CHAPTER IV

TWO-FLUID MODEL

In studying the theory of matter at low temperature, it passes to an extraordinary state which is characterized by the frictionless motion of electrons and helium atoms respectively. The two-fluid model is useful in describing phenomena that occurs between the absolute zero and transition temperature. In this model at absolute temperature one has a perfect frictionless fluid which flows frictionlessly with potential flow. If we heat it, the heat energy will excite the liquid. These excitations can make their ways from one place to another place, collide with each other until they give some properties of the so-called normal fluid component. That is there are two interpenetrating fluids, each possesing its own velocity and there is no or very little exchange of momentum between them. One of the fluids, the superfluid, bears frictionless motion and disappears at transition temperature.The other fluid, the normal fluid,behaves as a normal electron gas or normal liquid and disappears as temperature decreases(4).

4.1 TWO-FLUID MODEL OF SUPERCONDUCTIVITY (5).

From the interpretation of the two-fluid model of liquid helium by Landau(6) , it can be extended to serve as a basis for a two-fluid model of superconductivity. But first let us consider London equations where the current density $\vec{J}_s$ can be written as $\vec{J}_s = -e \rho_s \vec{v}_s / m$ where ρ_s is the density of the superfluid componentand $\vec{v}_s$ is the velocity of the superfluid component. At absolute zero ρ_s = nm and at some finite temperature

$$\frac{\rho_s}{\rho} = 1 - \frac{\rho_n}{\rho} = \frac{\Lambda}{\Lambda_T} \qquad (4.1)$$

where Λ_T is the London parameter which in a free electron approximation is equal to m/ne^2 at absolute zero and increase to ∞ as T is near the transition temperature.

By analogy to Landau's interpretation of the two-fluid model, we first consider the superfluid at rest and the excitations have a net momentum $\vec{J}_n$. Their distribution function f $(\vec{p})$ is chosen to minimize the free energy F subject to the condition that

$$\vec{J}_n = \sum \vec{p}\, f(\vec{p}) \qquad (4.2)$$

If $\vec{v}$ is the Lagrange multiplier for $\vec{J}_n$, then

$$\delta F - \vec{v}.\delta \vec{J}_n = 0 \qquad (4.3)$$

In the paper of BCS (2), minimizing F with respect to f $(\vec{p})$ subject to the condition (4.2); equation (4.3) becomes

$$E(\vec{p}) + \frac{1}{\beta} \ln \frac{f(\vec{p})}{1-f(\vec{p})} - \vec{v}.\vec{p} = 0$$

$$f(\vec{p}) = 1/\left\{ \exp\,(E(\vec{p}) - \vec{v}.\vec{p})\beta \right\} + 1 \qquad (4.4)$$

For small $\vec{v}$, $\vec{J}_n$ is proportional to $\vec{v}$ and the coefficient of proportionality is defined as the normal density ρ_n. Thus from Eq.(4.4), expanding to the first order of $\vec{v}$,

$$f(\vec{p}) = (\exp \beta E(\vec{p}) + 1)^{-1} - (\vec{v}.\vec{p}) \frac{\partial}{\partial E}(\exp \beta E(\vec{p}) + 1)^{-1} + \ldots$$

then, $$\vec{J}_n = \frac{1}{(2\pi\hbar)^3}\iint\left\{\vec{p}(\exp\beta E(\vec{p})+1)^{-1} - \vec{p}(\vec{v}.\vec{p})\frac{\partial}{\partial E}(\exp\beta E(\vec{p})+1)^{-1}\right\} d\vec{p}$$

The first term is zero since cos θ averages to zero over a sphere. The second term is

$$\vec{J}_n = \frac{-2}{(2\pi\hbar)^3}\int_0^{\infty}\int_0^{\pi} p \cos\theta \frac{\vec{v}}{v} pv \cos\theta \frac{\partial}{\partial E}\left(\exp\beta E(\vec{p}) + 1\right)^{-1} p^2 \sin\theta \, dp d\theta$$

$$= \frac{-2\pi}{(2\pi\hbar)^3} \quad \frac{2}{3}\vec{v}\int_0^{\infty} p^4 \frac{\partial}{\partial E}(\exp \beta E(\vec{p}) + 1)^{-1} dp$$

Then from definition;

$$\rho_n = \lim_{\vec{v}\to 0} \frac{\vec{J}n}{\vec{v}} = -\frac{4\pi}{3h^3}\int_0^{\infty} p^4 \frac{\partial}{\partial E}\left(\exp \beta E(\vec{p}) + 1)^{-1}\right) dp \qquad (4.5)$$

This we will see that it similars to that derives from (4.1) with the BCS value for $\frac{\Lambda}{\Lambda_T}$.

In momentum space, the system moves with the velocity $\vec{v}_s$ and the motion of the normal fluid depends on the difference of the velocity $\vec{v}_s - \vec{v}_n = \vec{v}$, Thus the total current density $\vec{J}$ is

$$\vec{J} = \rho\vec{v}_s - \rho_n(\vec{v}_s - \vec{v}_n) = (\rho - \rho_n)\vec{v}_s + \rho_n\vec{v}_n = \rho_s\vec{v}_s + \rho_n\vec{v}_n \qquad (4.6)$$

In equilibrium more excitations align oppositely to $\vec{v}_s$ than parallel to it, therefore $\vec{J}_n$ is directed oppositely to $\vec{v}_s$ and introducing $\rho - \rho_n = \rho_s$ we have the definition of the total current density. (7)

We can see that by following Landau's derivation of the two-fluid model we can show that the current of superconductivity is that of a mixture of two fluids, one of density ρ_s moving at velocity $\vec{v}_s$ and the other density ρ_n moving at velocity $\vec{v}_n$.

4.2 GALILEAN TRANSFORMATION FOR OPERATORS (8) (9).

Consider the state of thermodynamic equilibrium, which is characterized by the usual parameters; the particle number density ρ, temperature T and velocity $\vec{v}$. The dependence on the velocity $\vec{v}$ is trivial, by using the transformation

$$\psi \text{------}> \psi \exp im\, \vec{v}.\vec{r}/\hbar \qquad (4.7)$$

then we can write the expectation values in the states velocity $\vec{v}$ in terms of the expectation values in the states of the fluid at rest, i.e.

$$\langle .....\rangle_{\rho,T,\vec{v}} \text{------}> \langle ....\rangle_{\rho,T,o} \qquad (4.8)$$

But as in the first section we have two velocities, namely the velocity of the condensate $\vec{v}_s$ and the velocity of the normal component $\vec{v}_n$. The expectation values with two velocities $\langle ...\rangle_{\vec{v}_s,\vec{v}_n}$ can be expressed in terms of the expectation values in a coordinate system where the normal velocity is at rest, i.e.

$$\langle ...\rangle_{\vec{v}_s,\vec{v}_n} \text{----}> \langle ...\rangle_{\vec{v}_s-\vec{v}_n,\, o} \qquad (4.9)$$

To do this we perform a Galilean transformation as in Eq. (4.7) but replace $\vec{v}$ by $\vec{v}_n$.

Now we consider the free energy F ; from definition

$$F = -\frac{1}{\beta} \ln \sum_n \exp(-\beta E_n) = -\frac{1}{\beta} \ln \text{Tr} \exp(-\beta H)$$

where H is the Hamiltonian of the system and is given by

$$H = \frac{\hbar^2}{2m}\int \nabla\psi^+(\vec{x}).\nabla\psi(\vec{x})\,d\vec{x} + \frac{1}{2}\iint V(\vec{x}-\vec{y})\,\psi^+(\vec{x})\psi^+(\vec{y})\psi(\vec{y})\psi(\vec{x})\,d\vec{y}d\vec{x} \qquad (4.10)$$

When we take Galilean transformation of Hamiltonian, by choosing the coordinate moving with velocity $\vec{v}_s$ and then taking all averages only over states with $\vec{v}_s-\vec{v}_n = 0$, $\vec{v}'_n = -\vec{v}_s$, then Eq (4.10) becomes,

$$H = \frac{\hbar^2}{2m}\int\left\{\nabla\psi^+(\vec{x}) - \frac{im\vec{v}_s}{\hbar}\psi^+(\vec{x})\right\}\left\{\nabla\psi(\vec{x}) + \frac{im\vec{v}_s}{\hbar}(\vec{x})\right\} d\vec{x}$$

$$+ \frac{1}{2}\iint V(\vec{x}-\vec{y})\psi^+(\vec{x})\psi^+(\vec{y})\psi(\vec{y})\psi(\vec{x})\, d\vec{y}d\vec{x} \qquad (4.11)$$

Consider the expression $\rho\frac{\partial F}{\partial \vec{v}_s}$,

$$\rho\frac{\partial F}{\partial \vec{v}_s} = \rho\frac{\partial}{\partial \vec{v}_s}\left(\frac{-1}{\beta}\ln \mathrm{Tr}\exp(-\beta H)\right)$$

$$= \rho\,\mathrm{Tr}\left(\frac{\partial H}{\partial \vec{v}_s}\right)\exp(-\beta H)/\mathrm{Tr}\exp(-\beta H)$$

$$= \rho\left\langle\frac{\partial H}{\partial \vec{v}_s}\right\rangle$$

From Eq. (4.11)

$$\rho\frac{\partial F}{\partial \vec{v}_s} = \rho\left\langle\frac{\hbar^2}{2m}\int\left[-\frac{im}{\hbar}\left\{\psi^+(\vec{x})\nabla\psi(\vec{x})-\psi(\vec{x})\nabla\psi^+(\vec{x})\right\} + \frac{2m^2}{\hbar^2}\vec{v}_s\psi^+(\vec{x})\psi(\vec{x})\right] d\vec{x}\right\rangle_{0,\vec{v}'_n}$$

$$= m\vec{J}_s$$

We introduce the current density.

$$m\vec{J}_s = \rho\frac{\partial F}{\partial \vec{v}_s} = \rho\frac{\partial F}{\partial u}\vec{v}_s$$

where $u = \frac{1}{2}\vec{v}_s^{\,2}$ and define $\frac{\rho}{M}\frac{\partial F}{\partial u} = \rho_s$, where $M = 2m$.

Hence

$$\vec{J}_s = 2\rho_s\vec{v}_s \qquad (4.12)$$

Next we consider the expression of the total current density

$$\vec{J} = \frac{\hbar}{2mi}\left\langle\psi^+(\vec{x})\nabla\psi(\vec{x}) - \psi(\vec{x})\nabla\psi^+(\vec{x})\right\rangle$$

Taking Galilean transformation $\psi(\vec{x}) \dashrightarrow \psi(\vec{x})\exp im\,\vec{v}_n\cdot\vec{x}/\hbar$:

$$\vec{J} = \frac{\hbar}{2mi}\left\langle\psi^+(\vec{x})\nabla\psi(\vec{x}) - \psi(\vec{x})\nabla\psi^+(\vec{x}) + \frac{2mi}{\hbar}\vec{v}_n\psi^+(\vec{x})\psi(\vec{x})\right\rangle_{\vec{v}'_s,0}$$

$$= 2\rho_s(\vec{v}_s-\vec{v}_n) + \rho\vec{v}_n$$

where we use Eq. (4.12) and introducing the definition

$\rho_n = \rho - 2\rho_s$ we get the expression of the total current density

$$\vec{J} = 2\rho_s \vec{v}_s + \rho_n \vec{v}_n \qquad (4.13)$$

From consideration in this section we see that we get a different expression for $\vec{J}$, this difference comes from the fact that the condensate coupled pair have two space variables and instead of introducing two functions we can reduce these by considering the condensate coupled pair in the coordinates of center of mass. The total mass M and reduced mass μ that are defined by $M = m_1 + m_2$ and $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$. But since the coupled pair have the same mass we have $M = 2m$ and $\mu = m/2$. These will play their roles in the next chapter.