CHAPTER IV

SOME FURTHER REMARKS

First, we recall the following notation: Let X be a set and let

B_X = the semigroup of binary relations on X,

M_X = the transformation semigroup of all 1-1 transformations of X,

O_X = the transformation semigroup of all onto transformations of X,

AM_X = the transformation semigroup of all almost 1-1 transformations of X,

AO_X = the transformation semigroup of all almost onto transformations of X and

A_α = $\{x \in X \mid \alpha \text{ is not 1-1 at } x\}$ for all $\alpha \in T_X$ (the full transformation semigroup on X).

It is unknown whether the transformation semigroups M_X, O_X, AM_X and AO_X are absolutely closed. The aim of this chapter is to show that M_X, O_X, AM_X and AO_X are closed in B_X. It follows that if $S = M_X$, O_X, AM_X or AO_X, then S is closed in every subsemigroup of B_X which contains S (see (iii) page 8). In particular, M_X, O_X, AM_X and AO_X are closed in T_X (the full transformation semigroup on X) and P_X (the partial transformation semigroup on X). We leave as an open problem whether M_X, O_X, AM_X and AO_X are absolutely closed for any set X.

The following lemma is required to show that for any set X, M_X is closed in B_X.

Lemma 4.1. Let X be a set. If $\alpha \in B_X$ is such that $\alpha\beta \in M_X$ for some $\beta \in M_X$, then $\alpha \in M_X$.

Proof: Let $\alpha \in B_X$ and $\beta \in M_X$ be such that $\alpha\beta \in M_X$. Then $\alpha\beta = \gamma$ for some $\gamma \in M_X$. Since $\beta \in M_X$, the inverse map of β, β^{-1}, is in I_X where I_X is the symmetric inverse semigroup on X. Then $\alpha = \gamma\beta^{-1} \in I_X$. But $\Delta\alpha \supseteq \Delta\alpha\beta = \Delta\gamma = X$, so $\alpha \in M_X$. #

Theorem 4.2. For any set X, M_X is closed in B_X.

Proof: Let X be a set. To show that $\mathrm{Dom}(M_X,B_X) = M_X$, let $\alpha \in \mathrm{Dom}(M_X,B_X)$. Then by Corollary 1.2, there exist $\beta_0,\beta_1,\ldots,\beta_{2m} \in M_X$, $\gamma_1,\ldots,\gamma_m,\lambda_1,\ldots,\lambda_m \in B_X$ such that

$$\alpha = \beta_0\lambda_1,\ \beta_0 = \gamma_1\beta_1,$$

$$\gamma_i\beta_{2i} = \gamma_{i+1}\beta_{2i+1},$$

$$\beta_{2i-1}\lambda_i = \beta_{2i}\lambda_{i+1} \quad (i=1,\ldots,m-1),$$

$$\beta_{2m-1}\lambda_m = \beta_{2m}\ .$$

Then $\alpha = \gamma_m\beta_{2m}$. Since $\beta_1 \in M_X$ and $\gamma_1\beta_1 = \beta_0 \in M_X$, by Lemma 4.1, $\gamma_1 \in M_X$. Since $\gamma_2\beta_3 = \gamma_1\beta_2 \in M_X$, by Lemma 4.1, $\gamma_2 \in M_X$. From $\gamma_{i+1}\beta_{2i+1} = \gamma_i\beta_{2i}$ for all $i \in \{1,\ldots,m-1\}$, it follows by Lemma 4.1 inductively that $\gamma_i \in M_X$ for all $i \in \{1,\ldots,m\}$. Then $\gamma_m \in M_X$. But $\alpha = \gamma_m\beta_{2m}$, so $\alpha \in M_X$. Hence $\mathrm{Dom}(M_X,B_X) = M_X$. Therefore M_X is closed in B_X. #

To prove that for any set X, O_X and AO_X are closed in B_X, the following lemma is required.

Lemma 4.3. Let X be a set and U a subsemigroup of T_X. Assume that for each $\alpha \in U$, $\beta \in T_X$, $\alpha\beta \in U$ implies $\beta \in U$. Then U is closed in B_X.

Proof: If $X = \emptyset$, then $|U| = 1$, so U is closed in B_X. Assume that $X \neq \emptyset$. To show that $Dom(U,B_X) = U$, let $\alpha \in Dom(U,B_X)$. By assumption, we have that $1_X \in U$. By Corollary 1.2, there exist $\beta_0,\beta_1,\ldots,\beta_{2m} \in U$, $\gamma_1,\ldots,\gamma_m,\lambda_1,\ldots,\lambda_m \in B_X$ such that

$$\alpha = \beta_0\lambda_1,\ \beta_0 = \gamma_1\beta_1,$$

$$\gamma_i\beta_{2i} = \gamma_{i+1}\beta_{2i+1},$$

$$\beta_{2i-1}\lambda_i = \beta_{2i}\lambda_{i+1} \quad (i=1,\ldots,m-1),$$

$$\beta_{2m-1}\lambda_m = \beta_{2m}.$$

Then $\alpha = \gamma_m\beta_{2m}$. Let $q \in X$ and $V = U \cup CT_X$. Since CT_X is an ideal of T_X, V is a subsemigroup of T_X containing CT_X. For each $\lambda \in B_X$, define $\varphi_\lambda \in T_X$ as follows: For $p \in X$,

$$p\varphi_\lambda = \begin{cases} p(X_p\lambda) & \text{if } X_p\lambda \in V, \\ q & \text{if } X_p\lambda \notin V. \end{cases}$$

Then from $\beta_{2i-1}\lambda_i = \beta_{2i}\lambda_{i+1}$ $(i = 1,\ldots,m-1)$ and $\beta_{2m-1}\lambda_m = \beta_{2m}$, we have by Lemma 3.2 that

$$\beta_{2i-1}\varphi_{\lambda_i} = \beta_{2i}\varphi_{\lambda_{i+1}} \qquad (i = 1,\ldots,m-1),$$

$$\beta_{2m-1}\varphi_{\lambda_m} = \beta_{2m}.$$

Now, we have the following system of equalities:

$$\beta_0 = \gamma_1\beta_1,$$

$$\gamma_i\beta_{2i} = \gamma_{i+1}\beta_{2i+1},\ \beta_{2i-1}\varphi_{\lambda_i} = \beta_{2i}\varphi_{\lambda_{i+1}}\ (i=1,\ldots,m-1),$$

$$\beta_{2m-1}\varphi_{\lambda_m} = \beta_{2m}\ .$$

Then by Lemma 3.1, $\beta_0\varphi_{\lambda_1} = \gamma_m\beta_{2m}$. From $\alpha = \gamma_m\beta_{2m}$, we have $\alpha = \beta_0\varphi_{\lambda_1}$. Since $\beta_{2m-1}\varphi_{\lambda_m} = \beta_{2m}$, by assumption, $\varphi_{\lambda_m} \varepsilon U$. Since $\beta_{2m-3}\varphi_{\lambda_{m-1}} = \beta_{2m-2}\varphi_{\lambda_m} \varepsilon U$, by assumption, $\varphi_{\lambda_{m-1}} \varepsilon U$. From $\beta_{2i-1}\varphi_{\lambda_i} = \beta_{2i}\varphi_{\lambda_{i+1}}$ for all $i \varepsilon \{1,\ldots,m-1\}$, it follows by assumption inductively that $\varphi_{\lambda_i} \varepsilon U$ for all $i \varepsilon \{1,\ldots,m\}$. Then $\varphi_{\lambda_1} \varepsilon U$. But $\alpha = \beta_0\varphi_{\lambda_1}$, so $\alpha \varepsilon U$. This proves that $Dom(U,B_X) = U$. Therefore U is closed in B_X. #

<u>Theorem 4.4</u>. For any set X, O_X is closed in B_X.

<u>Proof</u>: Let X be a set. It is known that for $\alpha,\beta : X \to X$, if $\alpha\beta$ is onto, then β is onto. This implies that for $\alpha \varepsilon O_X$, $\beta \varepsilon T_X$, $\alpha\beta \varepsilon O_X$ implies $\beta \varepsilon O_X$. Then by Lemma 4.3, O_X is closed in B_X. #

<u>Theorem 4.5</u>. For any set X, AO_X is closed in B_X.

<u>Proof</u>: Let X be a set. Let $\alpha \varepsilon AO_X$ and $\beta \varepsilon T_X$ be such that $\alpha\beta \varepsilon AO_X$. Then $X \setminus \nabla\alpha\beta$ is finite. Since $\nabla\alpha\beta \subseteq \nabla\beta$, $X \setminus \nabla\beta \subseteq X \setminus \nabla\alpha\beta$, which implies that $X \setminus \nabla\beta$ is finite. Therefore $\beta \varepsilon AO_X$. Hence by Lemma 4.3, AO_X is closed in B_X. #

The next two lemmas are required to prove that for any set X, AM_X is closed in B_X.

<u>Lemma 4.6</u>. Let X be a set. If $\alpha \in B_X$ and $\beta \in T_X$ are such that $\alpha\beta \in T_X$, then for $x \in X \setminus A_{\alpha\beta}, t,z \in X$, $(x,z),(t,z) \in \alpha$ implies $x = t$.

<u>Proof</u>: Let $\alpha \in B_X$ and $\beta \in T_X$ be such that $\alpha\beta \in T_X$. Let $x \in X \setminus A_{\alpha\beta}$ and $t, z \in X$ be such that $(x,z),(t,z) \in \alpha$. Then $(x,z\beta),(t,z\beta) \in \alpha\beta$. Since $x \in X \setminus A_{\alpha\beta}$ (that is, $\alpha\beta$ is 1-1 at x), $x = t$. #

<u>Lemma 4.7</u>. Let X be a set and $\alpha \in B_X$. Suppose that $F \subseteq X$ which satisfies the following property: for $x \in X \setminus F, t,z \in X$, $(x,z),(t,z) \in \alpha$ implies $x = t$. Then the following statements hold:

(1) If $\beta \in T_X$, then for $x \in X \setminus (F \cup A_\beta\alpha^{-1}), t,z \in X$, $(x,z),(t,z) \in \alpha\beta$ implies $x = t$, where $A_\beta\alpha^{-1} = \{w \in X \mid (w,y) \in \alpha \text{ for some } y \in A_\beta\}$.

(2) If $\beta \in AM_X$ and F is finite, then $A_\beta\alpha^{-1}$ is finite, and hence $F \cup A_\beta\alpha^{-1}$ is finite.

<u>Proof</u>: To prove (1), let $\beta \in T_X$, $x \in X \setminus (F \cup A_\beta\alpha^{-1})$ and $t,z \in X$ be such that $(x,z),(t,z) \in \alpha\beta$. Then there exist $w,w' \in X$ such that $(x,w),(t,w') \in \alpha$ and $(w,z),(w',z) \in \beta$. Since $x \in X \setminus A_\beta\alpha^{-1}$ and $(x,w) \in \alpha$, it follows that $w \in X \setminus A_\beta$. But $(w,z),(w',z) \in \beta$, so $w = w'$. Therefore $(x,w),(t,w) \in \alpha$. Since $x \in X \setminus F$, by assumption, $x = t$.

To prove (2), assume that F is finite and $\beta \in AM_X$. For each $y \in X$, let $y\alpha^{-1} = \{x \in X \mid (x,y) \in \alpha\}$. Then $A_\beta\alpha^{-1} = \bigcup_{y \in A_\beta} y\alpha^{-1}$. Claim that for each $y \in X$, $y\alpha^{-1}$ is finite. To show the claim, let $y \in X$.

Case 1: $y\alpha^{-1}\cap(X\setminus F)\neq\emptyset$. Let $x\in y\alpha^{-1}\cap(X\setminus F)$. Then $(x,y)\in\alpha$ and $x\in X\setminus F$. If $w\in y\alpha^{-1}$, then $(w,y)\in\alpha$, so by assumption, $x = w$. Therefore $y\alpha^{-1} = \{x\}$.

Case 2: $y\alpha^{-1}\cap(X\setminus F)=\emptyset$. Then $y\alpha^{-1} = (y\alpha^{-1}\cap(X\setminus F))\cup(y\alpha^{-1}\cap F) = y\alpha^{-1}\cap F$ and therefore $y\alpha^{-1}\subseteq F$. Since F is finite, $y\alpha^{-1}$ is finite.

Hence we have the claim. Since $\beta\in AM_X$, A_β is finite. Then we have by the claim that $\bigcup_{y\in A_\beta} y\alpha^{-1}$ is finite. Hence $A_\beta\alpha^{-1}$ is finite. #

Corollary 4.8. Let X be a set. Let $\alpha,\beta\in T_X$ and $\gamma,\mu\in B_X$ be such that $\gamma\alpha = \mu\beta$. Suppose that $F\subseteq X$ which satisfies the following property: for $x\in X\setminus F, t,z\in X$, $(x,z),(t,z)\in\gamma$ implies $x = t$. Then for $x\in X\setminus(F\cup A_\alpha\gamma^{-1}), t,z\in X$, $(x,z),(t,z)\in\mu$ implies $x = t$.

Proof: Assume that the assumption holds. Let $x\in X\setminus(F\cup A_\alpha\gamma^{-1})$ and $t,z\in X$ be such that $(x,z),(t,z)\in\mu$. Then $(x,z\beta),(t,z\beta)\in\mu\beta$. But $\gamma\alpha = \mu\beta$, so $(x,z\beta),(t,z\beta)\in\gamma\alpha$. By Lemma 4.7(1), we get that $x = t$.

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Theorem 4.9. For any set X, AM_X is closed in B_X.

Proof: Let X be a set. If $X=\emptyset$, then $|AM_X| = 1$, so AM_X is closed in B_X. Assume that $X\neq\emptyset$. To show that $\mathrm{Dom}(AM_X,B_X) = AM_X$, let $\alpha\in\mathrm{Dom}(AM_X,B_X)$. Since $1_X\in AM_X$, by Corollary 1.2, there exist $\beta_0,\beta_1,\ldots,\beta_{2m}\in AM_X$, $\gamma_1,\ldots,\gamma_m,\lambda_1,\ldots,\lambda_m\in B_X$ such that

$$\alpha = \beta_0\lambda_1,\ \beta_0 = \gamma_1\beta_1,$$

$$\gamma_i\beta_{2i} = \gamma_{i+1}\beta_{2i+1},$$

$$\beta_{2i-1}\lambda_i = \beta_{2i}\lambda_{i+1}\quad (i=1,\ldots,m-1),$$

$$\beta_{2m-1}\lambda_m = \beta_{2m}\ .$$

Then $\alpha = \gamma_m\beta_{2m}$. Let $q \in X$ and $U = AM_X \cup CT_X$. Since CT_X is an ideal of T_X, U is a subsemigroup of T_X containing CT_X. For each $\lambda \in B_X$, define $\varphi_\lambda \in T_X$ as follows: For $p \in X$,

$$p\varphi_\lambda = \begin{cases} p(X_p\lambda) & \text{if } X_p\lambda \in U, \\ q & \text{if } X_p\lambda \notin U. \end{cases}$$

Then from $\beta_{2i-1}\lambda_i = \beta_{2i}\lambda_{i+1}$ $(i = 1,\ldots,m-1)$ and $\beta_{2m-1}\lambda_m = \beta_{2m}$, we have by Lemma 3.2 that

$$\beta_{2i-1}\varphi_{\lambda_i} = \beta_{2i}\varphi_{\lambda_{i+1}} \quad (i = 1,\ldots,m-1),$$

$$\beta_{2m-1}\varphi_{\lambda_m} = \beta_{2m}.$$

Now, we have the following system of equalities:

$$\beta_0 = \gamma_1\beta_1,$$

$$\gamma_i\beta_{2i} = \gamma_{i+1}\beta_{2i+1},\ \beta_{2i-1}\varphi_{\lambda_i} = \beta_{2i}\varphi_{\lambda_{i+1}} \quad (i=1,\ldots,m-1),$$

$$\beta_{2m-1}\varphi_{\lambda_m} = \beta_{2m},$$

so by Lemma 3.1, we get that $\beta_0\varphi_{\lambda_1} = \gamma_m\beta_{2m}$. Since $\alpha = \gamma_m\beta_{2m}$, $\alpha = \beta_0\varphi_{\lambda_1} \in T_X$.

Next, we shall show that A_α is finite. From $\gamma_1\beta_1 = \beta_0 \in AM_X$, it follows by Lemma 4.6 that for each $x \in X \setminus A_{\beta_0}$, $t,z \in X$, $(x,z),(t,z) \in \gamma_1$ implies $x = t$. Let $F_1 = A_{\beta_0}$. Since $\beta_0 \in AM_X$, F_1 is finite. Since $\gamma_1\beta_2 = \gamma_2\beta_3$, by Corollary 4.8, we have that for each $x \in X \setminus (F_1 \cup A_{\beta_2}\gamma_1^{-1})$, $t,z \in X$, $(x,z),(t,z) \in \gamma_2$ implies $x = t$. Let $F_2 = F_1 \cup A_{\beta_2}\gamma_1^{-1}$. Then we have by Lemma 4.7(2) that F_2 is finite since $\beta_2 \in AM_X$. Since $\gamma_2\beta_4 = \gamma_3\beta_5$, by Corollary 4.8, we have that for each $x \in X \setminus (F_2 \cup A_{\beta_4}\gamma_2^{-1})$, $t,z \in X$, $(x,z),(t,z) \in \gamma_3$ implies $x = t$. Let $F_3 = F_2 \cup A_{\beta_4}\gamma_2^{-1}$. By Lemma 4.7(2), F_3 is finite. From $\gamma_i\beta_{2i} = \gamma_{i+1}\beta_{2i+1}$ for all $i \in \{1,\ldots,m-1\}$, it follows by Corollary 4.8 and Lemma 4.7(2) inductively that for each $i \in \{1,\ldots,m-1\}$, there exists a finite subset F_{i+1} of X such that for each $x \in X \setminus F_{i+1}$, $(x,z),(t,z) \in \gamma_{i+1}$ implies $x = t$. Hence there exists a finite subset F_m of X such that for each $x \in X \setminus F_m$, $t,z \in X$, $(x,z),(t,z) \in \gamma_m$ implies $x = t$. By Lemma 4.7(1), we have that for each $x \in X \setminus (F_m \cup A_{\beta_{2m}}\gamma_m^{-1})$, $t,z \in X$, $(x,z),(t,z) \in \gamma_m\beta_{2m}$ implies $x = t$. Let $F = F_m \cup A_{\beta_{2m}}\gamma_m^{-1}$. By Lemma 4.7(2), F is finite. Since $\gamma_m\beta_{2m} = \alpha \in T_X$, α is 1-1 at every point in $X \setminus F$. Then $A_\alpha \subseteq F$, so A_α is finite. Hence $\alpha \in AM_X$.

This proves that $Dom(AM_X, B_X) = AM_X$. Hence AM_X is closed in B_X. #