Chapter III

Quantum Logic

3.1 Introduction

One of the alternatives to the Copenhagen interpretation is the quantum logic approach (see, for example, Jammer, 1974, Chapter 8). Attempts to revise classical logic dated back some time ago but it can be said that quantum logic originated from the work of Birkhoff and von Neumann in 1936. They concluded that the logical structure of the propositional calculus of classical mechanics is that of a Boolean lattice, while the logical structure of the propositional calculus of quantum mechanics is non-Boolean (an orthocomplemented modular lattice, to be specific). Other approaches include the many-valued logic and the axiomatic approach. Efforts have also been made to construct a general theory which includes both quantum and classical mechanics. This chapter ends with the unanswered question of the relationship between quantum logic and logic.

3.2 Early Works

L. E. J. Brouwer and his student, Arend Heyting, suggested in 1908 a revision of classical logic, called "intuitionism," in order to avoid some antinomies in the set theory such as the "Russell paradox" (Encyclopaedia Britannica, 1978, Macropeaedia, V.16, p.571) (In 1902, Bertrand Russell stated that if "x is a set and ($x \notin x$)"

defines a set R of of all sets not members of themselves then R ϵ R and R $\notin$ R, a contradiction). Different logical systems were also proposed, e.g. by Russell and A. N. Whitehead; S. Leśniewski; W. V. Quine.

The traditional Aristotelian or Chrysippean logic obeys the law of bivalence which states that there are only two truth values "true" and "false". In 1920 Jan Łukasiewicz (1920) proposed a third truth value 1/2, different from 0 (false), and from 1 (true). His rules (see Table 3.1) are as follows:

1) if only 0 and 1 are involved, the old rules survive,

2) the truth-value of Np ("non p") is 1/2 if that of p is 1/2,

3) the implication pCq ("p implies q") is evaluated according to the rule that if the value of the antecedent p is less than or equal to the value of the consequent q, its value is 1 and otherwise 1/2.

4) pOq, defined by pCq.C.q, corresponds to "p or q"; pAq, defined by N (Np.O.Nq), corresponds to "p and q"; and pEq, defined by pCq.A.qCp, corresponds to "p is equivalent to q." It can be seen that the law of the excluded middle (either p or not p) is no longer valid.

He also hinted at generalizing his three-valued logic into an infinitely-many-valued logic. If [P] denotes the truth value of P, lying in [0,1] and if

$$[Np] = 1 - [p],$$

p	Np
1	0
1/2	1/2
0	1

p	q	pCq
1	1	1
1	1/2	1/2
1	0	0
1/2	1	1
1/2	1/2	1
1/2	0	1/2
0	1	1
0	1/2	1
0	0	1

Table 3.1 Truth tables of Łukasiewicz's system

$$\text{and } [pCq] = \begin{cases} 1 & \text{if } [p] \leq [q], \\ 1 - [p] + [q] & \text{if } [p] > [q] \end{cases}$$

then clearly his new system contained the two-valued logic and his three-valued logic as special cases.

In 1931 Zygmunt Zawirski (1931) suggested the use of Łucasiewicz's three-valued logic in quantum mechanics to avoid the difficulty arising from the wave-particle duality which, he said, is a self-contradictory but valid statement.

3.3 Nondistributive Logic

Garrett Birkhoff and John von Neumann (1936) have considered the possibility that the logic of quantum mechanics may be different from that of classical mechanics. They first analysed the propositional calculus of classical dynamics. They then identified each subset of phase space with a proposition and conversely. Each experimental proposition a thus corresponds to a subset S_a of phase space Γ and is true if P, the point representing the state referred to in a, lies in S_a. They have also defined conjunction, disjunction, negation, implication and equivalence. The conjuction $a \cap b$ of the two propositions a and b is true if P lies in the intersection of S_a and S_b, the disjunction $a \cup b$ is true if P lies in the union of S_a and S_b, while the negation (or complementation) a' of a asserts that P does not lie in S_a. If whenever a is true b is also true, we say that "a implies b" and denote this by a b'; then S_a is a subset of S_b. The implication ⊆ is reflexive, transitive and antisymmetric (see Appendix D). "a is equivalent to b" or a = b if a ⊆ b and b ⊆ a.

A physical quality is defined as the set of all experimental propositions equivalent to a given experimental proposition. So the physical qualities attributable to any physical system form a partially ordered system. Since the distributive identity holds in classical mechanics, they have concluded that the propositional calculus of classical mechanics forms a Boolean lattice (see Appendix E).

In quantum mechanics we replace subsets of Γ by subspaces of a Hilbert space and the truth value of a proposition a corresponds to the eigenvalues of the projection operator associated with the subspace referred to in a. In the terminology of Birkhoff and von Neumann, the propositional calculus of quantum mechanics is a complemented (nowadays we use the term orthocomplemented) lattice.

Moreover, the distributive identity holds in classical mechanics but not in quantum mechanics. We have to replace it by "modular identity" :

$$\text{If } a \subseteq c, \text{ then } a \cup (b \cap c) = (a \cup b) \cap c \qquad (3.1)$$

Obviously every distributive lattice is modular but the reverse is generally not true.

An example to show that the distributive identity is not valid in quantum mechanics : Let b denote the proposition "$S_z = \hbar/2$," a the proposition "$S_x = \hbar/2$," and hence a' the proposition "$S_x = -\hbar/2$," where S_z and S_x represent spin components for a spin - 1/2 particle. Clearly, $b \cap (a \cup a') = b$, whereas $(b \cap a) \cup (b \cap a') = 0$

Birkhoff and von Neumann conclude that whereas the logical structure of the propositional calculus of classical mechanics is that of a Boolean lattice, the logical structure of the propositional calculus of quantum mechanics is that of an orthocomplemented modular lattice.

The most important critisms of the work of Birkhoff and von Neumann on quantum logic come from Karl R. Popper in 1968 (Popper, 1968). He stated the following theorems (discovered after 1936) : Any uniquely complemented lattice L is Boolean, provided it satisfies at least one of the following four conditions :

a) L is modular (in fact, "weak" modularity is sufficient);
b) L is finite;
c) L is orthocomplemented;
d) L is measurable and the "admissible" measures of L are isotone.

Since the lattice proposed by Birkhoff and von Neumann satisfies each of the four conditions (a) to (d) it is, in fact, Boolean.

3.4 Many-Valued Logic

Some philosophers, e.g. Paul Hertz; Louis Rougier and others, believe that logic is empirical. So newly discovered experiments may lead to the revision of the system of logic.

Paulette Février (1937) has proposed to apply many-valued logic to quantum mechanics. She suggested that Heisenberg's indeterminacy relations should be considered as laws basic to the construction of logic for microscopic objects. She gave an example to show why her logic is three-valued. Let a denote the proposition "the energy E has the value E_o" (assuming discrete energy spectrum); if E_o lies in the spectrum (i.e. E_o can be obtained as a value of E), then a is "true" ("V" for "vraie, nécessairement ou de façon contingente") provided a measurement of E yields E_o, and is "false" ("F" for "fausse de manière contingente") provided it does not yield E_o; the proposition a is "absolutely (necessarily) false" ("A" for "fausse nécessairement") if E_o does not belong to the spectrum (i.e. E_o cannot be a value of E). Table 3.2 a and b are the truth tables for the conjunction of "non-conjugate" propositions, i.e. propositions for compatible measurements and another truth table for the conjunction of "conjugate" propositions.

&	V	F	A
V	V	F	A
F	F	F	A
A	A	A	A

a) "non-conjugate" propositions

&	V	F	A
V	A	A	A
F	A	A	A
A	A	A	A

b) "conjugate" propositions

Table 3.2 Truth tables of Février's system

Note that Table 3.2 a agrees with the truth table for p A q in Łukasiewicz's three-valued logic (see Table 3.1).

Hans Reichenbach (1965) invented a three-valued logic. He introduced the terms <u>phenomena</u> and <u>interphenomena</u> in microphysics. <u>Phenomena</u> are all those occurrences which are connected with macrocosmic occurrences by rather short causal chains and are verifiable by devices. <u>Interphenomena</u> are occurrences which happen between the coincidences and are introduced by inferential chains of a much more complicated sort in the form of an interpolation within the world of phenomena. Thus he divides interpretations of quantum mechanics into two classes :

1. <u>exhaustive</u> interpretations -- provide a description of interphenomena as well as phenomena.

2. <u>restrictive</u> interpretations -- restrict the assertion of quantum mechanics to statements about phenomena.

He claims that in quantum mechanics each exhaustive interpretation leads to causal anomalies. While causal anomalies can be avoided in restrictive interpretations according to which statements about interphenomena are discarded as meaningless by using a three-valued logic : true (T), false (F) and indeterminate (I) which characterizes statements that are regarded as meaningless in the Bohr-Heisenberg interpretation. There are three kinds of negation (see Table 3.3). The disjunction V, conjunction Λ, alternative implication → and standard equivalence ≡ are defined by Table 3.4 (for nonconjugate propositions).

proposition a	T	I	F
cyclical negation ~ a	I	F	T
diametrical negation - a	F	I	T
complete negation $\bar{a}$	I	T	T

Table 3.3 Reichenbach's three kinds of negation

a	T	T	T	I	I	I	F	F	F
b	T	I	F	T	I	F	T	I	F
$a \vee b$	T	T	T	T	I	I	T	I	F
$a \wedge b$	T	I	F	I	I	F	F	F	F
$a \rightarrow b$	T	F	F	T	T	T	T	T	T
$a \equiv b$	T	I	F	I	T	I	F	I	T

Table 3.4 Truth tables of Reichenbach's system

The rule of complementarity can be formulated as follows. Let U denote the statement "The physical quantity X has the value u" and let V denote the statement "The physical quantity Y (complementary to X) has the value v," (X and Y noncommutative), then the rule of complementarity reads

$$U \vee \sim U \rightarrow \sim \sim V,$$

which has the value "true" (T) if and only if at least one of the two statements U, V has the "indeterminate" (I).

Statements which lead to causal anomalies can be avoided because in his three-valued logic they become "indeterminate."

Carl Friedrich Freiherr von Weizsäcker (1955) proposed "complementarity logic" as a modification of the logic of contingent propositions. He defined elementary propositions as those propositions which describe pure cases. His idea is that every elementary proposition can have, apart from 1 and 0, a complex number as its truth value. If we associate the two-component vector (u,v) with the question of whether a_1 or a_2 is true so that (1,0) corresponds to the truth of a_1 and (0,1) to the truth of a_2, then he says that for every vector (u,v) (normalized) there exists a proposition which is true if the proposition of the original alternative has the truth values u and v. Every proposition, characterized by (u,v) which differs from a_1 and from a_2, is called complementary to a_1 and a_2. If one of the two complementary propositions is true or false, the other is neither true nor false.

3.5 The Axiomatic Approach

George Whitelaw Mackey (1963) has studied the axiomatic and quantum logical approaches to quantum mechanics. He bases his axiomatization of quantum mechanics on two undefined notions : observables and states. He defines "question" as an observable whose probability measure is concentrated in the points 0 and 1 of R. His first six postulates imply that with every physical system a partially ordered orthocomplemented set S can be associated which allows us to identify observables with S-valued measures on the Borel sets of the real line and states with probability measures on S and which allows us to deduce some theorems of quantum mechanics without resorting to the notion of a Hilbert space. His seventh and eighth postulates express the distinguishing feature of quantum mechanics, and establish the one-to-one correspondence of observables with self-adjoint operators in a separable Hilbert space and of pure states with its one-dimensional subspaces. His ninth (and last) postulate concerns quantum dynamics and leads to the time-dependent Schrödinger equation.

Josef Jauch (1968) has found that the propositional system of all quantum mechanical "yes-no experiments" is a complete, orthocomplemented, weakly modular atomic lattice, which, moreover, is irreducible and satisfies the covering law.

Peter Mittelstaedt (1961 a, b) has noticed that if a quantum system is known with certainty to have a property A and if a property B, incompatible with A, is measured and it is found that the system has property B, then the probability that it has A is now less than 1 (certainty). The knowledge, originally possessed by the observer about the system, has thus been lost through the acquisition of additional information, a conclusion that contradicts the principle of "unrestricted availability" of classical logic. Mittelstaedt thus suggests that we should use a logic in which this principle is not presupposed, namely, the "operative logic" of Paul Lorenzen (1955). Lorenzen says that the laws of logic are rules whose evidence follows from an examination of the possibilities to prove the assertions. If somebody asserts **a → b** (a, **b** are elementary propositions, → is the implication) he is committed to prove **b** if a can be, or has been proved by the "opponent." The proof thus assumes the form of a dialogue between the proponent and the opponent. If the proponent wins, then his assertion is an "effective-logical" statement. For example, a → (b → a) is an "effective-logical statement :

PROPONENT	OPPONENT
1. a → (b → a)	
	2. a
3. Why a ?	
	4. Proof of a
5. b → a	
	6. b
7. Why b ?	
	8. Proof of b
9. a	
	10. Why a ?
11. See 4.	

Since the proponent has won whatever the particular contents of the proposition a and b, 1 above is an "effective-logical" statement or logical statement. Clearly, the principle of unrestricted availability has been employed.

The following 10 statements L_1 to L_{10}, which can always be successfully defended by a proponent, constitute the so-called affirmative logical calculus (whereas → is part of the proposition, ⇒ belongs to the metalanguage, i.e., X ⇒ Y denotes that if the proposition X is derivable then also the proposition Y is derivable) :

L_1 . $a \to a$

L_2 . $a \to b,\ b \to c \ \Rightarrow\ a \to c$

L_3 . $a \wedge b \to a$

L_4 . $a \wedge b \to b$

L_5 . $c \to a,\ c \to b \ \Rightarrow\ c \to a \wedge b$

L_6 . $a \to a \vee b$

L_7 . $b \to a \vee b$

L_8 . $a \to c,\ b \to c \ \Rightarrow\ a \vee b \to c$

L_9 . $(a \wedge (a \to b)) \to b$

L_{10} . $a \wedge c \to b \ \Rightarrow\ c \to (a \to b)$

Lorenzen then defines the (trivial) assertion V which can never be questioned and the (absurd) assertion Λ which if asserted, makes one lose the dialogue, $\neg a$ (non-a) which is $a \to \Lambda$.

The calculus of effective logic consists of L_1 - L_{10} and the following L_{11} - L_{12}.

L_{11} . $a \wedge \neg a \to \Lambda$

L_{12} . $a \wedge b \to \Lambda \ \Rightarrow\ c \to \neg a$

The classical logic L_c consists of L_1 - L_{12} and the following L_{13} (which is not dialogically demonstrable). Its structure is that of a Boolean lattice with $\neg a$ as the complement of a.

L_{13} . $V \to a \vee \neg a$ (law of the excluded third)

(tertium non datur)

Mittelstaedt says that some of the laws of L_c lose their validity in quantum mechanics because the principle of unrestricted availability does not hold for incompatible propositions. A quantum mechanical proposition **a**, in contrast to a proposition in classical physics, has only restricted availability and may be "quoted" in dialogical demonstrations only if between the proof of **a** and its subsequent quotation all propositions proved were compatible with **a**. He calls this the "commensurability rule," and dialogically demonstrable implications, such as L_1 to L_9, which satisfy this rule, he calls quantum-dialogically demonstrable.

The affirmative quantum logic consists of L_1 - L_9 and the following Q_{10}.

$$Q_{10} . \quad a \wedge c \to b \quad \Rightarrow \quad (a \to c) \to (a \to b)$$

The effective quantum logic consists of L_1 - L_9, Q_{10}, L_{11} and the following Q_{12}.

$$Q_{12} . \quad a \wedge a \to \Lambda \quad \Rightarrow \quad (a \to c) \to \neg a .$$

The quantum logic L_p consists of the effective plus L_{13}. Its structure is that of an orthocomplemented modular lattice.

Mittelstaedt gives the double-slit experiment as an example. Let **a** be the proposition "the particle arrives somewhere on the screen" and **b** the proposition "the particle passes through the upper slit." Then

$a \rightarrow (a \wedge b) \vee (a \wedge \neg b)$	**tertium non datur** relative to a not valid in quantum logic.
$V \rightarrow b \vee \neg b$	"absolute" **tertium non datur** valid in quantum logic.

Mittelstaedt thinks that the confusion of the "relative" with the "absolute" **tertium non datur** leads to the faulty abandonment of the two-valued logic as proposed by Février or Reichenbach.

3.6 Generalizations

The more recent investigators try to construct, within a unified conceptual framework, a general theory of physics which comprises classical as well as quantum mechanics.

Constantin Piron (1972, 1976) produces a formal system of propositions which can be summarized as follows (D is for definition).

D_1 (physical system). A physical system is a part of the real world, thought of as existing in spacetime and external to the physicist.

D_2 (question). A question is any experiment leading to an alternative of which the terms are "yes" and "no".

D_3 (opposite or inverse question). if α is a question, $\alpha^{\sim}$ is the question obtained by exchanging the terms of the alternative.

D_4 (product of questions). If $\{\alpha_i\}_{i\in J}$ is a family of questions, $\pi_J\alpha_i$ is the question defined in the following manner : one measures an arbitrary one of the α_i and attributes to $\pi_J\alpha_i$ the answer thus obtained.

Rule R_1 (opposite of a product question). By starting from the definitions, one can verify the following rule : $(\pi_J\alpha_i)^{\sim} = \pi_J\alpha_i^{\sim}$.

D_5 (trivial question). There exists a trivial question I which consists in measuring anything (or doing nothing) and stating that the answer is "yes" each time.

D_6 (certain or true question). When the physical system has been prepared in such a way that the physicist can affirm that in the event of an experiment corresponding to a question α the result will be "yes," the question α is certain or the question α is true.

D_7 (preorder relation between β and γ). If the question γ is true whenever the question β is true, the question β is stronger than the question γ, which is symbolized by $\beta < \gamma$, the relation D_7 is transitive.

D_8 (equivalent questions). If one has $\beta < \gamma$ and $\gamma < \beta$, then β and γ are equivalent questions, which is denoted by $\beta \approx \gamma$.

D_9 (proposition). The equivalence class containing the question β is called proposition and is denoted by **b**. The set of all the propositions defined for a system is symbolized by $\mathcal{L}$.

D_{10} (true proposition). The proposition **b** is true if and only if the question β of which **b** is the equivalence class is true (in the sense of D_6).

Theorem T_1. The set of propositions $\mathcal{L}$ is a complete lattice, i.e., there exists for any family of propositions $\{b_i\}_{i \in J}$ a proposition $\wedge_J b_i$ such that

$$x < b_i, \ \forall i \in J \ \Leftrightarrow \ x < \wedge_J b_i$$

N. Hadjisavvas et al. (1980) remark that the formulation of T_1 as well as its proof imply certain (usual) notations and concepts that have not been explicitly defined previously ($\wedge_J b_i$, $x < b_i$, $V_J b_i$ for propositions). The structure of the proof of T_1 implies the assumption of the following well-known meanings for these concepts and notations (Piron, 1972, p.291) :

D_{11} (order relation between propositions). If one has $\forall \beta \in$ **b**, $\forall \gamma \in$ **c** . $\beta < \gamma$, then the proposition **b** is stronger than the proposition **c**, which is symbolized by **b** < **c**.

D_{12} ("product" or "conjunction" of propositions). Given any family of propositions $\{b_i\}_{i \in J}$ from $\mathcal{L}$, $\wedge_J b_i$ denotes the equivalence class containing the question $\pi_J \beta_i$, where $\beta_i \in b_i$.

D_{13} <u>("sum" of propositions)</u>. Given a family $\{b_i\}_{i\in J}$ of propositions from $\mathcal{L}$, $V_J b_i$ denotes the product $\Lambda_\alpha x_\alpha$ of all the propositions $x_\alpha \in \mathcal{L}$ such that $b_i < x_\alpha$, $\forall i$.

D_{14} <u>(absurd proposition, trivial proposition)</u>. (not explicitly defined in Piron (1976). Theorem T_1 entails the existence of an absurd proposition $\Lambda_{b\in\mathcal{L}} b = 0$. The equivalence class of the trivial question I defines a trivial proposition I (same notation as for the trivial question).

D_{15} <u>(complementary proposition for b)</u>. The proposition c is a complementary proposition for a given proposition b if $b \vee c = 1$ and $b \wedge c = 0$.

D_{16} <u>(compatible complement for b)</u>. The proposition c is a compatible complement $c = b'$ of a given proposition b if it is a complementary proposition for b and if furthermore there exists a question β such that $\beta \in b$ and $\beta^\sim \in c$.

Axiom C <u>(existence of a compatible complement)</u>. For each proposition b there exists at least one compatible complement b'.

The well-known concepts of a lattice and of lattice generated by a family of propositions (see Appendix E) are then used for the following axiom :

Axiom P. If b $<$ c are propositions from $\mathcal{L}$ and if b' is the compatible complement for b, and c' is the compatible complement for c, then the sublattice generated by (b, b', c, c') is a distributive lattice.

Axiom P entails :

(1) the uniqueness of the compatible complement b' for any b $\in \mathcal{L}$;

(2) orthocomplementation :

$\forall\, b \in \mathcal{L} : (b')' = 0, \quad b \vee b' = I, \quad b \wedge b' = 0$

$\forall\, (b, c) \in \mathcal{L} \times \mathcal{L} : \; b < c \;\Rightarrow\; b' > c'$

and (3) weak modularity :

$\forall\, (b, c) \in \mathcal{L} \times \mathcal{L}$, if $b < c$, then

$c \wedge (c' \vee b) = b, \quad b \vee (b' \wedge c) = c$

D_{17} <u>(covering law and atom)</u> (Piron, 1976, p.9). If $b \neq p$ and $b < p$ then p covers b when $b < x < p \Rightarrow x = b$ or $x = p$. An element which covers 0 is called an atom i.e. if p is such that $0 < x < p \Rightarrow x = 0$ or $x = p$, then p is called an atom.

Axiom A <u>(atomicity, covering law)</u>. If $b \in \mathcal{L}$, $b \neq 0$ then there exists an atom $p : p < b$. If p is an atom and if $p \wedge b = 0$, then $p \vee b$ covers b.

D_{18} <u>(orthogonal propositions)</u>. $b \in \mathcal{L}$ is orthogonal to $c \in \mathcal{L}$ and is symbolized by $b \perp c$ if $b < c'$.

D_{19} <u>(propositional system)</u>. A complete lattice satisfying axioms C, P and A is a propositional system, i.e. a complete, orthocomplemented weakly modular and atomic lattice.

(The classical propositional systems are propositional systems which, moreover, are distributive).

N. Hadjisavvas et al. (1980) have pointed out that Piron's formalism is built on two interconnected levels : the level of questions and the level of propositions, so they call it a questions-propositions system (qp - s). Now, on the level of propositions, the logicomathematical structure which emerges is that of a complete, orthocomplemented, weakly modular, and atomic lattice (D_{19}) and is known to be self-consistent.

They investigate the structure introduced by two levels and show that the qp - s is self-consistent in the abstract theory of models, i.e., they show that it does admit a model. But they criticize Axiom C in that the existence of a compatible complement a' for any a $\in \mathcal{L}$ is questionable. They conclude that Piron's system contains semantic obstacles and thus does not qualify as a generator of quantum mechanics.

C.F. von Weizsäcker and his school (Drieschner et al. 1988) interpret quantum theory as a universal theory of prediction and reconstruct "abstract" quantum theory (see also 2.9). "Abstract" means the general frame of quantum theory, without reference to a three-dimensional position space, to concepts like particle or field, or to special laws of dynamics. "Reconstruction" is the attempt to do this by formulating simple and plausible postulates on prediction in order to derive the basic concepts of quantum theory from them. Thereby no law of "classical" physics is presupposed which would then have to be "quantized." They make the hypothesis that only quantum theory can be the basis for the whole of physics (Görnitz, 1988a).

Abstract quantum theory comprises only four basic concepts (Drieschner et al., 1988) :

1. Hilbert space as a state space.

2. Probability relations between states as defined by Hilbert metric.

3. Composition of objects by the tensor product of their state spaces.

4. Dynamics as a unitary state space representation of the additive real group of time translations.

To reconstruct quantum theory, von Weizsäcker introduces three postulates (Görnitz, 1988) :

(A) Separable alternatives. There exist separable, finite, empirically decidable alternatives.

(B) Indeterminism. With any pair of mutually exclusive states x, y in an alternative, there exist states z with conditional symmetric probabilities different from 0 or 1 to find z for given x or y :

$$p\ (x,\ z) \neq (0 \text{ or } 1),\ p\ (y,\ z) \neq (0 \text{ or } 1)$$

(C) Kinematics. States of a given alternative develop in time in such a way that their relative probabilities remain unchanged. From these postulates some consequences follow :

(i) State space. The set of states for every n-fold alternative constitutes an n-dimensional vector space.

(ii) Symmetry. No state of an alternative is distinguished. There exists a probability-preserving symmetry group. The probabilities bring in the continuum, so the symmetry group will be a Lie group.

(iii) Dynamics. The states develop under the action of a one-dimensional subgroup of the symmetry group with time as its parameter.

(iv) Preservation of states. If a state is to be recognizable, there must exist a dynamics that keeps this state constant.

If a dynamics is to be observable, first it has to hold the alternative separated. Second, if it has no eigenstates, no state could be observed. So it follows :

(v) <u>Vector space</u>. The state space has to be a vector space over the complex numbers, moreover an n-dimensional Hilbert space.

Of the skew fields under consideration, **R**, **C** and **H**, only the complex numbers are algebraically closed, so in this case we can always have diagonalizable self-adjoint generators for the possible dynamics.

(vi) <u>Composition</u>. Two alternatives are decided by deciding their Cartesian product. The state space of the product alternative is the tensor product of the state spaces of the two subalternatives.

They hope that the whole concrete quantum theory and, with it the basic laws of physics, can be derived by means of the additional postulate of urs :

a) Every n-fold alternatives can be decomposed into a product of binary alternatives.

b) Every state space can be understood as a subspace of a tensor product of two-dimensional spaces.

Then they set up the central dynamical postulate :

c) For any object there is at least one decomposition into binary (sub)objects -- called urs (from German **Ur-Alternativen** = original alternatives (Görnitz, 1986)) -- such that its dynamics is invariant under the symmetry group of the urs.

This postulate -- all objects "arise from" or "consist of" urs -- constitutes a radical abstract atomism.

So the ur is introduced as a (sub)object, quantum-theoretically described in a two-dimensional complex Hilbert space. The probability-preserving symmetry group for its states is built up from the U (1) as the dynamical subgroup, from SU (2) and the complex conjugation.

3.7 Quantum Logic and Logic

The problem of the relationship between quantum logic and logic concerns two main questions : "What is logic ?" and "Is quantum logic a logic ?". Below is listed some discussions on this issue.

3.7.1 What is logic ?

According to Charles Peirce (Baldwin, 1925; Copi, 1979), "Nearly a hundred definitions of it have been given." We reproduce here a survey of the literature.

Morris R. Cohen and Ernest Nagel (1934) : "Logic is correct reasoning. To be logical is to argue reasonably. By means of logic we can find out what follows if we accept a given statement as true." and "Logic ... concerned with the question of the adequacy of different kinds of evidence."

Irving M. Copi (1972) : "Logic is the study of the methods and principles used to distinguish good (correct) from bad (incorrect) reasoning." Logic is not " 'the' science of the laws of thought," because it is "not a branch of psychology" and also "not all thinking is reasoning." Another definition "the science of reasoning" will not do because "logician is not ... concerned with the dark ways by which the mind arrives at its conclusions. ... He is concerned only with the correctness of the completed process."

K. Jaakko J. Hintikka (Encyclopaedia Britannica, Macropaedia, 1978, V.11, p.73) : "... the study of truths based completely on the meanings of the terms they contain. ... According to the wider interpretation, all truths depending only on meanings belong to logic. ... According to the narrower conception, logical truths obtain (or hold) in virtue of ... logical constants."

Karl R. Popper (1972) : "... the three main views of the nature of logic...

(A) The rules of logic are laws of thought.

(A1) They are natural laws of thought -- they describe how we actually do think; and we cannot think otherwise.

(A2) They are normative laws -- they tell us how we ought to think.

(B) The rules of logic are the most general laws of nature -- they are descriptive law holding for any object whatsoever.

(C) The rules of logic are laws of certain descriptive languages -- of the use of words and especially of sentences."

Patrick Suppes (1957) : "In the narrow sense, logic is the theory of valid arguments or the theory of deductive inference. A slightly broader sense includes the theory of definition. A still broader sense includes the general theory of sets."

3.7.2 Is Quantum Logic a Logic ?

Birkhoff and von Neumann (1936) claimed to have laid the foundations of a new logic. But how can mathematics which uses standard logic be applicable to quantum mechanics which uses nonstandard logic ?

Piron (1964) suggests that his lattice L_q of quantum mechanical propositions could not qualify as a logic because although $a \leq b$ is the analogue of the logical implication $a \rightarrow b$, $a \leq b$ cannot be considered a proposition, since it is not a yes-no experiment. Jauch (1968) has pointed out that quantum logic is the formalization of empirical facts obtained by induction while ordinary logic is an analysis of the meaning of propositional structures which is true under all circumstances and even tautologically so. Likewise, Mittelstaedt (1968b) stresses that the fact that for certain propositions in quantum mechanics, formal logic ceases to be applicable does not affect the **a priori** nature of logic.

By contrast, David Finkelstein (1969) argues that there is no such thing as an **a priori** universally valid logic. Logic, like geometry, undergoes a process of evolution. Hilary Putnam (1969) also uses the same analogy. Just as the rejection of Euclidean geometry enabled Einstein to get rid of macroscopic anomalies, he says, so microphysical anomalies can be dissolved as soon as the distributive law of classical logic is given up. Patrick Heelan (1970) has pointed out that Putnam uses **modus tollens** (if **p** U **q**, ~ **q**, then **p**) which is invalid in quantum logic.

Michael Satosi Watanabe (1966) has claimed to derive quantum mechanics from the new logic (while Finkelstein and Putnam derived the new logic from quantum mechanics). He says that in quantum mechanics the <u>Frege Principle</u> (assumption that each predicate corresponds one-to-one at each time point, to a well-defined (fixed) set of objects that satisfy the predicate) has to be replaced by the <u>Peirce Principle</u> which states that implication is the most important operation of human reasoning. He concludes that <u>probability</u> precedes <u>logic</u>. He also says that the new logic, if restricted to certain domains, reduces to the usual Boolean logic and there exists a domain of inference where the usual logic remains valid.

David Bohm (1951 pp. 168-172) has pointed out that there is an anology between the thought processes and quantum processes, for example, between the production of new ideas and the quantum jump etc. The reason for these analogies may be, according to Bohr, that although the mechanism of the brain is classical, certain key points controlling this mechanism are so sensitive that they must be described quantum-mechanically. So if the Bohm-Bohr hypothesis is correct, the brain may be governed by quantum logic and computers may be qualitatively different from the brain (Adler and Wirth, 1983). (The present impact of quantum logic on computer function comes from the radioactive decay originating in the material of the IC).

Lotfi A. Zadeh (Wang et al., 1983) has introduced the idea of fuzzy logic from his theory of fuzzy set. He observes that the human capability to understand and analyze imprecise concepts is not properly understood by existing analytical methods. Essentially, fuzziness is a type of imprecision that stems from a grouping of elements into classes (called fuzzy sets) that do not have sharply defined boundaries (Kandel, 1986; Haack, 1978). Whereas in abstract set theory an object either is or is not a member of a given set, in fuzzy set theory membership is a matter of degree. The theory of fuzzy sets deals with a subset A of the universe of discourse X, where the transition between full membership and no membership is gradual rather than abrupt. The "fuzzy subset" has no well-defined boundaries where the universe of discourse (the universe X) covers a definite range of objects. An example of fuzzy class of objects is "the set of long streets in Bangkok." Traditionally, the grade of membership 1 is assigned to those objects that fully and completely belong to A, while 0 is assigned to objects that do not belong to A at all. The more an object x belongs to A, the closer to 1 is its grade of membership $X_A(x)$

Fuzzy-set theory is a generalization of abstract set theory. Because of this generalization, fuzzy-set theory has a wider scope of applicability than abstract set theory in solving problems that involve, to some degree, subjective evaluation.

Intuitively, a <u>fuzzy set</u> is a class that admits the possibility of partial membership in it. Let X denote a space of objects. Then a fuzzy set A in X is a set of ordered pairs

$$A = \{x, X_A(x)\}, \quad x \in X,$$

where $X_A(x)$ is termed "the grade of membership of x in A" and for simplicity it is assumed that $X_A(x)$ is a number in the interval $[0,1]$, with the grades 1 and 0 representing, respectively, full membership and nonmembership in a fuzzy set, as discussed before. It is also possible to interpret $X_A(x)$ as the degree of possibility that x is the value of a parameter fuzzily restricted by A.

In general, we distinguish three kinds of inexactness : <u>generality</u>, that a concept applies to a variety of situations; <u>ambiguity</u>, that it describes more than one distinguishable subconcept; and <u>vagueness</u>, that precise boundaries are not defined. All three types of inexactness are represented by a fuzzy set : Generality occurs when the universe is not just one point, ambiguity occurs when there is more than one local maximum of a membership function; and vagueness occurs when the function takes values other than just 0 and 1.

Now fuzzy set theory can be used to characterize a non-standard logic called fuzzy logic. We may view fuzzy logic as a special kind of many-valued logic. In fuzzy logic, the truth value of a formula, instead of assuming two values (0 and 1), can assume any value in the interval [0, 1] and is used to indicate the degree of truth represented by the formula. Note that two-valued logic is a special case of fuzzy logic. Fuzzy logic is not just a logic for handling arguments in which vague terms occur essentially; it is itself imprecise. For this reason Zadeh's proposal is much more radical for it challenges deeply entrenched ideas about the characteristic objectives and methods of logic.

At present fuzzy set theory has a wide range of application (Wang et al., 1983) including pattern recognition, decision analysis and approximate deductive reasoning. It has the greatest applicability for problems not adequately addressed by current parametric models especially systems charaterized by ill-defined and difficult-to-observe-parameters or imprecise goals.