is one-to-one, $b\lambda^\dagger_{a,b} = a$. Similarly, $a\lambda^\dagger_{a,b} = b$. If $x \in X\setminus\{a, b\}$, then $x = x\lambda_{a,b} = x\lambda_{a,b}\lambda^\dagger_{a,b}\lambda_{a,b} = (x\lambda^\dagger_{a,b})\lambda_{a,b}$, so $x\lambda^\dagger_{a,b} = x$. Thus $\lambda^\dagger_{a,b} = \lambda_{a,b}$.

Next, we show that $a\alpha^\dagger_{a,b,b} = a$ or $b$, $x\alpha^\dagger_{a,b,b} = x$ for all $x \in X\setminus\{a, b\}$. Since $a = a\alpha_{a,b,b} = a\alpha_{a,b,b}\alpha^\dagger_{a,b,b}\alpha_{a,b,b} = (a\alpha^\dagger_{a,b,b})\alpha_{a,b,b}$, by the definition of $\alpha_{a,b,b}$, we have that $a\alpha^\dagger_{a,b,b} =$ a or b. If $x \in X\setminus\{a, b\}$, then $x\alpha_{a,b,b} = x\alpha_{a,b,b}\alpha^\dagger_{a,b,b}\alpha_{a,b,b} = (x\alpha^\dagger_{a,b,b})\alpha_{a,b,b}$, hence $x\alpha^\dagger_{a,b,b} = x$. Similarly, we can show that $b\alpha^\dagger_{b,a,a} =$ a or b, and $x\alpha^\dagger_{b,a,a} = x$ for all $x \in X\setminus\{a, b\}$.

<u>Case $a\alpha^\dagger_{a,b,b} = a$ and $b\alpha^\dagger_{b,a,a} = b$.</u> Then $\alpha_{a,b,b}\alpha^\dagger_{a,b,b} = \alpha_{a,b,b}$, $\alpha_{b,a,a}\alpha^\dagger_{b,a,a} = \alpha_{b,a,a}$, so $\alpha^*_{a,b,b} = \alpha_{a,b,b}$ and $\alpha^*_{b,a,a} = \alpha_{b,a,a}$. But $\alpha_{a,b,b} = \alpha_{b,a,a}\alpha_{a,b,b}$ and $\alpha_{a,b,b}\alpha_{b,a,a} = \alpha_{b,a,a}$, then $\alpha_{a,b,b} = \alpha^*_{b,a,a}\alpha^*_{a,b,b} = (\alpha_{a,b,b}\alpha_{b,a,a})^* = \alpha^*_{b,a,a} = \alpha_{b,a,a}$ which is a contradiction because $b\alpha_{a,b,b} = a$ but $b\alpha_{b,a,a} = b$.

<u>Case $a\alpha^\dagger_{a,b,b} = b$ and $b\alpha^\dagger_{b,a,a} = a$.</u> Then $\alpha_{a,b,b}\alpha^\dagger_{a,b,b} = \alpha_{b,a,a}$, $\alpha_{b,a,a}\alpha^\dagger_{b,a,a} = \alpha_{a,b,b}$, so $\alpha^*_{b,a,a} = \alpha_{b,a,a}$ and $\alpha^*_{a,b,b} = \alpha_{a,b,b}$. Thus $\alpha_{a,b,b} = \alpha_{b,a,a}\alpha_{a,b,b} = \alpha^*_{b,a,a}\alpha^*_{a,b,b} = (\alpha_{a,b,b}\alpha_{b,a,a})^* = \alpha^*_{b,a,a} = \alpha_{b,a,a}$, a contradiction.

<u>Case $a\alpha^\dagger_{a,b,b} = a$ and $b\alpha^\dagger_{b,a,a} = a$.</u> Then $\alpha_{a,b,b}\alpha^\dagger_{a,b,b} = \alpha_{a,b,b}$, so $\alpha^*_{a,b,b} = \alpha_{a,b,b}$. Hence $\alpha^\dagger_{a,b,b} = \alpha^\dagger_{a,b,b}\alpha_{a,b,b}\alpha^\dagger_{a,b,b} = \alpha^\dagger_{a,b,b}\alpha_{a,b,b}$,

so $(\alpha^\dagger_{a,b,b})^* = \alpha^\dagger_{a,b,b}$. It follows that $\alpha^\dagger_{a,b,b} = (\alpha^\dagger_{a,b,b})^* = (\alpha^\dagger_{a,b,b}\alpha_{a,b,b})^* = \alpha^*_{a,b,b}(\alpha^\dagger_{a,b,b})^* = \alpha_{a,b,b}\alpha^\dagger_{a,b,b} = \alpha_{a,b,b}$. If $a\alpha^\dagger_{b,a,a} = a$, then $\alpha^\dagger_{b,a,a} = \alpha_{a,b,b} = \alpha^\dagger_{a,b,b}$, so $\alpha_{b,a,a} = \alpha_{a,b,b}$ which is a contradiction. If $a\alpha^\dagger_{b,a,a} = b$, then $\alpha^\dagger_{b,a,a} = \lambda_{a,b} = \lambda^\dagger_{a,b}$, so $\alpha_{b,a,a} = \lambda_{a,b}$ which is a contradiction. Then $a\alpha^\dagger_{b,a,a} = c$ for some $c \in X\setminus\{a, b\}$. Thus $\alpha^\dagger_{b,a,a}\alpha_{b,a,a} = \alpha_{c,a,a}$, so $\alpha^*_{c,a,a} = \alpha_{c,a,a}$. Since $b = b\alpha_{b,c,c}\alpha^\dagger_{b,c,c}\alpha_{b,c,c} = (b\alpha^\dagger_{b,c,c})\alpha_{b,c,c}$, it follows that $b\alpha^\dagger_{b,c,c} = b$ or $c$. If $x \in X\setminus\{b, c\}$, then $x\alpha_{b,c,c} = x\alpha_{b,c,c}\alpha^\dagger_{b,c,c}\alpha_{b,c,c} = (x\alpha^\dagger_{b,c,c})\alpha_{b,c,c}$, hence $x\alpha^\dagger_{b,c,c} = x$.

If $b\alpha^\dagger_{b,c,c} = c$, then $\alpha_{b,c,c}\alpha^\dagger_{b,c,c} = \alpha_{c,b,b}$, and therefore $\alpha_{c,b,b} = \alpha^*_{c,b,b} = (\alpha_{c,b,b}\alpha_{a,b,b})^* = \alpha^*_{a,b,b}\alpha^*_{c,b,b} = \alpha_{a,b,b}\alpha_{c,b,b} = \alpha_{a,b,b}$, which is a contradiction since $c \neq a$.

Assume that $b\alpha^\dagger_{b,c,c} = b$. Then $\alpha_{b,c,c}\alpha^\dagger_{b,c,c} = \alpha_{b,c,c}$, so $\alpha^*_{b,c,c} = \alpha_{b,c,c}$ and hence $\alpha^*_{a,b,c} = (\alpha_{b,c,c}\alpha_{a,b,b})^* = \alpha^*_{a,b,b}\alpha^*_{b,c,c} = \alpha_{a,b,b}\alpha_{b,c,c}$. Thus $c\alpha^*_{a,b,c} = b$. Since $\alpha_{c,a,b} = \alpha_{a,b,c}\alpha_{c,a,a}$, $\alpha^*_{c,a,b} = \alpha^*_{c,a,a}\alpha^*_{a,b,c} = \alpha_{c,a,a}\alpha_{a,b,b}\alpha_{b,c,c}$. But $\alpha_{a,b,c} = \alpha_{c,a,b}\alpha_{a,b,c}$, so $\alpha^*_{a,b,c} = \alpha^*_{a,b,c}\alpha^*_{c,a,b} = \alpha^*_{a,b,c}\alpha_{c,a,a}\alpha_{a,b,b}\alpha_{b,c,c}$. Hence $b = c\alpha^*_{a,b,c} = c\alpha^*_{a,b,c}\alpha_{c,a,a}\alpha_{a,b,b}\alpha_{b,c,c} = b\alpha_{c,a,a}\alpha_{a,b,b}\alpha_{b,c,c} = a$, a contradiction.

Case $a\alpha^\dagger_{a,b,b} = b$ and $b\alpha^\dagger_{b,a,a} = b$. A proof to get a contradiction can be given identically to the proof for the case $a\alpha^\dagger_{a,b,b} = a$ and $b\alpha^\dagger_{b,a,a} = a$, only replacing a by b and b by a; respectively.

Hence, the theorem is completely proved. #