CHAPTER I

PRELIMINARIES

In this chapter we shall give some notations, definitions and theorems used in this thesis. Our notations are :

$\mathbb{Z}$ is the set of all integers,

$\mathbb{Z}^+$ is the set of all positive integers,

$\mathbb{Z}^-$ is the set of all negative integers,

$\mathbb{Q}$ is the set of all rational numbers ,

$\mathbb{Q}^+$ is the set of all positive rational numbers,

$\mathbb{R}$ is the set of all real numbers,

$\mathbb{R}^+$ is the set of all positive real numbers,

$\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$,

$\mathbb{Q}_0^+ = \mathbb{Q}^+ \cup \{0\}$,

$\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$,

$\mathbb{R}_\infty^+ = \mathbb{R}^+ \cup \{\infty\}$,

$m\mathbb{Z} = \{mn \mid n \in \mathbb{Z}\}$, $m \in \mathbb{Z}^+$,

$\mathbb{Z}_n$, $n \in \mathbb{Z}^+$ is the set of congruence classes modulo n in $\mathbb{Z}$,

$|A|$ will denote the cardinality of the set A,

$(\mathbb{Z},+,\cdot,\leqslant)$, $(\mathbb{Q}^+,+,\cdot,\leqslant)$, $(\mathbb{Q}_0^+,+,\cdot,\leqslant)$, $(\mathbb{R}^+,+,\cdot,\leqslant)$, $(\mathbb{R}_0^+,+,\cdot,\leqslant)$, $(\mathbb{R}_\infty^+,+,\cdot,\leqslant)$, $(\mathbb{R},+,\cdot,\leqslant)$, $(m\mathbb{Z},+,\cdot,\leqslant)$ where $m \in \mathbb{Z}_0^+$ will mean that $\mathbb{Z}$, $\mathbb{Q}^+$, $\mathbb{Q}_0^+$, $\mathbb{R}^+$, $\mathbb{R}_0^+$, $\mathbb{R}_\infty^+$, $\mathbb{R}$, $m\mathbb{Z}$ where $m \in \mathbb{Z}_0^+$ have the usual addition, multiplication and order.

$(\mathbb{Z}^+,\min,\cdot,\leqslant)$,$(\{2^n|n \in \mathbb{Z}\}, \min,\cdot,\leqslant)$, $(\mathbb{Q}^+,\min,\cdot,\leqslant)$, $(\mathbb{R}^+,\min,\cdot,\leqslant)$ will mean that $\mathbb{Z}^+$, $\{2^n|n \in \mathbb{Z}\}$, $\mathbb{Q}^+$, $\mathbb{R}^+$ have the usual multiplication and order and $x+y = \min\{x,y\}$ (minimum of x,y).

$(\mathbb{Z}^+,\max,\cdot,\leqslant)$, $(\{2^n|n \in \mathbb{Z}\},\max,\cdot,\leqslant)$, $(\{2^n|n \in \mathbb{Z}\} \cup \{0\},\max,\cdot,\leqslant)$ $(\{2^n|n \in \mathbb{Z}\} \cup \{\infty\},\max,\cdot,\leqslant)$, $(\mathbb{Q}^+,\max,\cdot,\leqslant)$, $(\mathbb{R}^+,\max,\cdot,\leqslant)$, $(\mathbb{R}^+ \cup \{\infty\},\max,\cdot,\leqslant)$ will mean that $\mathbb{Z}^+$, $\{2^n|n \in \mathbb{Z}\}$, $\{2^n|n \in \mathbb{Z}\} \cup \{0\}$, $\{2^n|n \in \mathbb{Z}\} \cup \{\infty\}$, $\mathbb{Q}^+$, $\mathbb{R}^+$ have the usual multiplication and order and $x+y = \max\{x,y\}$ (maximum of x,y).

Definition 1.1. A binary relation $\leqslant$ on a nonempty set X is called an order on X iff for every a, b, c $\in$ X:

(i) $a \leqslant a$,

(ii) $a \leqslant b$ and $b \leqslant a$ implies that $a = b$,

(iii) $a \leqslant b$ and $b \leqslant c$ implies that $a \leqslant c$,

(iv) $a < b$ or $a = b$ or $a > b$ where $a < b$ will mean that $a \leqslant b$ and $a \neq b$, and $a > b$ means that $b < a$.

An ordered set X ia a set X with an order $\leqslant$ on X. We shall denote it by $(X, \leqslant)$.

Definition 1.2. Let $(X, \leqslant)$ be an ordered set. Then the opposite order on X, denoted by $\leqslant_{opp}$ is defined by $x \leqslant_{opp} y$ iff $y \leqslant x$ for all x, y $\in$ X.

Definition 1.3. Let $(X, \leqslant)$ be an ordered set and $B \subseteq X$ a nonempty set. Then $b \in B$ is a minimum (maximum) of B iff $b \leqslant x$ ($x \leqslant b$) for all $x \in B$, we denote this by $b = \min(B)$ ($\max(B)$).

Definition 1.4. Let $(X, \leqslant)$ be an ordered set and $x \in X$. Then :

(i) $y \in X$ is called an immediate predecessor (successor) of x iff $y < x$ $(x < y)$ and there does not exist $z \in X$ such that $y < z < x$ $(x < z < y)$, we shall denote it by $x^-(x^+)$,

(ii) x is called lower (upper) discrete iff either x is a minimum (maximum) or x has an immediate predecessor (successor),

(iii) x is called discrete iff x is both lower and upper discrete,

(iv) X is called lower (upper) discrete iff for every $x \in X$, x is lower (upper) discrete,

(v) X is called discrete iff for every $x \in X$, x is discrete.

Definition 1.5. Let $(X, \leqslant)$ be an ordered set and $x \in X$. Then :

(i) x is called lower (upper) dense iff x is not a lower (upper) discrete,

(ii) x is called dense iff x is both lower and upper dense,

(iii) X is called lower (upper) dense iff for every $x \in X$, x is lower (upper) dense,

(iv) X is called dense iff for every $x \in X$, x is dense,

(v) A nonempty subset B of X is called dense in X iff for every $x, y \in X$, $x < y$ implies that there exists a $t \in B$ such that $x \leqslant t \leqslant y$.

(vi) A nonempty subset B of X is called strongly dense in X iff for every $x, y \in X$, $x < y$ implies that there exists a t ε B such that $x < t < y$.

Definition 1.6. Let $(X, \leqslant)$ be an ordered set and $B \subseteq X$ a nonempty set. Then

(i) An upper (lower) bound of B in X is an element $b \varepsilon X$ such that $x \leqslant b$ $(b \leqslant x)$ for every $x \varepsilon B$,

(ii) If the set of upper (lower) bounds of B in X has a minimum (maximum) element, then this element is called the least upper bound (greatest lower bound) of B in X, it is also called the supremum of B in X (abbreviated sup (B) (inf(B))).

Proposiition 1.7. Let $(X, \leqslant)$ be an ordered set. Then the following are equivalent:

(i) every subset of X which has an upper bound has a supremum,

(ii) every subset of X which has a lower bound has an infimum.

Proof: We shall show that (i) implies (ii). Suppose that (i) holds. Let $A \subseteq X$ be a nonempty set having a lower bound. To show that A has an infimum, let t be a lower bound of A. Let $C = \{x \varepsilon X \mid x \text{ is a lower bound of } A\}$. Then $C \neq \emptyset$ since $t \varepsilon C$. Fix $a \varepsilon A$. Then a is an upper bound of C. By assumption, C has a supremum. Let $z = \sup(C)$. If $a \varepsilon A$, then a is an upper bound of C, so $z \leqslant a$. Hence z itself is a lower bound for A. For any lower bound b of A we have that $b \varepsilon C$ and therefore $b \leqslant z$. This shows that z is the infimum of A.

We have thus shown that (i) implies (ii), and obviously a similar argument will prove that (ii) implies (i). #

Definition 1.8. An ordered set is called complete iff it has either property (i) or (ii) in Proposition 1.7.

Theorem 1.9. Let $(X, \leqslant)$ be a complete discretely ordered set. Then the following properties hold:

(i) every set A in X which has an upper bound has a maximum,

(ii) every set B in X which has a lower bound has a minimum.

Proof: To show (i), suppose that $A \subseteq X$ has an upper bound. By Proposition 1.7, A has a supremum. Let $z = \sup(A)$. To show that $z \in A$, suppose not. Then $z \notin A$. Thus $a < z$ for all $a \in A$. Therefore $a \leqslant z^- < z$ for all $a \in A$. Then z^- is an upper bound of A, hence $z \leqslant z^-$, a contradiction. Therefore $z \in A$. This shows that z is a maximum of A. Hence (i) holds and obviously a similar argument will prove that (ii) holds. #

Definition 1.10. Let $(X, \leqslant)$ and $(Y, \leqslant^*)$ be ordered sets and $f: X \to Y$ be a map. Then f is called an order map iff for every $x, y \in X$, $x \leqslant y$ implies that $f(x) \leqslant^* f(y)$.

If an order map is an injection, we call it an order injection.

If an order map is a surjection, we call it an order surjection.

If an order map is a bijection, we call it an isomorphism.

f is called an anti-order map iff for every $x, y \in X$, $x \leqslant y$ implies that $f(y) \leqslant^* f(x)$, anti-order injection, surjection and isomorphism are defined similarly.

<u>Proposition 1.11.</u> Let $(X, \leqslant)$ be an ordered set. Then $(X, \leqslant)$ and $(X, \leqslant_{opp})$ are anti-isomorphic.

<u>Proposition 1.12.</u> Let $(X, \leqslant)$ and $(Y, \leqslant^*)$ be ordered sets. Let $f: X \to Y$ be an isomorphism. Then X is complete iff Y is complete.

<u>Proposition 1.13.</u> Suppose that $(X, \leqslant)$, $(Y, \leqslant^*)$ and $(Z, \leqslant^{**})$ are ordered sets. Let $f: X \to Y$ and $g: Y \to Z$ be anti-isomorphisms. Then $g \circ f: X \to Z$ is an isomorphism.

<u>Definition 1.14.</u> Let $(X, \leqslant)$ be an ordered set. A <u>cut</u> in X is a nonempty proper subset A of X such that

(i) if $a \in A$ and $x < a$, then $x \in A$,

(ii) if $b \in A$, then there exists a $t \in A$ such that $b < t$.

<u>Definition 1.15</u> A system $(S,+,\leqslant)$ is called an <u>ordered</u> <u>semigroup</u> iff $(S, +)$ is a semigroup and $\leqslant$ is an order on S satisfying the property that for every $x, y \in S$, $x \leqslant y$ implies that $x+z \leqslant y+z$ and $z+x \leqslant z+y$ for all $z \in S$.

<u>Remark 1.16</u> $(S,+,\leqslant)$ is an ordered semigroup iff $(S,+,\leqslant_{opp})$ is an ordered semigroup.

<u>Definition 1.17.</u> An ordered semigroup $(G,\cdot,\leqslant)$ is called an <u>ordered</u> <u>group</u> iff $(G,\cdot)$ is a group.

<u>Remark 1.18</u> $(G,\cdot,\leqslant)$ is an ordered group iff $(G,\cdot,\leqslant_{opp})$ is an ordered

group. Furthermore,

(i) For any x, y ε G, x < y iff xz < yz and zx < zy for all z ε G,

(ii) For any x, y ε G, x < y iff $y^{-1} < x^{-1}$,

(iii) If (G,·) is an abelian group, then (G,·,$\leqslant$) is isomorphic to (G,·,$\leqslant_{opp}$).

Proposition 1.19 Let (G,·,$\leqslant$) be an ordered group and let x, y ε G be such that x < y. Then the following properties hold:

(1) $x^n < y^n$ for all n ε $\mathbb{Z}^+$,

(2) 1 < x implies that $1 < x^n$ for all n ε $\mathbb{Z}^+$,

(3) For every m, n ε $\mathbb{Z}$, m < n iff $z^m < z^n$ for all z > 1.

Proof: Assume that x, y ε G are such that x < y. To show (1), let n ε $\mathbb{Z}^+$ be arbitrary. We shall prove this by using induction on n. If n = 1, then we are done. Suppose that (1) is true for n-1 $\geqslant$ 1. Therefore $x^{n-1} < y^{n-1}$. By Remark 1.18 (i), $x(x^{n-1}) < x(y^{n-1})$. By assumption, $x(y^{n-1}) < yy^{n-1} = y^n$. Thus $x^n < y^n$. Hence $x^n < y^n$ for all n ε $\mathbb{Z}^+$.

The proof of (2) is obvious and the proof of (3) follows from (2). #

Proposition 1.20 Let (G,·,$\leqslant$) be an ordered group. If $|G| > 1$ then G has no maximum element and no minimum element.

Proof: Assume that $|G| > 1$. Now, we shall show that G has no maximum element. Let x ε G be arbitrary. It suffices to show that there exists a $y \in G$ be such that $x < y$. If $x < 1$, then we are done. If $1 < x$, then by Remark 1.18 (i), $x < x^2$, so we are done. Suppose that $x = 1$. Let $y \in G \setminus \{1\}$. If $y > 1$, then done. If $y < 1$, then $y^{-1} > 1$ so we are done.

Similary, we can show that G has no minimum element. #

<u>Proposition 1.21</u> Let $(G, \cdot, \leqslant)$ be an ordered group. If there is $x \in G$ such that either x is lower dense or x is upper dense, then $(G, \leqslant)$ is densely ordered. (Hence if $x \in G$ is either lower discrete or upper discrete, then $(G, \leqslant)$ is discretely ordered.)

<u>Proof</u>: Assume that there is $x \in G$ such that x is lower dense or x is upper dense. Suppose that x is upper dense. Let $g \in G$ be arbitrary. First, we shall show that g is upper dense. Let $y \in G$ be such that $g < y$. We must show that there exists a $z \in G$ such that $g < z < y$. By Remark 1.18 (i), $g(g^{-1}x) < y(g^{-1}x)$. Thus $x < yg^{-1}x$. Since x is upper dense, there is a $z \in G$ such that $x < z < yg^{-1}x$ which implies that $g < zx^{-1}g < y$. Therefore g is upper dense.

Next, we shall show that x^{-1} is lower dense. Let $s < x^{-1}$. By Remark 1.18 (ii), $x < s^{-1}$. Since x is upper dense, there exists a $v \in G$ such that $x < v < s^{-1}$ which implies that $s < v < x^{-1}$. Therefore x^{-1} is lower dense.

Finally, we shall show that g is lower dense. Let $t \in G$ be such that $t < g$. Then by Remark 1.18 (i), $t(g^{-1}x^{-1}) < g(g^{-1}x^{-1})$. Thus $tg^{-1}x^{-1} < x^{-1}$. Since x^{-1} is lower dense, there exists a $w \in G$ such that $tg^{-1}x^{-1} < w < x^{-1}$. By Remark 1.18 (i), $tg^{-1}x^{-1}(xg) < w(xg) < x^{-1}(xg)$ which implies that $t < wxg < g$.

Therefore g is lower dense.

This shows that g is dense element in G. Since $g \in G$ is arbitrary, $(G, \leqslant)$ is densely ordered. If x is lower dense we can use a similar proof to show that $(G, \leqslant)$ is densely ordered.

Using the same proof that we just used above we get that if there is an $x \in G$ which x is either lower discrete or upper discrete then $(G, \leqslant)$ is discretely ordered. #

<u>Remark 1.22</u> Let $(G,\cdot,\leqslant)$ be an ordered group. Fix $g_0 \in G$. Then g_0 is upper dense or upper discrete. We see that $(G, \leqslant)$ is either densely ordered or discretely ordered.

<u>Proposition 1.23</u> Let $(G,\cdot,\leqslant)$ be an ordered group and $x, y \in G \setminus \{1\}$ be such that $x < y$. Then the following are equivalent:

(i) there exists $n \in \mathbb{Z}$ such that $x^n > y$,

(ii) there exists $m \in \mathbb{Z}$ such that $y^m < x$.

<u>Proof</u>: To show (i) implies (ii), suppose that (i) holds. We must show that there exists an $m \in \mathbb{Z}$ such that $y^m < x$.

<u>Case 1</u>: $1 < x < y$. Then by Remark 1.18 (ii), $y^{-1} < x^{-1} < 1 < x$, so we are done.

<u>Case 2</u>: $x < 1 < y$. Then by assumption, $x^n > y$ for some $n \in \mathbb{Z}^-$. By assumption again, $y^m > x^n$ for some $m \in \mathbb{Z}$. Therefore $y^{-m} < x^{-n}$ and $-n \in Z^+$, it follows that $y^{-m} < x^{-n} < x$.

<u>Case 3</u>: $x < y < 1$. This proof is similar to the proof of Case 2.

We have thus ckecked that (i) implies (ii), and obviously the same argument, will prove that (ii) implies (i). #

Definition 1.24 An ordered group of order > 1 is called Archimedean iff it has either property (i) or (ii) in Proposition 1.23.

Proposition 1.25. Let $(G,\cdot,\leqslant)$ be a complete ordered group of order > 1. Then $(G,\cdot,\leqslant)$ is Archimedean.

Proof: Assume that $x, y \in G \setminus \{1\}$ are such that $x < y$. Suppose that $y^m \nless x$ for all $m \in \mathbb{Z}$. We must show that there exists an $n \in \mathbb{Z}$ such that $x^n > y$. To prove this, suppose not. Then $x^n \leqslant y$ for all $n \in \mathbb{Z}$. Let $L = \{x^n \mid n \in \mathbb{Z}\}$. Then y is an upper bound of L. Since $L \subseteq G$ and G is complete, L has a least upper bound. Let $z = \sup(L)$.

Case 1: $x < 1$. Now, we have that $x^n < z$ for all $n \in \mathbb{Z}$. Let $m \in \mathbb{Z}$. Then $m-1 \in \mathbb{Z}$. Thus $x^{m-1} < z$, it follows that $x^m < zx$. Therefore $x^m < zx$ for all $m \in \mathbb{Z}$, so zx is an upper of L. Thus $z < zx$. By Remark 1.18 (i), $(z^{-1})z < (z^{-1})zx$ which implies that $1 < x$, a contradiction.

Case 2: $1 < x$. This proof is similar to the proof of Case 1.

Therefore we get that $(G,\cdot,\leqslant)$ is Archimedean. #

Theorem 1.26 Let $(G,\cdot,\leqslant)$ be an Archimedean discretely ordered group. Then $(G,\leqslant)$ is complete and $(G,\cdot,\leqslant)$ is isomorphic to $(\mathbb{Z},+,\leqslant)$.

Proof: We shall show that $(G,\cdot,\leqslant)$ is isomorphic to $(\mathbb{Z},+,\leqslant)$. Since $(G,\leqslant)$ is discretely ordered set and $|G| > 1$ there exists a $g \in G$ such that $g > 1$ and there does not exists a $z \in G$ such that $g > z > 1$. ....(*)

We claim that $G = \langle g\rangle$ where $\langle g\rangle = \{g^n \mid n \in \mathbb{Z}\}$. To prove the claim, let $x \in G$ be arbitrary. We shall show that $x \in \langle g\rangle$. If $x = g$, then $x \in \langle g\rangle$, so we are done. Suppose that $x \neq g$. Either $x < g$ or $g < x$.

Case 1: $x < g$. By (*), $x \leqslant 1 < g$. If $x = 1$, then $x = 1 = g^0 \in \langle g\rangle$. Suppose that $x \neq 1$. Then $x < 1 < g$. Since G is Archimedean, $g^{m_0} < x$ for some $m_0 \in \mathbb{Z}^-$. Let $A = \{m \in \mathbb{Z}^- \mid g^m < x\}$. Then $m_0 \in A$, so $A \neq \emptyset$. Let $N = \max(A)$. If $N = -1$, then $g^{-1} < x < 1$. Thus $1 < xg < g$ which contradicts (*). Then $N < -1$, so $N+1 \in \mathbb{Z}^-$. Therefore $x \leqslant g^{N+1}$. If $x < g^{N+1}$, then $g^N < x < g^{N+1}$. Thus $(g^{-N})g^N < (g^{-N})x < (g^{-N})g^{N+1}$. Then $1 < g^{-N}x < g$ which contradicts (*). Therefore we get that $x = g^{N+1}$. Hence $x \in \langle g\rangle$.

Case 2: $g < x$. A proof similar to the proof of Case 1 shows that $x \in \langle g\rangle$.

Therefore we get that $G \subseteq \langle g\rangle$. Clearly, $\langle g\rangle \subseteq G$. Thus $G = \langle g\rangle$, so we have the claim. Hence $(G,\cdot)$ is a cyclic group. By Proposition 1.19(3), $m < n$ implies that $g^m < g^n$. Then $(G,\cdot,\leqslant)$ is an ordered infinite cyclic group. Hence $(G,\cdot,\leqslant)$ is isomorphic to $(\mathbb{Z},+,\leqslant)$. It follows from Proposition 1.12 that $(G,\leqslant)$ is complete.#

Lemma 1.27 Let $(G,\cdot,\leqslant)$ be an ordered group and $a, b \in G$. Then the following properties hold:

(i) If $a^m \leqslant b^n$ and $\frac{m}{n} = \frac{m_1}{n_1}$ where $m, m_1 \in \mathbb{Z}$ and $n, n_1 \in \mathbb{Z}^+$, then $a^{m_1} \leqslant b^{n_1}$,

(ii) If $a^m > b^n$ and $\frac{m}{n} = \frac{m_1}{n_1}$ where $m, m_1 \in \mathbb{Z}$ and $n, n_1 \in \mathbb{Z}^+$, then $a^{m_1} > b^{n_1}$.

Proof: To show (i), suppose that $a^m \leqslant b^n$ and $\frac{m}{n} = \frac{m_1}{n_1}$ where $m, m_1 \in \mathbb{Z}$ and $n, n_1 \in \mathbb{Z}^+$. By Proposition 1.19 (1), we have that $(a^m)^{n_1} \leqslant (b^n)^{n_1}$, so $a^{mn_1} \leqslant b^{nn_1}$. ....(1)

Now, we have that $mn_1 = nm_1$. From (1), we have that $a^{nm_1} \leqslant b^{nn_1}$, so $(a^{m_1})^n \leqslant (b^{n_1})^n$. .... (2)

If $b^{n_1} < a^{m_1}$, then by Proposition 1.19((1), $(b^{n_1})^n < (a^{m_1})^n$ which contradicts (2). Therefore we get that $a^{m_1} \leqslant b^{n_1}$.

The proof of (ii) is similar to the proof of (i). #

Theorem 1.28. Let $(G,\cdot,\leqslant)$ be an Archimedean densely ordered group. Then $(G,\cdot,\leqslant)$ can be embedded into $(\mathbb{R},+,\leqslant)$.

Proof: Since $G \neq \{1\}$ we can fix an $a > 1$. Let $b \in G$ be arbitrary.

Let $V_b = \{\frac{m}{n} \in \mathbb{Q} \mid a^m \leqslant b^n \text{ and } n > 0\}$ and

$U_b = \{\frac{m}{n} \in \mathbb{Q} \mid a^m > b^n \text{ and } n > 0\}$.

Step 1. We shall show that $V_b \neq \emptyset$ and $U_b \neq \emptyset$.

Case 1: $a = b$. Then $1 \in V_b$, so $V_b \neq \emptyset$. Since $1 < a$, $b = a < a^2$.

Thus $2 \varepsilon U_b$. Therefore $U_b \neq \emptyset$.

<u>Case 2</u>: $a < b$. Tnen $1 \varepsilon V_b$, so $V_b \neq \emptyset$. By the Archimedean property $a^n > b$ for some $n \varepsilon \mathbb{Z}$. Thus $n \varepsilon U_b$. Then $U_b \neq \emptyset$.

<u>Case 3</u>: $b < a$. This proof is similar to the proof of Case 2.

Therefore $V_b \neq \emptyset$ and $U_b \neq \emptyset$.

<u>Step 2</u>. We shall show that $V_b \cap U_b = \emptyset$. To prove this, suppose not. Then $V_b \cap U_b \neq \emptyset$. Let $t \varepsilon V_b \cap U_b$. Then $t \varepsilon V_b$ and $t \varepsilon U_b$. There are $m, r \varepsilon \mathbb{Z}$ and $n, s \varepsilon \mathbb{Z}^+$ such that $t = \frac{m}{n}$ and $t = \frac{r}{s}$ and $a^m \leqslant b^n$ and $a^r > b^s$. ....(1)

By (1) and Proposition 1.19 (1), $a^{ms} \leqslant b^{ns}$ and $b^{ns} < a^{nr}$, it follows that $a^{ms} < a^{nr}$. But we have that $ms = nr$, this implies that $a^{ms} < a^{ms}$, a contradiction. Hence $V_b \cap U_b = \emptyset$.

<u>Step 3</u>. We shall show that for every $s \varepsilon V_b$ and for every $t \varepsilon U_b$, $s < t$. Let $s \varepsilon V_b$ and $t \varepsilon U_b$. Then $s = \frac{m}{n}$ and $t = \frac{\ell}{q}$ for some $\ell, m \varepsilon \mathbb{Z}$ and $n, q \varepsilon \mathbb{Z}^+$. Therefore $a^m \leqslant b^n$ and $a^\ell > b^q$. By Proposition 1.19 (1), $a^{mq} \leqslant b^{nq} < a^{\ell n}$. By $1 < a$ and Proposition 1.19 (3), $mq < \ell n$. Thus $\frac{m}{n} < \frac{\ell}{q}$. Hence $s < t$.

<u>Step 4</u>. We shall show that $\sup(V_b) = \inf(U_b)$. Let $t_0 \varepsilon U_b$. By Step 3, $s < t_0$ for all $s \varepsilon V_b$. Then t_0 is an upper bound of V_b. Thus $\sup(V_b) < t_0$. Since $t_0 \varepsilon U_b$ is arbitrary, $\sup(V_b) \leqslant \inf(U_b)$. If $\sup(V_b) < \inf(U_b)$, then there exists an $r \varepsilon \mathbb{Q}$ such that $\sup(V_b) < r < \inf(U_b)$. Thus $r \notin V_b$ and $r \notin U_b$ which is a contradiction. Therefore we get that $\sup(V_b) = \inf(U_b)$.

Define $F: G \to \mathbb{R}$ by $F(b) = \sup(V_b)$ for all $b \in G$. Clearly, F is well-defined.

Step 5. We shall show that F is an order map. Let $x, y \in G$ be such that $x < y$. We claim that $V_x \subseteq V_y$. To prove the claim, let $w \in V_x$. Then $w = \frac{m}{n}$ for some $m \in \mathbb{Z}$ and for some $n \in \mathbb{Z}^+$. Therefore $a^m \leqslant x^n$. By Proposition 1.19 (1), $a^m \leqslant x^n \leqslant y^n$. Thus $w = \frac{m}{n} \in V_y$. Hence $V_x \subseteq V_y$, so we have the claim. Therefore $\sup(V_x) \leqslant \sup(V_y)$. Hence $F(x) \leqslant F(y)$.

Step 6. We shall show that $(G,\cdot)$ is an abelian group. Let $a, b \in G$ be arbitrary. We claim that for every $x \in G$, $x > 1$ implies that there is a $z \in G$ such that $x > z > 1$ and $z^2 \leqslant x$. To prove the claim, let $x \in G$ be such that $x > 1$. Since $(G,\cdot)$ is densely ordered, x is a lower dense. Then there exists a $y \in G$ such that $x > y > 1$. Thus

$$1 < y^{-1}x < x. \qquad \text{.......................(2)}$$

If $(y^{-1}x)^2 < x$, then let $z = y^{-1}x$ and we have the claim by (2). Suppose that $x \leqslant (y^{-1}x)^2$. Then $x \leqslant y^{-1}xy^{-1}x$, so $1 < y^{-1}xy^{-1}$. Thus $y < xy^{-1}$ which implies that $y^2 \leqslant x$. Let $z = y$, so we have the claim.

Suppose that $ab \neq ba$. Without loss of generality, suppose that $ba < ab$. Then $1 < aba^{-1}b^{-1}$. Let $x = aba^{-1}b^{-1}$. By the claim, there exists a $z \in G$ such that $1 < z < x$ and $z^2 \leqslant x$. ....(3)

By Proposition 1.25, $(G,\cdot,\leqslant)$ is Archimedean. Then there are $m, n \in \mathbb{Z}$ such that $z^m \leqslant a < z^{m+1}$ and $z^n \leqslant b < z^{n+1}$. ....(4)

From (4), we have that $a^{-1} < z^{-m}$ and $b^{-1} < z^{-n}$. Therefore

$$x = aba^{-1}b^{-1} < (z^{m+1})(z^{n+1})(z^{-m})(z^{-n}) = (z^{(m+n)+2})(z^{-(m+n)}) = z^2$$

which contradicts (3). Thus $ab = ba$ for all $a, b \in G$. Hence $(G,\cdot)$

is an abelian group.

Step 7. We shall show that $F(xy) = F(x) + F(y)$ for all $x, y \in G$. Let $x, y \in G$ be arbitrary. We shall show that the following properties hold:

(i) $V_x + V_y \subseteq V_{xy}$ where $V_x + V_y = \{s+t \mid s \in V_x \text{ and } t \in V_y\}$.

(ii) $U_x + U_y \subseteq U_{xy}$ where $U_x + U_y = \{v+w \mid v \in U_x \text{ and } w \in U_y\}$.

To show (i), let $w \in V_x + V_y$. Then $w = s+t$ for some $s \in V_x$ and $t \in V_y$. Then $s = \frac{m}{n}$ and $t = \frac{\ell}{n}$ for some $n \in \mathbb{Z}^+$. By Lemma 1.27, $a^m \leqslant x^n$ and $a^\ell \leqslant y^n$ which implies that $a^{m+\ell} \leqslant x^n a^\ell \leqslant x^n y^n$. ......(5)

By Step 6, $x^n y^n = (xy)^n$. From (5), we have that $a^{m+\ell} \leqslant (xy)^n$. Therefore $\frac{m+\ell}{n} \in V_{xy}$, so $w = \frac{m}{n} + \frac{\ell}{n} \in V_{xy}$. Hence $V_x + V_y \subseteq V_{xy}$.

The proof of (ii) is similar to the proof of (i). From (i), we have that $\sup(V_x + V_y) \leqslant \sup(V_{xy})$. ....(6)

From (6), we have that $\sup(V_x) + \sup(V_y) \leqslant \sup(V_{xy})$. Thus

$F(x) + F(y) \leqslant F(xy)$. ....(7)

From (ii), we have that $\inf(U_x + U_y) \geqslant \inf(U_{xy})$. ....(8)

From (8), we have that $\inf(U_x) + \inf(U_y) \geqslant \inf(U_{xy})$. Thus

$F(x) + F(y) \geqslant F(xy)$. ....(9)

By (7) and (9), $F(xy) = F(x) + F(y)$.

Step 9. We shall show that F is an injection. We want to show that $\ker(F) = \{1\}$. Suppose not. Then $\ker(F) \neq \{1\}$. Let $b \in \ker(F) \setminus \{1\}$. Then $b < 1$ or $1 < b$.

Case 1: $b < 1$. Then $b < 1 < a$. By the Archimedean property, $a^m < b$ for some $m \in \mathbb{Z}^-$, so $a^m < b < 1$. By the Archimedean property again, $b^n < a^m$ for some $n \in \mathbb{Z}^+$. Therefore $\frac{m}{n} \in U_b$ and $\frac{m}{n} < 0$. But we have that $b \in \ker(F)$, this implies that $F(b) = 0 = \sup(V_b) = \inf(U_b)$. Thus $0 \leqslant \frac{m}{n}$, a contradiction.

Case 2: $1 < b$. Then $1 < a < ab$ and $1 < b < ab$. By the Atchimedean property, $ab < a^m$ for some $m \in \mathbb{Z}^+$. Thus $1 < b < ab < a^m$. By the Archimedean property, $a^m < b^n$ for some $n \in \mathbb{Z}^+$. Thus $\frac{m}{n} \in V_b$ and $\frac{m}{n} > 0$. But we have that $b \in \ker(F)$, this implies that $F(b) = 0 = \sup(V_b)$. Thus $\frac{m}{n} \leqslant 0$, a contradiction.

Therefore we get that $\ker(F) = \{1\}$. Hence F is an injection.

This show that F is an order monomorphism. #

Lemma 1.29. Let $(G,\cdot,\leqslant)$ be a densely ordered group and let $f: (G,\cdot,\leqslant) \to (\mathbb{R},+,\leqslant)$ be an order monomorphism. Suppose that $x \in \mathbb{R} \setminus \{0\}$ is arbitrary. Then the following properties hold :

(i) If $0 < x$, then there exists a $g \in G$ such that $0 < f(g) < x$.

(ii) If $x < 0$, then there exists a $h \in G$ such that $x < f(h) < 0$.

Proof: To show (i), suppose that $0 < x$. We shall show that there exists a $g \in G$ such that $0 < f(g) < x$. Suppose not. Then there does not exist a $g \in G$ such that $0 < f(g) < x$. Let
$A = \{a \in \mathbb{R}^+ \mid$ there does not exist $g \in G$ such that $0 < f(g) < a\}$.
$A \neq \emptyset$ since $x \in A$.

We shall show that A has an upper bound. Let $g \in G$ be such that $g > 1$. Then $f(g) > f(1) = 0$. Since $(G, \leqslant)$ is densely ordered, 1 is upper dense. Thus there exists an $h \in G$ such that $1 < h < g$. Then $0 = f(1) < f(h) < f(g)$. Therefore $f(g) \notin A$. We claim that $f(g)$ is an upper bound of A. Suppose not. Then $f(g) < b$ for some $b \in A$. Thus $0 < f(g) < b$, a contradiction. Hence we have the claim. Since $\mathbb{R}$ is complete, A has a supremum. Let $z = \sup(A)$. If $z \notin A$, then there exists a $y \in G$ such that $0 < f(y) < z$. Thus there exists an $r \in A$ such that $f(y) < r < z$, it follows that $0 < f(y) < r$. Thus $r \notin A$, a contradiction. Therefore $z \in A$. Since $\frac{3}{2} z > z$, $\frac{3}{2} z \notin A$. Then there exists a $t \in G$ such that $\frac{3}{2} z > f(t) > 0$ which implies that

$$\frac{3}{2} z > f(t) \geqslant z. \qquad \text{..........................(1)}$$

If $f(t) = z$, then $f(t) > 0 = f(1)$. Thus $t > 1$. Since $(G, \leqslant)$ is densely ordered, 1 is upper dense. Then there exists an $s \in G$ such that $t > s > 1$. Therefore $0 = f(1) < f(s) < f(t) = z$. Thus $z \notin A$, a contradiction. Then $f(t) \neq z$. From (1), we have that

$$\frac{3}{2} z > f(t) > z > 0 \qquad \text{..........................(2)}$$

Now, we shall consider the sequence $((\frac{n+1}{n})z)_{n\in\mathbb{Z}^+}$ in $\mathbb{R}^+$. Note that $\lim\limits_{n\to\infty} (\frac{n+1}{n} z) = z$. By (2), $(0, f(t))$ is an open set containing z. Then there exists an $N \in \mathbb{Z}^+$ such that $n \geqslant N$ implies that $(\frac{n+1}{n})z \in (0, f(t))$. Let $n = N$. Therefore $(\frac{N+1}{N})z \in (0, f(t))$, so

$$f(t) > (\frac{N+1}{N})z > 0. \qquad \text{..........................(3)}$$

Now, we have that $(\frac{N+1}{N})z > z$. By (2) and (3), $\frac{3}{2}z > f(t) > (\frac{N+1}{N})z > z$.

..........................(4)

Since $(\frac{N+1}{N})z \notin A$, there exists a $v \in G$ such that $(\frac{N+1}{N})z > f(v) > 0$.

.......................(5)

<u>Case 1</u>: $f(v) = z$. Then $f(v) > 0 = f(1)$, so $v > 1$. Since $(G, \leqslant)$ is densely ordered, 1 is upper dense. Then there exists a $w \in G$ such that $v > w > 1$. Thus $f(v) > f(w) > f(1) = 0$, so $z > f(w) > 0$. Then $z \notin A$, a contradiction.

<u>Case 2</u>: $f(v) < z$. Then $0 < f(v) < z$ by (5). Thus $z \notin A$, a contradiction.

<u>Case 3</u>: $z < f(v)$. From (4) and (5), we have that

$\frac{3}{2} z > f(t) > (\frac{N+1}{N})z > f(v) > z$. Thus $\frac{3}{2} z - z > f(t) - f(v) > 0$,

so $\frac{1}{2} z > f(t) - f(v) > 0$. ....(6)

Since f is homomorphism, $0 = f(1) = f(vv^{-1}) = f(v) + f(v^{-1})$.

Thus $f(v^{-1}) = -(f(v))$. ....(7)

From (6) and (7), we have that $\frac{1}{2} z > f(t) + f(v^{-1}) > 0$ which implies that $\frac{1}{2} z > f(tv^{-1}) > 0$. Therefore $z > \frac{1}{2} z > f(tv^{-1}) > 0$. Hence $z \notin A$, a contradiction.

This show that there exists a $g \in G$ such that $0 < f(g) < x$.

(ii) follows easily from (i) so we have proven the lemma. #

<u>Theorem 1.30</u>. Let $(G,\cdot,\leqslant)$ be a densely ordered group and let $f: (G,\cdot,\leqslant) \rightarrow (\mathbb{R},+,\leqslant)$ be an order monomorphism. Then $f(G)$ is strongly dense in $\mathbb{R}$.

Proof: Assume that $y, z \in R$ are such that $y < z$. We shall show that there exists a $g \in G$ such that $y < f(g) < z$. If $y = 0$ or $z = 0$, then by Lemma 1.29, so we are done. Suppose that $y \neq 0$ and $z \neq 0$.

Case 1: $y < 0 < z$. Then $y < f(1) < z$.

Case 2: $0 < y < z$. Then $0 < z-y$, it follows that $0 < \frac{1}{n} < z-y$ for some $n \in \mathbb{Z}^+$. By Lemma 1.29, there exists a $g \in G$ such that

$0 < f(g) < \frac{1}{n}$ . ...(1)

By the Archimedean property, there exists an $m \in \mathbb{Z}^+$ such that $mf(g) > y$. Let $A = \{\ell \in \mathbb{Z}^+ \mid \ell f(g) > y\}$. $A \neq \emptyset$ since $m \in A$. Let $N = \min(A)$. If $N = 1$, then $y < f(g)$. From (1), we have that $y < f(g) < \frac{1}{n} < z-y < z$, so we are done. Suppose that $N > 1$. Then $N-1 \in \mathbb{Z}^+$. Thus $0 < (N-1)f(g) < y$ which implies that $-y < -(N-1)f(g) < 0$. Therefore we get that $Nf(g)-y < Nf(g)-(N-1)f(g) = f(g) < \frac{1}{n} < z-y$. Hence $Nf(g) < z$. But we have that $y < Nf(g)$, this implies that $y < Nf(g) < z$. Hence $y < f(g^N) < z$.

Case 3: $y < z < 0$. This can be easily proven using Case 2.

Hence $f(G)$ is strongly dense in $\mathbb{R}$. #

Theorem 1.31. Let $(G,\cdot,\leqslant)$ be a complete densely ordered group. Then $(G,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,\cdot,\leqslant)$.

Proof: We have that $(\mathbb{R},+,\leqslant)$ is isomorphic to $(\mathbb{R}^+,\cdot,\leqslant)$. By Proposition 1.25 and Theorem 1.28, $(G,\cdot,\leqslant)$ can be embedded into $(\mathbb{R}^+,\cdot,\leqslant)$. Let $F: G \to \mathbb{R}^+$ be an order monomorphism. By Theorem 1.30, F(G) is strongly dense in $\mathbb{R}^+$. ....(*)

We shall show that F is a surjection. Let $b \in \mathbb{R}^+$ be arbitrary. We must show that there exists a $g_0 \in G$ such that $F(g_0) = b$. Suppose not. Let $g \in G$ be arbitrary. Then $F(g) \neq b$, so $F(g) < b$ or $b < F(g)$. We claim that there exists an $h \in G$ such that $F(h) < b$. If $F(g) < b$, then we have the claim. If $b < F(g)$, then by the Archimedean property, there exists an $m \in \mathbb{Z}$ such that $m(F(g)) < b$. But we have that F is homomorphism, this implies that $F(g^m) < b$, so we have the claim.

Let $A = \{h \in G \mid F(h) < b\}$. By the claim, $A \neq \emptyset$. A proof similar to the proof of claim we can show that there exists a $w \in G$ such that $b < F(w)$. Thus $h < w$ for all $h \in A$. Then A has an upper bound, so A has a supremum. Let $z = \sup(A)$. Clearly, $F(z) \neq b$

Case 1: $F(z) < b$. Then $z \in A$. By (*), there exists a $v \in G$ such that $F(z) < F(v) < b$. Thus $v \in A$. By F is an order injection, $z < v$. Then $v \notin A$, a contradiction.

Case 2: $b < F(z)$. A proof similar to the proof of Case 1 gives a contradiction.

Therefore there exists a $g \in G$ such that $F(g) = b$. Thus F is a surjection. Hence F is isomorphism. #

Definition 1.32. A triple $(S,+,\cdot)$ is called a semiring iff

(i) $(S,+)$ is a commutative semigroup,

(ii) $(S,\cdot)$ is a semigroup

and (iii) $x\cdot(y+z) = x\cdot y + x\cdot z$ and $(x+y)\cdot z = x\cdot z + y\cdot z$ for all $x,y,z \in S$. The operations + and $\cdot$ are called the addition and multiplication of the semiring, respectively.

Definition 1.33. A semiring $(D,+,\cdot)$ is called a skew ratio semiring iff $(D,\cdot)$ is a group.

Theorem 1.34. ([2]) If $(D,+,\cdot)$ is a skew ratio semiring, then the smallest skew ratio semiring of D (called the prime skew ratio semiring of D) is either isomorphic to

(1) $(\mathbb{Q}^+,+,\cdot)$ if $1+1 \neq 1$

or (2) $(\{1\},+,\cdot)$ if $1+1 = 1$.

Definition 1.35. A semiring $(K,+,\cdot)$ is said to be a skew semifield iff $(K,\cdot)$ is a group with zero.

Theorem 1.36. ([2]) Let K be a skew semifield and $a \in K$ be such that $(K \setminus \{a\},\cdot)$ is a group. Then either a is an additive identity or a is an additive zero.

(The proof of this theorem in [2] does not use the commutativity of multiplication.)

Remark 1.37. Let K be a skew semifield and let $a \in K$ be as in Theorem 1.36,

(1) if a is an additive identity, then we shall denote it by 0 and we shall call K a 0-skew semifield,

(2) if a is an additive zero, then we shall denote it by ∞ and we shall call K an ∞-skew semifield.

Example 1.38. If $K = \{0,1\}$ is a 0-skew semifield, then K must have the multiplication table

$\cdot$	0	1
0	0	0
1	0	1

and one of the following two addition tables;

+	0	1
0	0	1
1	1	0

(1)

+	0	1
0	0	1
1	1	1

(2)

K with the addition in table (1) is the field $\mathbb{Z}_2$ and K with the addition in table (2) is called the Boolean semifield.

If $K = \{1,\infty\}$ is an ∞-skew semifield, then K must have the multiplication table

$\cdot$	1	∞
1	1	∞
∞	∞	∞

and one of the following two addition tables;

+	1	∞
1	∞	∞
∞	∞	∞

(3)

+	1	∞
1	1	∞
∞	∞	∞

(4)

K with the addition in table (3) is called the ∞-skew semifield with the trivial addition of order 2 and K with the addition in table (4) is called the ∞-skew semifield with the almost trivial addition of order 2.

Theorem 1.39. ([2]) If K is a 0-skew semifield, then the smallest 0-skew semifield of K (called the prime 0-skew semifield of K) is either isomorphic to $\mathbb{Q}_0^+$ with the usual addition and multiplication or

$\mathbb{Z}_p$ where p is a prime number or the Boolean semifield.

(The proof of this theorem in [2] does not use the commutativity multiplication.)

Definition 1.40. Let K be an ∞-skew semifield. If $x+y = \infty$ for all $x, y \in K$ we say that K has the trivial addition. If

$$x+y = \begin{cases} \infty & \text{when } x \neq y \\ x & \text{when } x = y \end{cases}$$

we say that K has the almost trivial addition.

Definition 1.41. A semiring $(R,+,\circ)$ is called a skew ring iff $(R,+)$ is a group.