CHAPTER II

Manipulator Mathematical Model

2.1 Introduction

The basics of the manipulator path tracking control study are the investigation of dynamical behaviors of manipulator mechanisms. This can be done through the study of dynamic equations or the mathematical model of manipulators derived from conventional principles of the classical solid mechanics. The general manipulator model presents some specific complexness of a system of differential equations featuring complicated, time-variant and highly nonlinear dynamical terms. In the context of modern manipulator control, with affordable computing power, a manipulator dynamic model has usually been used in various control scheme to compensate for nonlinearities and decouple force interference between joints leaving servo error regulating task to the servo portion of a controller that is of various types depending upon choice of the control designer. The utilization and application of dynamic knowledge from a manipulator model in control design and synthesis had conducted inspiration and founded basis of embedding adaptive mechanism in manipulator control schemes.

In this chapter the basics of manipulator model derivation will be briefly presented in order to address what the scope of our manipulator mathematical modeling will be, and what dynamics should be considered outside the model that we intend to construct. Begin with mathematical representation of geometrical relation between manipulator linkages, kinematic equations describing physical motion of a manipulator will be detailed to support dynamic model derivation in the sequel. Energy approach based on formulation of Lagragian terms is exploited as it will give very systematic way to the derivation of manipulator dynamics.

We shall state first the basic assumptions underlining the mathematical model derivation methodology as a platform that will be considered true throughout this treatise.

1. Mechanical manipulators are considered to be serial link mechanism that consist of rigid links sequentially connected together by actuated joints. Each joint generally exhibits one degree of freedom as joint rotation or joint-axis translation. Link flexibility is considered to have very small amplitude with respect to joint displacement. Vibration signals caused from link flexibility are therefore to be considered outside the total control system bandwidth and neglected in all our discussion.

2. The manipulator model is formulated in the absence of any friction and other internal and external disturbances and we assume no backlash and any mechanical clearance in all joint composition.

3. The manipulator model does not include external loading effects. This can be apparently seen that some of them are either unpredictable caused from their random nature such as crashing with unexpectedly obstacle or hardly to determine because of geometrical complexity of the body involved such as the end effector or load to be picked up

2.2 Kinematical Background

The first step into studying science of motion of a physical system is to explore its kinematical properties. Kinematic study involves mathematical representation of the geometric configuration of physical systems under motion without regarding to any forces that cause the motion. We will present briefly some kinematics of a robot manipulator in order for having enough background to support the dynamic model derivation thereafter.

2.2.1 Kinematical Representations

The Homogeneous Transformation is used as the basic data type representing position and orientation of the manipulator body in working space. The transformation utilizes a combination of a rotation matrix and a position vector resulting to a four-by-four matrix which has the important property that the composition of coordinates can be represented by the operation of matrix multiplication.

The transformation matrix ${}^{1}\mathbf{T}_2$ that describes relative position and orientation of the second coordinate frame with respect to and in terms of the first coordinate frame can be expressed as

$$ {}^{1}\mathbf{T}_2 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.1} $$

The second frame orientation is described by three orthogonal vectors $\mathbf{o},\mathbf{a},\mathbf{n}$ which form three axes of the frame, its components expressed in terms of the first frame. Position of the described frame is presented as vector $\mathbf{p}$ pointing toward from the origin of the first coordinate frame to the one of the second coordinate frame.

The inverse of the transformation can be easily derived from orthogonal property of its vector elements and is indeed the matrix inverse of the transformation ${}^{1}\mathbf{T}_2$

$$ {}^{2}\mathbf{T}_1 = \begin{bmatrix} n_x & n_y & n_z & -\mathbf{p}\cdot\mathbf{n} \\ o_x & o_y & o_z & -\mathbf{p}\cdot\mathbf{o} \\ a_x & a_y & a_z & -\mathbf{p}\cdot\mathbf{a} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2} $$

where elements in the above matrix represent the position and orientation vectors as in the previous transformation matrix except that the reference is now the second frame and the described frame is then the first.

With this homogeneous transformation, we have got a very systematic tool to be applied to describe configuration of serial linkage mechanisms of robot manipulators the geometrical constraints of rigid links make terms in the transformation more easy to be derived.

Based on the well-known systematic method that was pioneered into robotic science by Roberts (5) and Pieper (6), appeared in the work of R. Paul and C. N. Stevenson (3), we can define some general rules to affix a coordinate frame to each link at the corresponding joint axis. The method parametrically characterizes each link and describes the relationship between two consecutive joints by four fixed parameters consisted in mathematical terms of elements of the specific transformation matrix which we usually designate as, in the context of robotic, $\mathbf{A}$ matrix.

Figure 2.1 illustrates the way to affix a coordinate frame to a joint axis and corresponding link and definition of link parameters and variable for revolute and prismatic joint respectively using the Denavit-Hatenberg notation (4). With this definition, the size and shape of links can be described by the measurement of two quantities, the first is called the *link length*, a_n defined to be the length of the common normal between two axes of link n, and the second is the *twist angle*, α_n defined by the angle between the orthogonal projections of joint axes n and $n+1$ onto a plane normal to the common normal.

The relative position between links $n-1$ and n at joint n can be measured by the distance d_n between two common normal intersections with the joint axis n. This quantity is called the *joint distance* and defined to be a fixed parameter when the joint is revolute or to be a joint variable when the joint is prismatic.

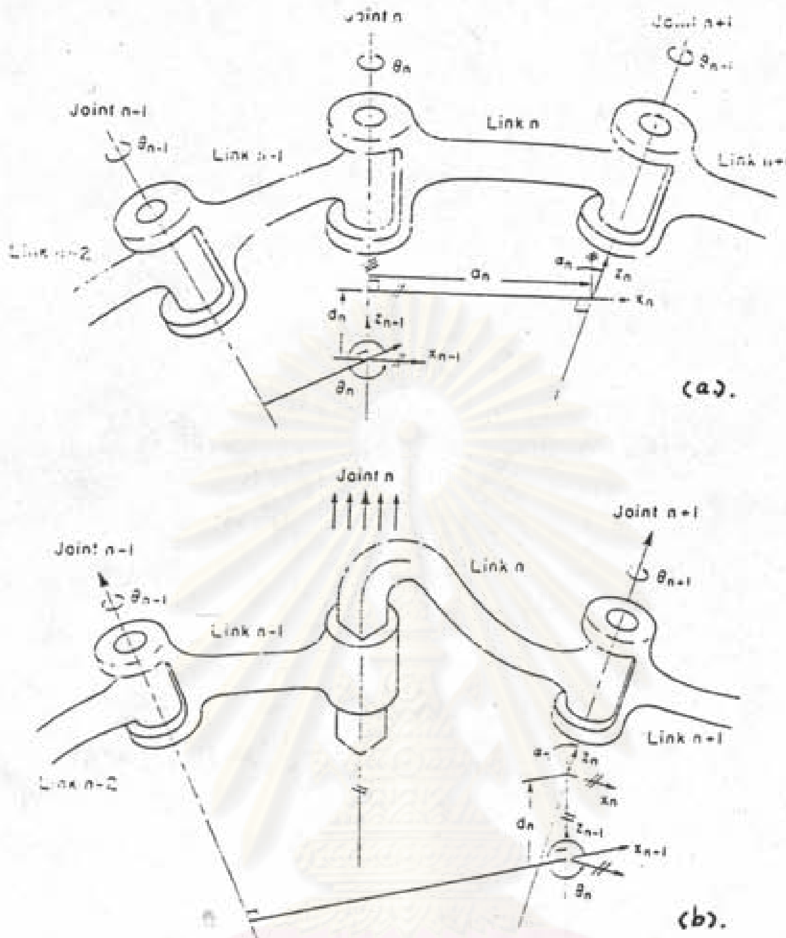

Figure 2.1: Affixing coordinate frames to joint axes, Definition of link parameters and variable: (a). for revolute joint n; (b). for prismatic joint n.

The final quantity is the *joint angle*, θ_n used as a joint variable instead of a joint fixed parameter, if the joint discussed is revolute and considered to be a fixed parameter in a prismatic joint type. Its value is measured as an angle between projections of the two common normal onto the plane normal to the joint axis n.

The transformation matrix $\mathbf{A}$ that represents geometrical relation between manipulator link $n - 1$ and link n at joint n has its form as Equation 2.1 and 2.2. When applied with this coordinate frame system, its expression will be as follows.

for link n and joint n of revolute type

$$\mathbf{A}_n = \begin{bmatrix} \cos\theta_n & -\sin\theta_n\cos\alpha_n & \sin\theta_n\sin\alpha_n & a_n\cos\theta_n \\ \sin\theta_n & \cos\theta_n\cos\alpha_n & -\cos\theta_n\sin\alpha_n & a_n\sin\theta_n \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.3)$$

for prismatic joint type

$$\mathbf{A}_n = \begin{bmatrix} \cos\theta_n & -\sin\theta_n\cos\alpha_n & \sin\theta_n\sin\alpha_n & 0 \\ \sin\theta_n & \cos\theta_n\cos\alpha_n & -\cos\theta_n\sin\alpha_n & 0 \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.4)$$

We can see that the matrix is a function of the corresponding joint coordinates.

2.2.2 Kinematic Equations

With the information of $\mathbf{A}_n$ at link n, for a manipulator having n links and n joints, the methodology of the homogeneous transformation enable us to express the position and orientation of the end of the manipulator mechanism in terms of the mechanism base coordinate by applying recursive matrix multiplication to obtain a coordinate combination description. For an n-degree-of-freedom manipulator, given $\mathbf{A}_1, \mathbf{A}_2, \ldots \mathbf{A}_n$, $\mathbf{A}_1$ describes the position and orientation of the first link, $\mathbf{A}_2$ describes the position and orientation of the second link with respect to the first and so on. Thus the position and orientation of the second link in manipulator base coordinate are given by the matrix product

$$\mathbf{T}_2 = \mathbf{A}_1\mathbf{A}_2 \tag{2.5}$$

Similarly, $\mathbf{A}_3$ describes the third link in terms of the second and we have

$$\mathbf{T}_3 = \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 \tag{2.6}$$

These matrix products can be obtained sequentially to the description of the link n as follows

$$\mathbf{T}_n = \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\ldots\mathbf{A}_n \tag{2.7}$$

These equations are generally called the kinematic equations of robot manipulators. $\mathbf{T}$ matrix notion is conventionally used with the leading superscript omitted, if it is 0 as referred to the base coordinate.

In this way, every part of or attached objects to the link mechanism can be mathematically recognized in terms of the base coordinate which we usually use as a coordinate of the manipulator working space. This means that any position vector can be transformed from one coordinate to its equivalence in another coordinates. This can be simply done by a matrix multiplication of the transformation and the vector. Given a position vector pointing to an object, its expression is written consisting of x, y, z components in the coordinate n which ,for instance, may be the coordinate frame of a video camera attached at somewhere in the link mechanism. The position vector is the information of the relative distance obtained from an image processor equipped with that camera. We want to register it into the manipulator working space, which configed by the base coordinate that indeed is used to characterize every object involved in manipulation program. Then we must transform the vector to be described in the base coordinate. This can be done by the following multiplication

$$\mathbf{r} = \mathbf{T}_n {}^{n}\mathbf{r} \tag{2.8}$$

Again, we omit the 0 superscription. ${}^{n}\mathbf{r}$ is the position vector in the coordinate n and $\mathbf{r}$ is the position vector in the base coordinate.

2.3 Manipulator Dynamics

Manipulator dynamics represent some special complexities of very complicated physical systems. This comes from being an active mechanism that can change configuration of itself during performing a task. Recursive configurations of manipulator mechanisms, however, permit us to develop a systematic way to derive the dynamic equations for general manipulators. We will make use of Lagragian mechanics as this method allows us to get the dynamic equations of very complex system in a simple manner. The method applying the classical mechanical energy concepts also provides us insight to the dynamical behaviors of a physical system and structure of terms in the equations can even be easily discused in a symbolic fashion.

2.3.1 Lagrangian Formulation

The Lagragian approach takes a different view of dynamic discussion of physical systems. In contrast to Newtonian approach that dynamic equations come from the time derivative of momentum property of all mass in a physical system, it looks for energy content of the overall system. For a system in motion, We can define, an energy term, the Lagragian L as the different between the kinetic energy K and the potential energy P of the whole system

$$L = K - P \tag{2.9}$$

The terms can be expressed in arbitrary *n* coordinates for a system of *n* degrees of freedom based on the concept of *generalized coordinates* in analytical mechanics. And the *generalized forces* corresponding to selected generalized coordinates can be obtained from the *Lagrange's Equations* that indeed are the dynamic equations of the system.

$$Q_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \tag{2.10}$$

where Q_i, q_i are the generalized forces and generalized coordinates respectively. If a generalized coordinate is selected to be an angular coordinate the corresponding generalized force will be a generalized torque. This can be easily clarified by consideration of the equations' dimensions.

2.3.2 Derivation of Manipulator Dynamic Equations

Based on the kinetic energy terms in the Lagragian formulation, we must first determine velocity of any point in a link mechanism with respect to the base coordinate which is considered to be the inertial frame of the consequent whole dynamic discussion. we will begin with a point described by a position vector ${}^{i}\mathbf{r}$ with respect to the coordinate frame *i* of link *i*. Its expression in the base coordinate is

$$\mathbf{r} = \mathbf{T}_i {}^{i}\mathbf{r} \tag{2.11}$$

Throughout this derivation the zero superscript will be omitted and this implies the base coordinate as reference. Differentiate (2.11) with respect to time to obtain the vector of velocity of the point

$$\dot{\mathbf{r}} = \left[\sum_{j=1}^{i} \frac{\partial \mathbf{T}_i}{\partial q_j}\dot{q}_j\right]{}^{i}\mathbf{r} + \mathbf{T}_i \frac{d}{dt}\left({}^{i}\mathbf{r}\right) \tag{2.12}$$

Since the vector is pointing to a fixed point with respect to the coordinate frame i, its expression is thereby invariant to time. The second term on the right side of (2.12) hence diminishes.

$$\dot{\mathbf{r}} = \left[\sum_{j=1}^{i} \frac{\partial \mathbf{T}_i}{\partial q_j}\dot{q}_j\right]{}^{i}\mathbf{r} \tag{2.13}$$

The square of the velocity is

$$\dot{\mathbf{r}}^2 = \mathbf{r}\cdot\mathbf{r} \tag{2.14}$$

which is a scalar quantity. (2.14) can be written in matrix-vector format that

$$\dot{\mathbf{r}}^2 = tr\left[\dot{\mathbf{r}}\dot{\mathbf{r}}^T\right] \tag{2.15}$$

where the trace of a matrix represents the summation of all diagonal elements of the matrix argument. Replace the expression for that

$$\dot{\mathbf{r}}^T = {}^{i}\mathbf{r}^T \sum_{k=1}^{i} \frac{\partial \mathbf{T}_i^T}{\partial q_k}\dot{q}_k \tag{2.16}$$

and into (2.15) , we get

$$\begin{aligned}\dot{\mathbf{r}}^2 &= tr\left[\sum_{j=1}^{i} \frac{\partial \mathbf{T}_i}{\partial q_j}\dot{q}_j\,{}^{i}\mathbf{r}\,{}^{i}\mathbf{r}^T \sum_{k=1}^{i} \frac{\partial \mathbf{T}_i^T}{\partial q_k}\dot{q}_k\right] \\ &= tr\left[\sum_{j=1}^{i}\sum_{k=1}^{i} \frac{\partial \mathbf{T}_i}{\partial q_j}{}^{i}\mathbf{r}\,{}^{i}\mathbf{r}^T \frac{\partial \mathbf{T}_i^T}{\partial q_k}\dot{q}_j\dot{q}_k\right]\end{aligned} \tag{2.17}$$

The kinetic energy of a particle of mass dm located on link n at ${}^{i}\mathbf{r}$ is

$$
\begin{aligned}
dK &= \frac{1}{2} dm\left(\dot{\mathbf{r}}^2\right) \\
&= \frac{1}{2} dm \times tr\left[\sum_{j=1}^{i}\sum_{k=1}^{i} \frac{\partial \mathbf{T}_i}{\partial q_j} {}^{i}\mathbf{r}^{i}\mathbf{r}^{T} \frac{\partial \mathbf{T}_i^T}{\partial q_k} \dot{q}_j \dot{q}_k\right] \\
&= \frac{1}{2} tr\left[\sum_{j=1}^{i}\sum_{k=1}^{i} \frac{\partial \mathbf{T}_i}{\partial q_j} {}^{i}\mathbf{r} dm {}^{i}\mathbf{r}^{T} \frac{\partial \mathbf{T}_i^T}{\partial q_k} \dot{q}_j \dot{q}_k\right]
\end{aligned} \tag{2.18}
$$

The total kinetic energy of link i is the summation of the kinetic energy of each particle in link i that is

$$
\begin{aligned}
K_i &= \int_{m_i} dK_i \\
&= \frac{1}{2} tr\left[\sum_{j=1}^{i}\sum_{k=1}^{i} \frac{\partial \mathbf{T}_i}{\partial q_j} \left(\int_{m_i} {}^{i}\mathbf{r}^{i}\mathbf{r}^{T} dm\right) \frac{\partial \mathbf{T}_i^T}{\partial q_k} \dot{q}_j \dot{q}_k\right] \\
&= \frac{1}{2} tr\left[\sum_{j=1}^{i}\sum_{k=1}^{i} \frac{\partial \mathbf{T}_i}{\partial q_j} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_k} \dot{q}_j \dot{q}_k\right]
\end{aligned} \tag{2.19}
$$

where

$$
J_i = \int_{m_i} {}^{i}\mathbf{r}^{i}\mathbf{r}^{T} dm
$$

represents the inertia tensor of link i that characterizes mass distribution of a rigid body in three-dimensional space.The elements of the matrix consist of the moment of inertia, cross product of inertia and the first moment of a link body taken at the center of mass with respect to the x, y, and z axis of the coordinate frame i that

$$J_i = \begin{bmatrix} \int_{m_i} {}^ix^2 dm & \int_{m_i} {}^ix^iydm & \int_{m_i} {}^ix^izdm & \int_{m_i} {}^ixdm \\ \int_{m_i} {}^ix^iy & \int_{m_i} {}^iy^2dm & \int_{m_i} {}^iy^izdm & \int_{m_i} {}^iydm \\ \int_{m_i} {}^ix^izdm & \int_{m_i} {}^iy^izdm & \int_{m_i} {}^iz^2dm & \int_{m_i} {}^izdm \\ \int_{m_i} {}^ixdm & \int_{m_i} {}^iydm & \int_{m_i} {}^izdm & \int_{m_i} dm \end{bmatrix} \tag{2.20}$$

and that

$$J_i = \begin{bmatrix} \frac{-I_{ixx} + I_{iyy} + I_{izz}}{2} & I_{ixy} & I_{ixz} & m_i\bar{x}_i \\ I_{ixy} & \frac{I_{ixx} - I_{iyy} + I_{izz}}{2} & I_{iyz} & m_i\bar{y}_i \\ I_{ixz} & I_{iyz} & \frac{I_{ixx} + I_{iyy} - I_{izz}}{2} & m_i\bar{z}_i \\ m_i\bar{x}_i & m_i\bar{y}_i & m_i\bar{z}_i & m_i \end{bmatrix} \tag{2.21}$$

where

$$I_{xx} = \int (y^2 + z^2) dm$$
$$I_{yy} = \int (x^2 + z^2) dm$$
$$I_{zz} = \int (x^2 + y^2) dm$$
$$I_{xy} = \int xy\ dm$$
$$I_{xz} = \int xz\ dm$$
$$I_{yz} = \int yz\ dm$$
$$m\bar{x} = \int x\ dm$$
$$m\bar{y} = \int y\ dm$$
$$m\bar{z} = \int z\ dm$$

with

$$\int x^2 dm = \frac{1}{2}\left(-I_{xx} + I_{yy} + I_{zz}\right)$$
$$\int y^2 dm = \frac{1}{2}\left(I_{xx} - I_{yy} + I_{zz}\right)$$
$$\int z^2 dm = \frac{1}{2}\left(I_{xx} + I_{yy} - I_{zz}\right)$$

The total kinetic energy of the manipulator is the sum of the kinematic energy of each link in the mechanism structure and of the actuators at joints. Finally we obtain the manipulator kinetic energy expression

$$\begin{aligned} K &= \sum_{i=1}^{n} K_i \\ &= \frac{1}{2}\sum_{i=1}^{n} tr\left[\sum_{j=1}^{i}\sum_{k=1}^{i} \frac{\partial \mathbf{T}_i}{\partial q_j} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_k} \dot{q}_j \dot{q}_k\right] \end{aligned} \quad (2.22)$$

where n specifies a number of linkages of a manipulator which also equals the number of degrees of freedom. This equation represents the kinetic energy of the manipulator structure alone excluding that of the actuators at each joints. However that kinetic energy contribution could be easily included into the kinetic equation if desire, its expression depends on joint types and actuators used.

For the potential energy of a manipulator, we derive its formula from the concept of the work done by moving an object in the gravity field of the earth. The potential energy of the manipulator is referred to summation of work done by every mass in the mechanism at a high of current configuration above some zero reference in the gravity field of acceleration g. Consider link i of mass whose center of mass can be specified by vector expressed in the coordinate frame i, the potential energy evaluated with an arbitrary zero reference described in the base coordinate is

$$P_i = -m_i \mathbf{g}^T \mathbf{T}_i \, {}^i\overline{\mathbf{r}}_i \quad (2.23)$$

where ${}^{i}\mathbf{r}$ is the vector pointing to the center of mass of link i, the superscript i denotes vector description with the coordinate of link i. The presence of T_n transformation determines that the potential energy is derived in terms of the base coordinate, the same as of the kinetic energy with local link parametric descriptions. And the total potential energy of the manipulator structure up to link n is

$$P = -\sum_{i=1}^{n} m_i \mathbf{g}^T \mathbf{T}_i {}^{i}\mathbf{r}_i \tag{2.24}$$

And now we can write the Lagrangian of a manipulator as

$$L = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} tr\left[\frac{\partial \mathbf{T}_i}{\partial q_j} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_k}\right]\dot{q}_j\dot{q}_k + \sum_{i=1}^{n} m_i \mathbf{g}^T \mathbf{T}_i {}^{i}\mathbf{r}_i \tag{2.25}$$

Replace this Lagrangian in (2.10) and manipulate the terms in the following steps, then, differentiate the Lagrangian with respect to the first derivative of the coordinate p

$$\frac{\partial L}{\partial \dot{q}_p} = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{i} tr\left[\frac{\partial \mathbf{T}_i}{\partial q_p} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_k}\right]\dot{q}_k + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i} tr\left[\frac{\partial \mathbf{T}_i}{\partial q_j} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_p}\right]\dot{q}_j \tag{2.26}$$

Because the trace operator is affected only by the terms on diagonal of a matrix. Hence the value of the trace is invariant to the transpose, then we can write

$$tr\left[\frac{\partial \mathbf{T}_i}{\partial q_j} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_k}\right] = tr\left[\left(\frac{\partial \mathbf{T}_i}{\partial q_j} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_k}\right)^T\right] \tag{2.27}$$

On the right side, the transpose of the product is the product of the transpose such that

$$tr\left[\left(\frac{\partial \mathbf{T}_i}{\partial q_j} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_k}\right)^T\right] = tr\left[\frac{\partial \mathbf{T}_i}{\partial q_k} J_i^T \frac{\partial \mathbf{T}_i^T}{\partial q_j}\right] \tag{2.28}$$

From (2.20) notice that J_i is symmetric. That is

$$J_i = J_i^T \tag{2.29}$$

We then obtain

$$tr\left[\frac{\partial \mathbf{T}_i}{\partial q_j} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_k}\right] = tr\left[\frac{\partial \mathbf{T}_i}{\partial q_k} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_j}\right] \tag{2.30}$$

Use this result to change the product of the argument of the trace in the second term of (2.29) as (2.34) and since j and k ,respectively, in the first and the second term are just dummy indices, we can change simply the index j to k in the second term of (2.29). And then we can obtain

$$\frac{\partial L}{\partial \dot{q}_p} = \sum_{i=1}^{n}\sum_{k=1}^{i} tr\left[\frac{\partial \mathbf{T}_i}{\partial q_k} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_p}\right]\dot{q}_k \tag{2.31}$$

Notice the fact that the transformation is a function of the joint variables up to i joints,i.e.

$$\frac{\partial \mathbf{T}_i}{\partial q_p} = 0 \quad , p > i$$

The final expression of $\frac{\partial L}{\partial \dot{q}_p}$ comes as

$$\frac{\partial L}{\partial \dot{q}_p} = \sum_{i=p}^{n}\sum_{k=1}^{i} tr\left[\frac{\partial \mathbf{T}_i}{\partial q_k} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_p}\right]\dot{q}_k \tag{2.32}$$

Differentiate (2.36) with respect to time we then obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_p}\right) = \sum_{i=p}^{n}\sum_{k=1}^{i} tr\left[\frac{\partial \mathbf{T}_i}{\partial q_k} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_p}\right]\ddot{q}_k + \sum_{i=p}^{n}\sum_{k=1}^{i}\sum_{m=1}^{i} tr\left[\frac{\partial^2 \mathbf{T}_i}{\partial q_k \partial q_m} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_p}\right]\dot{q}_k \dot{q}_m$$
$$+ \sum_{i=p}^{n}\sum_{k=1}^{i}\sum_{m=1}^{i} tr\left[\frac{\partial^2 \mathbf{T}_i}{\partial q_p \partial q_m} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_k}\right]\dot{q}_k \dot{q}_m \tag{2.33}$$

The further step is to derive the second term of (2.25) that

$$\frac{\partial L}{\partial q_p} = \frac{1}{2}\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} tr\left[\frac{\partial^2 \mathbf{T}_i}{\partial q_j \partial q_p} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_k}\right]\dot{q}_j \dot{q}_k + \frac{1}{2}\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} tr\left[\frac{\partial^2 \mathbf{T}_i}{\partial q_k \partial q_p} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_j}\right]\dot{q}_j \dot{q}_k$$
$$+ \sum_{i=p}^{n} m_i \mathbf{g}^T \frac{\partial \mathbf{T}_i}{\partial q_p}{}^i\bar{\mathbf{r}}_i \tag{2.34}$$

Again, notice that we can use the result of (2.30) to verify that the first and the second term on the right side of (2.34) are identical. Hence (2.34) then turns to be

$$\frac{\partial L}{\partial q_p} = \sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} tr\left[\frac{\partial^2 \mathbf{T}_i}{\partial q_j \partial q_p} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_k}\right]\dot{q}_j \dot{q}_k + \sum_{i=p}^{n} m_i \mathbf{g}^T \frac{\partial \mathbf{T}_i}{\partial q_p}{}^i\bar{\mathbf{r}}_i \tag{2.35}$$

At this point we already have all the terms for the Lagrange equation. Replace (2.33) and (2.35) into the equation, we observe that the right side, third term of (2. 33) and the right side, first term of (2.35) are indeed identical although the dummy indices are used. Therefore, they are diminish when substitute in (2.10). The rest then comes out as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_p}\right)-\frac{\partial L}{\partial q_p}=\sum_{i=p}^{n}\sum_{k=1}^{i} tr\left[\frac{\partial \mathbf{T}_i}{\partial q_k} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_p}\right]\ddot{q}_k+\sum_{i=p}^{n}\sum_{k=1}^{i}\sum_{m=1}^{i} tr\left[\frac{\partial^2 \mathbf{T}_i}{\partial q_k \partial q_m} J_i \frac{\partial \mathbf{T}_i^T}{\partial q_p}\right]\dot{q}_k \dot{q}_m - \sum_{i=p}^{n} m_i \mathbf{g}^T \mathbf{T}_i\, {}^i\overline{\mathbf{r}}_i \tag{2.36}$$

or we may equivalently rewrite as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=\sum_{j=i}^{n}\sum_{k=1}^{j} tr\left[\frac{\partial \mathbf{T}_j}{\partial q_k} J_j \frac{\partial \mathbf{T}_j^T}{\partial q_i}\right]\ddot{q}_k+\sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} tr\left[\frac{\partial^2 \mathbf{T}_j}{\partial q_k \partial q_m} J_j \frac{\partial \mathbf{T}_j^T}{\partial q_i}\right]\dot{q}_k \dot{q}_m - \sum_{j=i}^{n} m_j \mathbf{g}^T \frac{\partial \mathbf{T}_j}{\partial q_i}\, {}^j\overline{\mathbf{r}}_j \tag{2.37}$$

and

$$T_i=\sum_{j=i}^{n}\sum_{k=1}^{j} tr\left[\frac{\partial \mathbf{T}_j}{\partial q_k} J_j \frac{\partial \mathbf{T}_j^T}{\partial q_i}\right]\ddot{q}_k+\sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} tr\left[\frac{\partial^2 \mathbf{T}_j}{\partial q_k \partial q_m} J_j \frac{\partial \mathbf{T}_j^T}{\partial q_i}\right]\dot{q}_k \dot{q}_m - \sum_{j=i}^{n} m_j \mathbf{g}^T \frac{\partial \mathbf{T}_j}{\partial q_i}\, {}^j\overline{\mathbf{r}}_j \tag{2.38}$$

This is the dynamic equation expressing balance of dynamic forces occurred with link *i*.The equation relates amount of generalized force to the sum of an inertial term, the second term that exhibits colioris and centripetal effects and conservative force due to gravity. The complete equation for an n-degree-of freedom manipulator consists of *n* force-balanced equations for running at 1 to *n* and may be compactly expressed in matrix-vector form relating the generalized force vector as functions of the generalized coordinate vector. The complete equation of motion in matrix-vector form comes out as

$$\tau = \mathbf{D(q)\ddot{q}} + \mathbf{h(q,\dot{q})} + \mathbf{g(q)} \tag{2.39}$$

where

$$\tau = \begin{bmatrix} T_1 \\ T_2 \\ . \\ . \\ . \\ T_n \end{bmatrix}$$

$$\mathbf{D(q)} = \begin{bmatrix} D_{11} & D_{12} & . & . & . & D_{1n} \\ D_{21} & D_{22} & . & . & . & D_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ D_{n1} & D_{n2} & . & . & . & D_{nn} \end{bmatrix}$$

$$\mathbf{h(q,\dot{q})} = \begin{bmatrix} \sum_{j=1}^{n}\sum_{k=1}^{n} D_{1jk}\dot{q}_j\dot{q}_k \\ \sum_{j=1}^{n}\sum_{k=1}^{n} D_{2jk}\dot{q}_j\dot{q}_k \\ . \\ . \\ . \\ \sum_{j=1}^{n}\sum_{k=1}^{n} D_{njk}\dot{q}_j\dot{q}_k \end{bmatrix}$$

$$\mathbf{g(q)} = \begin{bmatrix} D_1 \\ D_2 \\ . \\ . \\ . \\ D_n \end{bmatrix}$$

and where

$$D_{ij} = \sum_{p=\max i,j}^{n} tr\left[\frac{\partial \mathbf{T}_p}{\partial q_j} J_p \frac{\partial \mathbf{T}_p^T}{\partial q_i}\right]$$

$$D_{ijk} = \sum_{p=\max i,j,k}^{n} tr\left[\frac{\partial^2 \mathbf{T}_p}{\partial q_j \partial q_k} J_p \frac{\partial \mathbf{T}_p^T}{\partial q_i}\right]$$

$$D_i = \sum_{p=i}^{n} -m_p g^T \frac{\partial \mathbf{T}_p}{\partial q_i} {}^p\bar{\mathbf{r}}_p$$