ขอบเขตการประมาณค่าด้วยการแจกแจงปัวซงสำหรับผลรวมสุ่มของตัวแปรสุ่มแบร์นูลลี

เรือตรีหญิงศศิธร คงอุดมทรัพย์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2554

BOUNDS IN POISSON APPROXIMATION FOR RANDOM SUMS OF BERNOULLI RANDOM VARIABLES

Sub-Lieutenant Sasithorn Kongudomthrap

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2011

Thesis Title	BOUNDS IN POISSON APPROXIMATION FOR RANDOM SUMS OF BERNOULLI RANDOM VARIABLES
By	SUB Lt. Sasithorn Kongudomthrap, WRTN
Field of Study	Mathematics
Thesis Advisor	Assistant Professor Nattakarn Chaidee, Ph.D.
Thesis Co-advisor	Professor Kritsana Neammanee, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

.................................Dean of the Faculty of Science

(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

.................................Chairman

(Associate Professor Imchit Termwuttipong, Ph.D.)

.................................Thesis Advisor

(Assistant Professor Nattakarn Chaidee, Ph.D.)

.................................Thesis Co-advisor

(Professor Kritsana Neammanee, Ph.D.)

.................................Examiner

(Kittipat Wong, Ph.D.)

.................................External Examiner

(Kittipong Laipaporn, Ph.D.)

ศศิธร คงอุดมทรัพย์ : ขอบเขตการประมาณค่าด้วยการแจกแจงปัวซงสำหรับผลรวมแบบสุ่มของตัวแปรสุ่มแบร์นูลลี. (BOUNDS IN POISSON APPROXIMATION FOR RANDOM SUMS OF BERNOULLI RANDOM VARIABLES) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ.ดร.ณัฐกาญจน์ ใจดี, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม : ศ.ดร. กฤษณะ เนียมมณี, 30 หน้า.

กำหนดให้ (X_n) เป็นลำดับของตัวแปรสุ่มแบร์นูลลีและ N เป็นตัวแปรสุ่มที่มีค่าเป็นจำนวนเต็มบวก กำหนดให้ $S_N = X_1 + X_2 + \cdots + X_N$ เป็นผลรวมสุ่ม สมมติให้ $N, X_1, X_2, \ldots$ เป็นตัวแปรสุ่มที่อิสระต่อกัน ในวิทยานิพนธ์ฉบับนี้ เราให้ขอบเขตการประมาณค่าในปัวซงแบบเอกรูปและไม่เอกรูปสำหรับ S_N

ภาควิชา : คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต.......................................

สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก.................

ปีการศึกษา 2554 ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์ร่วม ....

5272555423 : MAJOR MATHEMATICS

KEYWORDS : RANDOM SUMS / POISSON APPROXIMATION

SASITHORN KONGUDOMTHRAP : BOUNDS IN POISSON APPROXIMATION FOR RANDOM SUMS OF BERNOULLI RANDOM VARIABLES. ADVISOR : ASST.PROF.NATTAKARN CHAIDEE, Ph.D., CO-ADVISOR : PROF.KRITSANA NEAMMANEE, Ph.D., 30 pp.

Let (X_n) be a sequence of Bernoulli random variables and N a positive integer-valued random variable. Define $S_N = X_1 + X_2 + \cdots + X_N$ be random sums. Assume $N, X_1, X_2, \ldots$ are independent. In this thesis, we establish uniform and non-uniform bounds in Poisson approximation for S_N.

.

Department :	Mathematics and Computer Science	Student's Signature	
Field of Study :	Mathematics	Advisor's Signature	
Academic Year :	2011	Co-advisor's Signature	

ACKNOWLEDGEMENTS

I am grateful to Asst.Prof.Dr.Nattakarn Chaidee and Prof.Dr.Kritsana Neammanee, my thesis advisor and co-advisor, respectively, for all suggestions and helpful advice in preparing and writing this thesis. I am also sincerely grateful to the thesis committee, for their suggestions and valuable comments. Moreover, I would like to thank all of my teachers who have taught me and thank all my friends for giving me invaluable experiences at Chulalongkorn university. I would like to express my deep gratitude to my beloved family for their love and encouragement.

Finally, I would like to thank The Centre of Excellence in Mathematics which support scholarship for my research.

CONTENTS

CHAPTER I

INTRODUCTION

Fix $n \in \mathbb{N}$, let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with

$$P(X_i = 1) = p_i = 1 - P(X_i = 0)$$

for $i = 1, 2, \ldots, n$. Let U_λ denote a Poisson random variable with mean $\lambda > 0$, i.e.,

$$P(U_\lambda = x) = \frac{e^{-\lambda}\lambda^x}{x!} \text{ for } x = 0, 1, \ldots; \ \lambda_n = \sum_{i=1}^{n} p_i \text{ and } \mathbb{Z}_0^+ = \{0, 1, 2, \ldots\}.$$

Successively improved estimates of the total variation distance between the distribution of $S_n = X_1 + X_2 + \cdots + X_n$ and U_λ have been obtained by many mathematicians. The followings are examples of bounds of the difference between the distribution of S_n and U_λ.

In 1960, Le Cam[5] showed that

$$\sup_{x \in \mathbb{Z}_0^+} |P(S_n \leq x) - P(U_{\lambda_n} \leq x)| \leq \sum_{i=1}^{n} p_i^2.$$

Observe that the above bound does not depend on x. We call such a bound a **uniform bound**. The examples of uniform bounds in Poisson approximation for the distribution of S_n are the followings. Kerstan[4] gave his result in the form

$$\sup_{x \in \mathbb{Z}_0^+} |P(S_n \leq x) - P(U_{\lambda_n} \leq x)| \leq 1.05\lambda_n^{-1} \sum_{i=1}^{n} p_i^2, \quad \text{if } \max_{1 \leq i \leq n} p_i \leq 1/4.$$

Chen[2] used Stein method to obtain the following bound

$$\sup_{x \in \mathbb{Z}_0^+} |P(S_n \leq x) - P(U_{\lambda_n} \leq x)| \leq 5\lambda_n^{-1} \sum_{i=1}^{n} p_i^2$$

and then Barbour and Hall[1] improved the result of Chen[2] as follows.

$$\sup_{x \in \mathbb{Z}_0^+} |P(S_n \leq x) - P(U_{\lambda_n} \leq x)| \leq \lambda_n^{-1}(1 - e^{-\lambda_n}) \sum_{i=1}^{n} p_i^2.$$

In 2003, Neammanee[6] gave a bound in the form

$$|P(S_n = x) - P(U_{\lambda_n} = x)| \leq \frac{1}{x} \sum_{i=1}^{n} p_i^2$$

for $x = 1, 2, \ldots, n-1$ and $\lambda_n \in (0, 1]$.

Notice that the bound in Neammanee[6] depends on x. It is called a **non-uniform bound**. The following are examples of non-uniform bounds between the distribution of S_n and U_λ. In the same year, Neammanee[7] generalized his result to the case of any positive λ_n in the form

$$|P(S_n = x) - P(U_{\lambda_n} = x)| \leq \min\left\{\frac{1}{x}, \lambda_n^{-1}\right\} \sum_{i=1}^{n} p_i^2 \tag{1.1}$$

for $x = 1, 2, \ldots, n-1$.

Teerapabolarn and Neammanee [8] gave some result, in 2006, as follows.

$$|P(S_n \leq x) - P(U_{\lambda_n} \leq x)| \leq \lambda_n^{-1}(1 - e^{-\lambda_n}) \min\left\{1, \frac{e^{\lambda_n}}{x+1}\right\} \sum_{i=1}^{n} p_i^2 \tag{1.2}$$

for $x = 1, 2, \ldots, n$.

Let $X_1, X_2, \ldots$ be a sequence of independent Bernoulli random variables and N a positive integer-valued random variable. Assume $N, X_1, X_2, \ldots$ are independent. Define the **random sums** of the sequence (X_n) to be $S_N = X_1 + X_2 + \cdots + X_N$. Let $\lambda_N = \sum_{i=1}^{N} p_i$ and $\lambda = E\lambda_N$.

In 1991, Yannaros[9] gave uniform bounds of the difference of the distribution of S_N and U_λ. The following is the result.

Theorem 1.1. *[9] Let $X_1, X_2, \ldots$ be independent Bernoulli random variables with*

$$P(X_i = 1) = p_i = 1 - P(X_i = 0)$$

and N a positive integer-valued random variable which is independent of the X_i's. Then

$$\sup_{x \in \mathbb{Z}_0^+} |P(S_N \leq x) - P(U_\lambda \leq x)| \leq E\big|\lambda_N - \lambda\big| + E\left(\frac{1 - e^{-\lambda_N}}{\lambda_N} \sum_{i=1}^{N} p_i^2\right). \tag{1.3}$$

In his work, Yannaros[9] also gave the bound in (1.3) in the case that X_i's are indentically distributed.

Theorem 1.2. *[9] Let $X_1, X_2, \ldots$ be independent and identically distributed Bernoulli random variables with*

$$P(X_i = 1) = p = 1 - P(X_i = 0)$$

and N a positive integer-valued random variable which is independent of the X_i's. Then

$$\sup_{x \in \mathbb{Z}_0^+} |P(S_N \le x) - P(U_{pEN} \le x)|$$
$$\le \min\left\{\frac{p}{2\sqrt{1-p}}, pE(1 - e^{-pN})\right\} + \frac{1}{2}\sqrt{p\frac{\mathrm{Var}\,(N)}{EN}}\min\left\{1, 2\sqrt{pEN}\right\}.$$

In this work, uniform and non-uniform bounds in Poisson approximation for random sums of Bernoulli random variables are given. The followings are the results.

Theorem 1.3. *Let $X_1, X_2, \ldots$ be independent Bernoulli random variables with*

$$P(X_i = 1) = p_i = 1 - P(X_i = 0)$$

and N a positive integer-valued random variable which is independent of the X_i's. Then

1) $|P(S_N = x) - P(U_\lambda = x)| \le \frac{7\lambda}{2x}$ *where* $x \in \{1, 2, \ldots\}$,

2) $\sup_{x \in \mathbb{Z}^+} |P(S_N = x) - P(U_\lambda = x)| \le \dfrac{3\lambda}{2} + 2\min\{\lambda, E|\lambda - \lambda_N|\}$.

Note that, when $x = 0$ the exact probability can be explicitly computed, that is,

$$P(S_N = 0) = \sum_{n=1}^{\infty} P(N = n)P(S_n = 0) = \sum_{n=1}^{\infty} P(N = n)\prod_{i=1}^{n}(1 - p_i) = E\prod_{i=1}^{N}(1 - p_i).$$

If X_i's are identically distributed, we obtained the following corollary.

Corollary 1.4. *Let $X_1, X_2, \ldots$ be independent and identically distributed Bernoulli random variables with*

$$P(X_i = 1) = p = 1 - P(X_i = 0)$$

and N a non-negative integer-valued random variable which is independent of the X_i's. Then

1) $|P(S_N = x) - P(U_\lambda = x)| \leq \frac{7pEN}{2x}$ *where* $x \in \{1, 2, \ldots\}$,

2) $\sup_{x \in \mathbb{Z}^+} |P(S_N = x) - P(U_\lambda = x) \leq \dfrac{3pEN}{2} + 2p \min\{EN, E|N - EN|\}$.

Theorem 1.5. *Let* $X_1, X_2, \ldots$ *be independent Bernoulli random variables with*

$$P(X_i = 1) = p_i = 1 - P(X_i = 0)$$

and N *a positive integer-valued random variable which is independent of the* X_i*'s. Then*

$$|P(S_N \leq x) - P(U_\lambda \leq x)| \leq \frac{3\lambda}{x} + E\left[\lambda_N^{-1}(1 - e^{-\lambda_N}) \min\left\{1, \frac{e^{\lambda_N}}{x+1}\right\} \sum_{i=1}^{N} p_i^2\right]$$

where $x \in \{1, 2, \ldots\}$.

Corollary 1.6. *Let* $X_1, X_2, \ldots$ *be independent and identically distributed Bernoulli random variables with*

$$P(X_i = 1) = p = 1 - P(X_i = 0)$$

and N *a positive integer-valued random variable which is independent of the* X_i*'s. Then*

$$|P(S_N \leq x) - P(U_\lambda \leq x)| \leq \frac{3pEN}{x} + pE\left[(1 - e^{-pN}) \min\left\{1, \frac{e^{pN}}{x+1}\right\}\right]$$

where $x \in \{1, 2, \ldots\}$.

CHAPTER II

PRELIMINARIES

In this chapter, we review some basic knowledge in probability which will be used in our work.

Let $(\Omega, \mathcal{F}, P)$ be a measure space. If $P(\Omega) = 1$, then $(\Omega, \mathcal{F}, P)$ is called **a probability space** and P is called a **probability measure**. The set Ω will be refered as **sample space** and its elements are called **points** or **elementary events** and the elements of $\mathcal{F}$ are called **events**. For any event $A \in \mathcal{F}$, the value $P(A)$ is called **the probability of** A. We will use the notations $P(X \in B)$ in place of $P(\{\omega \in \Omega : X(\omega) \in B\})$. In the case where $B = (-\infty, a]$ or $[a, b]$, $P(X \in B)$ is denoted by $P(X \leq a)$ and $P(a \leq X \leq b)$, respectively. Let $X : \Omega \to \mathbb{R}$. If $\{\omega \in S | X(\omega) \leq x\}$ belong to $\mathcal{F}$ for all $x \in \mathbb{R}$, then X is called a **random variable**.

Let X be a random variable. A function $F : \mathbb{R} \to [0, 1]$ which is defined by

$$F(x) = P(X \leq x)$$

is called **the distribution function** of X.

A random variable X with its distribution function F is said to be a **discrete random variable** if the image of X is countable and said to be a **continuous random variable** if F can be written in the form

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

for some nonnegative integrable function f on $\mathbb{R}$. In this case, we say that f is the **probability function** of X.

Let $X_1, X_2, \ldots, X_n$ be random variables. Then $X_1, X_2, \ldots, X_n$ are **indepen-**

dent if and only if

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) = P(X_1 \leq x_1)P(X_2 \leq x_2)\cdots P(X_n \leq x_n)$$

for all $x_i \in \mathbb{R}$ where $i = 1, 2, \ldots, n$.

A sequence of random variables (X_n) is said to be **independent** if $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ are independent for all distinct $i_1, i_2, \ldots, i_k$ and for all $k \in \mathbb{N}$.

The followings are examples of discrete random variables.

Example 2.1. *Let X be a random variable with*

$$P(X = 1) = p \quad \textit{and} \quad P(X = 0) = 1 - p$$

*where $0 \leq p \leq 1$. Then X is called a **Bernoulli random variable** with parameter p, and denoted by $X \sim Ber(p)$.*

Example 2.2. *Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variable with parameter p. Then $X = X_1 + X_2 + \cdots + X_n$ is called a **binomial random variable** with parameter n, p, and denoted by $X \sim B(n, p)$.*

Example 2.3. *Let X be a random variable. If*

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

*where $k = 0, 1, 2, \ldots$, then X is called a **Poisson random variable** with parameter $\lambda > 0$, and denoted by $X \sim U_\lambda$.*

Let X be a discrete random variable. Assume $\sum_{x \in \mathrm{Im} X} |x| P(X = x) < \infty$. Then the **expected value** or **mean value** of X can be defined by

$$EX = \sum_{x \in \mathrm{Im} X} x P(X = x).$$

If $EX^2 < \infty$, then the **variance** of X is defined by

$$\mathrm{Var}\,(X) = E[X - EX]^2 = EX^2 - (EX)^2.$$

The following proposition is the properties of EX and $\mathrm{Var}\,(X)$.

Proposition 2.1. *Let X, Y be random variables and $a, b \in \mathbb{R}$. Then*

1. $E(X+Y) = EX + EY$,

2. $E(aX) = aEX$,

3. *If* $X \leq Y$, *then* $EX \leq EY$,

4. $|EX| \leq E|X|$,

5. $(EX)^2 \leq E(X^2)$,

6. *if* X, Y *are independent, then* $E(XY) = EXEY$,

7. $\mathrm{Var}\,(aX+b) = \mathrm{Var}\,(aX) = a^2\mathrm{Var}\,(X)$.

The following inequality is useful in our work.

Chebyshev's inequality : Let X be a random variable. Then

$$P(|X| \geq \epsilon) \leq \frac{E|X|^p}{\epsilon^p} \qquad \text{for all} \quad \epsilon, p > 0.$$

CHAPTER III

POINTWISE APPROXIMATION FOR RANDOM SUMS OF BERNOULLI RANDOM VARIABLES

Let (X_n) be a sequence of independent Bernoulli random variables with

$$P(X_i = 1) = p_i = 1 - P(X_i = 0)$$

where $p_i \in (0,1)$ and $i \in \mathbb{N}$, U_λ a Poisson random variable with mean $\lambda > 0$.

Let N be a positive integer-valued random variables. Assume $N, X_1, X_2, \dots$ are independent. Define $S_N = X_1 + X_2 + \cdots + X_N$, $\lambda_N = \sum_{i=1}^{N} p_i$ and $\lambda = E\lambda_N$.

In this chapter, we give bounds of $|P(S_N = x) - P(U_\lambda = x)|$. This approximation always called pointwise approximation. The followings are our results.

Theorem 3.1. *Let $X_1, X_2, \dots$ be independent Bernoulli random variables with*

$$P(X_i = 1) = p_i = 1 - P(X_i = 0)$$

and N a positive integer-valued random variable which is independent of the X_i 's. Then

1) $|P(S_N = x) - P(U_\lambda = x)| \le \frac{7\lambda}{2x}$ *for $x = 1, 2, \dots,$*

2) $\sup\limits_{x \in \mathbb{Z}^+} |P(S_N = x) - P(U_\lambda = x)| \le \dfrac{3\lambda}{2} + 2\min\{\lambda, E|\lambda - \lambda_N|\}$.

Note that, when $x = 0$ the exact probability can be explicitly computed, that is,

$$P(S_N = 0) = \sum_{n=1}^{\infty} P(N = n)P(S_n = 0) = \sum_{n=1}^{\infty} P(N = n)\prod_{i=1}^{n}(1 - p_i) = E\prod_{i=1}^{N}(1 - p_i).$$

If X_i's are identically distributed, we obtained the following corollary.

Corollary 3.2. *Let $X_1, X_2, \dots$ be independent and identically distributed Bernoulli random variables with*

$$P(X_i = 1) = p = 1 - P(X_i = 0)$$

and N a positive integer-valued random variable which is independent of the X_i 's. Then

1) $|P(S_N = x) - P(U_\lambda = x)| \le \dfrac{7pEN}{2x}$ *for* $x = 1, 2, \dots,$

2) $\sup_{x\in\mathbb{Z}^+} |P(S_N = x) - P(U_\lambda = x) \le \dfrac{3pEN}{2} + 2p\min\{EN, E|N - EN|\}.$

3.1 Proof of Theorem 3.1

Proof. 1) Let $\lambda_n = \sum_{i=1}^{n} p_i$ and $x \in \{1, 2, \dots\}$. Note that

$$|P(S_N = x) - P(U_\lambda = x)| \le A_1 + A_2 \tag{3.1}$$

where

$$A_1 = \sum_{n=1}^{\infty} P(N = n)|P(U_{\lambda_n} = x) - P(U_\lambda = x)|,$$
$$A_2 = \sum_{n=1}^{\infty} P(N = n)|P(S_n = x) - P(U_{\lambda_n} = x)|.$$

By Chebyshev 's inequality, we obtain

$$\begin{aligned} A_1 &\le \sum_{n=1}^{\infty} P(N = n)[P(U_{\lambda_n} \ge x) + P(U_\lambda \ge x)] \\ &\le \sum_{n=1}^{\infty} P(N = n)\left[\frac{EU_{\lambda_n}}{x} + \frac{EU_\lambda}{x}\right] \\ &= \frac{1}{x}\sum_{n=1}^{\infty} P(N = n)(\lambda_n + \lambda) \\ &= \frac{1}{x}(E\lambda_N + \lambda) \\ &= \frac{2\lambda}{x}. \end{aligned} \tag{3.2}$$

To bound A_2, we note that

$$A_2 = A_{21} + A_{22} \tag{3.3}$$

where

$$A_{21} = \sum_{\substack{n=1 \\ n\neq x}}^{\infty} P(N=n)|P(S_n=x) - P(U_{\lambda_n}=x)|,$$

$$A_{22} = P(N=x)|P(S_x=x) - P(U_{\lambda_x}=x)|.$$

From (1.1), Chebyshev's inequality and the fact that $P(S_n = x) = 0$ for $n = 1, 2, \ldots, x-1$, we have

$$\begin{aligned}
A_{21} &= \sum_{n=1}^{x-1} P(N=n)|P(S_n=x) - P(U_{\lambda_n}=x)| \\
&\quad + \sum_{n=x+1}^{\infty} P(N=n)|P(S_n=x) - P(U_{\lambda_n}=x)| \\
&\leq \sum_{n=1}^{x-1} P(N=n)P(U_{\lambda_n}=x) + \frac{1}{x}\sum_{n=x+1}^{\infty} P(N=n)\sum_{i=1}^{n} p_i^2 \\
&\leq \sum_{n=1}^{x-1} P(N=n)P(U_{\lambda_n}\geq x) + \frac{1}{x}\sum_{n=x+1}^{\infty} P(N=n)\lambda_n \\
&\leq \frac{1}{x}\sum_{n=1}^{x-1} P(N=n)EU_{\lambda_n} + \frac{1}{x}\sum_{n=x+1}^{\infty} P(N=n)\lambda_n \\
&= \frac{1}{x}\sum_{\substack{n=1 \\ n\neq x}}^{\infty} P(N=n)\lambda_n. \qquad (3.4)
\end{aligned}$$

By AM-GM inequality, it follows that

$$\prod_{i=1}^{x} p_i \leq \Big(\prod_{i=1}^{x} p_i\Big)^{\frac{1}{x}} \leq \frac{p_1+p_2+\cdots+p_x}{x} = \frac{\lambda_x}{x}. \qquad (3.5)$$

Observe that if $x = 1$, then

$$|P(S_x=x) - P(U_{\lambda_x}=x)| = |p_1 - e^{-p_1}p_1| = p_1|1-e^{-p_1}| \leq p_1 \leq \frac{3\lambda_1}{2}. \qquad (3.6)$$

Assume that $x \geq 2$. If $\lambda_x \leq x-1$, then

$$\begin{aligned}
e^{\lambda_x} &\geq \frac{\lambda_x^{x-2}}{(x-2)!} + \frac{\lambda_x^{x-1}}{(x-1)!} \\
&= \frac{\lambda_x^{x-2}(x-1)}{(x-1)!} + \frac{\lambda_x^{x-1}}{(x-1)!} \\
&= \frac{\lambda_x^{x-1}}{(x-1)!}\Big(\frac{x-1}{\lambda_x} + 1\Big) \\
&\geq \frac{2\lambda_x^{x-1}}{(x-1)!}
\end{aligned}$$

this implies that

$$\frac{e^{-\lambda_x}\lambda_x^x}{x!} \le \frac{\lambda_x^x(x-1)!}{2\lambda_x^{x-1}x!} = \frac{\lambda_x}{2x}. \tag{3.7}$$

For $\lambda_x = x$, we have

$$\begin{aligned} e^{\lambda_x} &\ge \frac{\lambda_x^{x-1}}{(x-1)!} + \frac{\lambda_x^x}{x!} \\ &= \frac{x\lambda_x^{x-1}}{(x!)} + \frac{\lambda_x^x}{x!} \\ &= \frac{\lambda_x^x}{x!}\Big(\frac{x}{\lambda_x} + 1\Big) \\ &= \frac{2\lambda_x^x}{x!}. \end{aligned}$$

Thus

$$\frac{e^{-\lambda_x}\lambda_x^x}{x!} \le \frac{\lambda_x^x x!}{2\lambda_x^x x!} = \frac{1}{2}. \tag{3.8}$$

From (3.7) and (3.8), we have

$$\frac{e^{-\lambda_x}\lambda_x^x}{x!} \le \frac{\lambda_x}{2x} \tag{3.9}$$

for $0 < \lambda_x \le x$ and $x = 2, 3, \ldots$.

By (3.5) and (3.9), we obtain

$$|P(S_x = x) - P(U_{\lambda_x} = x)| \le \prod_{i=1}^{x} p_i + \frac{e^{-\lambda_x}\lambda_x^x}{x!} \le \frac{\lambda_x}{x} + \frac{\lambda_x}{2x} = \frac{3\lambda_x}{2x} \tag{3.10}$$

for $x = 2, 3, \ldots$.

From (3.6) and (3.10), we have

$$A_{22} \le \frac{3}{2x}P(N = x)\lambda_x \tag{3.11}$$

for $x = 1, 2, \ldots$.

From (3.3), (3.4) and (3.11), we obtain

$$A_2 \le \frac{1}{x}\sum_{\substack{n=1 \\ n\ne x}}^{\infty} P(N = n)\lambda_n + \frac{3}{2x}P(N = x)\lambda_x \le \frac{3E\lambda_N}{2x} = \frac{3\lambda}{2x}. \tag{3.12}$$

Hence by (3.1), (3.2) and (3.12),

$$|P(S_N = x) - P(U_\lambda = x)| \le \frac{2\lambda}{x} + \frac{3\lambda}{2x} = \frac{7\lambda}{2x}.$$

2) Freedman ([3], pp. 260) showed that for any $\mu_1, \mu_2 > 0$,

$$\sup_{x\in\mathbb{Z}_0^+} |P(U_{\mu_1} \le x) - P(U_{\mu_2} \le x)| \le |\mu_1 - \mu_2|.$$

This implies that

$$\begin{aligned}
A_1 &= \sum_{n=1}^{\infty} P(N=n)|P(U_{\lambda_n} = x) - P(U_\lambda = x)| \\
&\le \sum_{n=1}^{\infty} P(N=n)\big\{|P(U_{\lambda_n} \le x) - P(U_\lambda \le x)| + |P(U_\lambda \le x-1) - P(U_{\lambda_n} \le x-1)|\big\} \\
&\le 2\sum_{n=1}^{\infty} P(N=n)|\lambda - \lambda_n| \\
&= 2E|\lambda - \lambda_N|.
\end{aligned}$$

From this fact and (3.2), we have

$$A_1 \le 2\min\big\{\lambda, E|\lambda - \lambda_N|\big\}. \tag{3.13}$$

From (3.1), (3.12) and (3.13), we obtain

$$\sup_{x\in\mathbb{Z}^+} |P(S_N = x) - P(U_\lambda = x)| \le \frac{3\lambda}{2} + 2\min\big\{\lambda, E|\lambda - \lambda_N|\big\}.$$

□

3.2 Examples

Example 3.1. *Fix $n \in \mathbb{N}$, let N be a random variable defined by*

$$P(N = n) = 1.$$

Let $X_1, X_2, \dots$ be independent Bernoulli random variables with

$$P(X_i = 1) = p_i = 1 - P(X_i = 0).$$

Assume $N, X_1, X_2, \dots$ are independent. Then

1) $|P(S_N = x) - P(U_{\lambda_n} = x)| \le \frac{7\lambda_n}{2x}$ *for $x = 1, 2, \dots,$*

2) $\sup_{x\in\mathbb{Z}^+} |P(S_N = x) - P(U_{\lambda_n} = x)| \le \frac{3\lambda_n}{2}.$

Furthermore if $p_1 = p_2 = \cdots = p$, *then*

1) $|P(S_N = x) - P(U_{np} = x)| \leq \frac{7np}{2x}$ *for* $x = 1, 2, \ldots,$

2) $\sup_{x \in \mathbb{Z}^+} |P(S_N = x) - P(U_{np} = x)| \leq \frac{3np}{2}$.

Proof. Note that

$$\lambda = E\lambda_N = P(N = n)\lambda_n = \lambda_n, \tag{3.14}$$

and

$$E|\lambda - \lambda_N| = P(N = n)|\lambda_n - \lambda_n| = 0.$$

By Theorem 3.1, we get

$$|P(S_N = x) - P(U_{\lambda_n} = x)| \leq \frac{7\lambda_n}{2x} \quad \text{for } x = 1, 2, \ldots$$

and

$$\sup_{x \in \mathbb{Z}^+} |P(S_N = x) - P(U_{\lambda_n} = x)| \leq \frac{3\lambda_n}{2} + 2\min\Big\{\lambda, E|\lambda - \lambda_N|\Big\} = \frac{3\lambda_n}{2}.$$

□

Example 3.2. *Fix* $n \in \mathbb{N}$, *let* N *be a random variable defined by*

$$P(N = n) = \frac{1}{2} \quad \textit{and} \quad P(N = 2n) = \frac{1}{2}.$$

Let $X_1, X_2, \ldots$ *be independent Bernoulli random variables with*

$$P(X_i = 1) = p_i = 1 - P(X_i = 0).$$

Assume $N, X_1, X_2, \ldots$ *are independent. Then*

1) $|P(S_N = x) - P(U_\lambda = x)| \leq \frac{7\big(\lambda_n + \lambda_{2n}\big)}{4x}$ *for* $x = 1, 2, \ldots,$

2) $\sup_{x \in \mathbb{Z}^+} |P(S_N = x) - P(U_\lambda = x)| \leq \frac{1}{4}(7\lambda_{2n} - \lambda_n)$.

Furthermore if $p_1 = p_2 = \cdots = p$, *then*

1) $|P(S_N = x) - P(U_\lambda = x)| \le \dfrac{21np}{4x}$ *for* $x = 1, 2, \ldots,$

2) $\sup\limits_{x \in \mathbb{Z}^+} |P(S_N = x) - P(U_\lambda = x)| \le \dfrac{13np}{4}.$

Proof. Since

$$\lambda = E\lambda_N = P(N = n)\lambda_n + P(N = 2n)\lambda_{2n} = \frac{\lambda_n}{2} + \frac{\lambda_{2n}}{2} = \frac{1}{2}\big(\lambda_n + \lambda_{2n}\big) \quad (3.15)$$

and

$$\begin{aligned} E|\lambda_N - \lambda| &= P(N = n)|\lambda_n - \lambda| + P(N = 2n)|\lambda_{2n} - \lambda| \\ &= \frac{1}{2}\big|\lambda_n - \frac{1}{2}\big(\lambda_n + \lambda_{2n}\big)\big| + \frac{1}{2}\big|\lambda_{2n} - \frac{1}{2}\big(\lambda_n + \lambda_{2n}\big)\big| \\ &= \frac{1}{2}\big|\frac{\lambda_n}{2} - \frac{\lambda_{2n}}{2}\big| + \frac{1}{2}\big|\frac{\lambda_{2n}}{2} - \frac{\lambda_n}{2}\big| \\ &= \frac{1}{2}\big|\lambda_{2n} - \lambda_n\big| \\ &= \frac{1}{2}(\lambda_{2n} - \lambda_n), \end{aligned}$$

we have

$$\min\Big\{\lambda, E|\lambda_N - \lambda|\Big\} = \min\Big\{\frac{1}{2}(\lambda_n + \lambda_{2n}), \frac{1}{2}(\lambda_{2n} - \lambda_n)\Big\} = \frac{1}{2}(\lambda_{2n} - \lambda_n).$$

By Theorem 3.1, we have

$$|P(S_N = x) - P(U_\lambda = x)| \le \frac{7\big(\lambda_n + \lambda_{2n}\big)}{4x} \quad \text{for } x = 1, 2, \ldots$$

and

$$\sup_{x \in \mathbb{Z}^+} |P(S_N = x) - P(U_\lambda = x)| \le \frac{3(\lambda_n + \lambda_{2n})}{4} + \lambda_{2n} - \lambda_n = \frac{1}{4}(7\lambda_{2n} - \lambda_n).$$

□

Example 3.3. *Let N be a random variable defined by*

$$P(N = n) = \frac{1}{2^n} \quad \textit{for } n = 1, 2, \ldots.$$

Let $X_1, X_2, \ldots$ be independent Bernoulli random variables with

$$P(X_i = 1) = p_i = 1 - P(X_i = 0).$$

Assume $N, X_1, X_2, \ldots$ are independent. Then

1) $|P(S_N = x) - P(U_{2p} = x)| \leq \dfrac{7p}{x}$ *for $x = 1, 2, \ldots,$*

2) $\sup_{x \in \mathbb{Z}^+} |P(S_N = x) - P(U_{2p} = x)| \leq 5p.$

Proof. Since $EN = \sum_{n=1}^{\infty} \frac{n}{2^n} = 2,$

$$\begin{aligned}
E|N - EN| &= \sum_{n=1}^{\infty} \frac{1}{2^n}|n-2| \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n}|n-2| \\
&= \frac{1}{2} + 0 + \frac{1}{2^3} + \frac{2}{2^4} + \frac{3}{2^5} + \cdots \\
&= \frac{1}{2} + \frac{1}{2^2}\left(\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots\right) \\
&= \frac{1}{2} + \frac{1}{2^2}\sum_{n=1}^{\infty} \frac{n}{2^n} \\
&= 1.
\end{aligned}$$

By Corollary 3.2, we get

$$|P(S_N = x) - P(U_{2p} = x)| \leq \frac{7p}{x} \quad \text{for } x = 1, 2, \ldots$$

and

$$\sup_{x \in \mathbb{Z}^+} |P(S_N = x) - P(U_{2p} = x)| \leq 3p + 2p \min\{2, 1\} = 5p.$$

□

Example 3.4. *Let* $0 < \mu \le 1$ *and let* N *be a random variable defined by*

$$P(N = n) = \frac{e^{-\mu}\mu^{n-1}}{(n-1)!} \quad \text{for } n = 1, 2, \ldots.$$

Let $X_1, X_2, \ldots$ *be independent Bernoulli random variables with*

$$P(X_i = 1) = p_i = 1 - P(X_i = 0).$$

Assume $N, X_1, X_2, \ldots$ *are independent. Then*

1) $|P(S_N = x) - P(U_{\mu p} = x)| \le \dfrac{7p(\mu+1)}{2x}$ *for* $x = 1, 2, \ldots,$

2) $\sup_{x \in \mathbb{Z}^+} |P(S_N = x) - P(U_{\mu p} = x)| \le \dfrac{7p(\mu+1)}{2}.$

Proof. Note that

$$\begin{aligned} EN &= \sum_{n=1}^{\infty} nP(N = n) \\ &= \sum_{n=1}^{\infty} \frac{ne^{-\mu}\mu^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{(n+1)e^{-\mu}\mu^{n}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{ne^{-\mu}\mu^{n}}{n!} + \sum_{n=0}^{\infty} \frac{e^{-\mu}\mu^{n}}{n!} \\ &= \mu + 1 \end{aligned}$$

and

$$\begin{aligned} E|N - EN| &= \sum_{n=1}^{\infty} \frac{e^{-\mu}\mu^{n-1}}{(n-1)!}\big|n - (\mu+1)\big| \\ &= \mu e^{-\mu} + \sum_{n=2}^{\infty} \frac{e^{-\mu}\mu^{n-1}}{(n-1)!}\big(n - (\mu+1)\big) \\ &= \mu e^{-\mu} + \sum_{n=2}^{\infty} \frac{ne^{-\mu}\mu^{n-1}}{(n-1)!} + (\mu+1)\sum_{n=2}^{\infty} \frac{e^{-\mu}\mu^{n-1}}{(n-1)!} \\ &= \mu e^{-\mu} + \left(\sum_{n=1}^{\infty} \frac{ne^{-\mu}\mu^{n-1}}{(n-1)!} - e^{-\mu}\right) + (\mu+1)\sum_{n=1}^{\infty} \frac{e^{-\mu}\mu^{n}}{n!} \\ &= \mu e^{-\mu} + \left(\mu + 1 - e^{-\mu}\right) + (\mu+1)\left(\sum_{n=0}^{\infty} \frac{e^{-\mu}\mu^{n}}{n!} - e^{-\mu}\right) \\ &= \mu e^{-\mu} + \mu + 1 - e^{-\mu} - (\mu+1)(1 - e^{-\mu}) \\ &= 2\mu e^{-\mu}. \end{aligned}$$

Then

$$\min\left\{EN, E|N-EN|\right\} = \min\{\mu+1, 2\mu e^{-mu}\} \le \mu+1.$$

By Corollary 3.2, we get

$$|P(S_N = x) - P(U_{\mu p} = x)| \le \frac{7p(\mu+1)}{2x} \quad \text{for } x = 1, 2, \ldots$$

and

$$\sup_{x\in\mathbb{Z}^+} |P(S_N = x) - P(U_{\mu p} = x)| \le \frac{3p(\mu+1)}{2} + 2p(\mu+1) = \frac{7p(\mu+1)}{2}.$$

□

CHAPTER IV

NON-UNIFORM BOUND IN POISSON APPROXIMATION FOR RANDOM SUMS OF BERNOULLI RANDOM VARIABLES

In this chapter we give the non-uniform bounds of $|P(S_N \le x) - P(U_\lambda \le x)|$. The notation in chapter 3 can be refered in this chapter.

In 1991, Yannaros[9] gave uniform bounds of the difference between the distribution of S_N and U_λ. The following is his result.

Theorem 4.1. *[9] Let $X_1, X_2, \ldots$ be independent Bernoulli random variables with*

$$P(X_i = 1) = p_i = 1 - P(X_i = 0)$$

and N a positive integer-valued random variable which is independent of the X_i 's. Then

$$\sup_{x \in \mathbb{Z}_0^+} |P(S_N \le x) - P(U_\lambda \le x)| \le E\big|\lambda_N - \lambda\big| + E\bigg(\frac{1 - e^{-\lambda_N}}{\lambda_N} \sum_{i=1}^{N} p_i^2\bigg). \tag{4.1}$$

In his work, Yannaros[9] improved (4.1) and obtained the bound as stated in the following theorem.

Theorem 4.2. *[9] Let $X_1, X_2, \ldots$ be independent and identically distributed Bernoulli random variables with*

$$P(X_i = 1) = p = 1 - P(X_i = 0)$$

and N a positive integer-valued random variable which is independent of the X_i's. Then we have

$$\sup_{x \in \mathbb{Z}_0^+} |P(S_N \le x) - P(U_{pEN} \le x)|$$
$$\le \min\left\{\frac{p}{2\sqrt{1-p}}, pE(1 - e^{-pN})\right\} + \frac{1}{2}\sqrt{p\frac{\mathrm{Var}\,(N)}{EN}} \min\left\{1, 2\sqrt{pEN}\right\}.$$

The following theorem is our main result.

Theorem 4.3. *Let $X_1, X_2, \ldots$ be independent Bernoulli random variables with*

$$P(X_i = 1) = p_i = 1 - P(X_i = 0)$$

and N a positive integer-valued random variable which is independent of the X_i's. Then

$$|P(S_N \leq x) - P(U_\lambda \leq x)| \leq \frac{3\lambda}{x} + E\left[\lambda_N^{-1}(1 - e^{-\lambda_N}) \min\left\{1, \frac{e^{\lambda_N}}{x+1}\right\} \sum_{i=1}^{N} p_i^2\right]$$

for $x = 1, 2, \ldots$.

Corollary 4.4. *Let $X_1, X_2, \ldots$ be independent and identically distributed Bernoulli random variables with*

$$P(X_i = 1) = p = 1 - P(X_i = 0)$$

and N a positive integer-valued random variable which is independent of the X_i's. Then

$$|P(S_N \leq x) - P(U_\lambda \leq x)| \leq \frac{3pEN}{x} + pE\left[(1 - e^{-pN}) \min\left\{1, \frac{e^{pN}}{x+1}\right\}\right]$$

for $x = 1, 2, \ldots$.

4.1 Proof of Theorem 4.3

Proof. We note that

$$|P(S_N \leq x) - P(U_\lambda \leq x)| \leq B_1 + B_2 \tag{4.2}$$

where

$$B_1 =: \sum_{n=1}^{\infty} P(N = n)|P(S_n \leq x) - P(U_{\lambda_n} \leq x)|$$
$$B_2 =: \sum_{n=1}^{\infty} P(N = n)|P(U_{\lambda_n} \leq x) - P(U_\lambda \leq x)|.$$

Using Chebyshev's inequality, we obtain

$$\begin{aligned}
&\sum_{n=1}^{x} P(N=n)|P(S_n \le x) - P(U_{\lambda_n} \le x)| \\
&= \sum_{n=1}^{x} P(N=n)[1 - P(U_{\lambda_n} \le x)] \\
&\le \sum_{n=1}^{x} P(N=n)P(U_{\lambda_n} \ge x) \\
&\le \sum_{n=1}^{x} P(N=n)\left[\frac{EU_{\lambda_n}}{x}\right] \\
&= \sum_{n=1}^{x} P(N=n)\left[\frac{\lambda_n}{x}\right] \\
&\le \sum_{n=1}^{\infty} P(N=n)\left[\frac{\lambda_n}{x}\right] \\
&= \frac{\lambda}{x},
\end{aligned}$$

and using (1.2) to get

$$\begin{aligned}
&\sum_{n=x+1}^{\infty} P(N=n)|P(S_n \le x) - P(U_{\lambda_n} \le x)| \\
&\le \sum_{n=x+1}^{\infty} P(N=n)\lambda_n^{-1}(1-e^{-\lambda_n}) \min\left\{1, \frac{e^{\lambda_n}}{x+1}\right\} \sum_{i=1}^{n} p_i^2 \\
&\le \sum_{n=1}^{\infty} P(N=n)\lambda_n^{-1}(1-e^{-\lambda_n}) \min\left\{1, \frac{e^{\lambda_n}}{x+1}\right\} \sum_{i=1}^{n} p_i^2 \\
&= E\lambda_N^{-1}(1-e^{-\lambda_N}) \min\left\{1, \frac{e^{\lambda_N}}{x+1}\right\} \sum_{i=1}^{N} p_i^2.
\end{aligned}$$

This implies that

$$B_1 \le \frac{\lambda}{x} + E\lambda_N^{-1}(1-e^{-\lambda_N}) \min\left\{1, \frac{e^{\lambda_N}}{x+1}\right\} \sum_{i=1}^{N} p_i^2. \tag{4.3}$$

Similar to (3.2), we can show that

$$\begin{aligned} B_2 &= \sum_{n=1}^{\infty} P(N=n)|P(U_\lambda > x) - P(U_{\lambda_n} > x)| \\ &\le \sum_{n=1}^{\infty} P(N=n)[P(U_\lambda \ge x) + P(U_{\lambda_n} \ge x)] \\ &= \frac{2\lambda}{x}. \end{aligned} \tag{4.4}$$

From (4.2), (4.3) and (4.4), we complete the proof. □

Example 4.1. *Fix $n \in \mathbb{N}$, let N be a random variable defined by*

$$P(N=n) = 1.$$

Let $X_1, X_2, \ldots$ be independent Bernoulli random variables with

$$P(X_i = 1) = p_i = 1 - P(X_i = 0).$$

Assume $N, X_1, X_2, \ldots$ are independent. Then

1) $|P(S_N \le x) - P(U_{\lambda_n} \le x)| \le \dfrac{3\lambda_n}{x} + \dfrac{e^{\lambda_n} - 1}{\lambda_n(x+1)} \displaystyle\sum_{i=1}^{n} p_i^2$
for $x = 1, 2, \ldots,$

2) $\displaystyle\sup_{x \in \mathbb{Z}_0^+} |P(S_N \le x) - P(U_{\lambda_n} \le x)| \le \frac{1 - e^{-\lambda_n}}{\lambda_n} \sum_{i=1}^{n} p_i^2.$

Furthermore if $p_1 = p_2 = \cdots = p$, then

(i) $|P(S_N \le x) - P(U_{np} \le x)| \le \dfrac{3np}{x} + \dfrac{p(e^{np} - 1)}{x+1}$ *for $x = 1, 2, \ldots,$*

(ii) $\displaystyle\sup_{x \in \mathbb{Z}_0^+} |P(S_N \le x) - P(U_{np} \le x)| \le \min\left\{\frac{p}{2\sqrt{1-p}}, p(1 - e^{-np})\right\}.$

Proof. 1) From Example 3.1, we have $\lambda = \lambda_n$ and $E|\lambda_N - \lambda| = 0$.

Note that

$$\begin{aligned} &E\left[\lambda_N^{-1}(1 - e^{-\lambda_N}) \min\left\{1, \frac{e^{\lambda_N}}{x+1}\right\} \sum_{i=1}^{N} p_i^2\right] \\ &= P(N=n)\left[\lambda_n^{-1}(1 - e^{-\lambda_n}) \min\left\{1, \frac{e^{\lambda_n}}{x+1}\right\} \sum_{i=1}^{n} p_i^2\right] \\ &= \lambda_n^{-1}(1 - e^{-\lambda_n}) \min\left\{1, \frac{e^{\lambda_n}}{x+1}\right\} \sum_{i=1}^{n} p_i^2 \\ &\le \frac{e^{\lambda_n} - 1}{\lambda_n} \sum_{i=1}^{n} p_i^2. \end{aligned}$$

By Theorem 4.3, we have

$$|P(S_N \le x) - P(U_{\lambda_n} \le x)| \le \frac{3\lambda_n}{x} + \frac{e^{\lambda_n} - 1}{\lambda_n(x+1)} \sum_{i=1}^{n} p_i^2.$$

2) Since $E|\lambda_N - \lambda| = 0$,

$$E\left(\frac{1 - e^{-\lambda_N}}{\lambda_N} \sum_{i=1}^{N} p_i^2\right) = P(N = n)\left(\frac{1 - e^{-\lambda_n}}{\lambda_n} \sum_{i=1}^{n} p_i^2\right) = \frac{1 - e^{-\lambda_n}}{\lambda_n} \sum_{i=1}^{n} p_i^2,$$

and Theorem 4.1, we have 2).

Note that (i) follows directly from Corollary 4.4.

To show (ii), note that $EN = P(N = n)n = n$ and

$$\mathrm{Var}\,(N) = E[N - EN]^2 = E[N - n]^2 = P(N = n)[n - n]^2 = 0.$$

By Theorem 4.2, we have

$$\sup_{x \in \mathbb{Z}_0^+} |P(S_N \le x) - P(U_{np} \le x)| \le \min\left\{\frac{p}{2\sqrt{1-p}}, p(1 - e^{-np})\right\}.$$

□

Example 4.2. *Fix $n \in \mathbb{N}$, let N be a random variable defined by*

$$P(N = n) = \frac{1}{2} \quad \text{and} \quad P(N = 2n) = \frac{1}{2}.$$

Let $X_1, X_2, \ldots$ be independent Bernoulli random variables with

$$P(X_i = 1) = p_i = 1 - P(X_i = 0).$$

Assume $N, X_1, X_2, \ldots$ are independent. Then

1) $|P(S_N \le x) - P(U_\lambda \le x)|$
$$\le \frac{3(\lambda_n + \lambda_{2n})}{2x} + \frac{1}{2(x+1)}\left\{\frac{e^{\lambda_n} - 1}{\lambda_n} \sum_{i=1}^{n} p_i^2 + \frac{e^{\lambda_{2n}} - 1}{\lambda_{2n}} \sum_{i=1}^{2n} p_i^2\right\} \text{ for } x = 1, 2, \ldots,$$

2) $\sup_{x \in \mathbb{Z}_0^+} |P(S_N \le x) - P(U_\lambda \le x)|$
$$\le \frac{1}{2}(\lambda_{2n} - \lambda_n) + \frac{1}{2}\left(\frac{1 - e^{-\lambda_n}}{\lambda_n} \sum_{i=1}^{n} p_i^2 + \frac{1 - e^{-\lambda_{2n}}}{\lambda_{2n}} \sum_{i=1}^{2n} p_i^2\right).$$

Furthermore if $p_1 = p_2 = \cdots = p$*, then*

(i) $|P(S_N \le x) - P(U_{\frac{3np}{2}} \le x)|$
$$\le \frac{9np}{2x} + \frac{p}{2(x+1)}\left\{e^{2np} + e^{np} - 2\right\} \text{ for } x = 1, 2, \ldots,$$

(ii) $$\sup_{x \in \mathbb{Z}_0^+} |P(S_N \le x) - P(U_{\frac{3np}{2}} \le x)| \le \frac{p}{2}\min\left\{\frac{1}{\sqrt{1-p}}, 2 - e^{-np} - e^{-2np}\right\} + \frac{np}{2}.$$

Proof. 1) By Example 3.2, we have $E\left|\lambda_N - \lambda\right| = \dfrac{1}{2}(\lambda_{2n} - \lambda_n)$.

Note that

$$E\left[\lambda_N^{-1}(1 - e^{-\lambda_N})\min\left\{1, \frac{e^{\lambda_N}}{x+1}\right\}\sum_{i=1}^{N} p_i^2\right]$$
$$= P(N = n)\lambda_n^{-1}(1 - e^{-\lambda_n})\min\left\{1, \frac{e^{\lambda_n}}{x+1}\right\}\sum_{i=1}^{n} p_i^2$$
$$+ P(N = 2n)\lambda_{2n}^{-1}(1 - e^{-\lambda_{2n}})\min\left\{1, \frac{e^{\lambda_{2n}}}{x+1}\right\}\sum_{i=1}^{2n} p_i^2$$
$$= \frac{1}{2}\left[\frac{1}{\lambda_n}(1 - e^{-\lambda_n})\min\left\{1, \frac{e^{\lambda_n}}{x+1}\right\}\sum_{i=1}^{n} p_i^2 + \frac{1}{\lambda_{2n}}(1 - e^{-\lambda_{2n}})\min\left\{1, \frac{e^{\lambda_{2n}}}{x+1}\right\}\sum_{i=1}^{2n} p_i^2\right]$$
$$\le \frac{1}{2(x+1)}\left[\frac{e^{\lambda_n} - 1}{\lambda_n}\sum_{i=1}^{n} p_i^2 + \frac{e^{\lambda_{2n}} - 1}{\lambda_{2n}}\sum_{i=1}^{2n} p_i^2\right].$$

From this fact and Theorem 4.3, we get

$$|P(S_N \le x) - P(U_\lambda \le x)|$$
$$\le \frac{3(\lambda_n + \lambda_{2n})}{2x} + \frac{1}{2(x+1)}\left\{\frac{e^{\lambda_n} - 1}{\lambda_n}\sum_{i=1}^{n} p_i^2 + \frac{e^{\lambda_{2n}} - 1}{\lambda_{2n}}\sum_{i=1}^{2n} p_i^2\right\} \text{ for } x = 1, 2, \ldots.$$

2) Observe that

$$E\left(\frac{1 - e^{-\lambda_N}}{\lambda_N}\sum_{i=1}^{N} p_i^2\right) = P(N = n)\left(\frac{1 - e^{-\lambda_n}}{\lambda_n}\sum_{i=1}^{n} p_i^2\right) + P(N = 2n)\left(\frac{1 - e^{-\lambda_{2n}}}{\lambda_{2n}}\sum_{i=1}^{2n} p_i^2\right)$$
$$= \frac{1}{2}\left(\frac{1 - e^{-\lambda_n}}{\lambda_n}\sum_{i=1}^{n} p_i^2 + \frac{1 - e^{-\lambda_{2n}}}{\lambda_{2n}}\sum_{i=1}^{2n} p_i^2\right). \tag{4.5}$$

From $E\left|\lambda_N - \lambda\right| = \dfrac{1}{2}(\lambda_{2n} - \lambda_n)$, (4.5) and Theorem 4.1, we obtain

$$\sup_{x \in \mathbb{Z}_0^+} |P(S_N \le x) - P(U_\lambda \le x)|$$
$$\le \frac{1}{2}(\lambda_{2n} - \lambda_n) + \frac{1}{2}\left(\frac{1 - e^{-\lambda_n}}{\lambda_n}\sum_{i=1}^{n} p_i^2 + \frac{1 - e^{-\lambda_{2n}}}{\lambda_{2n}}\sum_{i=1}^{2n} p_i^2\right).$$

(i) Note that $EN = nP(N = n) + 2nP(N = 2n) = \dfrac{3n}{2}$ and

$$E\left[(1-e^{-pN})\min\left\{1, \frac{e^{pN}}{x+1}\right\}\right]$$
$$= P(N=n)\left[(1-e^{-np})\min\left\{1, \frac{e^{np}}{x+1}\right\}\right] + P(N=2n)\left[(1-e^{-2np})\min\left\{1, \frac{e^{2np}}{x+1}\right\}\right]$$
$$\leq \frac{e^{np}-1}{2(x+1)} + \frac{e^{2np}-1}{2(x+1)}$$
$$= \frac{1}{2(x+1)}(e^{2np} + e^{np} - 2).$$

By Corollary 4.4, we get (i) holds.

(ii) We note that

$$\begin{aligned}
\text{Var}\,(N) &= E[N - EN]^2 \\
&= E\left[N - \frac{3n}{2}\right]^2 \\
&= P(N=n)\left[n - \frac{3n}{2}\right]^2 + P(N=2n)\left[2n - \frac{3n}{2}\right]^2 \\
&= \frac{n^2}{8} + \frac{n^2}{8} \\
&= \frac{n^2}{4}
\end{aligned}$$

and

$$\begin{aligned}
E(1-e^{-pN}) &= 1 - Ee^{-pN} \\
&= 1 - \left[P(N=n)e^{-np} + P(N=2n)e^{-2np}\right] \\
&= 1 - \frac{1}{2}\left[e^{-np} + e^{-2np}\right] \\
&= \frac{1}{2}[2 - e^{-np} - e^{-2np}].
\end{aligned}$$

By Theorem 4.2, we have

$$\sup_{x\in\mathbb{Z}_0^+} |P(S_N \leq x) - P(U_{\frac{3np}{2}} \leq x)|$$
$$\leq \frac{p}{2}\min\left\{\frac{1}{\sqrt{1-p}}, 2 - e^{-np} - e^{-2np}\right\} + \frac{1}{2}\sqrt{\frac{np}{6}}\min\left\{1, 2\sqrt{\frac{3np}{2}}\right\}$$
$$\leq \frac{p}{2}\min\left\{\frac{1}{\sqrt{1-p}}, 2 - e^{-np} - e^{-2np}\right\} + \frac{np}{2}.$$

□

Example 4.3. *Let N be a random variable defined by*

$$P(N=n)=\frac{1}{2^n}\quad \textit{for } n=1,2,\dots.$$

Assume $p_1=p_2=\cdots=p$ and $e^p<2$. Let $X_1,X_2,\dots$ be independent Bernoulli random variables with

$$P(X_i=1)=p_i=1-P(X_i=0).$$

Assume $N,X_1,X_2,\dots$ are independent. Then

1) $|P(S_N\le x)-P(U_{2p}\le x)|\le \frac{6p}{x}+\frac{2p(e^p-1)}{(2-e^p)(x+1)}$ *for $x=1,2,\dots,$*

2) $\sup_{x\in\mathbb{Z}_0^+}|P(S_N\le x)-P(U_{2p}\le x)|\le p\min\left\{\frac{1}{2\sqrt{1-p}},\frac{2(e^p-1)}{2e^p-1}\right\}+\sqrt{2}p.$

Proof. From Example 3.3, we have $EN=2$.

1) By Corollary 4.4 and the fact that

$$\begin{aligned}
E\left[(1-e^{-pN})\min\left\{1,\frac{e^{pN}}{x+1}\right\}\right] &= \sum_{n=1}^{\infty}\frac{1}{2^n}\left[(1-e^{-np})\min\left\{1,\frac{e^{np}}{x+1}\right\}\right]\\
&\le \sum_{n=1}^{\infty}\left[\frac{e^{np}-1}{2^n(x+1)}\right]\\
&= \frac{1}{x+1}\left[\sum_{n=1}^{\infty}\frac{e^{np}}{2^n}-\sum_{n=1}^{\infty}\frac{1}{2^n}\right]\\
&= \frac{1}{x+1}\left[\frac{e^p}{2-e^p}-1\right]\\
&= \frac{2(e^p-1)}{(2-e^p)(x+1)},
\end{aligned}$$

we obtain

$$|P(S_N\le x)-P(U_{2p}\le x)|\le \frac{6p}{x}+\frac{2p(e^p-1)}{(2-e^p)(x+1)}\quad\text{for } x=1,2,\dots.$$

2) Observe that

$$\begin{aligned}
\mathrm{Var}\,(N) &= E[N-EN]^2\\
&= E[N-2]^2\\
&= \sum_{n=1}^{\infty}\frac{(n-2)^2}{2^n}\\
&= 2
\end{aligned}$$

and

$$\begin{aligned}
E(1-e^{-pN}) &= 1-Ee^{-pN}\\
&= 1-\sum_{n=1}^{\infty}\frac{e^{-np}}{2^n}\\
&= 1-\sum_{n=1}^{\infty}\left(\frac{e^{-p}}{2}\right)^n\\
&= 1-\frac{1}{2e^p-1}\\
&= \frac{2(e^p-1)}{2e^p-1}.
\end{aligned}$$

Applying Theorem 4.2, we have

$$\begin{aligned}
&\sup_{x\in\mathbb{Z}_0^+}|P(S_N\le x)-P(U_{2p}\le x)|\\
&\le p\min\left\{\frac{1}{2\sqrt{1-p}},\frac{2(e^p-1)}{2e^p-1}\right\}+\frac{\sqrt{p}}{2}\min\{1,2\sqrt{2p}\}\\
&\le p\min\left\{\frac{1}{2\sqrt{1-p}},\frac{2(e^p-1)}{2e^p-1}\right\}+\sqrt{2}p.
\end{aligned}$$

□

Example 4.4. *Let $0<\mu\le 1$ and let N be a random variable defined by*

$$P(N=n)=\frac{e^{-\mu}\mu^{n-1}}{(n+1)!}\quad \textit{for } n=1,2,\ldots.$$

Let $X_1,X_2,\ldots$ be independent Bernoulli random variables with

$$P(X_i=1)=p_i=1-P(X_i=0).$$

Assume $N,X_1,X_2,\ldots$ are independent. Then

1) $|P(S_N\le x)-P(U_{\mu p}\le x)|\le\dfrac{3p(\mu+1)}{x}+\dfrac{p(e^{\mu e^p-\mu+p}-1)}{x+1}$ *for $x=1,2,\ldots,$*

2) $$\begin{aligned}&\sup_{x\in\mathbb{Z}_0^+}|P(S_N\le x)-P(U_{\mu p}\le x)|\\ &\le p\min\left\{\frac{1}{2\sqrt{1-p}},1-e^{\mu e^{-p}-\mu+p}\right\}+p\sqrt{\mu+2}.\end{aligned}$$

Proof. From Example 3.4, we have $EN = \mu + 1$.

1) Note that

$$
\begin{aligned}
E\Big[(1-e^{-pN})\min\Big\{1, \frac{e^{pN}}{x+1}\Big\}\Big] &= \sum_{n=1}^{\infty} \frac{e^{-\mu}\mu^{n-1}}{(n-1)!}\Big[(1-e^{-np})\min\Big\{1, \frac{e^{np}}{x+1}\Big\}\Big] \\
&\leq \sum_{n=1}^{\infty} \frac{e^{-\mu}\mu^{n-1}}{(n-1)!}\Big[(1-e^{-np})\frac{e^{np}}{x+1}\Big] \\
&= \frac{1}{x+1}\sum_{n=1}^{\infty} \frac{e^{-\mu}\mu^{n-1}}{(n-1)!}(e^{np}-1) \\
&= \frac{1}{x+1}\Big[\sum_{n=1}^{\infty} \frac{e^{-\mu}e^{np}\mu^{n-1}}{(n-1)!} - \sum_{n=1}^{\infty} \frac{e^{-\mu}\mu^{n-1}}{(n-1)!}\Big] \\
&= \frac{1}{x+1}\Big[\sum_{n=0}^{\infty} \frac{e^{-\mu}e^{(n+1)p}\mu^{n}}{n!} - \sum_{n=0}^{\infty} \frac{(\mu e^{-\mu})^{n}}{n!}\Big] \\
&= \frac{1}{x+1}\Big[e^{-\mu+p}\sum_{n=0}^{\infty} \frac{(e^{p}\mu)^{n}}{n!} - 1\Big] \\
&= \frac{e^{\mu e^{p}-\mu+p}-1}{x+1}. \qquad (4.6)
\end{aligned}
$$

By (4.6) and Corollary 4.4,

$$
|P(S_N \leq x) - P(U_{\mu p} \leq x)| \leq \frac{3p(\mu+1)}{x} + \frac{p(e^{\mu e^{p}-\mu+p}-1)}{x+1}.
$$

2) Note that

$$
\begin{aligned}
EN^2 &= \sum_{n=1}^{\infty} \frac{n^2 e^{-\mu} e^{n-1}}{(n-1)!} \\
&= \sum_{n=0}^{\infty} \frac{(n+1)^2 e^{-\mu} e^{n}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{n^2 e^{-\mu} e^{n}}{n!} + 2\sum_{n=0}^{\infty} \frac{n e^{-\mu} e^{n}}{n!} + \sum_{n=0}^{\infty} \frac{e^{-\mu} e^{n}}{n!} \\
&= \mu^2 + \mu + 2(\mu+1) + 1,
\end{aligned}
$$

then $\mathrm{Var}\,(N) = EX^2 - [EX]^2 = \mu^2 + \mu + 2(\mu+1) + 1 - (\mu+1)^2 = \mu + 2$.

Observe that

$$\begin{aligned} E(1-e^{-pN}) &= 1 - Ee^{-pN} \\ &= 1 - \sum_{n=1}^{\infty} \frac{e^{-np}e^{-\mu}\mu^{n-1}}{(n-1)!} \\ &= 1 - \sum_{n=0}^{\infty} \frac{e^{-(n+1)p}e^{-\mu}\mu^{n}}{n!} \\ &= 1 - e^{-\mu-p} \sum_{n=0}^{\infty} \frac{(\mu e^{-p})^n}{n!} \\ &= 1 - e^{\mu(e^{-p}-1)-p}. \end{aligned}$$

By Theorem 4.2, we obtain

$$\begin{aligned} &\sup_{x \in \mathbb{Z}_0^+} |P(S_N \leq x) - P(U_{\mu p} \leq x)| \\ &\leq p \min\left\{ \frac{1}{2\sqrt{1-p}}, 1 - e^{\mu e^{-p}-\mu+p} \right\} + \frac{1}{2}\sqrt{\frac{p(\mu+2)}{\mu+1}} \min\{1, 2\sqrt{p(\mu+1)}\} \\ &\leq p \min\left\{ \frac{1}{2\sqrt{1-p}}, 1 - e^{\mu e^{-p}-\mu+p} \right\} + p\sqrt{\mu+2}. \end{aligned}$$

□

REFERENCES

[1] Barbour, A.D., Hall, P.: On the rate of Poisson convergence, *Math. Proc. Cambridge Philos Soc.* 95, 473-480 (1984).

[2] Chen, L.H.Y.: On the convergence of Poisson binomial to Poisson distributions, *Ann. Probab.* 2, 178-180 (1974).

[3] Freedman, D.: The Poisson approximation for dependent events, *Ann. Probab.* 2, 256-269 (1974).

[4] Kerstan, J.: Verallgemeinerung eines Satzes von Procharov und Le Cam.Z., *Wahrsch. Verw. Febiete* 2, 173-179 (1964).

[5] Le Cam, L.: An approximation theorem for the Poisson binomial distribution, *Pacific J. Math.* 10, 1181-1197 (1960).

[6] Neammanee, K.: A nonuniform bound for the approximation of Poisson binomial by Poisson distribution, *Int. J. Math. Math. Sci.* 48, 3041-3046 (2003).

[7] Neammanee, K.: Pointwise Approximation of Poisson Binomial by Poisson Distribution, *Stochastic Model. Appl.* 6, 20-26 (2003).

[8] Teerapabolarn, K., Neammanee, K.: Poisson approximation for sums of dependent Bernoulli random variables, *Acta Math.* 22, 87-99 (2006).

[9] Yannaros, N.: Poisson approximation for random sums of Bernoulli random variables, *Statist. Probab. Lett.* 11, 161-165 (1991).

VITA

Name	:	SUB Lt. Sasithorn Kongudomthrap
Date of Birth	:	26 June 1986
Place of Birth	:	Bangkok, Thailand
Education	:	B.A. (Mathematics, second honor), Thammasart University, 2007
Scholarship	:	Science Achievement Scholarship of Thailand (SAST), 2003-2007 Centre of Excellence in Mathematics, 2009-2010
Work Experience	:	Royal Thai Naval Academy, 2011