CHAPTER III

A Uniform Bound in a Combinatorial Central Limit Theorem

Let (X_{ij}) be an $n \times n(n \geq 20)$ matrix of independent random variables with finite third moment and $\pi = (\pi(1), \pi(2), ..., \pi(n))$ be a random permutation of $\{1, 2, ..., n\}$ such that π and $X_{ij}{'}s$ are independent. In this chapter we give a uniform bound in a combinatorial central limit theorem of $\dfrac{W_n - EW_n}{\sqrt{VarW_n}}$ where $W_n = \sum_{i=1}^{n} X_{i\pi(i)}$.

Firstly, we will give some useful notations as follows:

For each $i, j \in \{1, 2, ..., n\}$, let μ_{ij} and σ_{ij}^2 be the expected value and variance of X_{ij}, respectively, and

$$\mu_{i.} = \frac{1}{n}\sum_j \mu_{ij}\ , \quad \mu_{.j} = \frac{1}{n}\sum_i \mu_{ij}, \quad \mu_{..} = \frac{1}{n^2}\sum_{i,j} \mu_{ij}$$

$$d^2 = \frac{1}{(n-1)}\sum_{i,j}(\mu_{ij} - \mu_{i.} - \mu_{.j} + \mu_{..})^2 \quad \text{and} \quad \sigma^2 = \frac{1}{n}\sum_{i,j}\sigma_{ij}^2.$$

From Ho and Chen(1978) we know that $VarW_n = d^2 + \sigma^2$.

Define

$$W = \frac{1}{\sqrt{d^2 + \sigma^2}}\sum_i (X_{i\pi(i)} - \mu_{..})$$

and let $\hat{X}_{ij} = \dfrac{1}{\sqrt{d^2 + \sigma^2}}(X_{ij} - \mu_{i.} - \mu_{.j} + \mu_{..}).$

We observe that

$$\begin{aligned}
\sum_i \hat{X}_{i\pi(i)} &= \sum_i \Big[\frac{1}{\sqrt{d^2+\sigma^2}}(X_{i\pi(i)} - \mu_{i.} - \mu_{.\pi(i)} + \mu_{..})\Big] \\
&= \frac{1}{\sqrt{d^2+\sigma^2}}\Big[\sum_i X_{i\pi(i)} - \frac{1}{n}\sum_{i,j}\mu_{ij} - \sum_i \mu_{.\pi(i)} + \sum_i \mu_{..}\Big] \\
&= \frac{1}{\sqrt{d^2+\sigma^2}}\Big[\sum_i X_{i\pi(i)} - n\mu_{..} - \sum_j \mu_{.j} + n\mu_{..}\Big] \\
&= \frac{1}{\sqrt{d^2+\sigma^2}}\Big[\sum_i X_{i\pi(i)} - n\mu_{..}\Big] \\
&= \frac{1}{\sqrt{d^2+\sigma^2}}\Big[\sum_i (X_{i\pi(i)} - \mu_{..})\Big] \\
&= W,
\end{aligned}$$

$$\begin{aligned}
EW &= E\Big[\frac{1}{\sqrt{d^2+\sigma^2}}\sum_i (X_{i\pi(i)} - \mu_{..})\Big] \\
&= \frac{1}{\sqrt{d^2+\sigma^2}}(\frac{1}{n}\sum_{i,j}\mu_{ij} - n\mu_{..}) \\
&= \frac{1}{\sqrt{d^2+\sigma^2}}(n\mu_{..} - n\mu_{..}) \\
&= 0
\end{aligned}$$

and $\qquad VarW = \dfrac{1}{d^2+\sigma^2}VarW_n = 1.$

Lemma 3.1 *Let $\hat{\mu}_{ij}$ be the expected value of $\hat{X}_{ij}$. Then*

1) $\sum_j \hat{\mu}_{ij} = 0$ *for every i and* $\sum_i \hat{\mu}_{ij} = 0$ *for every j,*

2) $\sum_{i,j} \hat{\mu}_{ij}^2 = (n-1)\Big(\dfrac{d^2}{d^2+\sigma^2}\Big).$

Proof.

1)
$$\sum_j \hat{\mu}_{ij} = \sum_j [E\hat{X}_{ij}]$$
$$= \frac{1}{\sqrt{d^2+\sigma^2}} \sum_j E[X_{ij} - \mu_{i.} - \mu_{.j} + \mu_{..}]$$
$$= \frac{1}{\sqrt{d^2+\sigma^2}} [\sum_j \mu_{ij} - \sum_j \mu_{i.} - \sum_j \mu_{.j} + n\mu_{..}]$$
$$= \frac{1}{\sqrt{d^2+\sigma^2}} [n\mu_{i.} - n\mu_{i.} - n\mu_{..} + n\mu_{..}]$$
$$= 0.$$

By the same process we can show that $\sum_i \hat{\mu}_{ij} = 0$ for every j.

2) Noting that,
$$d^2 = \frac{1}{n-1} \sum_{i,j} (\mu_{ij} - \mu_{i.} - \mu_{.j} + \mu_{..})^2$$
$$= \frac{1}{n-1} \sum_{i,j} (d^2+\sigma^2)\hat{\mu}_{ij}^2$$
$$= (\frac{d^2+\sigma^2}{n-1}) \sum_{i,j} \hat{\mu}_{ij}^2.$$

Then $\sum_{i,j} \hat{\mu}_{ij}^2 = (n-1)(\frac{d^2}{d^2+\sigma^2}).$ □

To prove the main result we need the following construction from Ho and Chen(1978).

Let I, K, L and M be random variables which are uniformly distributed on $\{1,2,...,n\}$ and $\pi = (\pi(1), \pi(2), ..., \pi(n))$, $\rho = (\rho(1), \rho(2), ..., \rho(n))$ and $\tau = (\tau(1), \tau(2), ..., \tau(n))$ are random permutations of $\{1,2,...,n\}$ i.e., $P(\pi = (l_1, l_2, ..., l_n)) = P(\rho = (l_1, l_2, ..., l_n)) = P(\tau = (l_1, l_2, ..., l_n)) = \frac{1}{n!}$ for all $(l_1, l_2, ..., l_n) \in S_n$ where S_n is the set of all permutations on $\{1,2,...,n\}$.

Assume that

$\{I, K, L, M, \pi, \rho, \tau\}$ is independent of $X_{ij}{}'s$, (3.1)

(I, K) and (L, M) are uniformly distributed on$\{(i, k) \mid i, k = 1, 2, ..., n$

and $i \neq k\}$, (3.2)

$(I, K), (L, M)$ and τ are mutually independent, (3.3)

(I, K) and ρ are independent, (3.4)

$$\rho(\alpha) = \begin{cases} \tau(\alpha) & \text{if} \quad \alpha \neq I, K, \tau^{-1}(L), \tau^{-1}(M), \\ L & \text{if} \quad \alpha = I, \\ M & \text{if} \quad \alpha = K, \\ \tau(I) & \text{if} \quad \alpha = \tau^{-1}(L), \\ \tau(K) & \text{if} \quad \alpha = \tau^{-1}(M), \end{cases} \tag{3.5}$$

where $\rho(\rho^{-1}(\alpha)) = \rho^{-1}(\rho(\alpha)) = \alpha$,

We note that there exists a system which satisfy the above conditions. (see for example, Ho,Soo-Thong(1978))

Let $S(\rho) = \sum_i \hat{X}_{i\rho(i)}$, $\tilde{S}(\rho) = S(\rho) - \hat{X}_{I\rho(I)} - \hat{X}_{K\rho(K)} + \hat{X}_{I\rho(K)} + \hat{X}_{K\rho(I)}$ and $S(\tau) = \sum_i \hat{X}_{i\tau(i)}$.

Proposition 3.2 *$(S(\rho), \tilde{S}(\rho))$ is an exchangeable pair, in the sense of*

$$P(S(\rho) \in B, \tilde{S}(\rho) \in \tilde{B}) = P(S(\rho) \in \tilde{B}, \tilde{S}(\rho) \in B)$$

for every Borel measurable subsets B and $\tilde{B}$ of $\mathbb{R}$.

Proof.

Let $a, b \in \mathbb{R}$ and S_n be the set of all permutations of $\{1, 2, ..., n\}$. Then

$$P(S(\rho) \leq a, \tilde{S}(\rho) \leq b)$$

$$= \sum_{\substack{i,k \\ i \neq k}} \sum_{(l_1, l_2, ..., l_n) \in S_n} P(\sum_j \hat{X}_{jl_j} \leq a, \hat{X}_{1l_1} + \cdots + \hat{X}_{il_k} + \cdots + \hat{X}_{kl_i} + \cdots + \hat{X}_{nl_n} \leq b,$$

$$(I, K) = (i, k), \rho = (l_1, l_2, ..., l_n))$$

$$= \sum_{\substack{i,k \\ i \neq k}} \sum_{(l_1, l_2, ..., l_n) \in S_n} P(\sum_j \hat{X}_{jl_j} \leq a, \hat{X}_{1l_1} + \cdots + \hat{X}_{il_k} + \cdots + \hat{X}_{kl_i} + \cdots + \hat{X}_{nl_n} \leq b,$$

$$(I, K) = (i, k), \rho = (l_1, ..., l_{i-1}, l_k, l_{i+1}, ..., l_{k-1}, l_i, l_{k+1}, ..., l_n))$$

$$= \sum_{\substack{i,k \\ i \neq k}} \sum_{(l_1, l_2, ..., l_n) \in S_n} P(\hat{X}_{1l_1} + \cdots + \hat{X}_{il_k} + \cdots + \hat{X}_{kl_i} + \cdots + \hat{X}_{nl_n} \leq a, \sum_j \hat{X}_{jl_j} \leq b,$$

$$(I, K) = (i, k), \rho = (l_1, l_2, ..., l_n))$$

$$= P(\tilde{S}(\rho) \leq a, S(\rho) \leq b).$$

So $(\tilde{S}(\rho), S(\rho))$ is an exchangeable pair. □

Lemma 3.3 *Let $\mathcal{B}$ be a σ-algebra generated by ρ and $X'_{ij}s$. Then*

1. $E\hat{X}_{I\rho(I)} = \frac{1}{n}\sum_i E(\hat{X}_{i\rho(i)})$ *and* $E\hat{X}_{I\rho(K)} = \frac{1}{n(n-1)}\sum_{\substack{i,k \\ i \neq k}} E(\hat{X}_{i\rho(k)})$

2. $E^{\mathcal{B}}\tilde{S}(\rho) = (1 - \frac{2}{n-1})S(\rho) + \frac{2}{n(n-1)}\sum_{i,j}\hat{X}_{ij}$

3. $E[\tilde{S}(\rho) - S(\rho)]^2 = \frac{4}{n-1}[1 - \frac{\sigma^2}{n(d^2 + \sigma^2)}]$

where $E^{\mathcal{B}}(X)$ is the conditional expectation of X with respect to $\mathcal{B}$.

Proof.

1. If I and $\{\hat{X}_{ij}, \rho\}$ are independent, we have

$$\begin{aligned} E\hat{X}_{I\rho(I)} &= \sum_i E(\hat{X}_{i\rho(i)}\chi_{B_i}) \\ &= \sum_i EE^I(\hat{X}_{i\rho(i)}\chi_{B_i}) \\ &= \sum_i E(\chi_{B_i}(E^I\hat{X}_{i\rho(i)})) \\ &= \sum_i E(\chi_{B_i})E(\hat{X}_{i\rho(i)}) \\ &= \frac{1}{n}\sum_i E(\hat{X}_{i\rho(i)}) \end{aligned}$$

where $B_i = \{\omega | I(\omega) = i\}$.

So, to prove 1 it suffices to show that I and $\{\hat{X}_{ij}, \rho\}$ are independent. To show I and $\{\hat{X}_{ij}, \rho\}$ are independent,it follows by the fact that

$$P(\hat{X}_{mn} \in A, I = i, \rho = (l_1, l_2, ..., l_n)) \quad \text{;which by (3.1)}$$

$$= P(\hat{X}_{mn} \in A)P(I = i, \rho = (l_1, l_2, ..., l_n))$$

$$= P(\hat{X}_{mn} \in A)\sum_{\substack{k \\ i\neq k}} P(I = i, K = k, \rho = (l_1, l_2, ..., l_n))$$

$$= P(\hat{X}_{mn} \in A)\sum_{\substack{k \\ i\neq k}} P((I, K) = (i, k), \rho = (l_1, l_2, ..., l_n)) \quad \text{; which by (3.4)}$$

$$= P(\hat{X}_{mn} \in A)P(\rho = (l_1, l_2, ..., l_n))\sum_{\substack{k \\ i\neq k}} P((I, K) = (i, k))$$

$$= P(\hat{X}_{mn} \in A, \rho = (l_1, l_2, ..., l_n))P(I = i).$$

Then I and $\{\hat{X}_{ij}, \rho\}$ are independent.

Similary we can show that $E\hat{X}_{I\rho(K)} = \dfrac{1}{n(n-1)} \sum\limits_{\substack{i,k \\ i \neq k}} E(\hat{X}_{i\rho(k)})$.

2. $$E^{\mathcal{B}}\tilde{S}(\rho) = E^{\mathcal{B}}[S(\rho) - \hat{X}_{I\rho(I)} - \hat{X}_{K\rho(K)} + \hat{X}_{I\rho(K)} + \hat{X}_{K\rho(I)}]$$

$$= S(\rho) - E^{\mathcal{B}}[\hat{X}_{I\rho(I)} + \hat{X}_{K\rho(K)} - \hat{X}_{I\rho(K)} - \hat{X}_{K\rho(I)}]$$

$$= S(\rho) - \frac{1}{n}\sum_{i} E^{\mathcal{B}}\hat{X}_{i\rho(i)} - \frac{1}{n}\sum_{k} E^{\mathcal{B}}\hat{X}_{k\rho(k)}$$

$$+ \frac{1}{n(n-1)}\sum_{\substack{i,k \\ i\neq k}} E^{\mathcal{B}}\hat{X}_{i\rho(k)} + \frac{1}{n(n-1)}\sum_{\substack{i,k \\ i\neq k}} E^{\mathcal{B}}\hat{X}_{k\rho(i)}$$

$$= S(\rho) - \frac{2}{n}\sum_{i} E^{\mathcal{B}}\hat{X}_{i\rho(i)} + \frac{2}{n(n-1)}\sum_{\substack{i,k \\ i\neq k}} E^{\mathcal{B}}\hat{X}_{i\rho(k)}$$

$$= S(\rho) - \frac{2}{n}\sum_{i} \hat{X}_{i\rho(i)} + \frac{2}{n(n-1)}[\sum_{i,k} \hat{X}_{i\rho(k)} - \sum_{i}\hat{X}_{i\rho(i)}]$$

$$= [1 - \frac{2}{n} - \frac{2}{n(n-1)}]S(\rho) + \frac{2}{n(n-1)}\sum_{i,j}\hat{X}_{ij}$$

$$= (1 - \frac{2}{n-1})S(\rho) + \frac{2}{n(n-1)}\sum_{i,j}\hat{X}_{ij}.$$

3. $$E[\tilde{S}(\rho) - S(\rho)]^2$$

$$= EE^{\mathcal{B}}[\tilde{S}(\rho) - S(\rho)]^2$$

$$= EE^{\mathcal{B}}[\hat{X}_{I\rho(K)} + \hat{X}_{K\rho(I)} - \hat{X}_{I\rho(I)} - \hat{X}_{K\rho(K)}]^2$$

$$= \frac{1}{n(n-1)}\sum_{\substack{i,k \\ i\neq k}} E[\hat{X}_{i\rho(k)} + \hat{X}_{k\rho(i)} - \hat{X}_{i\rho(i)} - \hat{X}_{k\rho(k)}]^2$$

$$= \frac{2}{n(n-1)}\sum_{\substack{i,k \\ i\neq k}}[E\hat{X}^2_{i\rho(i)} + E\hat{X}^2_{i\rho(k)} + E\hat{X}_{i\rho(k)}\hat{X}_{k\rho(i)} + E\hat{X}_{i\rho(i)}\hat{X}_{k\rho(k)}]$$

$$- \frac{4}{n(n-1)}\sum_{\substack{i,k \\ i\neq k}}[E\hat{X}_{i\rho(k)}\hat{X}_{i\rho(i)} + E\hat{X}_{i\rho(k)}\hat{X}_{k\rho(k)}]$$

$$= \frac{2}{n(n-1)}[(n-1)\sum_i E\hat{X}^2_{i\rho(i)} + \sum_{i,k} E\hat{X}^2_{i\rho(k)} - \sum_i E\hat{X}^2_{i\rho(i)}]$$

$$+ \frac{2}{n(n-1)} \sum_{\substack{i,k \\ i\neq k}} E\hat{X}_{i\rho(k)}\hat{X}_{k\rho(i)} + \frac{2}{n(n-1)} \sum_{\substack{i,k \\ i\neq k}} E\hat{X}_{i\rho(i)}\hat{X}_{k\rho(k)}$$

$$- \frac{4}{n(n-1)} \sum_{\substack{i,k \\ i\neq k}} E\hat{X}_{i\rho(k)}\hat{X}_{i\rho(i)} - \frac{4}{n(n-1)} \sum_{\substack{i,k \\ i\neq k}} E\hat{X}_{i\rho(k)}\hat{X}_{k\rho(k)}$$

$$= \frac{2}{n(n-1)} \sum_{i,k} E\hat{X}^2_{i\rho(k)} + \frac{2(n-2)}{n(n-1)} \sum_i E\hat{X}^2_{i\rho(i)}$$

$$+ \frac{2}{n(n-1)} \sum_{\substack{i,k \\ i\neq k}} E\hat{X}_{i\rho(k)}\hat{X}_{k\rho(i)} + \frac{2}{n(n-1)} \sum_{\substack{i,k \\ i\neq k}} E\hat{X}_{i\rho(i)}\hat{X}_{k\rho(k)}$$

$$- \frac{4}{n(n-1)} \sum_{\substack{i,k \\ i\neq k}} E\hat{X}_{i\rho(k)}\hat{X}_{i\rho(i)} - \frac{4}{n(n-1)} \sum_{\substack{i,k \\ i\neq k}} E\hat{X}_{i\rho(k)}\hat{X}_{k\rho(k)}$$

$$= \frac{2}{n(n-1)} \sum_{i,j} E\hat{X}^2_{ij} + \frac{2(n-2)}{n^2(n-1)} \sum_{i,j} E\hat{X}^2_{ij}$$

$$+ \frac{2}{n^2(n-1)^2} \sum_{\substack{i,k \\ i\neq k}} \sum_{\substack{l,m \\ l\neq m}} E\hat{X}_{im}\hat{X}_{kl} + \frac{2}{n^2(n-1)^2} \sum_{\substack{i,k \\ i\neq k}} \sum_{\substack{l,m \\ l\neq m}} E\hat{X}_{il}\hat{X}_{km}$$

$$- \frac{4}{n^2(n-1)} \sum_i \sum_{\substack{l,m \\ l\neq m}} E\hat{X}_{im}\hat{X}_{il} - \frac{4}{n^2(n-1)} \sum_{\substack{i,k \\ i\neq k}} \sum_l E\hat{X}_{il}\hat{X}_{kl}$$

$$= \frac{4}{n^2} \sum_{i,j} E\hat{X}^2_{ij} + \frac{4}{n^2(n-1)^2} \sum_{\substack{i,k \\ i\neq k}} \sum_{\substack{l,m \\ l\neq m}} E\hat{X}_{im}\hat{X}_{kl}$$

$$- \frac{4}{n^2(n-1)} \sum_i \sum_{\substack{l,m \\ l\neq m}} E\hat{X}_{im}\hat{X}_{il} - \frac{4}{n^2(n-1)} \sum_{\substack{i,k \\ i\neq k}} \sum_l E\hat{X}_{il}\hat{X}_{kl}. \tag{3.6}$$

Noting by Lemma 3.1 that ,

$$\sum_{i,j} E\hat{X}_{ij}^2 = \sum_{i,j}(Var\hat{X}_{ij} + \hat{\mu}_{ij}^2)$$

$$= \frac{1}{d^2+\sigma^2}(\sum_{i,j}\sigma_{ij}^2 + \sum_{i,j}\hat{\mu}_{ij}^2)$$

$$= \frac{n\sigma^2}{d^2+\sigma^2} + (n-1)\frac{d^2}{d^2+\sigma^2}$$

$$= n - \frac{d^2}{d^2+\sigma^2}, \tag{3.7}$$

$$\sum_{\substack{i,k \\ i\neq k}}\sum_{\substack{l,m \\ l\neq m}} E\hat{X}_{im}\hat{X}_{kl} = \sum_{\substack{i,k \\ i\neq k}}(\sum_{m,l}\hat{\mu}_{im}\hat{\mu}_{kl} - \sum_{l}\hat{\mu}_{il}\hat{\mu}_{kl})$$

$$= -\sum_{i,k,l}\hat{\mu}_{il}\hat{\mu}_{kl} + \sum_{i,l}\hat{\mu}_{il}^2$$

$$= -\sum_{i,l}\hat{\mu}_{il}\sum_{k}\hat{\mu}_{kl} + (n-1)\frac{d^2}{d^2+\sigma^2}$$

$$= (n-1)\frac{d^2}{d^2+\sigma^2}, \tag{3.8}$$

$$\sum_{i}\sum_{\substack{l,m \\ l\neq m}} E\hat{X}_{im}\hat{X}_{il} = \sum_{i}\sum_{\substack{l,m \\ l\neq m}}\hat{\mu}_{im}\hat{\mu}_{il}$$

$$= \sum_{i,l,m}\hat{\mu}_{im}\hat{\mu}_{il} - \sum_{i,l}\hat{\mu}_{il}^2$$

$$= -(n-1)\frac{d^2}{d^2+\sigma^2} \tag{3.9}$$

and,

$$\sum_{\substack{i,k \\ i\neq k}}\sum_{l} E\hat{X}_{il}\hat{X}_{kl} = \sum_{\substack{i,k \\ i\neq k}}\sum_{l}\hat{\mu}_{il}\hat{\mu}_{kl}$$

$$= \sum_{i,k,l}\hat{\mu}_{il}\hat{\mu}_{kl} - \sum_{i,l}\hat{\mu}_{il}^2$$

$$= -(n-1)\frac{d^2}{d^2+\sigma^2}. \tag{3.10}$$

From (3.6)-(3.10), we have that

$$\begin{aligned}
E[\tilde{S}(\rho)-S(\rho)]^2 &= \frac{4}{n^2}(n-\frac{d^2}{d^2+\sigma^2})+\frac{4}{n^2(n-1)}\frac{d^2}{d^2+\sigma^2}+\frac{8}{n^2}(\frac{d^2}{d^2+\sigma^2}) \\
&= \frac{4}{n}+\frac{4}{n^2(n-1)}\frac{d^2}{d^2+\sigma^2}+\frac{4}{n^2}(\frac{d^2}{d^2+\sigma^2}) \\
&= \frac{4}{n}+\frac{4}{n(n-1)}\frac{d^2}{d^2+\sigma^2} \\
&= \frac{4}{n-1}(1-\frac{1}{n}+(\frac{1}{n}\frac{d^2}{d^2+\sigma^2})) \\
&= \frac{4}{n-1}[1-\frac{\sigma^2}{n(d^2+\sigma^2)}].
\end{aligned}$$

□

Proposition 3.4((Stein,(1986), p.9) Let (X, X') be an exchangeable pair and $F: \Omega^2 \to \mathbb{R}$ be an anti-symmetric measurable function in the sense that $F(x, x') = -F(x', x)$ for all $x, x' \in \Omega$. If $E|F(X, X')| < \infty$, then $E[F(X, X')] = 0$.

Proposition 3.5

$$E[S(\rho)\sum_{i,j}\hat{X}_{ij}] = \frac{\sigma^2}{d^2+\sigma^2}.$$

Proof.

Let $F(w, w') = (w')^2 - w^2$. Hence F is anti-symmetric. By Proposition 3.2 Lemma 3.3 and Proposition 3.4 we have

$$\begin{aligned}
0 &= EF(S(\rho), \tilde{S}(\rho)) \\
&= E[\tilde{S}^2(\rho) - S^2(\rho)] \\
&= E\{2S(\rho)[\tilde{S}(\rho)-S(\rho)] + [\tilde{S}(\rho)-S(\rho)]^2\} \\
&= 2E\{E^{\mathcal{B}}[S(\rho)[\tilde{S}(\rho)-S(\rho)]]\} + E[\tilde{S}(\rho)-S(\rho)]^2 \\
&= 2ES(\rho)[E^{\mathcal{B}}\tilde{S}(\rho)-S(\rho)] + E[\tilde{S}(\rho)-S(\rho)]^2
\end{aligned}$$

$$= 2ES(\rho)\Big[(1 - \frac{2}{n-1})S(\rho) + \frac{2}{n(n-1)}\sum_{i,j}\hat{X}_{ij} - S(\rho)\Big] + E[\tilde{S}(\rho) - S(\rho)]^2$$

$$= -\frac{4}{n-1}ES^2(\rho) + \frac{4}{n(n-1)}E[S(\rho)\sum_{i,j}\hat{X}_{ij}] + E[\tilde{S}(\rho) - S(\rho)]^2. \qquad (3.11)$$

So, we have

$$E[S(\rho)\sum_{i,j}\hat{X}_{ij}] = nES^2(\rho) - \frac{n(n-1)}{4}E[\tilde{S}(\rho) - S(\rho)]^2$$

$$= n - (\frac{n(n-1)}{4})[\frac{4}{(n-1)}[1 - \frac{1}{n}(\frac{\sigma^2}{d^2+\sigma^2})]]$$

$$= \frac{\sigma^2}{d^2+\sigma^2} \qquad (3.12)$$

where we have use the fact that $ES^2(\rho) = VarW = 1$ in the second equality. □

Lemma 3.6. *Let $\delta = 10\beta$ where $\beta = \frac{1}{n}\sum_{i,j}E|\hat{X}_{ij}|^3$ and*

$$Y = \sum_{i\neq k}|\hat{X}_{i\rho(i)} + \hat{X}_{k\rho(k)} - \hat{X}_{i\rho(k)} - \hat{X}_{k\rho(i)}|\min(\delta, |\hat{X}_{i\rho(i)} + \hat{X}_{k\rho(k)} - \hat{X}_{i\rho(k)} - \hat{X}_{k\rho(i)}|).$$

Then
$$EY > \frac{11n}{5}.$$

Proof.

By the fact that

$$E|\tilde{S}(\rho) - S(\rho)|^3 = E|\hat{X}_{I\rho(I)} + \hat{X}_{K\rho(K)} - \hat{X}_{I\rho(K)} - \hat{X}_{K\rho(I)}|^3$$

$$\leq 64E|\hat{X}_{IM}|^3$$

$$= \frac{64}{n^2}\sum_{i,j}E|\hat{X}_{ij}|^3$$

$$= \frac{64\beta}{n},$$

$\min(a,b) \geq b - \dfrac{b^2}{4a}$ for $a, b > 0$ (see Appendix 2) and Lemma 3.3(2), we have

$$\begin{aligned}
EY &= n(n-1)E|\tilde{S}(\rho) - S(\rho)| \min(\delta, |\tilde{S}(\rho) - S(\rho)|) \\
&\geq n(n-1)\{E|\tilde{S}(\rho) - S(\rho)|^2 - \frac{1}{4\delta}E|\tilde{S}(\rho) - S(\rho)|^3\} \\
&\geq n(n-1)\{\frac{4}{n-1}[1 - \frac{\sigma^2}{n(d^2+\sigma^2)}] - \frac{64\beta}{4\delta n}\} \\
&\geq n(n-1)\{\frac{19}{5(n-1)} - \frac{8}{5n}\} \\
&> \frac{11n}{5}. \qquad \square
\end{aligned}$$

To prove Lemma 3.7, we introduce the following construction. Let $\overline{I}, \overline{K}, \overline{L}, \overline{M}$ be uniform distributed random variables on $\{1, 2, ..., n\}$ which satisfy the followings.

$(\overline{I}, \overline{K})and(\overline{L}, \overline{M})$are uniformly distributed random variables on

$$\{(i,k)|i,k = 1, 2, ..., n \ \text{ and } \ i \neq k\}, \tag{3.13}$$

$[(\overline{I}, \overline{K}), (\overline{L}, \overline{M})]$ is uniformly on $\{[(i,k),(l,m)]/i,k,l,m = 1, 2, ..., n$

$$\text{and } \ i \neq k, l \neq m \text{ and } (i,k) \neq (l,m)\} \tag{3.14}$$

$$\text{and } \ [(\overline{I}, \overline{K}), (\overline{L}, \overline{M})] \ \text{ and } \ \rho \ \text{ are mutually independent.} \tag{3.15}$$

Hence

$$P([(\overline{I}, \overline{K}), (\overline{L}, \overline{M})] = [(i,k),(l,m)]) = \frac{1}{n(n-1)[n(n-1)-1]}$$

for $i,k,l,m = 1, 2, ..., n, i \neq k, l \neq m$ and $(i,k) \neq (l,m)$.

Also,let

$$Z_{[(i,k),(l,m)]} = |\hat{X}_{il} + \hat{X}_{km} - \hat{X}_{im} - \hat{X}_{kl}| \min(\delta, |\hat{X}_{il} + \hat{X}_{km} - \hat{X}_{im} - \hat{X}_{kl}|),$$

$$\hat{Z}_{[(i,k),(l,m)]} = Z_{[(i,k),(l,m)]} - EZ_{[(i,k),(\rho(i),\rho(k))]},$$

$$Z(\rho) = \sum_{\substack{i,k \\ i\neq k}} \hat{Z}_{[(i,k),(\rho(i),\rho(k))]}$$

and

$$\tilde{Z}(\rho) = Z(\rho) - \hat{Z}_{[(\overline{I},\overline{K}),(\rho(\overline{I}),\rho(\overline{K}))]} - \hat{Z}_{[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))]} + \hat{Z}_{[(\overline{I},\overline{K}),(\rho(\overline{L}),\rho(\overline{M}))]} + \hat{Z}_{[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{K}))]}.$$

The following lemmas and propositions are properties of $Z(\rho)$ and Y where Y is the random variable defined in Lemma 3.6.

Lemma 3.7

1) $EZ^2(\rho) = VarY$

2) $\sum_{\substack{l,m \\ l\neq m}} E\hat{Z}_{[(i,k),(l,m)]} = 0$ for every (i,k) such that $i \neq k$

3) $(\tilde{Z}(\rho), Z(\rho))$ is an exchangeable pair.

Proof.

1) Note that

$$\begin{aligned}
Z(\rho) &= \sum_{\substack{i,k \\ i\neq k}} \hat{Z}_{[(i,k),(\rho(i),\rho(k))]} \\
&= \sum_{\substack{i,k \\ i\neq k}} (Z_{[(i,k),(\rho(i),\rho(k))]} - EZ_{[(i,k),(\rho(i),\rho(k))]}) \\
&= \sum_{\substack{i,k \\ i\neq k}} |\hat{X}_{i\rho(i)} + \hat{X}_{k\rho(k)} - \hat{X}_{i\rho(k)} - \hat{X}_{k\rho(i)}| \min(\delta, |\hat{X}_{i\rho(i)} + \hat{X}_{k\rho(k)} - \hat{X}_{i\rho(k)} - \hat{X}_{k\rho(i)}|) \\
&\quad - E(\sum_{\substack{i,k \\ i\neq k}} |\hat{X}_{i\rho(i)} + \hat{X}_{k\rho(k)} - \hat{X}_{i\rho(k)} - \hat{X}_{k\rho(i)}| \min(\delta, |\hat{X}_{i\rho(i)} + \hat{X}_{k\rho(k)} - \hat{X}_{i\rho(k)} - \hat{X}_{k\rho(i)}|))
\end{aligned}$$

$= Y - EY.$

Then we have $EZ^2(\rho) = E(Y - EY)^2 = VarY.$

2)
$$\sum_{\substack{l,m\\ l\neq m}} E\hat{Z}_{[(i,k),(l,m)]} = \sum_{\substack{l,m\\ l\neq m}} E[Z_{[(i,k),(l,m)]} - EZ_{[(i,k),(\rho(i),\rho(k))]}]$$
$$= \sum_{\substack{l,m\\ l\neq m}} EZ_{[(i,k),(l,m)]} - \sum_{\substack{l,m\\ l\neq m}} EZ_{[(i,k),(\rho(i),\rho(k))]}$$
$$= \sum_{\substack{l,m\\ l\neq m}} EZ_{[(i,k),(l,m)]} - \sum_{\substack{l,m\\ l\neq m}} EZ_{[(i,k),(l,m)]}$$
$$= 0.$$

3) By the same technique as Proposition 3.2. □

Lemma 3.8

1) $$E^{\mathcal{B}}(\tilde{Z}(\rho)) = (1 - \frac{2}{n(n-1)-1})Z(\rho) + \frac{2}{n(n-1)[n(n-1)-1]} \sum_{\substack{i,k\\ i\neq k}} \sum_{\substack{l,m\\ l\neq m}} \hat{Z}_{[(i,k),(l,m)]}$$

2) $$E[\tilde{Z}(\rho) - Z(\rho)]^2 \leq \frac{3204.4\beta^2}{n-1}.$$

Proof.

1) $E^{\mathcal{B}}(\tilde{Z}(\rho))$

$$= E^{\mathcal{B}}[Z(\rho) - \hat{Z}_{[(\overline{I},\overline{K}),(\rho(\overline{I}),\rho(\overline{K}))]} - \hat{Z}_{[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))]}$$
$$+ \hat{Z}_{[(\overline{I}.\overline{K}),(\rho(\overline{L}),\rho(\overline{M}))]} + \hat{Z}_{[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{K}))]}]$$
$$= Z(\rho) - E^{\mathcal{B}}[\hat{Z}_{[(\overline{I},\overline{K}),(\rho(\overline{I}),\rho(\overline{K}))]} + \hat{Z}_{[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))]}$$
$$- \hat{Z}_{[(\overline{I}.\overline{K}),(\rho(\overline{L}),\rho(\overline{M}))]} - \hat{Z}_{[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{K}))]}]$$

$$= Z(\rho) - \frac{2}{n(n-1)} \sum_{\substack{i,k \\ i \neq k}} E^{\mathcal{B}} \hat{Z}_{[(i,k),(\rho(i),\rho(k))]}$$

$$+ \frac{2}{n(n-1)[n(n-1)-1]} \sum_{\substack{i,k \\ i \neq k}} \sum_{\substack{l,m \\ l \neq m \\ (i,k) \neq (l,m)}} E^{\mathcal{B}} \hat{Z}_{[(i,k),(\rho(l),\rho(m))]}$$

$$= Z(\rho) - \frac{2}{n(n-1)} Z(\rho)$$

$$+ \frac{2}{n(n-1)[n(n-1)-1]} [\sum_{\substack{i,k \\ i \neq k}} \sum_{\substack{l,m \\ l \neq m}} \hat{Z}_{[(i,k),(\rho(l),\rho(m))]} - \sum_{\substack{i,k \\ i \neq k}} \hat{Z}_{[(i,k),(\rho(i),\rho(k))]}]$$

$$= [1 - \frac{2}{n(n-1)} - \frac{2}{n(n-1)[n(n-1)-1]}] Z(\rho)$$

$$+ \frac{2}{n(n-1)[n(n-1)-1]} \sum_{\substack{i,k \\ i \neq k}} \sum_{\substack{l,m \\ l \neq m}} \hat{Z}_{[(i,k),(\rho(l),\rho(m))]}$$

$$= (1 - \frac{2}{n(n-1)-1}) Z(\rho) + \frac{2}{n(n-1)[n(n-1)-1]} \sum_{\substack{i,k \\ i \neq k}} \sum_{\substack{l,m \\ l \neq m}} \hat{Z}_{[(i,k),(l,m)]}.$$

2) $E[\tilde{Z}(\rho) - Z(\rho)]^2$

$$= E[Z_{[(\overline{I},\overline{K}),(\rho(\overline{I}),\rho(\overline{K}))]} + Z_{[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))]}$$

$$- Z_{[(\overline{I},\overline{K}),(\rho(\overline{L}),\rho(\overline{M}))]} - Z_{[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{K}))]}]^2$$

$$\leq E[Z_{[(\overline{I},\overline{K}),(\rho(\overline{I}),\rho(\overline{K}))]} + Z_{[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))]}]^2$$

$$+ E[Z_{[(\overline{I},\overline{K}),(\rho(\overline{L}),\rho(\overline{M}))]} + Z_{[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{K}))]})]^2$$

$$\leq 2(EZ^2_{[(\overline{I},\overline{K}),(\rho(\overline{I}),\rho(\overline{K}))]} + EZ^2_{[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))]}$$

$$+ EZ^2_{[(\overline{I},\overline{K}),(\rho(\overline{L}),\rho(\overline{M}))]} + EZ^2_{[(\overline{L},\overline{M}),(\rho(\overline{I}),\rho(\overline{K}))]})$$

$$= 4EZ^2_{[(\overline{I},\overline{K}),(\rho(\overline{I}),\rho(\overline{K}))]} + 4EZ^2_{[(\overline{I},\overline{K}),(\rho(\overline{L}),\rho(\overline{M}))]}$$

$$= \frac{4}{n(n-1)} \sum_{\substack{i,k \\ i \neq k}} EZ^2_{[(i,k),(\rho(i),\rho(k))]}$$

$$+\frac{4}{n(n-1)[n(n-1)-1]}\sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m\\ (i,k)\neq(l,m)}} EZ^2_{[(i,k),(l,m)]}$$

$$\leq \frac{4}{n^2(n-1)^2}\sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}} EZ^2_{[(i,k),(l,m)]} + \frac{4.011}{n^2(n-1)^2}\sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}} EZ^2_{[(i,k),(l,m)]}$$

$$= \frac{8.011}{n^2(n-1)^2}\sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}} EZ^2_{[(i,k),(l,m)]}$$

$$\leq \frac{8.011\delta^2}{n^2(n-1)^2}\sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}} E|\hat{X}_{il}+\hat{X}_{km}-\hat{X}_{im}-\hat{X}_{kl}|^2$$

$$= 8.011\delta^2 E[\tilde{S}(\rho)-S(\rho)]^2 \quad \text{; which by Lemma 3.3 (2)}$$

$$\leq \frac{3204.4\beta^2}{n-1}. \qquad \square$$

Lemma 3.9. Assume that $\beta < \frac{1}{210}$. Then

$$E[\sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}} Z_{[(i,k),(l,m)]}Z(\rho)] \leq 43.779n^3(n-1)\beta.$$

Proof.

Note that

$$E[\sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}} Z_{[(i,k),(l,m)]}Z(\rho)] = E[(\sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}} \hat{Z}_{[(i,k),(l,m)]})(\sum_{\substack{p,q\\ p\neq q}} \hat{Z}_{[(p,q),(\rho(p),\rho(q))]})]$$

$$= \frac{1}{n(n-1)}\sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}}\sum_{\substack{p,q\\ p\neq q}}\sum_{\substack{r,s\\ r\neq s}} E(\hat{Z}_{[(i,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]})$$

$$= \frac{1}{n(n-1)}\sum_{(i,k,l,m,p,q,r,s)\in A} E(\hat{Z}_{[(i,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]}) \qquad (3.16)$$

where $A = \{(i,k,l,m,p,q,r,s)/i\neq k, l\neq m, p\neq q \text{ and } r\neq s\}$ and

$$A_1 = \{(i,k,l,m,p,q,r,s) \in A \;/ r \neq l, m \text{ and } s \neq l, m\}$$

$$A_2 = \{(i,k,l,m,p,q,r,s) \in A \;/ r = l, r \neq m, s \neq l \text{ and } s = m\}$$

$$A_3 = \{(i,k,l,m,p,q,r,s) \in A \;/ r \neq l, r = m, s = l \text{ and } s \neq m\}$$

$$A_4 = \{(i,k,l,m,p,q,r,s) \in A \;/ r = l, r \neq m, s \neq l \text{ and } s \neq m\}$$

$$A_5 = \{(i,k,l,m,p,q,r,s) \in A \;/ r \neq l, r = m, s \neq l \text{ and } s \neq m\}$$

$$A_6 = \{(i,k,l,m,p,q,r,s) \in A \;/ r \neq l, r \neq m, s = l \text{ and } s \neq m\}$$

$$A_7 = \{(i,k,l,m,p,q,r,s) \in A \;/ r \neq l, r \neq m, s \neq l \text{ and } s = m\}$$

$$A_8 = \{(i,k,l,m,p,q,r,s) \in A \;/ r \neq l, r = m, s = l \text{ and } s = m\}$$

$$A_9 = \{(i,k,l,m,p,q,r,s) \in A \;/ r = l, r \neq m, s = l \text{ and } s = m\}$$

$$A_{10} = \{(i,k,l,m,p,q,r,s) \in A \;/ r = l, r = m, s \neq l \text{ and } s = m\}$$

$$A_{11} = \{(i,k,l,m,p,q,r,s) \in A \;/ r = l, r = m, s = l \text{ and } s \neq m\}$$

$$A_{12} = \{(i,k,l,m,p,q,r,s) \in A \;/ r = l, r = m, s \neq l \text{ and } s \neq m\}$$

$$A_{13} = \{(i,k,l,m,p,q,r,s) \in A \;/ r = l, r \neq m, s = l \text{ and } s \neq m\}$$

$$A_{14} = \{(i,k,l,m,p,q,r,s) \in A \;/ r \neq l, r = m, s \neq l \text{ and } s = m\}$$

$$A_{15} = \{(i,k,l,m,p,q,r,s) \in A \;/ r \neq l, r \neq m, s = l \text{ and } s = m\}$$

$$A_{16} = \{(i,k,l,m,p,q,r,s) \in A \;/ r = l, r = m, s = l \text{ and } s = m\}$$

Observe that $A_8 - A_{16}$ are empty sets.

Firstly, consider the sum on A_1.

$$\sum_{A_1} E(\hat{Z}_{[(i,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]}) = \sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}}\sum_{\substack{p,q\\ p\neq q}}\sum_{\substack{r,s\\ r\neq s\\ r\neq l,m\\ s\neq l,m}} E\hat{Z}_{[(i,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]}$$

$$= \sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}}\Big(E\hat{Z}_{[(i,k),(l,m)]}\sum_{\substack{p,q\\ p\neq q}}\sum_{\substack{r,s\\ r\neq s\\ r\neq l,m\\ s\neq l,m}} E\hat{Z}_{[(p,q),(r,s)]}\Big)$$

$$\leq \sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}} E\hat{Z}_{[(i,k),(l,m)]}\sum_{\substack{p,q\\ p\neq q}}\sum_{\substack{r,s\\ r\neq s}} EZ_{[(p,q),(r,s)]}$$

$$= 0.$$

Next, we consider the sum on A_2. Note that

$$\sum_{A_2} E(\hat{Z}_{[(i,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]})$$

$$= \sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}}\sum_{\substack{p,q\\ p\neq q}} E[(Z_{[(i,k),(l,m)]} - EZ_{[(i,k),(\rho(i),\rho(k))]})(Z_{[(p,q),(l,m)]} - EZ_{[(p,q),(\rho(p),\rho(q))]})]$$

$$= \sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}}\sum_{\substack{p,q\\ p\neq q}} E[Z_{[(i,k),(l,m)]}Z_{[(p,q),(l,m)]}]$$

$$- \sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}}\sum_{\substack{p,q\\ p\neq q}}(EZ_{[(i,k),(\rho(i),\rho(k))]})(EZ_{[(p,q),(l,m)]})$$

$$- \sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}}\sum_{\substack{p,q\\ p\neq q}}(EZ_{[(p,q),(\rho(p),\rho(q))]})(EZ_{[(i,k),(l,m)]})$$

$$+ \sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}}\sum_{\substack{p,q\\ p\neq q}}(EZ_{[(i,k),(\rho(i),\rho(k))]})(EZ_{[(p,q),(\rho(p),\rho(q))]})$$

$$\leq \sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}}\sum_{\substack{p,q\\ p\neq q}} E[Z_{[(i,k),(l,m)]}Z_{[(p,q),(l,m)]}]$$

$$+ \sum_{\substack{i,k\\ i\neq k}}\sum_{\substack{l,m\\ l\neq m}}\sum_{\substack{p,q\\ p\neq q}}(EZ_{[(i,k),(\rho(i),\rho(k))]})(EZ_{[(p,q),(\rho(p),\rho(q))]}). \tag{3.17}$$

Consider the first term on the right hand side of (3.17). Follows from Hölder inequality and the fact that $a^\alpha b^\beta \leq \alpha a + \beta b$ $(a, b, \alpha, \beta > 0,\ \alpha + \beta = 1)$ we have

$$\sum_{\substack{i,k \\ i \neq k}} \sum_{\substack{l,m \\ l \neq m}} \sum_{\substack{p,q \\ p \neq q}} E[Z_{[(i,k),(l,m)]} Z_{[(p,q),(l,m)]}]$$

$$= \sum_{\substack{i,k \\ i \neq k}} \sum_{\substack{l,m \\ l \neq m}} \sum_{\substack{p,q \\ p \neq q}} E(|\hat{X}_{il} + \hat{X}_{km} - \hat{X}_{im} - \hat{X}_{kl}| \min(\delta, |\hat{X}_{il} + \hat{X}_{km} - \hat{X}_{im} - \hat{X}_{kl}|))$$

$$(|\hat{X}_{pl} + \hat{X}_{qm} - \hat{X}_{pm} - \hat{X}_{ql}| \min(\delta, |\hat{X}_{pl} + \hat{X}_{qm} - \hat{X}_{pm} - \hat{X}_{ql}|))$$

$$\leq \delta \sum_{\substack{i,k \\ i \neq k}} \sum_{\substack{l,m \\ l \neq m}} \sum_{\substack{p,q \\ p \neq q}} E(|\hat{X}_{il} + \hat{X}_{km} - \hat{X}_{im} - \hat{X}_{kl}| |\hat{X}_{pl} + \hat{X}_{qm} - \hat{X}_{pm} - \hat{X}_{ql}|^2)$$

$$\leq \delta \sum_{\substack{i,k \\ i \neq k}} \sum_{\substack{l,m \\ l \neq m}} \sum_{\substack{p,q \\ p \neq q}} (E|\hat{X}_{il} + \hat{X}_{km} - \hat{X}_{im} - \hat{X}_{kl}|^3)^{\frac{1}{3}} (E|\hat{X}_{pl} + \hat{X}_{qm} - \hat{X}_{pm} - \hat{X}_{ql}|^3)^{\frac{2}{3}}$$

$$\leq 16\delta \sum_{\substack{i,k \\ i \neq k}} \sum_{\substack{l,m \\ l \neq m}} \sum_{\substack{p,q \\ p \neq q}} (E|\hat{X}_{il}|^3 + E|\hat{X}_{km}|^3 + E|\hat{X}_{im}|^3 + E|\hat{X}_{kl}|^3)^{\frac{1}{3}}$$

$$(E|\hat{X}_{pl}|^3 + E|\hat{X}_{qm}|^3 + E|\hat{X}_{pm}|^3 + E|\hat{X}_{ql}|^3)^{\frac{2}{3}}$$

$$\leq \frac{16\delta}{3} \sum_{\substack{i,k \\ i \neq k}} \sum_{\substack{l,m \\ l \neq m}} \sum_{\substack{p,q \\ p \neq q}} (E|\hat{X}_{il}|^3 + E|\hat{X}_{km}|^3 + E|\hat{X}_{im}|^3 + E|\hat{X}_{kl}|^3)$$

$$+ \frac{32\delta}{3} \sum_{\substack{i,k \\ i \neq k}} \sum_{\substack{l,m \\ l \neq m}} \sum_{\substack{p,q \\ p \neq q}} (E|\hat{X}_{pl}|^3 + E|\hat{X}_{qm}|^3 + E|\hat{X}_{pm}|^3 + E|\hat{X}_{ql}|^3)$$

$$= 16\delta n(n-1) \sum_{\substack{i,k \\ i \neq k}} \sum_{\substack{l,m \\ l \neq m}} (E|\hat{X}_{il}|^3 + E|\hat{X}_{km}|^3 + E|\hat{X}_{im}|^3 + E|\hat{X}_{kl}|^3)$$

$$\leq 16\delta n(n-1)(4n^2 \sum_{i,j} E|\hat{X}_{ij}|^3)$$

$$= 64\delta n^4(n-1)\beta$$

$$\leq 3.048 n^4(n-1)\beta. \tag{3.18}$$

Next consider the second term on the right hand side of (3.17). We observe that

$$\Big[\sum_{\substack{i,k\\ i\neq k}} EZ_{[(i,k),(\rho(i),\rho(k))]}\Big]^2$$

$$= \Big[\sum_{\substack{i,k\\ i\neq k}} E|\hat{X}_{i\rho(i)} + \hat{X}_{k\rho(k)} - \hat{X}_{i\rho(k)} - \hat{X}_{k\rho(i)}| \min(\delta, |\hat{X}_{i\rho(i)} + \hat{X}_{k\rho(k)} - \hat{X}_{i\rho(k)} - \hat{X}_{k\rho(i)}|)\Big]^2$$

$$\leq \delta^2\Big[\sum_{\substack{i,k\\ i\neq k}} E|\hat{X}_{i\rho(i)} + \hat{X}_{k\rho(k)} - \hat{X}_{i\rho(k)} - \hat{X}_{k\rho(i)}|\Big]^2$$

$$\leq \delta^2[2(n-1)\sum_i E|\hat{X}_{i\rho(i)}| + 2\sum_{\substack{i,k\\ i\neq k}} E|\hat{X}_{i\rho(k)}|\Big]^2$$

$$= \delta^2[2(n-2)\sum_i E|\hat{X}_{i\rho(i)}| + 2\sum_{i,k} E|\hat{X}_{i\rho(k)}|\Big]^2$$

$$\leq \delta^2\Big[2n\sum_i E|\hat{X}_{i\rho(i)}| + 2\sum_{i,k} E|\hat{X}_{i\rho(k)}|\Big]^2$$

$$\leq 2\delta^2\Big[4n^2(\sum_i E|\hat{X}_{i\rho(i)}|)^2 + 4(\sum_{i,k} E|\hat{X}_{i\rho(k)}|)^2\Big]$$

$$\leq 2\delta^2\Big[4n^3\sum_i E^2|\hat{X}_{i\rho(i)}| + 4n^2\sum_{i,k} E^2|\hat{X}_{i\rho(k)}|\Big]$$

$$\leq 2\delta^2\Big[4n^3\sum_i(\sqrt{E\hat{X}^2_{i\rho(i)}})^2 + 4n^2\sum_{i,k}(\sqrt{E\hat{X}^2_{i\rho(k)}})^2\Big]$$

$$= 2\delta^2\Big[4n^3\sum_i E\hat{X}^2_{i\rho(i)} + 4n^2\sum_{i,k} E\hat{X}^2_{i\rho(k)}\Big] \quad \text{which by (3.7)}$$

$$= 2\delta^2\Big[4n^3(1 - \frac{1}{n}(\frac{d^2}{d^2+\sigma^2})) + 4n^2(n - \frac{d^2}{d^2+\sigma^2})\Big]$$

$$\leq 2\delta^2[4n^3 + 4n^3]$$

$$= 16\delta^2 n^3$$

$$\leq 7.620 n^3\beta.$$

Hence

$$\sum_{\substack{i,k \\ i\neq k}}\sum_{\substack{l,m \\ l\neq m}}\sum_{\substack{p,q \\ p\neq q}} EZ_{[(i,k),(\rho(i),\rho(k))]}EZ_{[(p,q),(\rho(p),\rho(q))]} = n(n-1)[\sum_{\substack{i,k \\ i\neq k}} EZ_{[(i,k),(\rho(i),\rho(k))]}]^2$$

$$\leq 7.6204n^4(n-1)\beta. \qquad (3.19)$$

From (3.17)-(3.19) we have

$$\sum_{A_2} E(\hat{Z}_{[(i,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]}) \leq 10.668n^4(n-1)\beta. \qquad (3.20)$$

We can use the same tecnique as (3.20) to find the sums on $A_3 - A_7$ and get the bounds as follow:

$$\sum_{A_3} E(\hat{Z}_{[(i,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]}) \leq 10.668n^4(n-1)\beta \qquad (3.21)$$

$$\sum_{A_i} E(\hat{Z}_{[(i,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]}) \leq 10.668n^4(n-1)^2\beta \quad \text{for} \quad i = 4,5,6,7 \qquad (3.22)$$

From (3.16) and (3.20)-(3.22) we have the lemma. □

Lemma 3.10. For $\delta = 10\beta$, we have

$$VarY \leq 43.970n^2\beta.$$

Proof.

Let F defined as in Proposition 3.5. So by Proposition 3.4 and Lemma 3.7 (3), we have

$$E(\tilde{Z}^2(\rho) - Z^2(\rho)) = 0. \qquad (3.23)$$

Hence, by Lemma 3.8 and Lemma 3.9,

$$
\begin{aligned}
0 &= E[(\tilde{Z}(\rho) - Z(\rho))(\tilde{Z}(\rho) + Z(\rho))] \\
&= E[(\tilde{Z}(\rho) - Z(\rho))(2Z(\rho) + \tilde{Z}(\rho) - Z(\rho))] \\
&= 2E[E^{\mathcal{B}}(\tilde{Z}(\rho) - Z(\rho))Z(\rho)] + E[\tilde{Z}(\rho) - Z(\rho)]^2 \\
&\le 2EZ(\rho)\Big[(1 - \frac{2}{[n(n-1)-1]})Z(\rho) \\
&\quad + \frac{2}{n(n-1)[n(n-1)-1]}\sum_{\substack{i,k \\ i\neq k}}\sum_{\substack{l,m \\ l\neq m}} \hat{Z}_{[(i,k),(l,m))]} - Z(\rho)\Big] + \frac{3204.4\beta^2}{n-1} \\
&= -\frac{4}{n(n-1)-1}EZ^2(\rho) + \frac{4}{n(n-1)[n(n-1)-1]}E(Z(\rho))\sum_{\substack{i,k \\ i\neq k}}\sum_{\substack{l,m \\ l\neq m}} \hat{Z}_{[(i,k),(l,m)]} \\
&\quad + \frac{3204.4\beta^2}{n-1}.
\end{aligned}
$$

Then
$$
\begin{aligned}
EZ^2(\rho) &\le \frac{1}{n(n-1)}E[Z(\rho)\sum_{\substack{i,k \\ i\neq k}}\sum_{\substack{l,m \\ l\neq m}} \hat{Z}_{[(i,k),(l,m)]}] + 3.805n\beta \\
&\le 43.779n^2\beta + 0.191n^2\beta \\
&\le 43.970n^2\beta.
\end{aligned}
$$

From this fact and Lemma 3.7 (1), $VarY \le 43.970n^2\beta.$ □

Proposition 3.11.(Concentration Inequality)

If $\beta < \frac{1}{210}$, then for $a < b$ we have

$$
P(a \le S(\rho) \le b) \le \frac{25}{12}(b-a) + 85.637\beta.
$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} -\frac{1}{2}(b-a) - \delta & if \quad t < a - \delta \\ t - \frac{1}{2}(b+a) & if \quad a - \delta \leq t \leq b + \delta \\ \frac{1}{2}(b-a) + \delta & if \quad t > b + \delta, \end{cases}$$

and

$$M(t) = (\frac{n-1}{4})(\tilde{S}(\rho) - S(\rho))[I(0 < t \leq \tilde{S}(\rho) - S(\rho)) - I(\tilde{S}(\rho) - S(\rho) \leq t < 0)]$$

where $I(E)$ is an indicator function on set E.

Let $F(w, \tilde{w}) = (\tilde{w} - w)(f(\tilde{w}) + f(w))$. Then F is antisymmetric.

Since $(S(\rho), \tilde{S}(\rho))$ is an exchangeable pair and F is antisymmetric, by Proposition 3.4 we have $EF(S(\rho), \tilde{S}(\rho)) = 0$. Thus

$$\begin{aligned} 0 &= E[\tilde{S}(\rho) - S(\rho)][f(\tilde{S}(\rho)) + f(S(\rho))] \\ &= E[\tilde{S}(\rho) - S(\rho)][2f(S(\rho)) + f(\tilde{S}(\rho)) - f(S(\rho))] \\ &= 2E[\tilde{S}(\rho) - S(\rho)]f(S(\rho)) + E[\tilde{S}(\rho) - S(\rho)][f(\tilde{S}(\rho)) - f(S(\rho))] \\ &= 2EE^{\mathcal{B}} f(S(\rho))[\tilde{S}(\rho) - S(\rho)] + E[\tilde{S}(\rho) - S(\rho)][f(\tilde{S}(\rho)) - f(S(\rho))] \\ &= 2Ef(S(\rho))[(1 - \frac{2}{n-1})S(\rho) + \frac{2}{n(n-1)} \sum_{i,j} \hat{X}_{ij} - S(\rho)] \\ &\quad + E[\tilde{S}(\rho) - S(\rho)][f(\tilde{S}(\rho)) - f(S(\rho))] \\ &= -\frac{4}{n-1} E[S(\rho) f(S(\rho))] + \frac{4}{n(n-1)} E[f(S(\rho)) \sum_{i,j} \hat{X}_{ij}] \\ &\quad + E[\tilde{S}(\rho) - S(\rho)][f(\tilde{S}(\rho)) - f(S(\rho))]. \end{aligned}$$

Let $R_0 = \frac{1}{n}E[f(S(\rho))\sum_{i,j}\hat{X}_{ij}]$. Then

$$E[S(\rho)f(S(\rho))]$$

$$= (\frac{n-1}{4})E[\tilde{S}(\rho) - S(\rho)][f(\tilde{S}(\rho)) - f(S(\rho))] + R_0$$

$$= (\frac{n-1}{4})E[\tilde{S}(\rho) - S(\rho)][\int_0^{\tilde{S}(\rho)-S(\rho)} f'(S(\rho)+t)dt] + R_0$$

$$= (\frac{n-1}{4})E[\tilde{S}(\rho) - S(\rho)]\int_{\mathbb{R}} f'(S(\rho)+t)[I(0 < t \leq \tilde{S}(\rho) - S(\rho))$$

$$- I(\tilde{S}(\rho) - S(\rho) \leq t < 0)dt] + R_0$$

$$= E[\int_{\mathbb{R}} f'(S(\rho)+t)M(t)dt] + R_0. \tag{3.24}$$

By definition of f and the fact that

$\{(x,y)|a \leq x \leq b \wedge -\delta \leq y \leq \delta\} \subseteq \{(x,y)|a-\delta \leq x+y \leq b+\delta\}$

we have

$$E[I(a \leq S(\rho) \leq b)\int_{|t|<\delta} M(t)dt] \leq E\int_{\mathbb{R}} I(a-\delta \leq S(\rho)+t \leq b+\delta)M(t)dt$$

$$= E[\int_{\mathbb{R}} f'(S(\rho)+t)M(t)dt].$$

Thus

$$E[S(\rho)f(S(\rho))]$$

$$\geq E[I(a \leq S(\rho) \leq b)\int_{|t|<\delta} M(t)dt + R_0$$

$$\geq (\frac{n-1}{4})E[I(a \leq S(\rho) \leq b)|\tilde{S}(\rho) - S(\rho)|\min(\delta, |\tilde{S}(\rho) - S(\rho)|)] + R_0$$

$$= (\frac{n-1}{4})E[I(a \leq S(\rho) \leq b)|\hat{X}_{I\rho(I)} + \hat{X}_{K\rho(K)} - \hat{X}_{I\rho(K)} - \hat{X}_{K\rho(I)}|$$

$$\min(\delta, |\hat{X}_{I\rho(I)} + \hat{X}_{K\rho(K)} - \hat{X}_{I\rho(K)} - \hat{X}_{K\rho(I)}|)] + R_0$$

$$
\begin{aligned}
&= \frac{1}{4n}\sum_{\substack{i,k\\ i\neq k}} E[I(a \le S(\rho) \le b)|\hat{X}_{i\rho(i)} + \hat{X}_{k\rho(k)} - \hat{X}_{i\rho(k)} - \hat{X}_{k\rho(i)}| \\
&\quad \min(\delta, |\hat{X}_{i\rho(i)} + \hat{X}_{k\rho(k)} - \hat{X}_{i\rho(k)} - \hat{X}_{k\rho(i)}|)] + R_0 \\
&= \frac{1}{4n}E[I(a \le S(\rho) \le b)Y] + R_0 \\
&\ge \frac{1}{4n}E[I(a \le S(\rho) \le b)YI(Y > \frac{6n}{5})] + R_0 \\
&\ge \frac{3}{10}E[I(a \le S(\rho) \le b)I(Y > \frac{6n}{5})] + R_0 \\
&\ge \frac{3}{10}E[I(a \le S(\rho) \le b) - I(a \le S(\rho) \le b, Y \le \frac{6n}{5})] + R_0 \\
&\ge \frac{3}{10}E[I(a \le S(\rho) \le b) - I(Y \le \frac{6n}{5})] + R_0 \\
&= \frac{3}{10}[P(a \le S(\rho) \le b) - P(Y \le \frac{6n}{5})] + R_0. \qquad (3.25)
\end{aligned}
$$

By Proposition 3.5,

$$
E(\sum_{i,j}\hat{X}_{ij})^2 = nE(S(\rho)\sum_{i,j}\hat{X}_{ij}) = \frac{n\sigma^2}{\sigma^2 + d^2}
$$

which implies

$$
\begin{aligned}
|R_0| &= \frac{1}{n}|E[f(S(\rho))\sum_{i,j}\hat{X}_{ij}]| \\
&\le \frac{1}{n}(\frac{1}{2}(b-a) + \delta)\sqrt{E(\sum_{i,j}\hat{X}_{ij})^2} \\
&\le \frac{1}{\sqrt{n}}(\frac{1}{2}(b-a) + \delta). \qquad (3.26)
\end{aligned}
$$

Hence, by (3.25)-(3.26),

$$\begin{aligned}
P(a \le S(\rho) \le b) &\le P(Y \le \frac{6n}{5}) + \frac{10}{3}E[S(\rho)f(S(\rho))] - \frac{10}{3}R_0 \\
&\le \frac{10}{3}E[S(\rho)f(S(\rho))] + P(EY - Y \ge n) - \frac{10}{3}R_0 \\
&\le \frac{10}{3}(\frac{1}{2}(b-a) + \delta) + \frac{VarY}{n^2} - \frac{10}{3}R_0 \\
&= \frac{25}{12}(b-a) + 85.637\beta. \qquad \square
\end{aligned}$$

Proof of Main Result

In this section, we will prove our main result by using Stein's method. In 1972, Stein gave a new method which free from Fourier transformation to find an error bound in normal approximation. His method based on the differential equation

$$f'(w) - wf(w) = h(w) - E(h(Z)) \tag{3.27}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function , $h : \mathbb{R} \to \mathbb{R}$ is a test function and Z is the standard normal random variable. We always call (3.27) Stein's equation for normal approximation. For any real number z, let h be an indicator function defined by

$$h_z(x) = \begin{cases} 1 & \text{if } x \le z, \\ 0 & \text{if } x > z, \end{cases} \tag{3.28}$$

then Stein's equation (3.27) has a unique solution $f_z : \mathbb{R} \to \mathbb{R}$ defined by

$$f_z(w) = \begin{cases} \sqrt{2\pi}e^{\frac{1}{2}w^2}\Phi(w)[1-\Phi(z)] & \text{if } w \le z, \\ \sqrt{2\pi}e^{\frac{1}{2}w^2}\Phi(z)[1-\Phi(w)] & \text{if } w > z \end{cases} \tag{3.29}$$

where Φ is the standard normal distribution function.

By (3.27)-(3.29) we have

$$f_z'(W) - Wf_z(W) = h(W) - \Phi(z)$$

which implies that

$$P(W \le z) - \Phi(z) = Ef_z'(W) - EWf_z(W). \tag{3.30}$$

So, we can find a bound of $Ef_z'(W) - EWf_z(W)$ instead of $P(W \le z) - \Phi(z)$. To do this, we also need the some propeties of the solution f_z of Stein's equation (3.27) as follows. For $s, t, w \in \mathbb{R}$

$$f_z'(w+s) - f_z'(w+t) \le \begin{cases} 1 & \text{if } w+s \le z, w+t > z, \\ (|w| + \frac{\sqrt{2\pi}}{4})(|s|+|t|) & \text{if } s \ge t, \\ 0 & \text{otherwise,} \end{cases} \tag{3.31}$$

$$f_z'(w+s) - f_z'(w+t) \ge \begin{cases} -1 & \text{if } w+s \ge z, w+t < z, \\ -(|w| + \frac{\sqrt{2\pi}}{4})(|s|+|t|) & \text{if } s < t, \\ 0 & \text{otherwise,} \end{cases} \tag{3.32}$$

(Chen and Shao (2001) ,pages 246-247)

$$|f_z'(w)| \le 1, \tag{3.33}$$

(Stein(1986) ,page 23)

$$|Ef_z'(S(\tau))E\int_{\mathbb{R}} M(t)dt - Ef_z'(S(\tau))\int_{\mathbb{R}} M(t)dt| < \frac{16}{n}, \tag{3.34}$$

and

$$|Ef_z(S(\rho))\sum_{i,j}\hat{X}_{ij}| \quad \leq \quad \frac{1}{n} \tag{3.35}$$

(Ho and Chen(1978) ,pages 243-245).

We also use the same notations as in the previous section and, without loss of generality, we assume that $\beta < \frac{1}{210}$. Using the same argument in the proof of (3.24) we can show that

$$E[S(\rho)f_z(S(\rho))] = E\int_{\mathbb{R}} f_z'(S(\rho)+t)M(t)dt + \frac{1}{n}E[f_z(S(\rho))\sum_{i,j}\hat{X}_{ij}] \tag{3.36}$$

and

$$E[S^2(\rho)] = E\int_{\mathbb{R}} M(t)dt + \frac{1}{n}E[S(\rho)\sum_{i,j}\hat{X}_{ij}]. \tag{3.37}$$

Hence

$$1 = E\int_{\mathbb{R}} M(t)dt + \frac{1}{n}E[S(\rho)\sum_{i,j}\hat{X}_{ij}]. \tag{3.38}$$

Then

$$\begin{aligned}
&P(W \leq z) - \Phi(z)\\
&= Ef_z'(W) - EWf_z(W)\\
&= Ef_z'(S(\tau)) - E(S(\rho)f_z(S(\rho))) \qquad \text{; which by (3.38)}\\
&= Ef_z'(S(\tau))E\int_{\mathbb{R}} M(t)dt + \frac{1}{n}E[S(\rho)\sum_{i,j}\hat{X}_{ij}]Ef_z'(S(\tau)) - E[S(\rho)f_z(S(\rho))]
\end{aligned}$$

; which by Proposition 3.5,(3.33),(3.34) and (3.36)

$$\leq E\int_{\mathbb{R}} f'_z(S(\tau))M(t)dt + \frac{17}{n} - E\int_{\mathbb{R}} f'_z(S(\rho)+t)M(t)dt$$

$$- \frac{1}{n}E[f_z(S(\rho))\sum_{i,j}\hat{X}_{ij}] \qquad \text{; which by (3.35)}$$

$$\leq E\int_{\mathbb{R}} (f'_z(S(\tau)) - f'_z(S(\rho)+t))M(t)dt + \frac{18}{n}$$

$$= E\int_{\mathbb{R}} (f'_z(S(\tau)) - f'_z(S(\tau)+\Delta S+t))M(t)dt + \frac{18}{n}$$

$$\text{where } \Delta S = S(\rho) - S(\tau) \quad \text{; which by (3.31)}$$

$$\leq E\int_{\substack{S(\tau)\leq z\\ S(\tau)+\Delta S+t>z}} M(t)dt$$

$$+ E\int_{\Delta S+t\leq 0} (|S(\tau)| + \frac{\sqrt{2\pi}}{4})(|\Delta S| + |t|)M(t)dt + \frac{18}{n}. \tag{3.39}$$

By (3.39) and the fact that

$$\{\omega|S(\tau)(\omega)\leq z, S(\tau)(\omega)+\Delta S(\omega)+t>z\}$$

$$= \{\omega|z-\Delta S(\omega)-t < S(\tau)(\omega)\leq z)\}\cap\{\omega|\Delta S(\omega)+t>0\}$$

we have

$$E\int_{\substack{S(\tau)\leq z\\ S(\tau)+\Delta S+t>z}} M(t)dt$$

$$= E\int_{\Delta S+t>0} I(z-\Delta S-t\leq S(\tau)\leq z)M(t)dt$$

$$= E(E^{\Delta S}\int_{\Delta S+t>0} I(z-\Delta S-t\leq S(\tau)\leq z)M(t)dt)$$

$$= E\int_{\Delta S+t>0} M(t)E^{\Delta S}I(z-\Delta S-t\leq S(\tau)\leq z)dt$$

$$= E\int_{\Delta S+t>0} P(z-\Delta S-t\leq S(\tau)\leq z|\Delta S)M(t)dt,$$

we have

$$
\begin{aligned}
&P(W \le z) - \Phi(z) \\
&\le E\int_{\Delta S+t>0} P(z-\Delta S - t \le S(\tau) \le z|\Delta S)M(t)dt \\
&\quad + E\int_{\Delta S+t\le 0}(|S(\tau)| + \frac{\sqrt{2\pi}}{4})(|\Delta S| + |t|)M(t)dt + \frac{18}{n} \\
&= E\int_{\Delta S+t>0} P(z - t \le S(\rho) \le z + \Delta S|\Delta S)M(t)dt \\
&\quad + E\int_{\Delta S+t\le 0}(|S(\tau)| + \frac{\sqrt{2\pi}}{4})(|\Delta S| + |t|)M(t)dt + \frac{18}{n} \quad \text{;which by Proposition 3.11} \\
&\le \frac{25}{12}E\int_{\Delta S+t>0}(|\Delta S| + |t|)M(t)dt + 85.637\beta\int_{\Delta S+t>0} EM(t)dt \\
&+ E\int_{\mathbb{R}}|S(\tau)|(|\Delta S| + |t|)M(t)dt + \frac{\sqrt{2\pi}}{4}E\int_{\Delta S+t\le 0}(|\Delta S| + |t|)M(t)dt + \frac{18}{n} \\
&\le \frac{25}{12}E\int_{\mathbb{R}}|\Delta S|M(t)dt + \frac{25}{12}E\int_{\mathbb{R}}|t|M(t)dt + 85.637\beta \\
&\quad + E\int_{\mathbb{R}}|S(\tau)||\Delta S|M(t)dt + E\int_{\mathbb{R}}|S(\tau)||t|M(t)dt + \frac{18}{n}. \qquad (3.40)
\end{aligned}
$$

Noting that

$$
\begin{aligned}
E\int_{\mathbb{R}}|t|M(t)dt &= (\frac{n-1}{8})E|\tilde{S}(\rho) - S(\rho)|^3 \\
&\le (\frac{n-1}{8})(\frac{64\beta}{n}) \\
&\le 8\beta, \qquad (3.41)
\end{aligned}
$$

$$
E\int_{\mathbb{R}}|S(\tau)||t|M(t)dt \;=\; E|S(\tau)|E\int_{\mathbb{R}}|t|M(t)dt \;\le\; 8\beta.
$$

By the fact that

$$E|\Delta S|^3 = E|\hat{X}_{\tau^{-1}(L)\tau(I)} + \hat{X}_{\tau^{-1}(M)\tau(K)} + \hat{X}_{IL} + \hat{X}_{KM} - \hat{X}_{I\tau(I)} - \hat{X}_{K\tau(K)} - \hat{X}_{\tau^{-1}(L)L} - \hat{X}_{\tau^{-1}(M)M}|^3$$

$$\leq 512E|\hat{X}_{IM}|^3$$

$$\leq \frac{512\beta}{n},$$

we have

$$E\int_{\mathbb{R}} |\Delta S|M(t)dt = (\frac{n-1}{4})E|\Delta S||\tilde{S}(\rho) - S(\rho)|^2$$

$$\leq (\frac{n-1}{4})\{E|\Delta S|^3\}^{\frac{1}{3}}\{E|\tilde{S}(\rho) - S(\rho)|^3\}^{\frac{2}{3}}$$

$$\leq 32\beta \tag{3.42}$$

and

$$E\int_{\mathbb{R}} |S(\tau)||\Delta S|M(t)dt \leq (\frac{n-1}{4})\{E|\Delta S|^3\}^{\frac{1}{3}}\{E|S(\tau)|^{\frac{3}{2}}E|\tilde{S}(\rho) - S(\rho)|^3\}^{\frac{2}{3}}$$

$$\leq 32\beta. \tag{3.43}$$

Hence, by (3.40)-(3.43),we have that

$$P(W \leq z) - \Phi(z) \leq 210\beta + \frac{18}{n}$$

and by the same method we can show that

$$P(W \leq z) - \Phi(z) \geq -210\beta - \frac{18}{n}.$$

Thus, We have $\sup_{\mathbb{R}} |F_n(z) - \Phi(z)| \leq 210\beta + \frac{18}{n}$. □