CHAPTER I

PRELIMINARIES

Let S be a semigroup. An element z of S is called a *right* [left] *zero* of S if xz = z [zx = z] for every x ε S. An element of S is called a *zero* of S if it is both a left and a right zero of S. An element e of S is called a *right* [left] *identity* of S if xe = x [ex = x] for every x ε S. An element of S is called an *identity* of S if it is both a left and a right identity of S. A semigroup can have at most one zero and at most one identity. The zero and the identity of a semigroup, if exist, are usually denote by 0 and 1, respectively.

A *right* [left] *zero semigroup* is a semigroup S in which xy = y [xy = x] for all x,y ε S. Then a semigroup S is a right [left] zero semigroup if and only if every element of S is a right [left] zero of S.

A semigroup S with zero 0 is called a *zero semigroup* if xy = 0 for all x,y ε S; and it is called a *Kronecker semigroup* if

$$xy = \begin{cases} x & \text{if } x = y, \\ 0 & \text{if } x \neq y \end{cases}$$

for all x,y ε S.

Let S be a semigroup, and let 0 be a symbol not representing any element of S. Let the notation S∪0 denote the semigroup obtained by extending the binary operation on S to 0 by defining 00 = 0 and 0a = a0 = 0 for all a ε S, and then let the notation S^0 denote the following semigroup :

$$S^{o} = \begin{cases} S & \text{if } S \text{ has a zero,} \\ S \cup 0 & \text{if } S \text{ has no zero.} \end{cases}$$

Similarly, let S be a semigroup and 1 a symbol not representing any element of S. Let the notation $S \cup 1$ denote the semigroup obtained by extending the binary operation on S to 1 by defining $11 = 1$ and $1a = a = a1$ for all $a \in S$, and let the notation S^1 denote the following semigroup :

$$S^{1} = \begin{cases} S & \text{if } S \text{ has an identity,} \\ S \cup 1 & \text{if } S \text{ has no identity.} \end{cases}$$

Let S be a semigroup. For $T \subseteq S$, T is called a <u>subsemigroup</u> of S if T forms a semigroup under the same operation on S. For a non-empty subset A of S, let

$$\langle A \rangle = \{a_1 a_2 \ldots a_n \mid a_i \in A,\ n \in \mathbb{N}\}$$

where $\mathbb{N}$ is the set of all positive integers. Then for $A \subseteq S$, $A \neq \phi$, $\langle A \rangle$ is a subsemigroup of S and it is called the <u>subsemigroup of S generated by A</u>. For $a \in S$, let $\langle a \rangle$ denotes $\langle \{a\} \rangle$ and it is called the <u>cyclic subsemigroup of S generated by a</u>. Hence $\langle a \rangle = \{a^n \mid n \in \mathbb{N}\}$ for every $a \in S$. If $S = \langle a \rangle$ for some $a \in S$, S is said to be a <u>cyclic semigroup</u>. For $a \in S$, if $\langle a \rangle$ is finite then there exists a positive integer m such that $\langle a \rangle = \{a, a^2, \ldots, a^m\}$ where $a, a^2, \ldots, a^m$ are all distinct.

Let S be a semigroup and A a nonempty subset of S. Then A is called a <u>right</u> [left] <u>ideal</u> of S if $AS \subseteq A$ [$SA \subseteq A$]. We call A an <u>ideal</u> of S if A is both a left and a right ideal of S.

A semigroup S is said to be <u>right simple</u> [left simple, simple] if S is the only right ideal [left ideal, ideal] of S.

A semigroup S is said to be <u>right</u> [left] <u>cancellative</u> if for $a,b,x \in S$, $ax = bx$ [$xa = xb$] implies $a = b$. A <u>cancellative semigroup</u> is a semigroup which is both left and right cancellative.

A semigroup S is called a <u>right group</u> if it is right simple and left cancellative. A <u>left group</u> is defined dually. A semigroup S is a right [left] group if and only if S is the product of $G \times E$ of a group G and a right [left] zero semigroup E [1, Theorem 1.27].

Let S and T be semigroups and ψ a map from S into T. The map ψ is said to be a <u>homomorphism</u> from S into T if

$$(ab)\psi = (a\psi)(b\psi)$$

for all $a,b \in S$. A homomorphism ψ from S into T is called an <u>isomorphism</u> if ψ is a 1-1 map. If there exists an isomorphism from S onto T, we say that the semigroups S and T are <u>isomorphic</u>, and we write $S \cong T$.

A <u>semiring</u> is a triple $(S,+,\cdot)$ such that $(S,+)$ and $(S,\cdot)$ are semigroups and $x\cdot(y + z) = x\cdot y + x\cdot z$, $(y + z)\cdot x = y\cdot x + z\cdot x$ for all $x,y,z \in S$. If $S = (S,+,\cdot)$ is a semiring, the operation + and $\cdot$ are called the <u>addition</u> and the <u>multiplication</u> of the semiring S, respectively. An element 0 of a semiring $(S,+,\cdot)$ is said to be a <u>zero</u> of the semiring $(S,+,\cdot)$ if $x\cdot 0 = 0\cdot x = 0$, $x + 0 = 0 + x = x$ for all

$x \varepsilon S$. A semiring $(S,+,\cdot)$ is additively commutative or AC if $(S,+)$ is commutative.

Let S be a semiring. For $T \subseteq S$, T is a subsemiring of S if T forms a semiring under the same operations on S. Observe that for every $x \varepsilon S$, xS and Sx are subsemirings of S.

A semigroup S is said to admit the structure of an additively commutative semiring with zero if there exists a binary operation + on the semigroup S^0 such that $(S^0,+,\cdot)$ is an additively commutative semiring with zero, where $\cdot$ is the operation on S^0. Hence the following clearly hold :

(1) A semigroup having only one element admits the structure of an AC semiring with zero.

(2) Any semigroup admitting a ring structure admits the structure of an AC semiring with zero.

(3) If a semigroup S admits the structure of an AC semiring with zero, then for any semigroup T with $T \cong S$, T also admits the structure of an AC semiring with zero.

For any set A, let $|A|$ denote the cardinality of A.

Let X be a set. A partial transformation of X is a map from a subset of X into (a subset of) X. The partial transformation of X with empty domain is called the empty transformation and it is denoted by 0. For a partial transformation α of X, the domain and the range of α are denoted by $\Delta\alpha$ and $\nabla\alpha$, respectively. Let P_X be the set of all partial transformations of X including the empty transformation 0. For $\alpha,\beta \ \varepsilon \ P_X$, define the product $\alpha\beta$ as follows : If $\nabla\alpha \cap \Delta\beta = \phi$, let $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \phi$, let

$$\alpha\beta = (\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}})(\beta|_{\nabla\alpha \cap \Delta\beta})$$

(the composite map) where $\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}}$ and $\beta|_{\nabla\alpha \cap \Delta\beta}$ denote the restrictions of α and β to $(\nabla\alpha \cap \Delta\beta)\alpha^{-1}$ and $\nabla\alpha \cap \Delta\beta$, respectively. Then P_X is a semigroup with zero 0 and identity 1_X where 1_X is the identity map on X and it is called the <u>partial transformation semigroup</u> on the set X. Observe that for $\alpha,\beta \in P_X$, $\Delta\alpha\beta = (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \subseteq \Delta\alpha$ and $\nabla\alpha\beta = (\nabla\alpha \cap \Delta\beta)\beta \subseteq \nabla\beta$. A partial transformation α of X is called a <u>1-1 partial transformation</u> of X if α is 1-1. Let I_X denote the set of all 1-1 partial transformations of X, that is,

$$I_X = \{\alpha \in P_X \mid \alpha \text{ is 1-1}\} .$$

Then I_X is a subsemigroup of P_X with identity 1_X and zero 0, and it is called the <u>1-1 partial transformation semigroup</u> or the <u>symmetric inverse semigroup</u> on the set X. By a <u>transformation</u> of a set X we mean a map of X into itself. Then an element $\alpha \in P_X$ is a transformation of X if and only if $\Delta\alpha = X$. Let T_X denote the set of all transformations of X, that is,

$$T_X = \{\alpha \in P_X \mid \Delta\alpha = X\} .$$

Then T_X is a subsemigroup of P_X with identity 1_X and it is called the <u>full transformation semigroup</u> on the set X. The permutation group on X is denoted by G_X. Then

$$G_X = \{\alpha \in P_X \mid \Delta\alpha = \nabla\alpha = X \text{ and } \alpha \text{ is 1-1}\}.$$

Observe that $G_X \subseteq I_X \subseteq P_X$ and $G_X \subseteq T_X \subseteq P_X$.

The semigroup of all 1-1 transformations of X and the semigroup of all onto transformations of X are denoted by M_X and E_X, respectively. Hence

$$M_X = \{\alpha : X \to X \mid \alpha \text{ is 1-1}\}$$
$$= \{\alpha \in I_X \mid \Delta\alpha = X\}$$

and

$$E_X = \{\alpha : X \to X \mid \alpha \text{ is onto}\}$$
$$= \{\alpha \in T_X \mid \nabla\alpha = X\}$$

It is known that for any set X, $M_X = G_X$ if and only if $|X| < \infty$, and $E_X = G_X$ if and only if $|X| < \infty$.

We denote the semigroup of all constant partial transformations of X and the semigroup of all constant transformations of X by C_X and F_X, respectively. Hence

$$C_X = \{\alpha \in P_X \mid |\nabla\alpha| = 1\} \cup \{0\} ,$$
$$F_X = \{\alpha \in T_X \mid |\nabla\alpha| = 1\} \quad \text{if } X \neq \phi ,$$
$$F_X = \{0\} \quad \text{if } X = \phi .$$

The shift of a partial transformation α of X, $S(\alpha)$, is defined to be the set $\{x \in \Delta\alpha \mid x\alpha \neq x\}$. A partial transformation α of X is said to be almost identical if the shift of α is finite, that is, $|S(\alpha)| < \infty$. Let

$$U_X = \{\alpha \in P_X \mid |S(\alpha)| < \infty\} ,$$
$$V_X = \{\alpha \in T_X \mid |S(\alpha)| < \infty\} ,$$

and

$$W_X = \{\alpha \in I_X \mid |S(\alpha)| < \infty\} .$$

If $\alpha,\beta \in P_X$, then $S(\alpha\beta) \subseteq S(\alpha) \cup S(\beta)$. Hence U_X, V_X and W_X are subsemigroups of P_X, T_X and I_X, respectively, and U_X, V_X and W_X are referred respectively as the semigroup of all almost identical partial transformations of X, the semigroup of all almost identical transformations of X, and the semigroup of all almost identical 1-1 partial transformations of X. Observe that if X is finite, then $U_X = P_X$, $V_X = T_X$ and $W_X = I_X$.

By a transformation semigroup on a set X, we mean a subsemigroup of the partial transformation semigroup of X.

By a multiplicative interval semigroup in $\mathbb{R}$, we mean an interval in $\mathbb{R}$ which is closed under usual multiplication.