CHAPTER 6

Non-local Harmonic Oscillator

In this chapter we apply our techniques to the calculation of the non-local harmonic oscillator propagator. In the first section we discuss briefly the non-local harmonic oscillator propagator. In sec. 6.2 we calculate the non-local harmonic oscillator propagator using our techniques. The conclusions and discussions are presented in the last section.

6.1 Preliminary

In chapter 1 we mentioned the non-local harmonic oscillator path integral and in chapter 3 we presented the Feynman's method of solving the Gaussian Path Integral. Since the non-local hrmonic oscillator path integral is the Gaussian Path Integral it can be written as the product of the prefactor and the exponent of the classical action. In order to obtain the non-local harmonic oscillator propagator one has to solve for the classical action and the prefactor.

To obtain the classical action one needs to find the classical path which can be obtained by making a variation on the action function in eq. (1.2) and then obtain a classical equation of motion. Unfortunately, this equation has the form of an integro-differential equation. To solve this equation Sa-yakanit (9) introduced the equation of motion for the non-local harmonic oscillator in the external field f(t) ;

$$\ddot{x}(t) + \frac{\omega^2}{T}\int_0^T d\sigma\,[x(t) - x(\sigma)] = \frac{f(t)}{m} \qquad (6.1)$$

with the boundary condition $x(0) = x_a$, $x(T) = x_b$.

By rearranging eq. (6.1) as

$$\left[\frac{d^2}{dt^2} + \omega^2\right] x(t) = \frac{\omega^2}{T}\int_0^T x(\sigma)\,d\sigma + \frac{f(t)}{m} \qquad (6.2)$$

he obtained the compound solution of the eq. (6.2) as

$$x_c(t) = \frac{1}{\sin\omega T}\left[x_b \sin\omega t + x_a \sin\omega(T-t)\right] \qquad (6.3)$$

The particular solution of the eq. (6.2) can be obtained by introducing the Green function which

$$\left[\frac{d^2}{dt^2} + \omega^2\right] g(t;\sigma) = \delta(t-\sigma) \qquad (6.4)$$

where $g(0\,;\sigma) = g(T\,;\sigma) = 0$. The particular solution can be written in the form

$$x_p(t) = \int_0^T d\sigma\, g(T;\sigma)\left[\frac{\omega^2}{T}\int_0^T d\sigma\, x(\sigma) + \frac{f(\sigma)}{m}\right] \qquad (6.5)$$

where $g(t\,;\sigma) = \frac{-1}{\omega \sin\omega T}\left[\sin\omega(T-t)\sin\omega\sigma\, H(t-\sigma) + \sin\omega(T-\sigma)\sin\omega t\, H(\sigma-t)\right]$

whith H denotes the heaviside step function.

The solution of eq. (6.1) can be written as

$$x(t) = x_c(t) + x_p(t)$$

$$= \frac{1}{\sin\omega T}\left[x_b \sin\omega t + x_a \sin\omega(T-t)\right] + \int_0^T d\sigma\, g(t;\sigma)\left[\frac{\omega^2}{T}\int_0^T d\sigma\, x(\sigma) + \frac{f(\sigma)}{m}\right] \qquad (6.6)$$

From this equation we see that by performing the integration over t, we obtain

$$\int_0^T x(\sigma)d\sigma = \left[\int_0^T x_c(t)dt + \int_0^T dt\int_0^T d\xi\, g(t;\xi)\frac{f(\xi)}{m}\right] \times \left[1 - \frac{\omega^2}{T}\int_0^T dt\int_0^T d\xi\, g(t;\xi)\right]^{-1}$$

$$= \frac{T}{2}\left[x_a + x_b\right] + \frac{T}{m\omega \sin\frac{\omega T}{2}}\int_0^T \sin\frac{\omega\xi}{2}\sin\frac{\omega}{2}(\xi - T)\, f(\xi)\, d\xi \qquad (6.7)$$

The classical path can be written as

$$x(t) = \frac{1}{\sin\omega T}\Big[\left\{x_b\sin\omega t + x_a\sin\omega(T-t)\right\} - 2\sin\frac{\omega t}{2}\sin\frac{\omega}{2}(T-t) \times \left\{(x_a+x_b)\sin\frac{\omega T}{2} - \frac{2}{m\omega}\int_0^T f(\sigma)\sin\frac{\omega\sigma}{2}\sin\frac{\omega}{2}(T-\sigma)d\sigma\right\}\Big] + \frac{1}{m}\int_0^T f(\sigma)g(t;\sigma)d\sigma \qquad (6.8)$$

Since we know the classical path, the classical action can be obtained by substituting the eq. (6.8) into the action function

$$S_{cl} = \frac{m}{2}\left[\int_0^T \dot{x}^2(t)\,dt - \frac{\omega^2}{2T}\int_0^T dt\int_0^T d\sigma\left[x(t)-x(\sigma)\right]^2\right] + \int_0^T f(t)\,x(t)\,dt \tag{6.9}$$

The classical action now has the form

$$\begin{aligned}S_{cl} = {} & \frac{m\omega}{4}\cot\frac{\omega T}{2}\left[x_b - x_a\right]^2 + \frac{x_b}{\sin\omega T}\int_0^T d\sigma\, f(\sigma)\Big[\sin\omega\sigma - 2\sin\frac{\omega}{2}(T-\sigma) \\ & \times \sin\frac{\omega\sigma}{2}\sin\frac{\omega T}{2}\Big] + \frac{x_a}{\sin\omega T}\int_0^T d\sigma\, f(\sigma)\Big[\sin\omega(T-\sigma) - 2\sin\frac{\omega\sigma}{2}\sin\frac{\omega}{2}(T-\sigma) \\ & \times \sin\frac{\omega T}{2}\Big] + \frac{1}{m\omega\sin\omega T}\int_0^T\int_0^t dt\,d\sigma\, f(t)f(\sigma)\Big[4\sin\frac{\omega}{2}(T-t)\sin\frac{\omega t}{2} \\ & \times \sin\frac{\omega\sigma}{2}\sin\frac{\omega}{2}(T-\sigma) - \sin\omega(T-t)\sin\omega\sigma\Big]\end{aligned} \tag{6.10}$$

the required classical action can be obtained by setting $f(t) = 0$ in eq. (6.10) ;

$$S_{cl} = \frac{m\omega}{4}\cot\frac{\omega T}{2}\left[x_b - x_a\right]^2 \tag{6.11}$$

To obtain the prefactor Sa-yakanit (9) generated the non-local harmonic oscillator path integral from a shifted-origin of the simple harmonic oscillator propagator and used the free-particle limit. He found the prefactor of the non-local harmonic oscillator propagator to be

$$F(T) = \left[\frac{m}{2\pi i\hbar T}\right]^{1/2}\left[\frac{\omega T}{\sin\omega T}\right]^{1/2}\left[\frac{\omega T}{2\tan\frac{\omega T}{2}}\right]^{1/2} \tag{6.12}$$

The non-local harmonic oscillator propagator can then be written as

$$K(x_b,T;x_a,0) = \left[\frac{m}{2\pi i\hbar T}\right]^{1/2}\left[\frac{\omega T}{\sin\omega T}\right]^{1/2}\left[\frac{\omega T}{2\tan\frac{\omega T}{2}}\right]^{1/2}\exp\left\{\frac{im\omega}{4\hbar}\cot\frac{\omega T}{2}\left[x_b-x_a\right]^2\right\} \tag{6.13}$$

In the following section we shall calculate the non-local harmonic oscillator propagator by using our techniques.

6.2 Calculating the Non-local Harmonic Oscillator Propagator

Since we are restricted ourselves to the discrete-time assumption, we can rewrite eq. (1.1) as

$$K(x_b,T;x_a,0) = \lim_{N\to\infty}\left(\frac{1}{A}\right)^N\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dx_0\ldots dx_N\exp\left\{\frac{-m}{2i\hbar\varepsilon}\left[\sum_{j=1}^{N}(x_j-x_{j-1})^2-\omega^2\varepsilon^2\times\sum_{j=0}^{N}x_j^2\right]-\frac{m\omega^2\varepsilon^2}{2i\hbar T}\left(\sum_{j=0}^{N}x_j\right)^2\right\}\delta(x_0-x_a)\,\delta(x_N-x_b) \tag{6.14}$$

By representing the paths as a cosine series we have

$$x_j = \frac{1}{\sqrt{N\varepsilon}}a_0 + \sqrt{\frac{2}{N\varepsilon}}\sum_{n=1}^{N-1}a_n\cos\frac{n\pi j}{N} + \frac{(-1)^j}{\sqrt{N\varepsilon}}a_N \tag{6.15}$$

The kinetic-energy and the potential-enerygy terms can be transformed to become (the derivations are presented in Appendix A)

$$\sum_{j=1}^{N}(x_j-x_{j-1})^2 = \sum_{m=0}^{N}\left(\frac{4}{\epsilon}\sin^2\frac{m\pi}{2N}\right)a_m^2 \qquad (6.16)$$

and

$$\sum_{j=0}^{N} x_j^2 = \sum_{n=0}^{N}\frac{1}{\epsilon}a_m^2 \qquad (6.17)$$

For the memory terms we have

$$\sum_{j=0}^{N} x_j = \frac{1}{\sqrt{N\epsilon}}\sum_{j=0}^{N} a_0 + \sqrt{\frac{2}{N\epsilon}}\sum_{m=1}^{N} a_m\left[\sum_{j=0}^{N}\cos\frac{m\pi}{N}j\right] + \frac{a_N}{\sqrt{N\epsilon}}\sum_{j=0}^{N}(-1)^j \qquad (6.18)$$

Using the identity

$$\sum_{j=0}^{N}\cos jx = \cos\frac{Nx}{2}\sin(N+1)\frac{x}{2}\csc\frac{x}{2} \qquad (6.19)$$

We get

$$\sum_{j=0}^{N}\cos j\frac{m\pi}{N} = \cos^2\frac{m\pi}{2} \qquad (6.20)$$

Eq. (6.18) can now be written as

$$\sum_{j=0}^{N} x_j = \frac{1}{\sqrt{N\epsilon}}(N+1)a_0 + \sqrt{\frac{2}{N\epsilon}}\sum_{m=1}^{N-1} a_m\cos^2\frac{m\pi}{2} + \frac{O(N)\,a_N}{\sqrt{N\epsilon}} \qquad (6.21)$$

where

$$O(N) = \begin{cases} 1 & \text{for } N = \text{even integer} \\ 0 & \text{for } N = \text{odd integer} \end{cases} \qquad (6.22)$$

The memory terms become

$$\left(\sum_{j=0}^{N} x_j\right)^2 = \frac{1}{T}\left[(N+1)a_0 + \sqrt{2}\sum_{n\text{-ev.}} a_{n\text{-ev.}} + O(N)\,a_N\right]^2 \qquad (6.23)$$

Eq. (6.14) can be further transformed to become

$$K(x_b,T;x_a,0) = \lim_{N\to\infty}\left(\frac{1}{A}\right)^N J\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} da_0\cdots da_N \exp\Big\{-\frac{m}{2i\hbar\epsilon^2}\sum_{n=0}^{N}\Big[4\sin^2\frac{n\pi}{2N} - \omega^2\epsilon^2\Big]a_n^2$$

$$-\frac{m\omega^2\epsilon^2}{2i\hbar T^2}\Big[(N+1)a_0 + \sqrt{2}\sum_{n\text{-ev.}} a_{n\text{-ev.}} + O(N)\,a_N\Big]\Big\}\,\delta\Big(\frac{1}{\sqrt{T}}\Big[a_0 + \sqrt{2}\sum_{n=1}^{N-1} a_n$$

$$+ a_N - \sqrt{T}x_a\Big]\Big)\,\delta\Big(\frac{1}{\sqrt{T}}\Big[a_0 + \sqrt{2}\sum_{n=1}^{N-1}(-1)^n a_n + (-1)^N a_N - \sqrt{T}x_b\Big]\Big) \qquad (6.24)$$

where $J = 2/(\epsilon)^{\frac{N+1}{2}}$ is the jacobian of transformation (the derivations are presented in Appendix B).

Using the properties of Dirac delta function eqs. (5.7) and (5.8) we can linearize the memory terms by using the fact that

$$\exp\Big\{-\frac{m\omega^2\epsilon^2}{2i\hbar T^2}\Big[(N+1)a_0 + \sqrt{2}\sum_{n\text{-ev.}} a_{n\text{-ev.}} + O(N)\,a_N\Big]\Big\}$$

$$= \left[\frac{-i\hbar T^2}{2\pi m\omega^2\epsilon^2}\right]^{1/2}\int_{-\infty}^{\infty} dk \exp\Big\{\frac{i\hbar T^2 k^2}{2m\omega^2\epsilon^2} + \Big[(N+1)a_0 + \sqrt{2}\sum_{n\text{-ev.}} a_{n\text{-ev.}} + O(N)\,a_N\Big]k\Big\} \qquad (6.25)$$

Thus we get

$$K(x_b,T;x_a,0) = \lim_{N\to\infty}\left(\frac{1}{A}\right)^N J\,\frac{T}{4\pi^2}\left[\frac{-i\hbar T^2}{2\pi m\omega^2\epsilon^2}\right]^{1/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} da_0\cdots da_N\,dk\,dp\,dq\,\exp\Big\{\frac{i\hbar T^2 k^2}{2m\omega^2\epsilon^2}$$

$$-ip\sqrt{T}x_a - iq\sqrt{T}x_b - \frac{m}{2i\hbar\epsilon^2}\sum_{n=0}^{N}\left(4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right)a_n^2 + \big[(N+1)k$$

$$+i(p+q)\big]a_0 + \left[O(N)k + i(p+(-1)^N q)\right]a_N + \sqrt{2}\left[i(p+q)+k\right]$$

$$\times\sum_{n\text{-}ev.} a_{n\text{-}ev.} + \sqrt{2}\,i(p-q)\sum_{n\text{-}odd} a_{n\text{-}odd}\Big\} \qquad (6.26)$$

Since the exponent can be separated into factors, the integral over each a_n can be done separately. The results of such integrations are

$$\int_{-\infty}^{\infty} da_0\exp\Big\{\frac{m\omega^2}{2i\hbar}a_0 + \left[(N+1)k + i(p+q)\right]a_0\Big\}$$

$$= \left[\frac{-2\pi i\hbar}{m\omega^2}\right]^{1/2}\exp\Big\{\frac{-i\hbar}{2m\omega^2}\left[(N+1)k + i(p+q)\right]^2\Big\} \qquad (6.27)$$

$$\int_{-\infty}^{\infty} da_N\exp\Big\{\frac{-m}{2i\hbar\epsilon^2}\left[4-\omega^2\epsilon^2\right]a_N^2 + \left[i(p+(-1)^N q) + k\,O(N)\right]a_N\Big\}$$

$$= \left[\frac{2\pi i\hbar\epsilon^2}{m[4-\omega^2\epsilon^2]}\right]^{1/2}\exp\Big\{\frac{i\hbar\epsilon^2}{2m[4-\omega^2\epsilon^2]}\left[i(p+(-1)^N q) + O(N)k\right]\Big\} \qquad (6.28)$$

$$\int_{-\infty}^{\infty} da_{n\text{-}ev.}\exp\Big\{\frac{-m}{2\pi i\hbar\epsilon^2}\left[4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right]a_{n\text{-}ev.}^2 + \sqrt{2}\left[i(p+q)+k\right]a_{n\text{-}ev.}\Big\}$$

$$= \left[\frac{2\pi i\hbar\epsilon^2}{m\left[4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right]}\right]^{1/2}\exp\Big\{\frac{i\hbar\epsilon^2}{m}\left[i(p+q)+k\right]^2\left[4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right]^{-1}\Big\}_{n\text{-}ev.} \qquad (6.29)$$

$$\int_{-\infty}^{\infty} da_{n\text{-}odd}\exp\Big\{\frac{-m}{2\pi i\hbar\epsilon^2}\left[4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right]a_{n\text{-}odd}^2 + \sqrt{2}\left[i(p-q)\right]a_{n\text{-}odd}\Big\}$$

$$= \left[\frac{2\pi i\hbar\epsilon^2}{m\left[4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right]}\right]^{1/2}\exp\Big\{\frac{-i\hbar\epsilon^2}{m}(p-q)^2\left[4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right]^{-1}\Big\} \qquad (6.30)$$

Thus

$$K(x_b,T;x_a,0) = \lim_{N\to\infty}\left(\frac{1}{A}\right)^N J\,\frac{T}{4\pi^2}\left[\frac{-i\hbar T^2}{2\pi m\omega^2\epsilon^2}\right]^{1/2}\left[\frac{-2\pi i\hbar}{m\omega^2}\right]^{1/2}\left[\frac{2\pi i\hbar\epsilon^2}{m[4-\omega^2\epsilon^2]}\right]^{1/2}$$

$$\times\prod_{n=1}^{N-1}\left[\frac{2\pi i\hbar\epsilon^2}{m[4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2]}\right]^{1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp\,dq\,dk\,\exp\Big\{\frac{i\hbar T^2k^2}{2m\omega^2\epsilon^2}$$

$$-i\sqrt{T}x_a p - i\sqrt{T}x_b q - \frac{i\hbar}{2m\omega^2}\left[(N+1)k + i(p+q)\right]^2 + \frac{i\hbar\epsilon^2}{m[4-\omega^2\epsilon^2]}$$

$$\times\left[i(p+(-1)^N q) + O(N)k\right]^2 + \frac{i\hbar\epsilon^2}{m}\left[i(p+q)+k\right]^2\sum_{n-ev.}\left[4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right]^{-1}$$

$$-\frac{i\hbar\epsilon^2}{m}(p-q)^2\sum_{n-odd}\left[4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right]^{-1}\Big\} \qquad (6.31)$$

Before performing the integration over p, q, and k we take the limit of N approaching infinity, using eqs. (5.19) and (5.20) we obtain

$$\lim_{N\to\infty}\frac{i\hbar\epsilon^2}{m}\left[i(p+q)+k\right]^2\sum_{n-ev.}\left[4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right]^{-1} = \left[i(p+q)+k\right]^2\left[\frac{i\hbar}{2m\omega^2} - \frac{i\hbar T}{4m\omega}\cot\frac{\omega T}{2}\right] \qquad (6.32)$$

and

$$\lim_{N\to\infty}\frac{i\hbar\epsilon^2}{m}(p-q)^2\sum_{n-odd}\left[4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right]^{-1} = \frac{i\hbar T}{4m\omega}(p-q)^2\tan\frac{\omega T}{2} \qquad (6.33)$$

The integral parts of eq. (6.31) become

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp\,dq\,dk\,\exp\Big\{\frac{i\hbar T^2k^2}{2m\omega^2\epsilon^2} - i\sqrt{T}x_a p - i\sqrt{T}x_b q - \frac{i\hbar}{2m\omega^2}\left[(N+1)k + i(p+q)\right]^2$$

$$+\frac{i\hbar\epsilon^2}{m[4-\omega^2\epsilon^2]}\left[i(p+(-1)^N q) + O(N)k\right] - \frac{i\hbar T}{4m\omega}(p-q)^2\tan\frac{\omega T}{2}$$

$$+\left[i(p+q)+k\right]^2\left[\frac{i\hbar}{2m\omega^2} - \frac{i\hbar T}{4m\omega}\cot\frac{\omega T}{2}\right]\Big\} \qquad (6.34)$$

We now rearrange eq. (6.34) as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp\,dq\,dk\,\exp\Big\{-\Big[\frac{i\hbar N}{m\omega^2}+\frac{i\hbar T}{4m\omega}\cot\frac{\omega T}{2}\Big]k^2+\Big[\frac{\hbar N}{m\omega^2}+\frac{\hbar T}{2m\omega}\cot\frac{\omega T}{2}\Big](p+q)k$$

$$+\frac{i\hbar T}{4m\omega}\cot\frac{\omega T}{2}(p+q)^2-\frac{i\hbar T}{4m\omega}\tan\frac{\omega T}{2}(p-q)^2 \quad -i\sqrt{T}x_a p-i\sqrt{T}x_b q\Big\} \qquad (6.35)$$

Performing the integration over k ;

$$\int_{-\infty}^{\infty} dk\,\exp\Big\{-\Big[\frac{i\hbar N}{m\omega^2}+\frac{i\hbar T}{4m\omega}\cot\frac{\omega T}{2}\Big]k^2+\Big[\frac{\hbar N}{m\omega^2}+\frac{\hbar T}{2m\omega}\cot\frac{\omega T}{2}\Big](p+q)k\Big\}$$

$$=\left[\frac{\pi m\omega^2}{i\hbar N\big[1+\frac{\omega T}{4N}\cot\frac{\omega T}{2}\big]}\right]^{1/2}\exp\Big\{\Big[\Big(\frac{\hbar N}{m\omega^2}\Big)^2+2\Big(\frac{\hbar N}{m\omega^2}\Big)\Big(\frac{\hbar T}{2m\omega}\cot\frac{\omega T}{2}\Big)+\Big(\frac{\hbar T}{2m\omega}\cot\frac{\omega T}{2}\Big)^2\Big]$$

$$\times(p+q)^2\frac{m\omega^2}{4i\hbar N\big[1+\frac{\omega T}{4N}\cot\frac{\omega T}{2}\big]}\Big\}$$

$$=\left[\frac{\pi m\omega^2}{i\hbar N}\right]^{1/2}\exp\Big\{-\frac{i\hbar N}{4m\omega^2}(p+q)^2-\frac{i\hbar T}{4m\omega}\cot\frac{\omega T}{2}(p+q)^2-\frac{im\omega^2}{4\hbar N}\Big(\frac{\hbar T}{2m\omega}\cot\frac{\omega T}{2}\Big)\Big\}$$

(6.36)

Then eq. (6.35) becomes

$$\left[\frac{\pi m\omega^2}{i\hbar N}\right]^{1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp\,dq\,\exp\Big\{-\Big[\frac{i\hbar N}{4m\omega^2}+\frac{i\hbar T}{4m\omega}\tan\frac{\omega T}{2}\Big]p^2$$

$$-\Big[\frac{i\hbar N}{4m\omega^2}+\frac{i\hbar T}{4m\omega}\tan\frac{\omega T}{2}\Big]q^2-\Big[\Big(\frac{i\hbar N}{2m\omega^2}-\frac{i\hbar T}{2m\omega}\tan\frac{\omega T}{2}\Big)q$$

$$+i\sqrt{T}x_a\Big]p \quad -i\sqrt{T}x_b q\Big\} \qquad (6.37)$$

Performing the integration over p ;

$$\int_{-\infty}^{\infty} dp \exp\left\{ -\left[\frac{i\hbar N}{4m\omega^2} + \frac{i\hbar T}{4m\omega} \tan\frac{\omega T}{2} \right] p^2 - \left[\left(\frac{i\hbar N}{4m\omega^2} - \frac{i\hbar T}{4m\omega} \tan\frac{\omega T}{2} \right) 2g \right.\right.$$

$$\left.\left. + i\sqrt{T} x_a \right] p \right\}$$

$$= \left[\frac{4\pi m\omega^2}{i\hbar N\left[1 + \frac{\omega T}{N}\tan\frac{\omega T}{2}\right]} \right]^{1/2} \exp\left\{ \left[\left(\frac{i\hbar N}{4m\omega^2} - \frac{i\hbar T}{4m\omega}\tan\frac{\omega T}{2} \right) 2g + i\sqrt{T} x_a \right]^2 \right.$$

$$\left. \times \left[\frac{i\hbar N}{m\omega^2} + \frac{i\hbar T}{m\omega}\tan\frac{\omega T}{2} \right]^{-1} \right\}$$

$$= \left[\frac{4\pi m\omega^2}{i\hbar N} \right]^{1/2} \exp\left\{ \left[\frac{1}{4} g^2 \left(\frac{i\hbar N}{m\omega^2} - \frac{i\hbar T}{m\omega}\tan\frac{\omega T}{2} \right) + i\sqrt{T} x_a g \left(\frac{i\hbar N}{m\omega^2} \right.\right.\right.$$

$$\left.\left.\left. - \frac{i\hbar T}{m\omega}\tan\frac{\omega T}{2} \right) - T x_a^2 \right] \left[\frac{i\hbar N}{m\omega^2} + \frac{i\hbar T}{m\omega}\tan\frac{\omega T}{2} \right]^{-1} \right\}$$

$$= \left[\frac{4\pi m\omega^2}{i\hbar N} \right]^{1/2} \exp\left\{ \left[\frac{1}{4} g^2 \left[\left(\frac{i\hbar N}{m\omega^2} + \frac{i\hbar T}{m\omega}\tan\frac{\omega T}{2} \right)^2 - 4\left(\frac{i\hbar N}{m\omega^2} \right) \frac{i\hbar T}{m\omega}\tan\frac{\omega T}{2} \right] \right.\right.$$

$$\left.\left. + i\sqrt{T} x_a g \left(\frac{i\hbar N}{m\omega^2} - \frac{i\hbar T}{m\omega}\tan\frac{\omega T}{2} \right) - T x_a^2 \right] \left(\frac{i\hbar N}{m\omega^2} + \frac{i\hbar T}{m\omega}\tan\frac{\omega T}{2} \right)^{-1} \right\}$$

$$= \left[\frac{4\pi m\omega^2}{i\hbar N} \right]^{1/2} \exp\left\{ \frac{1}{4} g^2 \left[\frac{i\hbar N}{m\omega^2} - 3\frac{i\hbar T}{m\omega}\tan\frac{\omega T}{2} \right] + i\sqrt{T} x_a g \right\} \tag{6.38}$$

Then eq. (6.37) becomes

$$\left[\frac{\pi m\omega^2}{i\hbar N}\right]^{1/2}\left[\frac{4\pi m\omega^2}{i\hbar N}\right]^{1/2}\int_{-\infty}^{\infty} dg\, \exp\left\{ -\frac{i\hbar T}{m\omega}\tan\frac{\omega T}{2}\, g^2 + i\sqrt{T}(x_b - x_a)\, g \right\}$$

$$= \left[\frac{\pi m\omega^2}{i\hbar N}\right]^{1/2}\left[\frac{4\pi m\omega^2}{i\hbar N}\right]^{1/2}\left[\frac{\pi m\omega}{i\hbar T\tan\frac{\omega T}{2}}\right]^{1/2}\exp\left\{ \frac{im\omega}{4\hbar}\cot\frac{\omega T}{2}(x_b - x_a)^2 \right\} \tag{6.39}$$

On substituting $(1/A) = (m/2\pi i\hbar\epsilon)^{1/2}$, $J = \frac{2}{(\epsilon)^{\frac{N+1}{2}}}$ and eq. (6.39) into the eq. (6.31) we obtain

$$K(x_b,T;x_a,0) = \lim_{N\to\infty}\left[\frac{m}{2\pi i\hbar\epsilon}\right]^{\frac{N}{2}}\frac{2}{(\epsilon)^{\frac{N+1}{2}}}\,\frac{T}{4\pi^2}\left[\frac{-i\hbar T^2}{2\pi m\omega^2\epsilon^2}\right]^{1/2}\left[\frac{-2\pi i\hbar}{m\omega^2}\right]^{1/2}$$

$$\times\left[\frac{2\pi i\hbar\epsilon^2}{m(4-\omega^2\epsilon^2)}\right]^{1/2}\left[\frac{\pi m\omega^2}{i\hbar N}\right]^{1/2}\left[\frac{4\pi m\omega}{i\hbar N}\right]^{1/2}\left[\frac{\pi m\omega}{i\hbar T\tan\frac{\omega T}{2}}\right]^{1/2}$$

$$\times\prod_{n=1}^{N-1}\left[\frac{2\pi i\hbar\epsilon^2}{m\left(4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right)}\right]^{1/2}\exp\left\{\frac{im\omega}{4\hbar}\cot\frac{\omega T}{2}(x_b-x_a)^2\right\} \tag{6.40}$$

Since we know from the previous chapter that

$$\lim_{N\to\infty}\prod_{n=1}^{N-1}\left[4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right]^{-1/2} = \frac{1}{\sqrt{N}}\left[\frac{\omega T}{\sin\omega T}\right]^{1/2} \tag{6.41}$$

Eq. (6.40) can be rewritten as

$$K(x_b,T;x_a,0) = \left[\frac{m}{2\pi i\hbar T}\right]^{1/2}\left[\frac{\omega T}{\sin\omega T}\right]^{1/2}\left[\frac{\omega T}{2\tan\frac{\omega T}{2}}\right]^{1/2}\exp\left\{\frac{im\omega}{4\hbar}\cot\frac{\omega T}{2}(x_b-x_a)^2\right\} \tag{6.42}$$

One can see that the non-local harmonic oscillator propagator can be obtained directly by using the same technique which we have developed in the previous chapter.

6.3 Conclusions

In this chapter we apply our techniques to the determination of the non-local harmonic oscillator propagator. We have linearized the memory term in the action function using an idea introduced by Stratonovich (12). After performing the integrations and taking the limit $N \to \infty$ the required result can be obtained.