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FUNCTIONAL EQUATION ANALOGOUS TO THE 2-DIMENSIONAL WAVE EQUATION

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$$c^2(\Delta^2_{1,h_1} f(x,y,t) + \Delta^2_{2,h_2} f(x,y,t)) = \Delta^2_{3,h_3} f(x,y,t)$$

ซึ่งเทียบคล้ายสมการคลื่น 2 มิติ

$$c^2(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}) = \frac{\partial^2 f}{\partial t^2}.$$

ภาควิชา คณิตศาสตร์ และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต.......................................

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In this thesis, we find the general solution of the following functional equation

$$c^2(\Delta_{1,h_1}^2 f(x,y,t) + \Delta_{2,h_2}^2 f(x,y,t)) = \Delta_{3,h_3}^2 f(x,y,t)$$

which is analogous to the 2-dimensional wave equation

$$c^2\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^2 f}{\partial t^2}.$$

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CHAPTER I

INTRODUCTION

A *functional equation* is an equation in which the unknowns are functions. The objective of studying a functional equation is either to find all the functions satisfying the equation, possibly with additional conditions, or to study its relation with other functional equations. The following example demonstrates a solution of a functional equation.

Example. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) - f(x) = y \qquad \text{for every } x, y \in \mathbb{R}. \tag{1.1}$$

<u>Solution</u>: Letting $x = 0$ in equation (1.1), we then have

$$f(y) - f(0) = y \qquad \text{for every } y \in \mathbb{R}.$$

Letting $c = f(0)$, the function f must be given by $f(y) = y + c$ for every $y \in \mathbb{R}$. Conversely, it is not hard to see that any function f defined by $f(x) = x + c$, where c is a constant, also satisfies equation (1.1). The function f is said to be the solution of equation (1.1). $\square$

Besides finding the solution of an equation, the equivalence among equations are also widely studied. For example, it was found ([3]) that, for given nonempty open interval $I \subset \mathbb{R}^+$ and $p \in [0,1]$, the functional equation

$$f(px + (1-p)y) + f(\frac{xy}{px + (1-p)y}) = f(x) + f(y) \qquad \text{for } x, y \in I \tag{1.2}$$

has the same set of solutions with the functional equation,

$$2f(\sqrt{xy}) = f(x) + f(y) \qquad \text{for } x, y \in I. \tag{1.3}$$

In other words, we can say that equation (1.2) is equivalent to equation (1.3).

Various functional equations have been studied. One of interesting results is the one concerning the functional equation

$$f(x+h,y)+f(x-h,y)-f(x,y+h)-f(x,y-h) = 0 \qquad \text{for } x, y \in \mathbb{R}. \tag{1.4}$$

This equation can be considered as a functional equation analogue of the wave equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2}.$$

In 1969, J.A. Baker [2] proved that all continuous solutions of equation (1.4) must be of the form

$$f(x,y) = \alpha(x+y) + \beta(x-y),$$

where $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are arbitrary continuous functions. Nevertheless, if the continuity of the function f is not assumed, McKiernan [6] found that each solution of equation (1.4) must be of the form

$$f(x,y) = \alpha(x+y) + \beta(x-y) + A(x,y),$$

where $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are arbitrary functions and $A : \mathbb{R}^2 \to \mathbb{R}$ is a *skew-symmetric bi-additive function*. More precisely, the function A satisfies the equations

$$A(x+z,y) = A(x,y) + A(z,y), \qquad A(x,y+z) = A(x,y) + A(x,z),$$
$$\text{and} \qquad A(x,y) = -A(y,x)$$

for all $x, y, z \in \mathbb{R}$.

In 1988, S. Haruki [5] has studied the functional equation

$$\frac{f(x+h_1,y)-2f(x,y)+f(x-h_1,y)}{h_1^2}=\frac{f(x,y+h_2)-2f(x,y)+f(x,y-h_2)}{h_2^2} \tag{1.5}$$

for all $x, y, h_1, h_2 \in \mathbb{R}$ such that $h_1, h_2 \neq 0$. This equation can be represented by using the divided symmetric partial difference operators as

$$\Delta^2_{1,h_1} f(x,y) = \Delta^2_{2,h_2} f(x,y).$$

Note that the operator Δ here is defined by

$$\Delta_{1,h} f(x,y) = \frac{f(x+\frac{h}{2},y)-f(x-\frac{h}{2},y)}{h}$$
$$\Delta_{2,h} f(x,y) = \frac{f(x,y+\frac{h}{2})-f(x,y-\frac{h}{2})}{h}.$$

Thus, the functional equation (1.5) can be regarded as another analogue of the wave equation.

Note that if the spans are restricted to $h_1 = h_2$, equation (1.5) becomes equation (1.4). So equation (1.4) is called the *symmetric case* of equation (1.5). The general solution of equation (1.4) is

$$\begin{aligned} f(x,y) =& a_0 + a_1(x^2+y^2) + a_2(3x^2y+y^3) + a_3(3xy^2+x^3) + a_4(x^3y+xy^3) \\ &+ A_1(x) + A_2(y) + B(x,y), \end{aligned}$$

where a_0, a_1, a_2, a_3, a_4 are constants in $\mathbb{R}$, A_1, A_2 are additive functions and B is a bi-additive function.

In this thesis, we will extend Haruki's work to the functional equation

$$\begin{aligned} \frac{f(x+h_1,y,t)-2f(x,y,t)+f(x-h_1,y,t)}{h_1^2}+&\frac{f(x,y+h_2,t)-2f(x,y,t)+f(x,y-h_2,t)}{h_2^2} \\ =&\frac{f(x,y,t+h_3)-2f(x,y,t)+f(x,y,t-h_3)}{h_3^2} \end{aligned}$$

for all $x, y, t \in \mathbb{R}$ and $h_1, h_2, h_3 \in \mathbb{R} \setminus \{0\}$. This equation can be written as

$$\Delta^2_{1,h_1} f(x,y,t) + \Delta^2_{2,h_2} f(x,y,t) = \Delta^2_{3,h_3} f(x,y,t). \tag{1.6}$$

Equation (1.6) can be considered as an analogue of the wave equation for wave motion in 2 dimensions,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial t^2}.$$

CHAPTER II

PRELIMINARIES

In this chapter, we will give basic concepts of additive and multi-additive functions and also recall Haruki's results (please refer to [5] for details).

Definition 2.1. Let $(G, +)$ be a commutative group. A function $A : G \to G$ is said to be *additive* if

$$A(x+y) = A(x) + A(y) \qquad \text{for all } x, y \in G.$$

Proposition 2.2. *Let* $A : \mathbb{R} \to \mathbb{R}$ *be an additive function*, $x \in \mathbb{R}$ *and* $r \in \mathbb{Q}$. *Then*

(i.) $A(0) = 0$,

(ii.) $A(-x) = -A(x)$,

(iii.) $A(rx) = rA(x)$.

A generalization of additivity to multi-variate functions is as follows:

Definition 2.3. Let G be a commutative group and $n \in \mathbb{N}$. A function $f : G^n \to G$ is additive in the i^{th} variable if

$$f(x_1, x_2, \ldots, x_i + y_i, \ldots, x_n) = f(x_1, x_2, \ldots, x_i, \ldots, x_n) + f(x_1, x_2, \ldots, y_i, \ldots x_n)$$

for every $x_1, x_2, \ldots, x_n, y_i \in G$.

If f is additive in each of its variables, f will be called an *n-additive* function. A 2-additive function is also known as being *bi-additive*.

Remark. It can be easily seen that if f is additive in the i^{th} variable, then $f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0$ for all $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in G$.

Next we recall the important results which were published by Haruki [5].

Theorem 2.4. (Haruki) *Let* $\phi, \varphi : \mathbb{R} \to \mathbb{R}$ *be functions satisfying* $(\Delta_h^2 \phi)(x) = \varphi(x)$*, where*

$$\Delta_h^2 \phi(x) = \frac{\phi(x+h) - 2\phi(x) + \phi(x-h)}{h^2}.$$

Then

$$\begin{aligned} \phi(x) &= a_1 + A(x) + a_2 x^2 + a_3 x^3 \\ \varphi(x) &= 2a_2 + 6a_3 x \end{aligned}$$

where a_1, a_2, a_3 *are constants and* A *is an additive function.*

Theorem 2.4 can be applied to any function ϕ whose second-order difference is independent of the span, h. Moreover, the following Haruki's solution for

$$(\Delta_{1,t}^2 g)(x,y) = (\Delta_{2,s}^2 g)(x,y) \tag{2.1}$$

for all $x, y \in \mathbb{R}$ and $s, t \in \mathbb{R} \setminus \{0\}$ is also required in our work.

Theorem 2.5. (Haruki) *Let* $g : \mathbb{R}^2 \to \mathbb{R}$ *satisfy equation (2.1). Then*

$$\begin{aligned} g(x,y) =& a_0 + a_1(x^2 + y^2) + a_2(3x^2 y + y^3) + a_3(3xy^2 + x^3) + a_4(x^3 y + xy^3) \\ &+ A_1(x) + A_2(y) + B(x,y) \end{aligned}$$

where a_0, a_1, a_2, a_3, a_4 *are constants in* $\mathbb{R}$*,* A_1, A_2 *are additive functions and* B *is a bi-additive function.*

CHAPTER III

FUNCTIONAL EQUATION ANALOGUE OF 2-DIMENSIONAL WAVE EQUATION

The divided symmetric partial difference operator is defined for $f : \mathbb{R}^3 \to \mathbb{R}$ as follows:

$$
\begin{aligned}
(\Delta_{1,h} f)(x,y,t) &= \frac{f(x+\frac{h}{2},y,t) - f(x-\frac{h}{2},y,t)}{h} \\
(\Delta_{2,h} f)(x,y,t) &= \frac{f(x,y+\frac{h}{2},t) - f(x,y-\frac{h}{2},t)}{h} \\
(\Delta_{3,h} f)(x,y,t) &= \frac{f(x,y,t+\frac{h}{2}) - f(x,y,t-\frac{h}{2})}{h}
\end{aligned}
$$

and

$$
(\Delta_{i,h}^{n+1} f)(x,y,t) = (\Delta(\Delta_{i,h}^{n} f))(x,y,t) \qquad \text{for } i \in \{1,2,3\}.
$$

Then a functional equation analogous to the wave equation with velocity c is

$$
c^2((\Delta_{1,h_1}^2 f)(x,y,t) + (\Delta_{2,h_2}^2 f)(x,y,t)) = (\Delta_{3,h_3}^2 f)(x,y,t) \tag{3.1}
$$

for $x,y,t \in \mathbb{R}$ and $h_1,h_2,h_3 \in \mathbb{R} \setminus \{0\}$.

The next theorem gives a relationship among solutions of equation (3.1).

Theorem 3.1. *Let* $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ *and* $c \in \mathbb{R} \setminus \{0\}$. *Assume that* $f_2(x,y,t) = f_1(x,y,ct)$ *for* $x,y,t \in \mathbb{R}$. *Then the equation*

$$
\Delta_{1,h_1}^2 f_1(x,y,t) + \Delta_{2,h_2}^2 f_1(x,y,t) = \Delta_{3,h_3}^2 f_1(x,y,t)
$$

for $x, y, t \in \mathbb{R}$ *and* $h_1, h_2, h_3 \in \mathbb{R} \setminus \{0\}$, *is equivalent to*

$$c^2(\Delta^2_{1,h_1} f_2(x,y,t) + \Delta^2_{2,h_2} f_2(x,y,t)) = \Delta^2_{3,h_3} f_2(x,y,t)$$

for $x, y, t \in \mathbb{R}$ *and* $h_1, h_2, h_3 \in \mathbb{R} \setminus \{0\}$.

Proof. The statement follows from the following observation:

$$\begin{aligned}
& c^2(\Delta^2_{1,h_1} f_2(x,y,t) + \Delta^2_{2,h_2} f_2(x,y,t)) - \Delta^2_{3,h_3} f_2(x,y,t) \\
= \ & c^2(\Delta^2_{1,h_1} f_2(x,y,t) + \Delta^2_{2,h_2} f_2(x,y,t)) \qquad (3.2) \\
& - (\frac{f_2(x,y,t+h_3) - 2f_2(x,y,t) + f_2(x,y,t-h_3)}{h_3^2}) \\
= \ & c^2(\Delta^2_{1,h_1} f_1(x,y,ct) + \Delta^2_{2,h_2} f_1(x,y,ct)) \qquad (3.3) \\
& - c^2(\frac{f_1(x,y,ct+ch_3) - 2f_1(x,y,ct) + f_1(x,y,at-ch_3)}{(ch_3)^2}) \\
= \ & c^2(\Delta^2_{1,h_1} f_1(x,y,t') + \Delta^2_{2,h_2} f_1(x,y,t')) - c^2\Delta^2_{3,h_3'} f_1(x,y,t')
\end{aligned}$$

where $t' = ct$ and $h_3' = ch_3$. □

Theorem 3.1 allows us to concentrate on solving the equation

$$(\Delta^2_{1,h_1} f)(x,y,t) + (\Delta^2_{2,h_2} f)(x,y,t) = (\Delta^2_{3,h_3} f)(x,y,t). \qquad (3.4)$$

When expanded, the equation (3.4) is

$$\begin{aligned}
& \frac{f(x+h_1,y,t) - 2f(x,y,t) + f(x-h_1,y,t)}{h_1^2} \\
& + \frac{f(x,y+h_2,t) - 2f(x,y,t) + f(x,y-h_2,t)}{h_2^2} \\
= \ & \frac{f(x,y,t+h_3) - 2f(x,y,t) + f(x,y,t-h_3)}{h_3^2}.
\end{aligned}$$

We need the following lemma before approaching our main result.

Lemma 3.2. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ and $f : \mathbb{R}^3 \to \mathbb{R}$ such that f is additive in the first variable. Then f and g satisfy the system of equations*

$$\begin{aligned} \Delta^2_{1,h_1} g(y,t) &= \Delta^2_{2,h_2} g(y,t) \\ xg(y,t) + \Delta^2_{2,h_1} f(x,y,t) &= \Delta^2_{3,h_2} f(x,y,t) \end{aligned} \tag{3.5}$$

for all $x, y, t \in \mathbb{R}$ and $h_1, h_2 \in \mathbb{R} \setminus \{0\}$ if and only if

$$\begin{aligned} f(x,y,t) = &\ C_0(x) + (y^2+t^2)C_1(x) + (t^3+3y^2t)C_2(x) + (y^3+3yt^2)C_3(x) \\ &+ (yt^3+y^3t)C_4(x) + b_0xt^2 + A_1(x,t) + A_2(x,y) + A_3(x,y,t) \\ &+ xt^2B_1(y) + xt^3B_3(y) - xy^2B_2(t) - xy^3B_4(t) \\ g(y,t) = &\ 2b_0 + 2B_1(y) + 2B_2(t) + 6tB_3(y) + 6yB_4(t) \end{aligned}$$

where B_i's and C_i's are additive functions, A_1, A_2 are bi-additive, A_3 is 3-additive and b_0 is a constant.

Proof. From the solution of one-dimensional wave functional equation in Theorem 2.5, we have

$$\begin{aligned} g(y,t) = &\ b_0 + b_1(y^2+t^2) + b_2(y^3+3yt^2) + b_3(t^3+3y^2t) + b_4(y^3t+yt^3) \\ &+ B_1(y) + B_2(t) + B^*(y,t) \end{aligned}$$

where b_i's are constants, B_1, B_2 are additive and B^* is bi-additive. From equation (3.5), substituting $h_2 = 1$ and then apply Theorem 2.4, we have

$$f(x,y,t) = D_0(x,t) + D_1(x,y,t) + y^2D_2(x,t) + y^3D_3(x,t)$$

where D_1 is additive in the second variable. Substituting this and $g(y,t)$ into equation (3.5), we get

$$\begin{aligned} & b_0x + b_1x(y^2+t^2) + b_2x(y^3+3yt^2) + b_3x(t^3+3y^2t) + b_4x(y^3t+yt^3) + xB_1(y) \\ & + xB_2(t) + xB^*(y,t) + 2D_2(x,t) + 6yD_3(x,t) \\ = \; & \Delta^2_{3,h_3}(D_0(x,t) + D_1(x,y,t) + y^2D_2(x,t) + y^3D_3(x,t)). \end{aligned} \tag{3.6}$$

Observe that whenever $r \in \mathbb{Q}$, substituting ry for y in equation (3.6) yields a polynomial of variable r as following

$$\begin{aligned} & xb_0 + xb_1(r^2y^2+t^2) + xb_2(r^3y^3+3ryt^2) + xb_3(t^3+3r^2y^2t) + xb_4(r^3y^3t+ryt^3) \\ & + rxB_1(y) + xB_2(t) + rxB^*(y,t)) + 2D_2(x,t) + 6ryD_3(x,t) \\ & - \Delta^2_{3,h_3}(D_0(x,t) + rD_1(x,y,t) + r^2y^2D_2(x,t) + r^3y^3D_3(x,t)) = 0, \end{aligned}$$

with all rational numbers being its roots. Hence all the coefficients of the polynomial must vanish, that is,

$$b_0x + b_1xt^2 + b_3xt^3 + xB_2(t) + 2D_2(x,t) = \Delta^2_{t,h_2}D_0(x,t), \tag{3.7}$$

$$3b_2xyt^2 + b_4xyt^3 + xB_1(y) + xB^*(y,t) + 6yD_3(x,t) = \Delta^2_{t,h_2}D_1(x,y,t), \tag{3.8}$$

$$b_1xy^2 + 3b_3xy^2t = \Delta^2_{t,h_2}y^2D_2(x,t), \tag{3.9}$$

$$b_2xy^3 + b_4xy^3t = \Delta^2_{t,h_2}y^3D_3(x,t). \tag{3.10}$$

By equation (3.9), equation (3.10) and using Theorem 2.4, we have

$$D_2(x,t) = C_1(x) + E_1(x,t) + \frac{b_1}{2}xt^2 + \frac{b_3}{2}xt^3, \tag{3.11}$$

$$D_3(x,t) = C_3(x) + E_2(x,t) + \frac{b_2}{2}xt^2 + \frac{b_4}{6}xt^3 \tag{3.12}$$

where E_1 and E_2 are additive in the second variable. Now equation (3.7) becomes

$$\Delta^2_{t,h_2}D_0(x,t) = b_0x + 2C_1(x) + xB_2(t) + 2E_1(x,t) + 2b_1xt^2 + 2b_3xt^3. \tag{3.13}$$

Therefore, by Theorem 2.4, we get

$$\Delta^2_{t,h_2} D_0(x,t) = k(x) + tC_2(x) \tag{3.14}$$

for some $k, C_2 : \mathbb{R} \to \mathbb{R}$. Thus

$$b_0 x + 2C_1(x) + xB_2(t) + 2E_1(x,t) + 2b_1 xt^2 + 2b_3 xt^3 = k(x) + tC_2(x).$$

By replacing t with rt and use the fact about polynomial roots as previous, we get $b_1 = 0 = b_3$ and $b_0 x + 2C_1(x) = k(x)$ and $xB_2(t) + 2E_1(x,t) = tC_2(x)$. Hence, by Theorem 2.4 and the equation (3.14), we obtain

$$D_0(x,t) = C_0(x) + A_1(x,t) + \frac{b_0 x + 2C_1(x)}{2} t^2 + \frac{C_2(x)}{6} t^3$$

where A_1 is additive in the second variable. And since $b_1 = 0$ and $b_3 = 0$, equation (3.11) becomes

$$D_2(x,t) = C_1(x) + E_1(x,t) = C_1(x) + \frac{tC_2(x) - xB_2(t)}{2}.$$

We get back to equations (3.8) and (3.12), we have

$$\Delta^2_{t,h_2} D_1(x,y,t) = xB_1(y) + 6yC_3(x) + xB^*(y,t) + 6yE_2(x,t) + 6b_2 xyt^2 + 2b_4 xyt^3.$$

By similar reasoning as previous, $b_2 = 0 = b_4$, $xB^*(y,t) + 6yE_2(x,t) = tE_3(x,y)$ and

$$D_1(x,y,t) = A_2(x,y) + A_3(x,y,t) + \frac{xB_1(y) + 6yC_3(x)}{2} t^2 + \frac{E_3(x,y)}{6} t^3, \tag{3.15}$$

$$D_3(x,t) = C_3(x) + E_2(x,t)$$

where A_3 is additive in the third variable. Note that E_3 is additive in the second variable since $E_3(x,y) = xB^*(y,1) + 6yE_2(x,1)$. If we substitute $t = 0$ into equation (3.15), we get $A_2(x,y) = D_1(x,y,0)$, which is additive in the second variable. Since all other functions in equation (3.15) are additive functions of y, we can see that A_3 is additive in the second variable. Thus

$$\begin{aligned} f(x,y,t) = \; & C_0(x) + A_1(x,t) + \frac{b_0 x + 2C_1(x)}{2} t^2 + \frac{C_2(x)}{6} t^3 \\ & + A_2(x,y) + A_3(x,y,t) + \frac{xB_1(y) + 6yC_3(x)}{2} t^2 + \frac{E_3(x,y)}{6} t^3 \\ & + y^2 (C_1(x) + \frac{tC_2(x) - xB_2(t)}{2}) \\ & + y^3 (C_3(x) + E_2(x,t)) \\ = & \; C_0(x) + (y^2+t^2)C_1(x) + (t^3 + 3y^2 t)\frac{C_2(x)}{6} + (y^3 + 3yt^2)C_3(x) + \frac{b_0}{2} x t^2 \\ & + A_1(x,t) + A_2(x,y) + A_3(x,y,t) + xt^2 \frac{B_1(y)}{2} + t^3 \frac{E_3(x,y)}{6} \\ & - xy^2 \frac{B_2(t)}{2} + y^3 E_2(x,t), \end{aligned} \tag{3.16}$$

and $$g(y,t) = b_0 + B_1(y) + B_2(t) + tE_3(1,y) - 6yE_2(1,t). \tag{3.17}$$

Now we want to show that each function in the solution is additive in the first variable. Substituting $y = 0 = t$ in equation (3.16), we get that $C_0(x) = f(x,0,0)$ is additive by an assumption on f. Let

$$\varphi_y(x) = y^2 C_1(x) + y^3 C_3(x) + A_2(x,y). \tag{3.18}$$

If we substitute $t = 0$ in equation (3.16), we get $\varphi_y(x) = f(x,y,0) - C_0(x)$. Hence for each $y \in \mathbb{R}$, φ_y is additive. One can verify (from equation (3.18)) that

$$\begin{aligned} C_1(x) = & -\frac{5}{2}\varphi_1(x) + 2\varphi_2(x) - \frac{1}{2}\varphi_3(x), \\ C_3(x) = & \; \frac{1}{2}\varphi_1(x) - \frac{1}{2}\varphi_2(x) + \frac{1}{6}\varphi_3(x), \\ A_2(x,y) = & \; 3\varphi_y(x) - \frac{3}{2}\varphi_{2y}(x) + \frac{1}{3}\varphi_{3y}(x). \end{aligned}$$

Hence we have that C_1 and C_3 are additive and A_2 is bi-additive. Next, let $y = 0$

in equation (3.16). We get

$$f(x,0,t) = C_0(x) + t^2 C_1(x) + t^3 \frac{C_2(x)}{6} + \frac{b_0}{2} x t^2 + A_1(x,t).$$

We let $\phi_t(x) = t^3 \frac{C_2(x)}{6} + A_1(x,t)$. Then $\phi_t(x) = f(x,0,t) - C_0(x) - t^2 C_1(x) - \frac{b_0}{2} x t^2$, an additive function of x. From the definition of ϕ_t, we have

$$\begin{aligned} C_2(x) = & -2\phi_1(x) + \phi_2(x), \\ A_1(x,t) = & \frac{8}{6}\phi_t(x) - \frac{1}{6}\phi_{2t}(x). \end{aligned}$$

Hence C_2 is additive and A_1 is bi-additive (it is already additive in the second variable). Next we let

$$\psi_{y,t}(x) = A_3(x,y,t) + t^3 \frac{E_3(x,y)}{6} + y^3 E_2(x,t). \tag{3.19}$$

Then we also have

$$\begin{aligned} \psi_{y,t}(x) = & \; f(x,y,t) - C_0(x) - (y^2+t^2)C_1(x) - (t^3 + 3y^2 t)\frac{C_2(x)}{6} - (y^3 + 3yt^2)C_3(x) \\ & - \frac{b_0}{2} x t^2 - A_1(x,t) - A_2(x,y) - x t^2 \frac{B_1(y)}{2} + \frac{x y^2 B_2(t)}{2} \end{aligned}$$

which is additive in x. From equation (3.19),

$$\begin{aligned} A_3(x,y,t) = & \; \frac{5}{3}\psi_{y,t}(x) - \frac{1}{6}\psi_{y,2t}(x) - \frac{1}{6}\psi_{2y,t}(x), \\ E_3(x,y) = & -2\psi_{y,1}(x) + \psi_{y,2}(x), \\ E_2(x,t) = & -\frac{1}{3}\psi_{1,t}(x) + \frac{1}{6}\psi_{2,t}(x). \end{aligned}$$

Hence every functions in equation (3.16) is additive with respect to x.

Now we are ready to finalize the solution for f, substitute equations (3.16) and (3.17) into equation (3.5), we obtain

$$tE_3(x,y) - xtE_3(1,y) = 6yE_2(x,t) - 6xyE_2(1,t). \tag{3.20}$$

Define $T(x,y) = E_3(x,y) - xE_3(1,y)$. By equation (3.20), we obtain

$$T(x,y) = 6yE_2(x,1) - 6xyE_2(1,1) = yC_4(x)$$

where $C_4(x) = 6E_2(x,1) - 6xE_2(1,1)$. Note that C_4 is additive. So we have

$$\begin{aligned} tE_3(x,y) &= xtE_3(1,y) + ytC_4(x), \\ 6yE_2(x,t) &= 6xyE_2(1,t) + ytC_4(x). \end{aligned}$$

Thus, from equation (3.16),

$$\begin{aligned} f(x,y,t) = \ & C_0(x) + (y^2+t^2)C_1(x) + (t^3+3y^2t)\frac{C_2(x)}{6} + (y^3+3yt^2)C_3(x) \\ & + (y^3t+t^3y)\frac{C_4(x)}{6} + \frac{b_0}{2}xt^2 + A_1(x,t) + A_2(x,y) + A_3(x,y,t) \\ & + xt^2\frac{B_1(y)}{2} - xy^2\frac{B_2(t)}{2} + xt^3\frac{E_3(1,y)}{6} + xy^3E_2(1,t). \end{aligned}$$

Also recall that

$$g(y,t) = b_0 + B_1(y) + B_2(t) + tE_3(1,y) - 6yE_2(1,t).$$

This completes the proof. □

We can obtain a simpler version of this Lemma, which is also required for the main theorem.

Corollary 3.3. *Let $f, g : \mathbb{R}^2 \to \mathbb{R}$. Then f and g satisfy the system of equations*

$$\begin{aligned} \Delta^2_{1,h_1} g(y,t) &= \Delta^2_{2,h_2} g(y,t) \\ g(y,t) + \Delta^2_{1,h_1} f(y,t) &= \Delta^2_{2,h_2} f(y,t) \end{aligned} \tag{3.21}$$

for all $y, t \in \mathbb{R}$ *and* $h_1, h_2 \in \mathbb{R} \setminus \{0\}$ *if and only if*

$$\begin{aligned} f(x,y) = &\ c_0 + c_1(y^2+t^2) + c_2(t^3+3y^2t) + c_3(y^3+3yt^2) \\ &+ c_4(yt^3+y^3t) + b_0t^2 + A_1(t) + A_2(y) + A_3(y,t) \\ &+ t^2B_1(y) + t^3B_3(y) - y^2B_2(t) - y^3B_4(t) \\ g(y,t) = &\ 2b_0 + 2B_1(y) + 2B_2(t) + 6tB_3(y) + 6yB_4(t) \end{aligned}$$

where $A_1, A_2, B_1, B_2, B_3, B_4$ *are additive functions,* A_3 *is bi-additive and* $b_0, c_0, c_1, \ldots, c_4$ *are constants.*

Proof. Note that the equation (3.21) can be rewritten as

$$xg(y,t) + x\Delta^2_{1,h_1} f(y,t) = x\Delta^2_{2,h_2} f(y,t).$$

Now we let $f^*(x,y,t) = xf(y,t)$. Then f^* is additive in the first variable and

$$\begin{aligned} \Delta^2_{1,h_1} g(y,t) &= \Delta^2_{2,h_2} g(y,t), \\ xg(y,t) + \Delta^2_{2,h_1} f^*(x,y,t) &= \Delta^2_{3,h_2} f^*(x,y,t). \end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned} f^*(x,y,t) = &\ C_0(x) + (y^2+t^2)C_1(x) + (t^3+3y^2t)C_2(x) + (y^3+3yt^2)C_3(x) \\ &+ (yt^3+y^3t)C_4(x) + b_0xt^2 + A_1(x,t) + A_2(x,y) + A_3(x,y,t) \\ &+ xt^2B_1(y) + xt^3B_3(y) - xy^2B_2(t) - xy^3B_4(t) \\ g(y,t) = &\ 2b_0 + 2B_1(y) + 2B_2(t) + 6tB_3(y) + 6yB_4(t) \end{aligned}$$

where B_i's are additive, A_1, a_2 are bi-additive and A_3 is 3-additive. The desired result follows from $f(y,t) = f^*(1,y,t)$. □

Having Lemma 3.2 and Corollary 3.3, we are ready to solve the functional equation (3.4).

Theorem 3.4. *Let $f : \mathbb{R}^3 \to \mathbb{R}$. Then f satisfies equation (3.4) if and only if*

$$\begin{aligned}
f(x,y,t) = \ & a_0 + a_1(y^2+t^2) + a_2(t^3+3y^2t) + a_3(y^3+3yt^2) + a_4(y^3t+yt^3) \\
& + a_5(x^2+t^2) + a_6(x^3+3xt^2) + (y^2+t^2)A_1(x) + (t^3+3y^2t)A_2(x) \\
& + (y^3+3yt^2)A_3(x) + (y^3t+yt^3)A_4(x) + (x^2+t^2)A_5(y) + (x^2-y^2)A_6(t) \\
& + (t^3+3x^2t)A_7(y) + (x^3+3xt^2)A_8(y) + 3(x^3t+xt^3)A_9(y) \\
& - (y^3-3x^2y)A_{10}(t) + (x^3-3xy^2)A_{11}(t) + 3(x^3y-xy^3)A_{12}(t) \\
& + A_{13}(x) + A_{14}(t) + A_{15}(y) + B_1(x,y) + B_2(y,t) + B_3(x,t) + T_3(x,y,t)
\end{aligned}$$

where a_i's are constants, A_i's are additive, B_i's are bi-additive and T is 3-additive.

Proof. Firstly, we put $h_2 = 1 = h_3$ in equation (3.4) and then apply Theorem 2.4. We obtain

$$f(x,y,t) = A(y,t) + B(x,y,t) + x^2C(y,t) + x^3D(y,t)$$

where B is additive in first variable. Substitute this into equation (3.4), we have

$$\begin{aligned}
2C(y,t) + 6xD(y,t) + \Delta^2_{y,h_2}(A(y,t) + B(x,y,t) + x^2C(y,t) + x^3D(y,t)) \\
= \Delta^2_{t,h_3}(A(y,t) + B(x,y,t) + x^2C(y,t) + x^3D(y,t)).
\end{aligned}$$

By replacing x with rx, where $r \in \mathbb{Q}$, we get a polynomial (of r) with infinite number of roots, and hence its coefficients must be all zero;

$$2C(y,t) + \Delta^2_{y,h_2}A(y,t) = \Delta^2_{t,h_3}A(y,t), \tag{3.22}$$

$$6xD(y,t) + \Delta^2_{y,h_2}B(x,y,t) = \Delta^2_{t,h_3}B(x,y,t), \tag{3.23}$$

$$x^2\Delta^2_{y,h_2}C(y,t) = x^2\Delta^2_{t,h_3}C(y,t), \tag{3.24}$$

$$x^3\Delta^2_{y,h_2}D(y,t) = x^3\Delta^2_{t,h_3}D(y,t). \tag{3.25}$$

From Corollary 3.3 and equations (3.22) and (3.24) we have

$$\begin{aligned} A(y,t) = \; & a_0 + a_1(y^2+t^2) + a_2(t^3+3y^2t) + a_3(y^3+3yt^2) + a_4(y^3t+yt^3) + c_0t^2 \\ & + A_1(t) + A_2(y) + A_3(y,t) + t^2C_1(y) + t^3C_3(y) - y^2C_2(t) - y^3C_4(t), \\ C(y,t) = \; & c_0 + C_1(y) + C_2(t) + 3tC_3(y) + 3yC_4(t). \end{aligned}$$

From Lemma 3.2, equations (3.23) and (3.25), we obtain

$$\begin{aligned} B(x,y,t) = \; & B_0(x) + (y^2+t^2)B_1(x) + (t^3+3y^2t)B_2(x) + (y^3+3yt^2)B_3(x) \\ & + (y^3t+yt^3)B_4(x) + 3d_0xt^2 + E_1(x,t) + E_2(x,y) + E_3(x,y,t) \\ & + 3xt^2D_1(y) + 3xt^3D_3(y) - 3xy^2D_2(t) - 3xy^3D_4(t), \\ D(y,t) = \; & d_0 + D_1(y) + D_2(t) + 3tD_3(y) + 3yD_4(t), \end{aligned}$$

where each unknown function is additive in every variable. Hence

$$
\begin{aligned}
f(x,y,t) = \; & a_0 + a_1(y^2+t^2) + a_2(t^3+3y^2t) + a_3(y^3+3yt^2) + a_4(y^3t+yt^3) + c_0t^2 \\
& + A_1(t) + A_2(y) + A_3(y,t) + t^2C_1(y) + t^3C_3(y) - y^2C_2(t) - y^3C_4(t) \\
& + B_0(x) + (y^2+t^2)B_1(x) + (t^3+3y^2t)B_2(x) + (y^3+3yt^2)B_3(x) \\
& + (y^3t+yt^3)B_4(x) + 3d_0xt^2 + E_1(x,t) + E_2(x,y) + E_3(x,y,t) \\
& + 3xt^2D_1(y) + 3xt^3D_3(y) - 3xy^2D_2(t) - 3xy^3D_4(t) \\
& + x^2(c_0 + C_1(y) + C_2(t) + 3tC_3(y) + 3yC_4(t)) \\
& + x^3(d_0 + D_1(y) + D_2(t) + 3tD_3(y) + 3yD_4(t)) \\
= \; & a_0 + a_1(y^2+t^2) + a_2(t^3+3y^2t) + a_3(y^3+3yt^2) + a_4(y^3t+yt^3) \\
& + c_0(x^2+t^2) + d_0(x^3+3xt^2) + (y^2+t^2)B_1(x) + (t^3+3y^2t)B_2(x) \\
& + (y^3+3yt^2)B_3(x) + (y^3t+yt^3)B_4(x) + (x^2+t^2)C_1(y) \\
& + (t^3+3x^2t)C_3(y) + (x^3+3xt^2)D_1(y) + 3(x^3t+xt^3)D_3(y) \\
& + (x^2-y^2)C_2(t) - (y^3-3x^2y)C_4(t) + (x^3-3xy^2)D_2(t) \\
& + 3(x^3y-xy^3)D_4(t) + B_0(x) + A_1(t) + A_2(y) + E_2(x,y) \\
& + A_3(y,t) + E_1(x,t) + E_3(x,y,t).
\end{aligned}
$$

It is straightforward to verify that a function of the above form is a solution of equation (3.4). Hence the proof is completed. □

REFERENCES

[1] Aczél, J. , Haruki, H., McKiernan, M.A., and Sakovič, G.N., General and regular solutions of functional equations characterizing harmonic polynomials. *Aequationes Mathematicae*, 1 (1968): 37-53.

[2] Baker, J.A., An Analogue of the Wave Equation and Certain Related Functional Equations. *Canad. Math. Bull.*, 6 (1969): 837-846.

[3] Bural, P., On the Equivalence of Functions Involving Means and Solution to a Problem of Daróczy. *Aequationes Mathematicae*, 75 (2008): 314-319.

[4] Czerwik, S., *Functional Equations and Inequalities in Several Variables*. World Scientific Publishing Co. Pte. Ltd. 2002.

[5] Haruki, S., On the general solution of a nonsymmetric partial difference functional equation analogous to the wave equation. *Aequationes Mathematicae*, 36 (1988): 20-31.

[6] McKiernan, M.A., The General Solution of Some Finite Difference Equations Analogous to the Wave Equation. *Aequationes Mathematicae*, 8 (1972): 263-266.

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