CHAPTER II

SKEW RATIO SEMIRINGS

In this chapter, we shall classify complete ordered skew ratio semirings up to isomorphism. First we shall classify those complete ordered skew ratio semirings satisfying the property that $1+1 = 1$. Then we shall consider the case where $1+1 \neq 1$.

<u>Definition 2.1</u>. A system $(D,+,\cdot,\leqslant)$ is called an <u>ordered skew ratio semiring</u> iff $(D,+,\cdot)$ is a skew ratio semiring and $\leqslant$ is an order on D satisfying the following properties:

(i) For any $x, y \in D$, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all $z \in D$,

(ii) For any $x, y \in D$, $x \leqslant y$ implies that $xz \leqslant yz$ and $zx \leqslant zy$ for all $z \in D$.

If equations (i) and (ii) hold for an order on D, then we say that the order is compatible with the addition and multiplication.

<u>Remark 2.2</u>. If $(D,+,\cdot,\leqslant)$ is an ordered skew ratio semiring. Then D has a maximum element or a minimum element iff $|D| = 1$.

<u>Proposition 2.3</u>. $(D,+,\cdot,\leqslant)$ is an ordered skew ratio semiring iff $(D,+,\cdot,\leqslant_{opp})$ is an ordered skew ratio semiring.

Proof: Assume that $(D,+,\cdot,\leqslant)$ is an ordered skew ratio semiring. By Remark 1.16, $(D,+,\leqslant)$ is an ordered semigroup iff $(D,+,\leqslant_{opp})$ is an ordered semigroup. By Remark 1.18, $(D,\cdot,\leqslant)$ is an ordered group iff $(D,\cdot,\leqslant_{opp})$ is an ordered group. The proposition is proved. #

Proposition 2.4. Let $(D,\cdot,\leqslant)$ be an ordered group. Then $(D,\min,\cdot,\leqslant)$ and $(D,\max,\cdot,\leqslant)$ are order skew ratio semirings. Furthermore, if $(D,\cdot)$ is an abelian group then $(D,\min,\cdot,\leqslant) \cong (D,\min,\cdot,\leqslant_{opp})$ and $(D,\max,\cdot,\leqslant) \cong (D,\max,\cdot,\leqslant_{opp})$

Proof: Clearly that $(D,\min)$ and $(D,\max)$ are semigroup. Thus we have that $(D,\min,\cdot,\leqslant)$ and $(D,\max,\cdot,\leqslant)$ are order skew ratio semirings.

Assume that $(D,\cdot)$ is an abelian group. By Remark 1.18, $(D,\cdot,\leqslant) \cong (D,\cdot,\leqslant_{opp})$. We get that $(D,\min,\cdot,\leqslant)$ is isomorphic to $(D,\min,\cdot,\leqslant_{opp})$ and $(D,\max,\cdot,\leqslant)$ is isomorphic to $(D,\max,\cdot,\leqslant_{opp})$. #

Theorem 2.5. Let $(D,+,\cdot,\leqslant)$ be a discrete complete ordered skew ratio semiring such that $1+1 = 1$. Then $(D,+,\cdot,\leqslant)$ is isomorphic to exactly one of the followings ratio semirings:

(1) $(\{1\},+,\cdot,\leqslant)$.

(2) $(\{2^n \mid n \in \mathbb{Z}\},\min,\cdot,\leqslant)$.

(3) $(\{2^n \mid n \in \mathbb{Z}\},\max,\cdot,\leqslant)$.

Proof: If $|D| = 1$ then $(D,+,\cdot,\leqslant) \cong (\{1\},+,\cdot,\leqslant)$. Suppose that $|D| > 1$. Let g be the immediate successor of 1. Then by Proposition 1.25 and Theorem 1.26, $(D,\cdot,\leqslant) \cong (\mathbb{Z},+,\leqslant)$ and g generates $(D,\cdot)$.

Let $x \in D$ and $I(x) = \{y \in D \mid x+y = x\}$. Since $1+1 = 1$, $x+x = x$. Thus $x \in I(x)$, it follows that $I(x) \neq \emptyset$ for all $x \in D$.

<u>Case 1</u>: $g^2 \in I(g)$. Thus $g^2 + g = g$ which implies that $g+1 = 1$. We claim that $g^n+1 = 1$ for all $n \in \mathbb{Z}_o^+$. We shall prove this by using induction on $n \in \mathbb{Z}_o^+$. Let $n \in \mathbb{Z}_o^+$. If $n = 0,1$ then done. Assume that the claim is true for $n-1 \geqslant 1$. Hence $g^n + g = g$, it follows $g^n + g + 1 = g+1$. We get that $g^n + 1 = 1$, so we have the claim.

Let $m, n \in \mathbb{Z}_o^+$. Thus $m \leqslant n$ or $n \leqslant m$, say $m \leqslant n$. So $n-m \in \mathbb{Z}_o^+$. We have that $g^n + g^m = g^m(g^{n-m}+1) = g^m\, 1 = g^m = g^{\min(n,m)}$. Therefore we get that $(D,+,\cdot,\leqslant) \cong (\mathbb{Z},\min,+,\leqslant)$.

<u>Case 2</u>: $g^2 \notin I(g)$. Thus $g^2 + g \neq g$.

<u>Subcase 2.1</u>: $1 \notin I(g)$. Thus $g+1 \neq g$. Suppose that $1+g < g$. Since $1 < g$, $1+1 \leqslant 1+g$. Therefore $1 \leqslant 1+g < g$ which implies that $1 = g+1$. So $g = g^2 + g$, a contradiction. Thus $g < 1+g$. We have that $1 < g^{-1} + 1$. Since $1 < g$, $g^{-1} < 1$. Therefore $g^{-1} + 1 \leqslant 1+1 = 1$ which is a contradiction. So this subcase cannot occur.

<u>Subcase 2.2</u>: $1 \in I(g)$. Thus $g+1 = g$. We claim that $g^n+1 = g^n$ for all $n \in \mathbb{Z}_o^+$. We shall prove this by using induction on $n \in \mathbb{Z}_o^+$. Let $n \in \mathbb{Z}_o^+$. If $n = 0$ and 1, then done. Assume that the claim is true for $n-1 \geqslant 1$, so we have that $g^{n-1} + g = g^{n-1}$. It follows that $g^n + g = g^n$. Therefore $g^n + g + 1 = g^n + 1$. Thus $g^n + g = g^n + 1$. So $g(g^{n-1}+1) = g^n + 1$. Then by assumption we have that $g(g^{n-1}) = g^n + 1$, so $g^n = g^n + 1$. Hence we have the claim. Let $m, n \in \mathbb{Z}_o^+$. So we have that $m \leqslant n$ or $n \leqslant m$, say $m \leqslant n$. Therefore $n-m \in \mathbb{Z}_o^+$. By the claim we get that $g^n + g^m = g^m(g^{n-m}+1) = g^m(g^{n-m}) = g^n = g^{\max(n,m)}$. Hence we have that $(D,+,\cdot,\leqslant) \cong (\mathbb{Z},\max,+,\leqslant)$.

Lastly, we must show that (1),(2) and (3) are not isomorphic to each other. Clearly (1) is neither isomorphic to (2) nor (3). To show that (2) is not isomorphic to (3), suppose that (2)is isomorphic phic to (3). Let $f: (\{2^n \mid n\in\mathbb{Z}\},\min,\cdot,\leq) \to (\{2^n \mid n\in\mathbb{Z}\},\max,\cdot,\leq)$ be an isomorphism and we let $x, y \in \{2^n \mid n\in\mathbb{Z}\}$ be such that $x < y$. Thus $x+y = x$, so $f(x+y) = f(x)$. By assumption, $f(x) < f(y)$. Therefore $f(x)+f(y) = f(y)$ which implies that $f(x+y) \neq f(x)+f(y)$, a contradiction. Thus (2) is not isomorphic to (3).

Hence, the theorem is proved. #

<u>Theorem 2.6</u>. Let $(D,+,\cdot,\leq)$ be a dense complete ordered skew ratio semiring such that $1+1 = 1$. Then $(D,+,\cdot,\leq)$ is isomorphic to exactly one of the followings ratio semirings:

(1) $(\mathbb{R}^+,\min,\cdot,\leq)$.

(2) $(\mathbb{R}^+,\max,\cdot,\leq)$.

<u>Proof</u>: Since D is dense, by Theorem 1.31 $(D,\cdot,\leq) \cong (\mathbb{R}^+,\cdot,\leq)$. Let $x \in D$ and $I(x) = \{y \in D \mid x+y = x\}$. Since $1+1 = 1$, $x+x = x$. Thus $I(x) \neq \emptyset$ for all $x \in D$, in particular $I(1) \neq \emptyset$. We shall show that $I(x) = x\cdot I(1)$ for all $x \in D$. Let $x \in D$ be arbitrary and let $y \in x\cdot I(1)$. Then there exists a $z \in I(1)$ sucht that $y = xz$. Thus $x+y = x+xz = x(1+z) = x$, so we have that $y \in I(x)$. Hence $x\cdot I(1) \subseteq I(1)$. On the other hand, let $y \in I(x)$. Therefore $x+y = x$. So we have that $1+x^{-1}y=1$. Thus $x^{-1}y \in I(1)$ which implies that $y \in x\cdot I(1)$, hence $I(x) \subseteq x\cdot I(1)$. So we get that $I(x) = x\cdot I(1)$.

Suppose that $I(x) = D$. Therefore $x\cdot I(1) = D$, it follows that $I(1) = x^{-1}D = D$. Thus for every $y \in D, I(y) = y\cdot I(1) = y\cdot D = D$. Let $a, b \in D$ be such that $a \neq b$. So we have that $a \in I(b)$ and $b \in I(a)$, it follows that $a+b = b$ and $a+b = a$. Hence $a = b$ which contradicts $a \neq b$. Therefore we get that $\emptyset \subset I(x) \subset D$ for all $x \in D$, in particular $\emptyset \subset I(1) \subset D$. Suppose that $x, y \in I(1)$ are such that $x < y$. We shall prove that $x < z < y$ implies that $z \in I(1)$. Let $z \in D$ be such that $x < z < y$. Thus $1 = 1+x \leqslant 1+z \leqslant 1+y = 1$. Therefore $1+z = 1$, so we have have that $z \in I(1)$.

Let $d \in I(1)$. We claim that $d^n \in I(1)$ for all $n \in \mathbb{Z}^+$. We shall shall prove this by using induction on $n \in \mathbb{Z}^+$. Let $n \in \mathbb{Z}^+$. If $n = 1$, then done. Assume that the claim is true for $n-1 \geqslant 1$. Therefore $d^{n-1} \in I(1)$. Thus $d^{n-1}+1 = 1$, it follows that $d^n+d = d$. So $d^n+d+1 = d+1$ and so $d^n+1 = 1$. We get that $d^n \in I(1)$. Hence we have the claim.

Case 1: $I(1)$ has an upper bound. By hypothesis $I(1)$ has a least upper bounded. Let $z = \sup(I(1))$. Since $1 \in I(1)$, $1 \leqslant z$.

Subcase 1.1: $z > 1$. We shall prove that $\{t \in D \mid t \geqslant 1\} = I(1)$. To prove this, note that there exists a $v \in I(1)$ such that $z > v > 1$. Let $w \in \{t \in D \mid t \geqslant 1\}$ be arbitrary. Thus $w \geqslant 1$. By Proposition 1.25, there exists an $n \in \mathbb{Z}^+$ such that $v^n > w$ and $v^n \in I(1)$. Now we have that v^n and $1 \in I(1)$. Hence $w \in I(1)$. Therefore $\{t \in D \mid t \geqslant 1\} \subset I(1)$. Suppose that $\{t \in D \mid t \geqslant 1\} \neq I(1)$. Then there exists a $u \in I(1)$ such that $u < 1$. Let $s < 1$ be arbitrary. By Prpposition 1.25, there exists an $m \in \mathbb{Z}^+$ such that $u^m < s$ and $u^m \in I(1)$. Thus $s \in I(1)$ so we have that $\{s \in D \mid s < 1\} \subseteq I(1)$. Therefore $D = \{t \in D \mid t \geqslant 1\} \cup \{s \in D \mid s < 1\} \subseteq I(1)$ which implies that $I(1) = D$, a contradiction. Hence $\{t \in D \mid t \geqslant 1\} = I(1)$. Let $d \in D$ be arbitrary. Now we shall show that $d \cdot I(1) = \{b \in D \mid b \geqslant d\}$. To prove this, let $z \in d \cdot I(1)$. Thus $d^{-1}z \in I(1) = \{t \in D \mid t \geqslant 1\}$. Therefore $d^{-1}z \geqslant 1$. So $z \geqslant d$. Hence $z \in \{b \in D \mid b \geqslant d\}$.

Therefore we get that $d\cdot I(1) \subseteq \{b\varepsilon D \mid b \geqslant d\}$. On the other hand, let $z \varepsilon \{b\varepsilon D \mid b \geqslant d\}$. So $z \geqslant d$ and so $d^{-1}z \geqslant 1$. Thus we have that $z = d(d^{-1}z) \varepsilon d\cdot\{t\varepsilon D \mid t \geqslant 1\} = d\cdot I(1)$. Hence $\{b\varepsilon D \mid b \geqslant d\} \subseteq d\cdot I(1)$. We showed that $d\cdot I(1) = \{b\varepsilon D \mid b \geqslant d\}$.

Let $x, y \varepsilon D$ be arbitrary. So $x \leqslant y$ or $y \leqslant x$, say $x \leqslant y$. Therefore we get that $I(x) = x\cdot I(1) = \{b\varepsilon D \mid b \geqslant x\}$. Thus $y \varepsilon I(x)$. It follows that $x+y = x$. Hence $x+y = \min\{x,y\}$. Therefore $x+y = \min\{x,y\}$ for all $x, y \varepsilon D$.

<u>Subcase 1.2</u>: $z = 1$. Suppose that $I(1) = \{1\}$. Therefore $I(x) = x\cdot I(1) = x\cdot\{1\} = \{x\}$ for all $x \varepsilon D$. Let $a, b \varepsilon D$ be such that $a \neq b$. Thus $a+b \neq a$ and $a+b \neq b$, so $a+b = d$ for some $d \varepsilon D \setminus \{a,b\}$. We get that $d+b = (a+b)+b) = a+b = d$. So we have that $b \varepsilon I(d) = \{d\}$. Hence $b = d$, a contradiction. Therefore $I(1) \neq \{1\}$. Hence there exists an $s \varepsilon I(1) \setminus \{1\}$. If $s > 1$ then $s > z = \sup\,(I(1))$ which is a contradiction. So $s < 1$. We claim that $\{r\varepsilon D \mid r \leqslant 1\} = I(1)$. Let $w\varepsilon\{r\varepsilon D \mid r \leqslant 1\}$ be arbitrary. Thus $w \leqslant 1$. By Proposition 1.25, there exists an $n \varepsilon \mathbb{Z}^+$ such that $s^n < w$ and $s^n \varepsilon I(1)$. So we have that $w \varepsilon I(1)$. Hence $\{r\varepsilon D \mid r \leqslant 1\} \subseteq I(1)$. Suppose that $\{r\varepsilon D \mid r \leqslant 1\} \neq I(1)$. Then there exists an $a \varepsilon I(1)$ such that $a > 1 = z = \sup\,(I(1))$, a contradiction. Hence we have the claim.

Let $x, y \varepsilon D$ be arbitrary. So $x \leqslant y$ or $y \leqslant x$, say $x \leqslant y$. Similarly proof as before we get that $y\cdot\{r\varepsilon D \mid r \leqslant 1\} = \{c\varepsilon D \mid c \leqslant y\}$. Therefore $I(y) = y\cdot I(1) = y\cdot\{r\varepsilon D \mid r \leqslant 1\} = \{c\varepsilon D \mid c \leqslant y\}$. Hence $x \varepsilon I(y)$. It follows that $x+y = y = \max\{x,y\}$. We get that $x+y = \max\{x,y\}$ for all $x, y \varepsilon D$.

<u>Case 2</u>: $I(1)$ has no upper bound. Then for every $x \geqslant 1$ there exists a $y \varepsilon I(1)$ such that $y > x$. Thus $x \varepsilon I(1)$. Therefore we get that

$\{t \varepsilon D | t \geqslant 1\} \subseteq I(1)$.

Subcase 2.1. $\{t \varepsilon D | t \geqslant 1\} = I(1)$. Using the same proof as before we get that $x+y = \min\{x,y\}$ for all $x, y \varepsilon D$.

Subcase 2.2. $\{t \varepsilon D | t \geqslant 1\} \subset I(1)$. Then there exists a $z \varepsilon I(1)$ such that $z < 1$. So $1 < z^{-1}$ which implies that $z^{-1} \varepsilon I(1)$. Thus $z^{-1}+1 = 1$. Therefore $1+z = z \neq 1$. Hence $z \notin I(1)$, a contradiction. Therefore this case can not occur.

We have shown that either $(D,+,\cdot,\leqslant) \cong (\mathbb{R}^+,\min,\cdot,\leqslant)$ or $(D,+,\cdot,\leqslant) \cong (\mathbb{R}^+,\max,\cdot,\leqslant)$.

Lastly, a proof similar to the one given in Theorem 2.5 shows that (1) is not isomorphic to (2). Hence we have theorem. #

Since we have just classified all complete ordered skew ratio semiring such that $1+1 = 1$ we shall now turn to the case where $1+1 \neq 1$. In [2] it was shown that the prime skew ratio semiring of such a skew ratio semiring with $1+1 \neq 1$ is isomorphic to $\mathbb{Q}^+$ with the usual addition and multiplication. Our first problem is to shown how many orders there are on $\mathbb{Q}^+$ compatible with the usual addition and multiplication.

Proposition 2.7. The only orders on $\mathbb{Q}^+$ compatible with the usual addition and multiplication are the usual order and the opposite of the usual order.

Proof: Let $\leqslant^*$ be an order on $\mathbb{Q}^+$ compatible with the usual addition and multiplication. Since $1, 2 \varepsilon \mathbb{Q}^+$ and $1 \neq 2$, $1 <^* 2$ or $2 <^* 1$.

<u>Case 1</u>: $1 <^* 2$. We shall show that in this case $<^*$ is the usual order on $\mathbb{Q}^+$. First, we claim that for every $n \in \mathbb{Z}^+$, $n <^* n+1$. We shall prove this by using induction on $n \in \mathbb{Z}^+$, let $n \in \mathbb{Z}^+$ be arbitrary. If $n = 1$, then we are done by assumption. Suppose that $n-1 <^* n$. Therefore $(n-1)+1 \leqslant^* n+1$. If $(n-1)+1 = n+1$ then $n = n+1$ which is a contradiction. Thus $n <^* n+1$. Hence we have the claim.

Let $m,n \in \mathbb{Z}^+$ be arbitrary. Next, we shall show that $m <^* n$ iff $m < n$. Suppose that $m < n$. Then there exist an $\ell \in \mathbb{Z}^+$ such that $n = m+\ell$. We shall prove this by using induction on ℓ. If $\ell = 1$, then by the claim we are done. Suppose that $m <^* m+(\ell-1)$. Therefore, by the claim, we get that $m+(\ell-1) <^* m+(\ell-1)+1 = m+\ell = n$. Hence $m <^* n$. On the other hand, suppose that $m \leqslant^* n$. To show that $m < n$, suppose that $n \leqslant m$. If $n < m$, then by the same proof as before $n <^* m$, a contradiction. If $n = m$, then $n <^* n$, a contradiction. Thus we get that $m < n$. This shows that for every $m, n \in \mathbb{Z}^+$, $m < n$ iff $m <^* n$.

Let $x, y \in \mathbb{Q}^+$ be arbitrary. Then there are $p,q,r,s \in \mathbb{Z}^+$ such that $x = \frac{p}{q}$ and $y = \frac{r}{s}$. Then $\frac{p}{q} <^* \frac{r}{s}$ iff $ps <^* rq$ iff $ps < rq$ iff $\frac{p}{q} < \frac{r}{s}$. We proved that $x \leqslant^* y$ iff $x \leqslant y$. Hence $\leqslant^*$ is the usual order.

<u>Case 2</u>: $2 <^* 1$. Using a proof similar to the one used in Case 1 we can show that $\leqslant^*$ is the opposite of the usual order.

The proposition is proved. #

It follows from Proposition 2.3 that, if P is a prime skew ratio semiring of such a skew ratio semiring D then P is isomorphic to $\mathbb{Q}^+$ with the usual addition, multiplication an order or P is isomorphic to $\mathbb{Q}^+$ with the usual addition and multiplication and the opposite of the usual order.

Notation: Let D be a skew ratio semiring. Let $n \in \mathbb{Z}^+$ and $x \in D$. Then we shall denote x+x+...+x (n times) by nx. Also for simplicity we shall denote 1+1+...+1 (n times) by n.

Definition 2.8. Let $(D,+,\cdot,\leqslant)$ be an ordered skew ratio semiring such that $1+1 \neq 1$. Then D is called Archimedian iff for every $x, y \in D$, $x < y$ implies that either

a) there exist an $n \in \mathbb{Z}^+$ such that $nx > y$ or

b) there exist an $n \in \mathbb{Z}^+$ such that $ny < x$.

Remark 2.9. Let D be an ordered skew ratio semiring and $P \subseteq D$ the prime skew ratio semiring. Then

(i) a) in Definition 2.8 holds if P is isomorphic to $\mathbb{Q}^+$ with the usual addition, multiplication and order.

(ii) b) in Definition 2.8 holds if P is isomorphic to $\mathbb{Q}^+$ with the usual addition and multiplication and the opposite of the usual order.

Proposition 2.10. If D is a complete ordered skew ratio semiring such that $1+1 \neq 1$ then D is Archimedian.

Proof: Assume that $x, y \in D$ are such that $x < y$, let P be the prime skew ratio semiring of D.

Case 1: P is isomorphic to $\mathbb{Q}^+$ with the usual order. To show that there exists an $n \in \mathbb{Z}^+$ such that $nx > y$, suppose that $nx \leqslant y$ for all

$n \in \mathbb{Z}^+$. Let $L = \{nx | n \in \mathbb{Z}^+\}$. Therefore y is an upper bound of L. Since $L \subseteq D$ and D complete, we get that L has a least upper bound. Let $z = \sup(L)$ and let $k \in \mathbb{Z}^+$ be arbitrary. Therefore $2kx \in L$, it follows that $2kx \leqslant z$. Thus $kx \leqslant 2^{-1}z$ for all $k \in \mathbb{Z}^+$ which implies that $2^{-1}z$ is an upper bound of L. We get that $z \leqslant 2^{-1}z$. Hence $1 \leqslant 2^{-1}$, a contradiction. Therefore we have that there exists an $n \in \mathbb{Z}^+$ such that $nx > y$.

Case 2: P is isomorphic to $\mathbb{Q}^+$ with the opposite of the usual order. Using a proof similar to the one used in Case 1 we can show that there exists an $m \in \mathbb{Z}^+$ such that $my < x$.

This shows that D is an Archimedian. #

Proposition 2.11. Let D be an Archimedian ordered skew ratio semiring such that $1+1 \neq 1$ and $P \subseteq D$, the prime skew ratio semiring of D is isomorphic to $\mathbb{Q}^+$ with the usual order. Then the sequence (n^{-1}) in D has the property that for every $d \in D$ there exists an N in $\mathbb{Z}^+$ such that $n \geqslant N$ implies that $n^{-1} < d$.

Proof: Let $d \in D$ be arbitrary. We first claim that there exists an $M \in \mathbb{Z}^+$ such that $M > d^{-1}$. To prove this, suppose that $n \leqslant d^{-1}$ for all $n \in \mathbb{Z}^+$. If there exists an $m \in \mathbb{Z}^+$ such that $m = d^{-1}$, then $m+1 > m = d^{-1}$, a contradiction. Thus $n^{-1} < d$ for all $n \in \mathbb{Z}^+$.

Let $M \in \mathbb{Z}^+$, so $M < d^{-1}$. Therefore, by hypothesis, there exists an $\ell \in \mathbb{Z}^+$ such that $\ell M > d^{-1}$, a contradiction. Hence we have the claim. Now, we can choose $N \in \mathbb{Z}^+$ such that $N > d^{-1}$. Let $n \in \mathbb{Z}^+$ be such that $n \geqslant N$ which implies that $n^{-1} \leqslant N^{-1} < d$. The proposition is proved. #

Proposition 2.12. Let D be a complete ordered skew ratio semiring such that $1+1 \neq 1$ and $P \subseteq D$, the prime skew ratio semiring of D is isomorphic to $\mathbb{Q}^+$ with the usual order. Then $1 = \inf\{1+n^{-1} \mid n \in \mathbb{Z}^+\}$.

Proof: Let P be the prime skew ratio semiring of D. We claim that for every $M \in P$ there exists an $N \in \mathbb{Z}^+$ such that $n \geqslant N$ implies that $n^2(2n+1)^{-1} > M$. To prove this, let $M \in P$ be arbitrary. Then there exists an $r, s \in \mathbb{Z}^+$ such that $M = rs^{-1}$. Let $N = 4(r+s)$. Therefore $N \in \mathbb{Z}^+$. Let $n \in \mathbb{Z}^+$ be such that $n > N$ which implies that $n > N = 4(r+s) > 3(r+s) = (r+s) + 2(r+s) > n^{-1}(r+s)(1+2n)$. Thus $n > n^{-1}(r+s)(1+2n)$, it follows that $n^2(1+2n)^{-1} > (r+s) > rs^{-1} = M$, so we have the claim.

Next, we shall prove that $1 = \inf\{1+n^{-1} \mid n \in \mathbb{Z}^+\}$. Let $L = \{1+n^{-1} \mid n \in \mathbb{Z}^+\}$. Now, we have that L has 1 as a lower bound. By hypothesis we get that L has a greatest lower bound, say z. Thus $1 \leqslant z$.

Suppose that $1 < z$. So $z \leqslant 1+n^{-1}$ for all $n \in \mathbb{Z}^+$. Since $z < z^2$, there exists an $N \in \mathbb{Z}^+$ such that $1+N^{-1} \leqslant z^2$. It follows that for all $n \in \mathbb{Z}^+$ and for all $m > N$, $1+m^{-1} < z^2 \leqslant (1+n^{-1})^2$. ....(1)

Fix $m > N$. By the claim, there exist an $N_m \in \mathbb{Z}^+$ such that $n \geqslant N_m$ implies that $n^2(1+2n)^{-1} > m$. Thus $(1+2n)n^{-2} < m^{-1}$ for all $n \geqslant N_m$, it follows that $1+(1+2n)n^{-2} \leqslant 1+m^{-1}$ for all $n \geqslant N_m$. Hence $n^{-2}+ 2n^{-1}+ 1 \leqslant 1+m^{-1}$ for all $n \geqslant N_m$. Thus we have that $(1+n^{-1})^2 \leqslant 1+m^{-1}$ for all $n \geqslant N_m$ which contradics (1). Therefore we get that $z = 1$.#

Proposition 2.13. Let D be a complete ordered skew ratio semiring such that $1+1 \neq 1$ and $P \subseteq D$, the prime skew ratio semiring of D is isomorphic to $\mathbb{Q}^+$ with the usual order. Then for every x, y ε D, $x < y$ implies that there exists an $n \in \mathbb{Z}^+$ such that $ny > nx+1$.

Proof: Let x, y ε D be such that $x < y$. To show that there exists an $n \in \mathbb{Z}^+$ such that $ny > nx+1$, suppose that $ny \leqslant nx+1$ for all $n \in \mathbb{Z}^+$. Thus $yx^{-1} \leqslant 1+(nx)^{-1}$ for all $n \in \mathbb{Z}^+$. ....(1)

Case 1: $x < 1$. Let $m \in \mathbb{Z}^+$ be arbitrary. Therefore $mx < m$. By Proposition 2.10, so there exists an $\ell \in \mathbb{Z}^+$ such that $\ell mx > m$. Thus $(\ell mx)^{-1} < m^{-1}$, it follows that $1+(\ell mx)^{-1} \leqslant 1+m^{-1}$. So by equation (1) we have that $yx^{-1} \leqslant 1+m^{-1}$ for all $m \in \mathbb{Z}^+$. By Proposition 2.12, we get that $yx^{-1} \leqslant 1$. Hence $y \leqslant x$, a contradiction.

Case 2: $1 \leqslant x$. Thus $n \leqslant nx$ for all $n \in \mathbb{Z}^+$ which implies that $1+(nx)^{-1} \leqslant 1+n^{-1}$ for all $n \in \mathbb{Z}^+$. By Proposition 2.12 and equation (1), we get that $yx^{-1} \leqslant 1$. Hence $y \leqslant x$, a contradiction.

We have shown that there exists an $n \in \mathbb{Z}^+$ such that $ny > nx+1$.#

Proposition 2.14. Let D be a complete ordered skew ratio semiring such that $1+1 \neq 1$ and $P \subseteq D$, the prime skew ratio semiring of D is isomorphic to $\mathbb{Q}^+$ with the usual order. Then P is strongly dense in D.

Proof: Let x, y ε D be such that $x < y$. We shall show that there exists an $r \in P$ such that $x < r < y$. By Proposition 2.13, let $n \in \mathbb{Z}^+$ be such that $ny > nx+1$. We claim that for every $k \geqslant n$, $ky > kx+1$.

If $n = k$, then we are done. Suppose that $k > n$. Then there exists an $\ell \in \mathbb{Z}^+$ such that $k = n+\ell$. Therefore we get that $ky = (n+\ell)y = ny+\ell y \geqslant (nx+1) + \ell y = nx+(1+\ell y) = nx+\ell y)+1 \geqslant (nx+\ell x)+1 = (n+\ell)x+1$

$= kx+1$. Hence $ky \geqslant kx+1$ for all $k > n$. If there exists a $k_1 > n$ such that $k_1y = k_1x+1$, then $y = x+k_1^{-1}$. Since $k_1 > n$, $k_1^{-1} < n^{-1}$, it follows that $x+k_1^{-1} \leqslant x+n^{-1}$ therefore $y \leqslant x+n^{-1}$. So we have that $ny \leqslant nx+1$, a contradiction. Therefore we get that $ky > kx+1$ for all $k \geqslant n$, so we have the claim. Since D is an Archimedian, there exists an $m \in \mathbb{Z}^+$ such that $mx > 1$. Consequently $pmx > p \geqslant 1$ for all $p \in \mathbb{Z}^+$. We can choose $p \in \mathbb{Z}^+$ such that $pm > n$. Let $q = pm$. Thus $qy > qx+1$ and $qx > 1$. Since D is an Archimedian, there exists an $r \in \mathbb{Z}^+$ such that $r > qx$. Let r_o be the smallest element in $\mathbb{Z}^+$ such that $r_o > qx$, it follows that $r_o > qx > 1$. Thus $r_o > qx \geqslant r_o-1$ which implies that $qy > qx+1 \geqslant (r_o-1)+1 = r_o > qx$. Therefore $y > q^{-1}r_o > x$ and $q^{-1}r_o \in P$, so done.#

<u>Notation</u>: Let D be an ordered skew ratio semiring and $P \subseteq D$, the prime skew ratio semiring of D. For each $z \in D$, we shall denote $A_z = \{a \in P | a < z\}$ and $B_z = \{b \in P | b > z\}$.

<u>Proposition 2.15</u>. Let $(D,+,\cdot,\leqslant)$ be a complete ordered skew ratio semiring such that $1+1 \neq 1$ and $P \subseteq D$, the prime skew ratio semiring of D is isomorphic to $\mathbb{Q}^+$ with the usual order. Then $\sup(A_z) = z = \inf(B_z)$ for all $z \in D$.

<u>Proof</u>: Let $z \in D$ be arbitrary. Since $z2^{-1} < z < 2z$, by Proposition 2.14, there are $r, s \in P$ such that $z2^{-1} < s < z < r < 2z$. Thus $A_z \neq \emptyset$ and $B_z \neq \emptyset$. Clearly z is an upper bound of A_z and z is a lower bound of B_z. Since D complete, A_z has a least upper bound and B_z has a greatest lower bound. So we have that $\sup(A_z) \leqslant z \leqslant \inf(B_z)$.

Suppose that $\sup(A_z) < z$. By Proposition 2.14, so there exists an $r \varepsilon P$ such that $\sup(A_z) < r < z$. Thus $r \varepsilon A_z$ which is a contradiction. Hence we have that $\sup(A_z) = z$. Similarly, we can show that $\inf(B_z) = z$.#

Proposition 2.16. Let D be a complete ordered skew ratio semiring such that $1+1 \neq 1$ and $P \subsetneq D$, the prime skew ratio semiring of D is isomorphic to $\mathbb{Q}^+$ with the usual order. Then $A_x + A_y = A_{x+y}$ for all $x, y \varepsilon D$.

Proof: Let $x, y \varepsilon D$ be arbitrary. To show that $A_x + A_y = A_{x+y}$, suppose that $r \varepsilon A_x + A_y$. So we have that $r = a+b$ for some $a \varepsilon A_x$ and $b \varepsilon A_y$, it follows that $a < x$ and $b < y$. By Proposition 2.14, there exists $s, t \varepsilon P$ such that $a < s < x$ and $b < t < y$. Since $a,b,s,t \varepsilon P$, $a+b < s+t$ which implies that $r = a+b < s+t \leqslant x+y$. So $r \varepsilon A_{x+y}$. Hence $A_x + A_y \subseteq A_{x+y}$. On the other hand, suppose that $p \varepsilon A_{x+y}$. Thus $p < x+y$. By Proposition 2.14, there exists a $u \varepsilon P$ such that $p < u < x+y$. Now, we have that $p = p(u \cdot u^{-1}) = (u^{-1}p)u < u^{-1}p(x+y) = u^{-1}px + u^{-1}py$. Since $p < u$, $u^{-1}p < 1$. It follows that $u^{-1}px < x$ and $u^{-1}py < y$. By Proposition 2.14, so there are, $v, w \varepsilon P$ such that $u^{-1}px < v < x$ and $u^{-1}py < w < y$, hence $p < u^{-1}px + u^{-1}py \leqslant v + w$. Now, we have that $p = p\cdot 1 = p(v+w)(v+w)^{-1} = ((v+w)^{-1}p)(v+w) = (v+w)^{-1}pv + (v+w)^{-1}pw$. Since $p < v + w$, $(v+w)^{-1}p < 1$. It follows that $(v+w)^{-1}pv < v < x$ and $(v+w)^{-1}pw < w < y$. Thus we get that $(v+w)^{-1}pv \varepsilon A_x$ and $(v+w)^{-1}pw \varepsilon A_y$. Therefore $p \varepsilon A_x + A_y$. Hence $A_{x+y} \subseteq A_x + A_y$. We have shown that $A_x + A_y = A_{x+y}$.#

Proposition 2.17. Let D be a complete ordered skew ratio semiring. Let P be the prime skew ratio semiring of D and suppose that P is isomorphic to $\mathbb{Q}^+$ with the usual order. If θ is an order map of D into itself such that for all $b \in P$, $\theta(b) = b$, then θ is the identity map of D.

Proof: Let $z \in D$ be arbitrary. By Proposition 2.15, $\sup(A_z) = z$. Let $b \in A_z$ be arbitrary. Therefore $z > b$. Since θ is an order map, $\theta(z) \geqslant \theta(b) = b$. So we have that $\theta(z)$ is an upper bound fo A_z which implies that $\sup(A_z) \leqslant \theta(z)$. Hence $z \leqslant \theta(z)$. If $z < \theta(z)$, then there exist an $r \in P$ such that $z < r < \theta(z)$. Now, $z < r$ and θ is an order map. Thus $\theta(z) \leqslant \theta(r) = r$ which is a contradiction. Therefore $\theta(z) = z$. Hence θ is the identity map of D.#

Theorem 2.18. Let $(D,+,\cdot,\leqslant)$ be a complete ordered skew ratio semiring such that $1+1 \neq 1$. Then $(D,+,\cdot,\leqslant)$ is isomorphic to exactly one of the followings ratio semirings:

(i) $(\mathbb{R}^+,+,\cdot,\leqslant)$.

(ii) $(\mathbb{R}^+,+,\cdot,\leqslant_{opp})$.

Proof: Let P be the prime skew ratio semiring of D.

Case 1: P is isomorphic to $\mathbb{Q}^+$ with the usual order. Let $f: P \to \mathbb{Q}^+$ be an order isomorphism. Define $F: D \to \mathbb{R}^+$ in the following way: Let $x \in D$ be arbitrary. By Proposition 2.15, $x = \sup(A_x) = \inf(B_x)$. Clearly $r < x$ for every $r \in A_x$. Since $x < 2x$, by Proposition 2.14, there exist $s \in P$ such that $x < s < 2x$. Thus $r < s$ for every $r \in A_x$. Since f is an order isomorphicm, $f(r) < f(s)$ for all $f(r) \in f(A_x)$. Therefore $f(s)$ is an upper bound of $F(A_x)$. By hypothesis, $f(A_x)$

has a least upper bound. Define $F(x) = \sup(f(A_x))$. Clearly F is well-defined.

Step 1: We shall show that F is extension of f. Assume that $b \in P$. Thus b is an upper bound of A_b which implies that f(b) is an upper bound of $f(A_b)$. Therefore $f(b) \geqslant \sup(f(A_b)) = F(b)$. Suppose that $f(b) > F(b)$. Thus there exist $q \in \mathbb{Q}^+$ such that $f(b) > q > F(b)$. Since f is surjection, so there exist $c \in P$ such that $f(c) = q$, hence $f(b) > f(c)$. Therefore $b > c$. So $c \in A_b$ and we get that $q = f(c) \in f(A_b)$, contradicting the fact that $q > F(b) = \sup(f(A_b))$. This shows that $F(b) = f(b)$. Therefore F is an extension of f.

Step 2. We shall show that $\inf(f(B_x)) = \sup(f(A_x))$. Now, we have that $r < x < s$ for all $r \in A_x$ and for all $s \in B_x$. Since f is an order isomorphism, $f(r) < f(s)$ for all $r \in A_x$ and for all $s \in B_x$. Using the same proof as in the first paragraph of this proof, we get that $\inf(f(B_x))$ exists. Since f is an order map, $\sup(f(A_x)) \leqslant \inf(f(B_x))$. If $\sup(f(A_x)) < \inf(f(B_x))$, then by Proposition 2.15 there exists an $\ell \in \mathbb{Q}^+$ such that $\sup(f(A_x)) < \ell < \inf(f(B_x))$. We can assume that $\ell \neq f(x)$. Since f is surjection, there exists a unique $h \in P$ such that $f(h) = \ell$. Now, we have that $f(r) \leqslant \sup(f(A_x)) < f(h) < \inf(f(B_x)) \leqslant f(s)$ for all $f(r) \in f(A_x)$, $f(s) \in f(B_x)$. Again using the fact that f is an order map, we see that $r < h < s$ for all $r \in A_x$ and for all $s \in B_x$. Furthermore, $h \neq x$.

Case 1: $h < x$. Thus $h \in A_x$ which implies that $h < h$, a contradiction.

Case 2: $x < h$. Thus $h \in B_x$ which implies that $h < h$, a contradiction.

<u>Step 3</u>: We shall show that F is an injective order map. Assume that $x, y \in D$ are such that $x < y$. It suffices to show that $F(x) < F(y)$. By Proposition 2.14, so there are $r, s \in P$ such that $x < r < s < y$. It follows that $r \in B_x$ and $s \in A_y$, so $f(r) \in f(B_x)$ and $f(s) \in f(A_y)$. By step 2, $F(x) = \inf(f(B_x)) \leqslant f(r) < f(s) \leqslant \sup(f(A_y)) = F(y)$. Hence we have that $F(x) < F(y)$.

<u>Step 4</u>: We shall show that F is a surjection. Assume that $r \in \mathbb{R}^+$, let $g: \mathbb{Q}^+ \to P$ is the inverse map of f. Define $G: \mathbb{R}^+ \to D$ by $G(r)$ $G(r) = \sup(g(A_r))$. Using the same proof as before we can show that G is an extension of g. Furthermore, $G \circ F(b) = G(F(b)) = G(F(b))$ $= G(f(b)) = g(f(b)) = g \circ f(b) = b$ for all $b \in P$. Thus $G \circ F$ is the identity map on P. By Proposition 2.17, $G \circ F$ is the identity map of D. Using a similar proof we can show that $F \circ G$ is the identity map of $\mathbb{R}^+$. Hence $G = F^{-1}$, so F is a bijection and therefore a surjection.

<u>Step 5</u>: We shall show that F is a homomorphism. Let $x, y \in D$ be arbitrary. First, we shall show that $f(A_x) = A_{F(x)}$. Suppose that $s \in A_{F(x)}$. Therefore $s \in \mathbb{Q}^+$ and $s < F(x)$. Then there exists an $r \in P$ such that $f(r) = s$, it follows that $F(r) = f(r) < F(x)$. Since F is an order map, $r < x$. Thus $r \in A_x$ which implies that $s = f(r) \in f(A_x)$. This shows that $A_{F(x)} \subseteq f(A_x)$. On the other hand, suppose that $u \in f(A_x)$. Then there exists an $r \in A_x$ such that $u = f(r)$. Now, we have that $r < x$. By step 3, $F(r) < F(x)$. Therefore $u = f(r) = F(r)$ $= F(r) < F(x)$, hence $u \in A_{F(x)}$. We get that $f(A_x) \subseteq A_{F(x)}$. Thus $f(A_x) = A_{F(x)}$.

Next, we shall show that $F(x+y) = F(x)+F(y)$. By Proposition 2.12, $A_{x+y} = A_x + A_y$. Therefore $F(x+y) = \sup(f(A_{x+y})) = \sup(f(A_x + A_y))$ $= \sup(f(A_x)+f(A_y)) = \sup(A_{F(x)}+A_{F(y)}) = \sup(A_{F(x)+F(y)})$. A proof similar to the one given in Proposition 2.15 shows that $\sup(A_{F(x)+F(y)})$ $= F(x) + F(y)$. Thus $F(x+y) = F(x) + F(y)$.

Finally, we must show that $F(xy) = F x)F(y)$. Let $r \in A_x$ and $s \in A_y$ be arbitrary. Then $rs < xy$. By Step 3, $F(rs) < F(xy)$ $= \sup(f(A_{xy}))$. Now, we have that $f(r)f(s) = f(rs) = F(rs) < \sup(f(A_{xy}))$. Thus $f(r) < \sup(f(A_{xy}))(f(s))^{-1}$. Since $f(r) \in f(A_y)$ is arbitrary, $\sup(f(A_y)) \leqslant \sup(f(A_{xy}))(f(s))^{-1}$, it follows that $f(s) \leqslant (\sup(f(A_y)))^{-1}\sup(f(A_{xy}))$. By the same argument, $\sup(f(A_x)) \leqslant (\sup(f(A_y)))^{-1}\sup(f(A_{xy}))$. Therefore $\sup(f(A_x))\sup$

$\sup(f(A_x))\sup(f(A_y)) \leqslant \sup(f(A_{xy}))$. ....(1)

Let $u \in B_x$ and $v \in B_y$ be arbitrary. Thus $xy < uv$. By Step3, $F(xy) < F(uv) = f(uv) = f(u)f(v)$. Therefore $F(xy)(f(v))^{-1} < f(u)$. Since $f(u) \in f(B_x)$ is arbitrary, $F(xy)(f(v))^{-1} \leqslant \inf(f(B_x))$, if follows that $(\inf(f(B_x)))^{-1}F(xy) \leqslant f(v)$. By the same argument, $(\inf(f(B_x)))^{-1}F(x\,y) < \inf(f(B_y))$. It follows that $\sup(f(A_{xy})) =$

$F(xy) \leqslant \inf(f(B_x))\inf(f(B_y))$. ....(2)

From (1), (2) and Step 2, $\sup(f(A_{xy})) = \sup(f(A_x))\sup(f(A_y))$. This shows that $F(x\,y) = F(x)F(y)$.

Thus F is an order isomorphism map, as required.

<u>Case 2</u>: P is isomorphic to $\mathbb{Q}^+$ with the opposite of the usual order. Let $f: (P,+,\cdot,\leqslant) \to (\mathbb{Q}^+,+,\cdot,\leqslant_{opp})$ be an order isomorphism and let $g: (P,+,\cdot,\leqslant_{opp}) \to (P,+,\cdot,\leqslant)$ and $h: (\mathbb{Q}^+,+,\cdot,\leqslant_{opp}) \to (\mathbb{Q}^+,+,\cdot,\leqslant)$ be the identity maps. Clearly, g and h are anti-isomorphisms. Therefore we

get that $h \circ f \circ g$ is a bijection map. To show that $h \circ f \circ g$ is an order map, let $x, y \in (P,+,\cdot,\leqslant_{opp})$ be such that $x \leqslant_{opp} y$. Since g is an anti-isomorphism, $g(y) \leqslant g(x)$, it follows that $f(g(y)) \leqslant_{opp} f(g(x))$. Since h is an anti-isomorphism, $h(f(g(x))) \leqslant h(f(g(y)))$. Thus $h \circ f \circ g$ is an order map, hence $(P,+,\cdot,\leqslant_{opp}) \cong (\mathbb{Q}^+,+,\cdot,\leqslant)$. By Proposition 2.3 and Case 1 of this proof, $(D,+,\cdot,\leqslant_{opp}) \cong (\mathbb{R}^+,+,\cdot,\leqslant)$. Clearly, $(D,+,\cdot,\leqslant)$ is anti-isomorphic to $(D,+,\cdot,\leqslant_{opp})$ and $(\mathbb{R}^+,+,\cdot,\leqslant)$ is anti-isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant_{opp})$ (the identity maps gives anti-isomorphisms) so using the same proof that we just used above we get that $(D,+,\cdot,\leqslant) \cong (\mathbb{R}^+,+,\cdot,\leqslant_{opp})$.#