CHAPTER I

PRELIMINARIES

Let X be a set. By a *transformation* of X, we mean a mapping of X into itself. Let $\mathcal{T}_X$ denote the set of all transformations of X. A *partial transformation* of X is a mapping from a subset of X into X. The *empty transformation* of X is the partial transformation with empty domain and is denoted by 0. For a partial transformation α of X, the domain and the range of α are denoted by $\Delta\alpha$ and $\nabla\alpha$, respectively. Let $\mathcal{PT}_X$ be the set of all partial transformations of X. For $\alpha, \beta \in \mathcal{PT}_X$, define the product $\alpha\beta$ as follows: If $\nabla\alpha \cap \Delta\beta = \varnothing$, let $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \varnothing$, let $\alpha\beta$ be the composition of mappings $\alpha|_{(\nabla\alpha\cap\Delta\beta)\alpha^{-1}}$ and $\beta|_{(\nabla\alpha\cap\Delta\beta)}$ where $\alpha|_{(\nabla\alpha\cap\Delta\beta)\alpha^{-1}}$ and $\beta|_{(\nabla\alpha\cap\Delta\beta)}$ denote the restrictions of α and β to $(\nabla\alpha \cap \Delta\beta)\alpha^{-1}$ and $\nabla\alpha \cap \Delta\beta$, respectively. Then for $\alpha, \beta \in \mathcal{PT}_X$, we have that

$$\Delta\alpha\beta = (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \subseteq \Delta\alpha, \quad \nabla\alpha\beta = (\nabla\alpha \cap \Delta\beta)\beta \subseteq \nabla\beta$$

and indeed,

$$|\nabla\alpha\beta| \leq \min\,\{|\nabla\alpha|, |\nabla\beta|\}$$

since $|(\nabla\alpha \cap \Delta\beta)\beta| \leq |\nabla\alpha \cap \Delta\beta| \leq |\nabla\alpha|$. If $\alpha \in \mathcal{PT}_X$ and $A \subseteq X$, we define $A\alpha = (A \cap \Delta\alpha)\alpha$. It then follows that

$$A\alpha \subseteq \nabla\alpha, \quad (A \cup B)\alpha = A\alpha \cup B\alpha,$$

$$\nabla\alpha\beta = (\nabla\alpha)\beta, \quad A\alpha\beta = (A\alpha)\beta$$

for all $\alpha, \beta \in \mathcal{PT}_X$ and $A, B \subseteq X$. Under the product defined above, $\mathcal{PT}_X$ is a semigroup with identity i_X and zero 0 where i_X is the identity mapping on X. By a *transformation semigroup* on X, we mean a subsemigroup of $\mathcal{PT}_X$. Let $\mathcal{I}_X$ denote the set of all 1–1 partial transformations of X. Then $0 \in \mathcal{I}_X$ and $\mathcal{T}_X$ and $\mathcal{I}_X$ are both transformation semigroups on X containing i_X. The semigroups $\mathcal{PT}_X$, $\mathcal{T}_X$ and $\mathcal{I}_X$ are called respectively as the *partial transformation semigroup* on X, the *full transformation semigroup* on X and the *symmetric inverse semigroup* on X. Let $\mathcal{G}_X$ denote the symmetric group on X. Then $\mathcal{G}_X$ is a subsemigroup of both $\mathcal{T}_X$ and $\mathcal{I}_X$.

We clearly have the following properties of mappings which will be used later.

1. For any nonempty sets A, B, if $\alpha: A \to B$ is 1–1, then there exists an onto mapping $\beta: B \to A$ such that $\alpha\beta = i_A$.
2. For any sets A, B, if $\alpha: A \to B$ is onto, then there exists a 1–1 mapping $\beta: B \to A$ such that $\beta\alpha = i_B$.
3. If X is finite and $\alpha \in \mathcal{PT}_X$ is such that $\nabla\alpha = X$, then $\alpha \in \mathcal{G}_X$.
4. If X is finite and $\alpha, \beta \in \mathcal{PT}_X$ are such that $\alpha\beta \in \mathcal{G}_X$, then $\alpha, \beta \in \mathcal{G}_X$.

Let X and Y be sets and let $\mathcal{PT}(X, Y)$ denote the set of all mappings from subsets of X into Y. Then $\mathcal{PT}(X,Y) \cup \mathcal{PT}(Y,X) \cup \mathcal{PT}(X,X) \cup \mathcal{PT}(Y,Y) \subseteq \mathcal{PT}_{X \cup Y}$ and we shall consider the product of two elements in $\mathcal{PT}(X,Y) \cup \mathcal{PT}(Y,X) \cup \mathcal{PT}(X,X) \cup \mathcal{PT}(Y,Y)$ as their product in $\mathcal{PT}_{X \cup Y}$. If S is a nonempty subset of $\mathcal{PT}(X,Y)$ and $\theta \in \mathcal{PT}(Y,X)$ such that $\alpha\theta\beta \in S$ for all $\alpha, \beta \in S$, then $(S, *)$ with $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in S$, is a semigroup and we denote this semigroup by (S, θ) which is called a *generalized transformation semigroup* of X into Y.

Let $\mathcal{T}(X,Y)$ and $\mathcal{I}(X,Y)$ be the set of all mappings from X into Y and the set of all 1–1 mappings from subsets of X into Y, respectively. Then $\mathcal{T}(X,Y)$ and $\mathcal{I}(X,Y)$ are subsets of $\mathcal{PT}(X,Y)$. If $\mathcal{S}(X,Y)$ is any one of $\mathcal{T}(X,Y)$, $\mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$, $\mathcal{S}(X,Y) \neq \varnothing$ and $\theta \in \mathcal{S}(Y,X)$, then $\alpha\theta\beta \in \mathcal{S}(X,Y)$ for all $\alpha, \beta \in \mathcal{S}(X,Y)$, that is, $(\mathcal{S}(X,Y),\theta)$ is a generalized transformation semigroup of X into Y. In this research, we consider generalized transformation semigroups of these types. Necessary and sufficient conditions for $\mathcal{S}(X,Y) \neq \varnothing$ and $\mathcal{S}(Y,X) \neq \varnothing$ are as follows:

(1) $\mathcal{T}(X,Y) \neq \varnothing$ and $\mathcal{T}(Y,X) \neq \varnothing$ if and only if $X \neq \varnothing$ and $Y \neq \varnothing$ or $X = Y = \varnothing$.

(2) All of $\mathcal{PT}(X,Y)$, $\mathcal{PT}(Y,X)$, $\mathcal{I}(X,Y)$ and $\mathcal{I}(Y,X)$ always contain 0.

The following theorem has been given by Vorobev in [11].

Theorem 1.1. *[11]. Let X be a finite set. If $\alpha \in \mathcal{T}_X$ is such that $|\nabla\alpha| = |X| - 1$, then $\mathcal{T}_X = \langle \mathcal{G}_X \cup \{\alpha\} \rangle$, the subsemigroup of $\mathcal{T}_X$ generated by $\mathcal{G}_X \cup \{\alpha\}$.*

We recall the following basic concepts on semigroups. A *congruence* on a semigroup S is an equivalence relation ρ on S such that for all $a, b, c \in S$, $a \,\rho\, b$ implies $ac \,\rho\, bc$ and $ca \,\rho\, cb$, and the *quotient semigroup of S modulo the congruence ρ* is the semigroup S/ρ with a binary operation defined by

$$(a\rho)(b\rho) = (ab)\rho \qquad (a, b \in S).$$

The semigroup S/ρ is a homomorphic image of S by the *natural homomorphism* $\rho^\natural \colon S \to S/\rho$ defined by

$$a\rho^\natural = a\rho \qquad (a \in S).$$

If T is a semigroup and $\varphi: S \to T$ is a homomorphism, then the relation ρ on S defined by

$$a \,\rho\, b \iff a\varphi = b\varphi \qquad (a, b \in S)$$

is a congruence on S and $S/\rho \cong S\varphi$ by the isomorphism $a\rho \mapsto a\varphi$.

Let U be a subsemigroup of a semigroup S. For any element $d \in S$, d is said to be *dominated* by U or U *dominates* d if for any semigroup T and for any homomorphisms $\varphi, \psi: S \to T, \varphi|_U = \psi|_U$ implies $d\varphi = d\psi$. The set of all elements of S dominated by U is called the *dominion* of U in S and is denoted by $\mathrm{Dom}(U, S)$. Then $U \subseteq \mathrm{Dom}(U, S) \subseteq S$. The subsemigroup U is said to be *dense* in S if $\mathrm{Dom}(U, S) = S$, that is, for any semigroup T and for any homomorphisms $\varphi, \psi: S \to T, \varphi|_U = \psi|_U$ implies $\varphi = \psi$ on S.

We give a remark here that if U is dense in S, S' is a semigroup and $\pi: S \to S'$ is a homomorphism, then $U\pi$ is dense in $S\pi$. To prove this, let T be a semigroup and $\varphi, \psi: S\pi \to T$ homomorphisms such that $\varphi|_{U\pi} = \psi|_{U\pi}$. Then $\pi\varphi, \pi\psi: S \to T$ are homomorphisms and for every $x \in U$, $x(\pi\varphi) = (x\pi)\varphi = (x\pi)\psi = x(\pi\psi)$. Since U is dense in S, it follows that $\pi\varphi = \pi\psi$ on S and thus for each $x \in S$, $(x\pi)\varphi = x(\pi\varphi) = x(\pi\psi) = (x\pi)\psi$. Hence $\varphi = \psi$ on $S\pi$. From this proof and the relationship between congruences and homomorphisms of semigroups mentioned above, we have the following proposition.

Proposition 1.2. *Let U be a dense subsemigroup of a semigroup S. If S' is a semigroup and $\phi: S \to S'$ is a homomorphism, then $U\phi$ is a dense subsemigroup of $S\phi$. Equivalently, if ρ is a congruence on S, then $\{x\rho \mid x \in U\}$ is a dense subsemigroup of the quotient semigroup S/ρ.*

Let U be a subsemigroup of a semigroup S. A *zigzag* of *length* m *in* S *over* U with *value* $d \in S$ is a system of equalities

$$d = u_0 y_1 \quad , \quad u_0 = x_1 u_1 \quad ,$$

$$u_{2i-1} y_i = u_{2i} y_{i+1} \quad , \quad x_i u_{2i} = x_{i+1} u_{2i+1} \quad (i = 1, \ldots, m-1),$$

$$u_{2m-1} y_m = u_{2m} \quad , \quad x_m u_{2m} = d$$

where $u_0, u_1, \ldots, u_{2m} \in U$ and $x_1, \ldots, x_m, y_1, \ldots, y_m \in S$, that is,

$$\begin{array}{lllll}
d = & u_0 y_1 & , u_0 \in U & , y_1 \in S & , \\
= & x_1 u_1 y_1 & , u_1 \in U & , x_1 \in S & , \quad u_0 = x_1 u_1, \\
= & x_1 u_2 y_2 & , u_2 \in U & , y_2 \in S & , u_1 y_1 = u_2 y_2, \\
& \cdots & \cdots & \cdots & \cdots \\
= & x_i u_{2i-1} y_i & , u_{2i-1} \in U & , x_i \in S & , x_{i-1} u_{2i-2} = x_i u_{2i-1}, \\
= & x_i u_{2i} y_{i+1} & , u_{2i} \in U & , y_{i+1} \in S & , u_{2i-1} y_i = u_{2i} y_{i+1}, \\
= & x_{i+1} u_{2i+1} y_{i+1} & , u_{2i+1} \in U & , x_{i+1} \in S & , x_i u_{2i} = x_{i+1} u_{2i+1}, \\
& \cdots & \cdots & \cdots & \cdots \\
= & x_m u_{2m-1} y_m & , u_{2m-1} \in U & , x_m \in S & , x_{m-1} u_{2m-2} = x_m u_{2m-1}, \\
= & x_m u_{2m} & , u_{2m} \in U & , u_{2m-2} y_m = u_{2m}. &
\end{array}$$

We give a remark here that if a zigzag in S over U with value d exists, then a zigzag Z of mininum length with value d always exists and for the case that $d \in S \setminus U$, we have that all x_i, y_i for the zigzag Z are in $S \setminus U$.

A very important tool for this research is the Zigzag Theorem given as follows:

Theorem 1.3. *[4] Let* U *be a subsemigroup of a semigroup* S. *Then* $d \in \mathrm{Dom}(U, S)$ *if and only if* $d \in U$ *or there is a zigzag in* S *over* U *with value* d.

The following corollary follows directly from the Zigzag Theorem and the remark given above.

Corollary 1.4. *Let U be a subsemigroup of a semigroup S and $d \in S \setminus U$. Then $d \in \mathrm{Dom}(U,S)$ if and only if there exists $u_0, u_1, \ldots, u_{2m} \in U, x_1, \ldots, x_m, y_1, \ldots, y_m \in S \setminus U$ such that*

$$d = u_0 y_1 \quad , \quad u_0 = x_1 u_1 \quad ,$$

$$u_{2i-1} y_i = u_{2i} y_{i+1} \quad , \quad x_i u_{2i} = x_{i+1} u_{2i+1} \quad (i = 1, \ldots, m-1),$$

$$u_{2m-1} y_m = u_{2m} \quad , \quad x_m u_{2m} = d.$$

It is clear that if S has an identity, then for every $u \in U$, there exists a zigzag in S over U with value u. Hence, the Zigzag Theorem under the assumption that S has an identity can be restated as follows:

> " Let S be a semigroup with identity and U a subsemigroup of S. Then for $d \in S$, $d \in \mathrm{Dom}(U,S)$ if and only if there exists a zigzag in S over U with value d."

A semigroup S is said to be *regular* if for each $x \in S$, there exists $y \in S$ such that $x = xyx$. Using the Zigzag Theorem, Hall has proved the following theorem in [1].

Theorem 1.5. *If U is a proper regular subsemigroup of a finite semigroup S, then* $\mathrm{Dom}(U,S) \neq S$.

The next theorem is a special case of a theorem given by Howie and Isbell in [3] which is also an application of the Zigzag Theorem.

Theorem 1.6. *If G is a subgroup of a semigroup S, then* $\mathrm{Dom}(G, S) = G$.

Since every subsemigroup of a finite group G is a subgroup of G, the following corollary is obtained from Theorem 1.6.

Corollary 1.7. *If U is a subsemigroup of a finite group G, then* $\mathrm{Dom}(U, G) = U$.