CHAPTER III

TRANSFORMATION SEMIGROUPS

In this chapter, we characterize various transformation semigroups, including well-known ones, which admit the structure of an AC semiring with zero.

Throughout this chapter, the following notation will be used. For any set X, let

G_X = the permutation group on X (the symmetric group on X),

P_X = the partial transformation semigroup on X,

T_X = the full transformation semigroup on X,

I_X = the 1-1 partial transformation semigroup on X (the symmetric inverse semigroup on X),

U_X = the semigroup of all almost identical partial transformations of X,

V_X = the semigroup of all almost identical transformations of X,

W_X = the semigroup of all almost identical 1-1 partial transformations of X,

M_X = the semigroup of all 1-1 transformations of X,

E_X = the semigroup of all onto transformations of X,

C_X = the semigroup of all constant partial transformations of X,

F_X = the semigroup of all constant transformations of X.

For any set A, let 1_A denote the identity map on A where 1_ϕ is the empty transformation 0. If X is a set, $\phi \neq A \subseteq X$ and $x \in X$, we let A_x denote the constant partial transformation of X with domain A and range $\{x\}$. Hence for any set X,

$$C_X = \{A_x \mid \phi \neq A \subseteq X,\ x \in X\} \cup \{0\} ,$$

$$F_X = \{X_x \mid x \in X\} .$$

If $a_1, a_2, \ldots, a_n$ are elements of a set X, then the notation $(a_1, a_2, \ldots, a_n)$ denotes the permutation of X defined by

$$x(a_1, a_2, \ldots, a_n) = \begin{cases} a_{i+1} & \text{if } x = a_i \ , \ i = 1,2,\ldots,n-1, \\ a_1 & \text{if } x = a_n \ , \\ x & \text{otherwise.} \end{cases}$$

For a set X and for a cardinal number ξ with $1 \leqslant \xi \leqslant |X|$, let R_ξ, D_ξ, $\bar{R}_\xi$ and $\bar{D}_\xi$ denote the following transformation semigroups :

$$R_\xi = \{\alpha \in P_X \mid |\nabla\alpha| < \xi\} ,$$

$$D_\xi = \{\alpha \in P_X \mid |\Delta\alpha| < \xi\} ,$$

$$\bar{R}_\xi = \{\alpha \in P_X \mid |\nabla\alpha| \leqslant \xi\} ,$$

$$\bar{D}_\xi = \{\alpha \in P_X \mid |\Delta\alpha| \leqslant \xi\} .$$

The first theorem gives a characterization of G_X, M_X and E_X which admit the structure of an AC semiring with zero.

3.1 <u>Theorem.</u> Let X be a set and let $S = G_X$, M_X or E_X. Then S admits the structure of an AC semiring with zero if and only if $|X| \leqslant 2$.

Proof : Assume that S admits the structure of an AC semiring with zero under an addition + . To show $|X| \leq 2$, suppose on the contrary that $|X| \geq 3$. Let a,b and c be three distinct elements in X. Then $(a,b,c),(a,c) \in S$, so

$$(a,b,c) + (a,c) = \alpha$$

for some α in S^0.

Case $\alpha = 0$. That is, $(a,b,c) + (a,c) = 0$. Then we have

$$(a,b,c)(a,c) + (a,c)(a,c) = 0$$

and

$$(a,c)(a,b,c) + (a,c)(a,c) = 0 ,$$

which imply that

$$(a,b) + 1_X = 0 ,$$

$$(b,c) + 1_X = 0 ,$$

respectively. Hence

$$(b,c) = 0 + (b,c) = (a,b) + 1_X + (b,c) = (a,b) + ((b,c) + 1_X)$$

$$= (a,b) + 0 = (a,b) \text{ , a contradiction.}$$

Case $\alpha \neq 0$. Then $\alpha : X \to X$ is 1-1 if $\alpha \in G_X$ or M_X and $\alpha : X \to X$ is onto if $\alpha \in G_X$ or E_X.

Subcase 1 : α is 1-1. Since

$$(a,b)(a,b,c) + (a,b)(a,c) = (a,b)((a,b,c) + (a,c)) = (a,b)\alpha ,$$

we have that

$$(a,c) + (a,b,c) = (a,b)\alpha$$

so $\alpha = (a,b)\alpha$ which implies $a\alpha = a(a,b)\alpha = b\alpha$. Since α is 1-1 ,

$a = b$, a contradiction.

<u>Subcase 2 : α is onto.</u> Since

$$(a,b,c)(a,c) + (a,c)(a,c) = ((a,b,c) + (a,c))(a,c) = \alpha(a,c) ,$$

we have that

$$(a,b) + 1_X = \alpha(a,c) .$$

Therefore

$$(a,b)(a,b) + 1_X(a,b) = \alpha(a,c)(a,b) = \alpha(a,c,b)$$

and thus

$$1_X + (a,b) = \alpha(a,c,b) .$$

It then follows that $\alpha(a,c) = \alpha(a,c,b)$. Since α is onto, there is an element x in X such that $x\alpha = c$. Then

$$a = c(a,c) = x\alpha(a,c) = x\alpha(a,c,b) = b ,$$

a contradiction.

Conversely, assume that $|X| \leqslant 2$. Then $S = G_X = M_X = E_X$. If $|X| = 0$ or 1, then $|S| = 1$, so S admits the structure of an AC semiring with zero. If $X = \{a,b\}$, $a \neq b$, then $S = \{1_X, (a,b)\}$, so S is a cyclic group of order 2. By Proposition 2.6, S admits the structure of an AC semiring with zero.

We shall characterize an alternating group admitting the structure of an AC semiring with zero in the following theorem.

3.2 <u>Theorem.</u> Let n be a positive integer such that $n \geqslant 2$ and A_n the alternating group of degree n. Then A_n admits the structure of an AC semiring with zero if and only if $n \leqslant 3$.

Proof : Assume A_n admits the structure of an AC semiring with zero under an addition + . Suppose $n \geq 4$, let a,b,c and d be four distinct elements in X. Therefore $(a,b)(c,d)$, $(a,d)(b,c) \in A_n$, so

$$(a,b)(c,d) + (a,d)(b,c) = \alpha$$

for some $\alpha \in A_n^0$.

Case $\alpha = 0$. Then $(a,b)(c,d) + (a,d)(b,c) = 0$. Since $(a,b)(c,d) = (a,b,c)(a,d,c)$ and $(a,d)(b,c) = (b,d,c)(a,d,c)$, we have that

$$(a,b,c)(a,d,c) + (b,d,c)(a,d,c) = 0 ,$$

so

$$((a,b,c) + (b,d,c))(a,d,c) = 0$$

which implies that

$$(a,b,c) + (b,d,c) = 0 .$$

Then

$$(a,c,b)((a,b,c) + (b,d,c)) = 0$$

and hence

$$1_X + (a,b)(c,d) = 0 .$$

Therefore

$$1_X + (a,b)(c,d) + (a,d)(b,c) = (a,d)(b,c)$$

Since $(a,b)(c,d) + (a,d)(b,c) = 0$, we have $1_X = (a,d)(b,c)$, a contradiction.

Case $\alpha \neq 0$. Then

$$(a,c)(b,d)((a,b)(c,d) + (a,d)(b,c)) = (a,c)(b,d)\alpha$$

which implies that

$$(a,d)(b,c) + (a,b)(c,d) = (a,c)(b,d)\alpha .$$

Since $(a,b)(c,d) + (a,d)(b,c) = \alpha$, $\alpha = (a,c)(b,d)\alpha$, so $(a,c)(b,d) = 1_X$ (since $\alpha \neq 0$) , a contradiction.

Conversely, assume $n \leq 3$. If $n = 2$, then $|A_2| = 1$, so A_2

admits the structure of an AC semiring with zero. If n = 3 , then A_3 is a cyclic group of order 3, so it admits the structure of an AC semiring with zero by Proposition 2.6. #

3.3 Theorem. For a set X, the full transformation semigroup on X, T_X , admits the structure of an AC semiring with zero if and only if $|X| \leqslant 1$.

Proof : Assume that there exists an operation + on T_X^o such that $(T_X^o,+,\cdot)$ is an AC semiring with zero where $\cdot$ is the operation of T_X^o. To show $|X| \leqslant 1$, suppose on the contrary that $|X| \geqslant 2$. Let a and b be two distinct elements in X. Then

$$X_a + X_b = \alpha$$

for some α in T_X^o.

Case $\alpha = 0$. That is, $X_a + X_b = 0$. Then we have

$$0 = 0X_a = (X_a + X_b)X_a = X_aX_a + X_bX_a = X_a + X_a$$

which implies that

$$X_a = X_a + 0 = X_a + (X_a + X_b) = (X_a + X_a) + X_b = 0 + X_b = X_b,$$

a contradiction.

Case $\alpha \neq 0$. Since

$$X_a(X_a + X_b) = X_a\alpha \quad , \quad (X_a + X_b)X_a = \alpha X_a \ ,$$

we have that

$$X_a + X_b = X_a\alpha \quad , \quad X_a + X_a = \alpha X_a \ ,$$

respectively. Since $X_a\alpha = X_{a\alpha}$ and $\alpha X_a = X_a$, it follows that

$$X_a + X_b = X_{a\alpha} \quad , \quad X_a + X_a = X_a .$$

From $X_a + X_b = X_{a\alpha}$, we have that

$$\begin{aligned} X_{a\alpha} &= X_a + X_b \\ &= X_b + X_a \\ &= X_a(a,b) + X_b(a,b) \\ &= (X_a + X_b)(a,b) \\ &= X_{a\alpha}(a,b) . \end{aligned}$$

Hence $X_{a\alpha} = X_{a\alpha}(a,b) = \begin{cases} X_b & \text{if } a\alpha = a , \\ X_a & \text{if } a\alpha = b , \\ X_{a\alpha} & \text{if } a\alpha \neq a \text{ and } a\alpha \neq b .. \end{cases}$

If $a\alpha = a$, then $X_a = X_{a\alpha} = X_b$, a contradiction. If $a\alpha = b$, then $X_b = X_{a\alpha} = X_a$, a contradiction. If $a\alpha \neq a$ and $a\alpha \neq b$, let β be an element of T_X defined by

$$x\beta = \begin{cases} a & \text{if } x = a \text{ or } x = b , \\ x & \text{otherwise} , \end{cases}$$

so we have that

$$X_a = X_a + X_a = X_a\beta + X_b\beta = (X_a + X_b)\beta = X_{a\alpha}\beta = X_{a\alpha} ,$$

which is a contradiction since $a\alpha \neq a$.

The converse is obvious since $|T_X| = 1$ if $|X| \leqslant 1$. #

The next five theorems give characterizations of the transformation semigroups P_X, I_X, U_X, W_X and C_X which admit the structure of an AC semiring with zero. To prove these theorems, the following lemma is required.

3.4 <u>Lemma</u>. Let a and b be two distinct elements of a set X. If S is a subsemigroup of the partial transformation semigroup of X, P_X, containing $\{a\}_a$, $\{a\}_b$, $\{b\}_a$, then S does not admit the structure of an AC semiring with zero.

<u>Proof</u> : Assume on the contrary that S admits the structure of an AC semiring with zero under an addition + . Then

$$\{a\}_a + \{a\}_b = \alpha$$

for some α in S^o. Then

$$\{a\}_a\alpha = \{a\}_a(\{a\}_a + \{a\}_b) = \{a\}_a + \{a\}_b = \alpha$$

and

$$\alpha\{a\}_a = (\{a\}_a + \{a\}_b)\{a\}_a = \{a\}_a + 0 = \{a\}_a .$$

If follows from $\{a\}_a = \alpha\{a\}_a$ that $\alpha \neq 0$, $a \in \Delta\alpha$, $a \in \nabla\alpha$ and $a\alpha = a$. Then $\{a\}_a\alpha = \{a\}_a$, and hence $\{a\}_a = \{a\}_a\alpha = \alpha = \{a\}_a + \{a\}_b$. Since $\{b\}_a \in S$, we have that

$$0 = \{a\}_a\{b\}_a = (\{a\}_a + \{a\}_b)\{b\}_a = 0 + \{a\}_a = \{a\}_a ,$$

a contradiction. #

If a set X contains two distinct elements a,b, then $\{a\}_a$, $\{a\}_b$, $\{b\}_a$ are elements of the transformation semigroups P_X, I_X, U_X, W_X and C_X. Hence the five following theorems are obtained directly from Lemma 3.4.

3.5 Theorem. For a set X, the partial transformation semigroup on X, P_X, admits the structure of an AC semiring with zero if and only if $|X| \leq 1$.

3.6 Theorem. For a set X, the symmetric inverse semigroup on X, I_X, admits the structure of an AC semiring with zero if and only if $|X| \leq 1$.

3.7 Theorem. For a set X, the semigroup of all almost identical partial transformations of X, U_X, admits the structure of an AC semiring with zero if and only if $|X| \leq 1$.

3.8 Theorem. For a set X, the semigroup of all almost identical 1-1 partial transformations of X, W_X, admits the structure of an AC semiring with zero if and only if $|X| \leq 1$.

3.9 Theorem. For a set X, the semigroup of all constant partial transformations of X, C_X, admits the structure of an AC semiring with zero if and only if $|X| \leq 1$.

The next theorem gives a characterization of the semigroup of all almost identical transformations of a set X, V_X, which admits the structure of an AC semiring with zero in term of the cardinality of X. First, we require the following lemma.

3.10 Lemma. If the semigroup of all almost identical transformations of a set X, V_X, admits the structure of an AC semiring with zero, then X is finite.

<u>Proof</u> : Assume that the semigroup V_X admits the structure of an AC semiring with zero under an addition + . To show the set X is finite, suppose on the contrary that X is infinite. Claim that $1_X + 1_X = 0$ or $1_X + 1_X = 1_X$. Suppose that $1_X + 1_X = \alpha \in V_X^0$, $\alpha \neq 0$ and $\alpha \neq 1_X$. Since $\alpha \neq 0$, $\alpha \in V_X$. Since $\alpha \neq 1_X$, $S(\alpha) \neq \phi$ where $S(\alpha) = \{x \in X \mid x\alpha \neq x\}$, the shift of α. Then $S(\alpha)$ contains some element of X, say a. Therefore $a\alpha \neq a$. Since $\alpha \in V_X$, $|S(\alpha)| < \infty$. Thus $X \setminus S(\alpha) \neq \phi$ because X is infinite. Let b be an element of $X \setminus S(\alpha)$. Then $b\alpha = b$. Hence

$$\begin{aligned} \alpha &= 1_X + 1_X \\ &= (a,b)1_X(a,b) + (a,b)1_X(a,b) \\ &= (a,b)(1_X + 1_X)(a,b) \\ &= (a,b)\alpha(a,b) , \end{aligned}$$

so $b = b\alpha = b(a,b)\alpha(a,b) = a\alpha(a,b)$. Since $a\alpha \neq a$, we have that either $a\alpha = b$ or $a\alpha \notin \{a,b\}$. If $a\alpha = b$, then $b = a\alpha(a,b) = b(a,b) = a$, a contradiction. If $a\alpha \notin \{a,b\}$, then $a\alpha = a\alpha(a,b) = b \in \{a,b\}$, a contradiction. Hence we prove the claim. It then follows that either $\beta + \beta = 0$ for all $\beta \in V_X$ or $\beta + \beta = \beta$ for all $\beta \in V_X$. In particular, $\beta + \beta = 0$ or $\beta + \beta = \beta$ for every $\beta \in V_X$.

For convenience, for x,y in X, let the notation $(x \to y)$ denote the element of V_X defined by

$$t(x \to y) = \begin{cases} y & \text{if } t = x , \\ t & \text{otherwise.} \end{cases}$$

Let a,b be two distinct elements in X. Then $(a \to b)$ and $(b \to a) \in V_X$, so

$$(a \to b) + (b \to a) = \alpha$$

for some α in V_X^o. Hence

$$(a \to b)(a \to b) + (b \to a)(a \to b) = \alpha(a \to b) ,$$

$$(a \to b)(a,b) + (b \to a)(a,b) = \alpha(a,b)$$

which imply that

$$(a \to b) + (a \to b) = \alpha(a \to b) ,$$

$$(b \to a) + (a \to b) = \alpha(a,b) ,$$

respectively. The last equality gives $\alpha = \alpha(a,b)$ since $(a \to b)+(b \to a) = \alpha$.

Case $\alpha = 0$. Then $(a \to b)+(b \to a) = \alpha = 0 = \alpha(a \to b) = (a \to b)+(a \to b)$. Hence

$$\begin{aligned} (a \to b) &= (a \to b) + 0 \\ &= (a \to b) + ((a \to b) + (b \to a)) \\ &= ((a \to b) + (a \to b)) + (b \to a) \\ &= 0 + (b \to a) \\ &= (b \to a) , \end{aligned}$$

a contradiction.

Case $\alpha \neq 0$. Then $(a \to b)+(a \to b) = \alpha(a \to b) \neq 0$, which implies that $(a \to b)+(a \to b) = (a \to b)$ since $\beta + \beta = 0$ or $\beta + \beta = \beta$ for every $\beta \in V_X$. Thus $(a \to b) = \alpha(a \to b)$. Since $\alpha = \alpha(a,b)$ and for $x \in X$,

$$x\alpha(a,b) = \begin{cases} b & \text{if } x\alpha = a , \\ a & \text{if } x\alpha = b , \end{cases}$$

it follows that $a,b \notin \nabla\alpha$. In particular, $a\alpha \neq a$. Since $(a \to b) = \alpha(a \to b)$ and $a\alpha \neq a$, we have that $b = a(a \to b) = a\alpha(a \to b) = (a\alpha)(a \to b) = a\alpha \in \nabla\alpha$, a contradiction. #

3.11 Theorem. For a set X, the semigroup of all almost identical transformations of a set X, V_X, admits the structure of an AC semiring with zero if and only if $|X| \leqslant 1$.

Proof : Assume that V_X admits the structure of an AC semiring with zero under an addition +. Then the set X is finite by **Lemma** 3.10. Hence $V_X = T_X$, the full transformation semigroup on X. It then follows from Theorem 3.3 that $|X| \leqslant 1$.

The converse is obvious since $|V_X| = 1$, if $|X| \leqslant 1$. #

If X is a set, then $F_X = \{X_x \mid x \in X\}$ and $X_x X_y = X_y$ for all $x, y \in X$, hence F_X is a right zero semigroup, so it admits the structure of an AC semiring with zero by Proposition 2.1. Therefore we have

3.12 Theorem. For a set X, the semigroup of all constant transformations of X, F_X, admits the structure of an AC semiring with zero.

Recall that for a set X and a cardinal number ξ with $1 \leqslant \xi \leqslant |X|$, R_ξ, D_ξ, $\bar{R}_\xi$ and $\bar{D}_\xi$ denote the following transformation semigroups :

$$R_\xi = \{\alpha \in P_X \mid |\nabla\alpha| < \xi\} ,$$

$$D_\xi = \{\alpha \in P_X \mid |\Delta\alpha| < \xi\} ,$$

$$\bar{R}_\xi = \{\alpha \in P_X \mid |\nabla\alpha| \leqslant \xi\} ,$$

$$\bar{D}_\xi = \{\alpha \in P_X \mid |\Delta\alpha| \leqslant \xi\} .$$

The next two theorems give characterizations of these transformation semigroups which admit the structure of an AC semiring with zero.

3.13 Theorem. Let X be a set and $1 \leqslant \xi \leqslant |X|$. Then

(1) R_ξ admits the structure of an AC semiring with zero if

and only if $\xi = 1$.

(2) D_ξ admits the structure of an AC semiring with zero if and only if $\xi = 1$.

Proof : Assume $\xi > 1$. Then $|X| > 1$, so there exist two distinct elements a,b in X. Hence $\{a\}_a$, $\{a\}_b$ and $\{b\}_a$ are elements of R_ξ and D_ξ. By Lemma 3.4 R_ξ and D_ξ do not admit the structure of an AC semiring with zero.

If $\xi = 1$, then $R_\xi = \{0\} = D_\xi$, so R_ξ and D_ξ admit the structure of an AC semiring with zero. #

3.14 Theorem. Let X be a set and $1 \leqslant \xi \leqslant |X|$. Then

(1) $\bar{R}_\xi$ admits the structure of an AC semiring with zero if and only if $|X| = 1$.

(2) $\bar{D}_\xi$ admits the structure of an AC semiring with zero if and only if $|X| = 1$.

Proof : Assume $|X| \geqslant 2$. Let a,b be two distinct elements in X. Then $\{a\}_a$, $\{a\}_b$ and $\{b\}_a$ are elements of $\bar{R}_\xi$ and $\bar{D}_\xi$. By Lemma 3.4, $\bar{R}_\xi$ and $\bar{D}_\xi$ do not admit the structure of an AC semiring with zero.

If $|X| = 1$, then $\bar{R}_\xi = \bar{D}_\xi = P_X$ which admits the structure of an AC semiring with zero by Theorem 3.5.