WEIGHTED COMPOSITION OPERATORS ON HOLOMORPHIC L^2−SPACES

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Let ψ be a real-valued smooth function on $\mathbb{C}$ such that $\Delta\psi = c$ for some $c > 0$. We give a necessary and sufficient condition on boundedness of a weighted composition operator on a holomorphic function space $\mathcal{H}L^2(\mathbb{C}, e^{-\psi})$.

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CONTENTS

CHAPTER I

INTRODUCTION

Let $\mathcal{H}$ be a Banach space of holomorphic functions on an open subset X of $\mathbb{C}^n$. Let $\varphi : X \to X$ be a holomorphic function. A composition operator C_φ is defined by $C_\varphi(f) = f \circ \varphi$ for any function $f \in \mathcal{H}$ such that $f \circ \varphi \in \mathcal{H}$. It has been extensively studied in various settings, in particular, on the Hardy, Bergman and Bloch spaces on the unit disk of the complex plane. In 2003, Carswell, MacCluer and Schuster [1] characterized boundedness of composition operators on the Segal-Bargmann space

$$\mathcal{H}L^2(\mathbb{C}^n, \tfrac{1}{2\pi}e^{-\frac{|z|^2}{2}}) = \Big\{F \in \mathcal{H}(\mathbb{C}^n)\Big| \int_{\mathbb{C}^n} |F(z)|^2 \frac{1}{2\pi} e^{-\frac{|z|^2}{2}} dz < \infty\Big\},$$

where $\mathcal{H}(\mathbb{C}^n)$ denotes the set of all holomorphic functions on $\mathbb{C}^n$. They established that C_φ is bounded if and only if $\varphi(z) = Az + B$, where A is an $n \times n$ matrix with $\|A\| \leq 1$ and B is an $n \times 1$ vector such that $\langle A\zeta, B\rangle = 0$ whenever $|A\zeta| = |\zeta|$.

In 2006, Ueki [4] considered a weighted composition operator on the Segal-Bargmann space defined by

$$uC_\varphi(f) = u \cdot (f \circ \varphi),$$

where u is an entire function. He characterized boundedness and compactness of the weighted composition operator on the Segal-Bargmann space. His results are written in term of a certain integral transform

$$B_\varphi(|u|^2)(w) = \frac{1}{2\pi} \int_{\mathbb{C}} |u(z)|^2 |e^{\frac{\langle \varphi(z), w\rangle}{2}}|^2 e^{-\frac{|w|^2}{2}} e^{-\frac{|z|^2}{2}}\, dz.$$

He obtained that

$$uC_\varphi \text{ is bounded } \text{ if and only if } \quad B_\varphi(|u|^2) \in L^\infty(\mathbb{C}).$$

Our objective of this work is to generalize Ueki's work to a space $\mathcal{H}L^2(\mathbb{C}, e^{-\psi})$ where ψ is a real-valued smooth function on $\mathbb{C}$ such that $\Delta\psi$ is a positive constant. Note that $\Delta(|z|^2/2) = 2 > 0$, so such a space is a generalization of the standard Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C}, \frac{1}{2\pi}e^{-|z|^2/2})$.

CHAPTER II

PRELIMINARIES

In this chapter, we first review the definition of holomorphic function space including its general properties that can be found in [3].

Let U be a non-empty open set in $\mathbb{C}$. Let $\mathcal{H}(U)$ denote the space of holomorphic (or complex analytic) functions on U. Let α be a continuous, strictly positive function on U.

Definition 2.1. Let $\mathcal{H}L^2(U,\alpha)$ denote the space of L^2 holomorphic functions with respect to the weight α, that is,

$$\mathcal{H}L^2(U,\alpha) = \left\{ F \in \mathcal{H}(U) \,\middle|\, \int_U |F(z)|^2 \,\alpha(z)\, dA(z) < \infty \right\},$$

where $dA(z)$ denotes 2-dimensional Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$. It is equipped with the inner product

$$\langle f, g \rangle = \int_U f(z)\overline{g(z)}\,\alpha(z)\, dA(z).$$

Theorem 2.2. *The space $\mathcal{H}L^2(U,\alpha)$ is a closed subspace of $L^2(U,\alpha)$, and therefore a Hilbert space.*

In fact, the pointwise evaluation is a continuous map from $\mathcal{H}L^2(U,\alpha)$ to $\mathbb{C}$. That is, for each $w \in U$, the map that takes a function $f \in \mathcal{H}L^2(U,\alpha)$ to the number $f(w)$ is a bounded linear functional on $\mathcal{H}L^2(U,\alpha)$. By the Riesz's theorem, this linear functional can be represented uniquely as an inner product with some $K_w \in \mathcal{H}L^2(U,\alpha)$. That is,

$$f(w) = \langle f, K_w \rangle = \int_U f(z)\overline{K_w(z)}\,\alpha(z)\, dA(z).$$

Define $K(z,w) = K_w(z)$ for any $z, w \in U$. We call K the *reproducing kernel* for the space $\mathcal{H}L^2(U,\alpha)$. Denote by k_w the *normalized kernel function*, that is, $k_w(z) = \frac{K_w(z)}{\|K_w\|}$.

Theorem 2.3. *Let $\{e_j\}$ be a countable orthonormal basis for $\mathcal{H}L^2(U,\alpha)$. Then for all $z, w \in U$*

$$\sum_j \left| e_j(z)\overline{e_j(w)} \right| < \infty$$

and

$$K(z,w) = \sum_j e_j(z)\overline{e_j(w)}.$$

Definition 2.4. A *Segal-Bargmann space* is a space $\mathcal{H}L^2(\mathbb{C}, \mu_t)$, where

$$\mu_t(z) = \frac{1}{\pi t} e^{-|z|^2/t}$$

for any $t > 0$.

Moreover, this space has

$$\left\{ \frac{z^n}{\sqrt{n! t^n}} \right\}_{n=0}^{\infty}$$

as an orthonormal basis. By Theorem 2.3, the reproducing kernel for the space $\mathcal{H}L^2(\mathbb{C}, \mu_t)$ is given by

$$K(z,w) = e^{\langle z,w \rangle / t},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{C}$.

Definition 2.5. Holomorphic function spaces $\mathcal{H}L^2(U,\alpha)$ and $\mathcal{H}L^2(U,\beta)$ are said to be *holomorphically equivalent spaces* if there exists a nowhere-zero holomorphic function ϕ on U such that

$$\beta(z) = \frac{\alpha(z)}{|\phi(z)|^2} \quad \text{for all } z \in U.$$

Proposition 2.6. *Let $\mathcal{H}L^2(U,\alpha)$ and $\mathcal{H}L^2(U,\beta)$ be holomorphically equivalent spaces and ϕ defined as above. Let $\Lambda : \mathcal{H}L^2(U,\alpha) \to \mathcal{H}L^2(U,\beta)$ be defined by $\Lambda(f) = \phi f$. Then Λ is unitary.*

Proof. It is obvious that Λ is linear. Let $g \in \mathcal{H}L^2(U,\beta)$. Then g/ϕ is holomorphic. Since

$$\int_U \frac{|g(w)|^2}{|\phi(w)|^2}\,\alpha(w)\,dA(w) = \int_U |g(w)|^2\,\beta(w)\,dA(w) < \infty,$$

we obtain $g/\phi \in \mathcal{H}L^2(U,\alpha)$. Thus Λ is surjective. Then for any $f \in \mathcal{H}L^2(U,\alpha)$,

$$\begin{aligned}\int_U |f(w)|^2\,\alpha(w)\,dA(w) &= \int_U |f(w)|^2|\phi(w)|^2\,\frac{\alpha(w)}{|\phi(w)|^2}\,dA(w)\\ &= \int_U |\Lambda f(w)|^2\,\beta(w)\,dA(w).\end{aligned}$$

That is, $\|f\|_\alpha = \|\Lambda f\|_\beta$, i.e. Λ preserves norm. Hence, Λ is unitary. □

Theorem 2.7. *Let $\mathcal{H}L^2(U,\alpha)$ and $\mathcal{H}L^2(U,\beta)$ be holomorphically equivalent spaces. Let K_α and K_β be their respective reproducing kernels. Then for each $z \in U$,*

$$K_\beta(z,w) = \phi(z)\overline{\phi(w)}K_\alpha(z,w).$$

Moreover, we get that

$$|K_\beta(z,w)| = |\phi(z)||\phi(w)||K_\alpha(z,w)|.$$

Proof. Let $\{e_j\}_{j=0}^\infty$ be an orthonormal basis for $\mathcal{H}L^2(U,\alpha)$. Since any unitary map preserves an orthonormal basis, $\{\phi e_j\}_{j=0}^\infty$ is an orthonormal basis for $\mathcal{H}L^2(U,\beta)$. Then, by Theorem 2.3,

$$\begin{aligned}K_\beta(z,w) &= \sum_{j=0}^\infty \phi(z)e_j(z)\overline{\phi(w)e_j(w)}\\ &= \phi(z)\overline{\phi(w)}\sum_{j=0}^\infty e_j(z)\overline{e_j(w)}\\ &= \phi(z)\overline{\phi(w)}K_\alpha(z,w).\end{aligned}$$

□

The next goal is to introduce a particular space of L^2 holomorphic functions that we are going to give attention throughout. Before that, let us recall some facts from complex analysis.

Definition 2.8. Let $z = x + iy \in \mathbb{C}$ and $f(z)$ be a complexed-valued function in an open set U such that f_{xx} and f_{yy} exist at every point of U. Then the *Laplacian* of f is defined by

$$\Delta f = f_{xx} + f_{yy}.$$

In the $(z, \overline{z})$-coordinate, the Laplacian is given by the formula

$$\Delta f = \frac{4\partial^2}{\partial z \partial \overline{z}} f.$$

If f is continuous and $\Delta f = 0$ at every point of an open set U, then f is said to be *harmonic* on U.

In this work we look at a space which is a generalization of the standard Segal-Bargmann space. Let ψ be a real-valued smooth function on $\mathbb{C}$ such that $\Delta\psi = c$ where c is a positive constant. Consider the holomorphic L^2-space $\mathcal{H}L^2(\mathbb{C}, e^{-\psi})$ equipped with the norm

$$\|f\|_\psi^2 = \int_{\mathbb{C}} |f(z)|^2 \, e^{-\psi(z)} \, dA(z).$$

Note that $\Delta(|z|^2/t) = 4/t > 0$, so such a space is a generalization of the standard Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C}, \mu_t)$.

Theorem 2.9. *Let U be an open simply connected set in $\mathbb{C}$ and α, β strictly positive smooth functions on U. Then $\mathcal{H}L^2(U, \alpha)$ and $\mathcal{H}L^2(U, \beta)$ are holomorphically equivalent spaces if and only if $\Delta \log \alpha(z) = \Delta \log \beta(z)$.*

Proof. See Proposition 5 in [2]. □

Corollary 2.10. *Let ψ a real-valued smooth function on $\mathbb{C}$ satisfying $\Delta\psi = c > 0$. Then $\mathcal{H}L^2(\mathbb{C}, e^{-\psi})$ and $\mathcal{H}L^2(\mathbb{C}, \mu_{4/c})$ are holomorphically equivalent.*

Proof. Since

$$\Delta \log e^{-\psi(z)} = -c$$

and

$$\Delta \log \mu_{4/c} = \Delta \log \frac{c}{4\pi} e^{-c\frac{|z|^2}{4}} = \Delta(\log \frac{c}{4\pi} + \log e^{-c\frac{|z|^2}{4}}) = -c,$$

By Theorem 2.9, we see that $\mathcal{H}L^2(\mathbb{C}, e^{-\psi})$ and $\mathcal{H}L^2(\mathbb{C}, \mu_{4/c})$ are holomorphically equivalent as desired. □

Lemma 2.11. *Let ψ be a real-valued smooth function on $\mathbb{C}$ satisfying $\Delta\psi = c > 0$. Then there exists a constant $M > 0$ such that for any $f \in \mathcal{H}L^2(\mathbb{C}, e^{-\psi})$,*

$$|f(0)|^2 \leq M\, e^{\psi(0)} \int_{D(0,1)} |f(w)|^2 e^{-\psi(w)} dA(w).$$

Proof. See Lemma 8 in [2]. □

CHAPTER III

BOUNDEDNESS OF WEIGHTED COMPOSITION OPERATOR

Definition 3.1. Let φ and u be entire functions on $\mathbb{C}$. The *weighted composition operator* uC_φ is defined by

$$uC_\varphi(f) = u \cdot (f \circ \varphi)$$

for an entire function f. In particular, if $u = 1$, then we call it the *composition operator* and denote it by C_φ.

Throughout this work, let ψ be a real-valued smooth function on $\mathbb{C}$ satisfying $\Delta\psi = c > 0$. In this chapter, we generalize the idea of S. Ueki (see [4]) to prove the boundedness of the weighted composition operator uC_φ on $\mathcal{H}L^2(\mathbb{C}, e^{-\psi})$. Our result will be expressed in terms of the integral transform

$$B_\varphi(|u|^2)(w) = \int_{\mathbb{C}} |u(z)|^2 \exp\Big(-\frac{c}{4}|\varphi(z) - w|^2 + \psi(\varphi(z)) - \psi(z)\Big) dA(z).$$

However we need several lemmas to reach our result.

Lemma 3.2. *There exists a constant $M > 0$ such that for any $f \in \mathcal{H}L^2(\mathbb{C}, e^{-\psi})$,*

$$|f(z)|^2 \leq M e^{\psi(z)} \int_{D(z,1)} |f(w)|^2 e^{-\psi(w)} dA(w).$$

Proof. Let $z \in \mathbb{C}$ and $g_z(w) = z + w$. Then $\Delta\psi = c = \Delta(\psi \circ g_z)$. Let $f \in$

$\mathcal{H}L^2(\mathbb{C}, e^{-\psi})$ and $h = f \circ g_z$. Then $h \in \mathcal{H}L^2(\mathbb{C}, e^{-\psi\circ g_z})$ and by Lemma 2.11,

$$\begin{aligned}
|f(z)|^2 &= |f \circ g_z(0)|^2 = |h(0)|^2 \\
&\leq M e^{\psi\circ g_z(0)} \int_{D(0,1)} |h(w)|^2 \, e^{-\psi\circ g_z(w)} \, dA(w) \\
&= M e^{\psi(z)} \int_{D(0,1)} |f \circ g_z(w)|^2 \, e^{-\psi\circ g_z(w)} \, dA(w) \\
&= M e^{\psi(z)} \int_{D(0,1)} |f(z+w)|^2 \, e^{-\psi(z+w)} \, dA(w) \\
&= M e^{\psi(z)} \int_{D(z,1)} |f(w)|^2 \, e^{-\psi(w)} \, dA(w)
\end{aligned}$$

□

Lemma 3.3. *For $\mathcal{H}L^2(\mathbb{C}, e^{-\psi})$, we have*

$$|K_{e^{-\psi}}(z,w)| = \frac{c}{4\pi} \exp\Big[\frac{1}{2}\Big(-\frac{c}{4}|z-w|^2 + \psi(z) + \psi(w)\Big)\Big].$$

Moreover,

$$\|K_w\|_\psi = \Big(\frac{c}{4\pi}\Big)^{1/2} e^{\frac{\psi(w)}{2}} \quad \textit{and} \quad |k_w(z)| = \Big(\frac{c}{4\pi}\Big)^{1/2} \exp\Big[\frac{1}{2}\Big(-\frac{c}{4}|z-w|^2 + \psi(z)\Big)\Big].$$

Proof. According to Corollary 2.10, $\mathcal{H}L^2(\mathbb{C}, e^{-\psi})$ and $\mathcal{H}L^2(\mathbb{C}, \mu_{4/c})$ are holomorphically equivalent, so there exists a nowhere-zero holomorphic function ϕ on $\mathbb{C}$ such that

$$e^{-\psi(z)} = \frac{\mu_{4/c}(z)}{|\phi(z)|^2}.$$

Then

$$|\phi(z)|^2 = \frac{\mu_{4/c}(z)}{e^{-\psi(z)}} = \frac{\frac{c}{4\pi} e^{-c\frac{|z|^2}{4}}}{e^{-\psi(z)}}.$$

We have

$$|\phi(z)| = \sqrt{\frac{\mu_{4/c}(z)}{e^{-\psi(z)}}} = \Big(\frac{c}{4\pi}\Big)^{1/2} \exp\Big[\frac{1}{2}\Big(-c\frac{|z|^2}{4} + \psi(z)\Big)\Big]. \tag{3.1}$$

Suppose that the reproducing kernels for $\mathcal{H}L^2(\mathbb{C}, e^{-\psi})$ and $\mathcal{H}L^2(\mathbb{C}, \mu_{4/c})$ are $K_{e^{-\psi}}$, $K_{\mu_{4/c}}$, respectively. By the property of holomorphic equivalence and the equation (3.1), it follows that

$$\begin{aligned}
&|K_{e^{-\psi}}(z,w)| \\
&\quad = |\phi(z)||\phi(w)||K_{\mu_{4/c}}(z,w)| \\
&\quad = \frac{c}{4\pi}\exp\left[\frac{1}{2}\Big(-c\frac{|z|^2}{4}+\psi(z)-c\frac{|w|^2}{4}+\psi(w)\Big)\right]|K_{\mu_{4/c}}(z,w)| \\
&\quad = \frac{c}{4\pi}\exp\left[\frac{1}{2}\Big(-c\frac{|z|^2}{4}+\psi(z)-c\frac{|w|^2}{4}+\psi(w)\Big)\right]\Big|\exp[\langle z,w\rangle c/4]\Big| \\
&\quad = \frac{c}{4\pi}\exp\left[\frac{1}{2}\Big(-c\frac{|z|^2}{4}+\psi(z)-c\frac{|w|^2}{4}+\psi(w)\Big)\right]\exp[\mathrm{Re}\langle z,w\rangle c/4] \\
&\quad = \frac{c}{4\pi}\exp\left[\frac{1}{2}\Big(-\frac{c}{4}(|z|^2+|w|^2-2\mathrm{Re}\langle z,w\rangle)+\psi(z)+\psi(w)\Big)\right] \\
&\quad = \frac{c}{4\pi}\exp\left[\frac{1}{2}\Big(-\frac{c}{4}|z-w|^2+\psi(z)+\psi(w)\Big)\right].
\end{aligned}$$

For simplicity, we write $K(z,w) = K_{e^{-\psi}}(z,w)$. Then

$$K(w,w) = \langle K_w, K_w\rangle_\psi = \|K_w\|_\psi^2$$

which implies that $K(w,w)$ is nonnegative. Hence

$$|K(w,w)| = \|K_w\|_\psi^2.$$

Thus

$$\begin{aligned}
\|K_w\|_\psi^2 &= \frac{c}{4\pi}e^{\psi(w)} \\
\|K_w\|_\psi &= \Big(\frac{c}{4\pi}\Big)^{1/2} e^{\frac{\psi(w)}{2}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
|k_w(z)| = \frac{|K_w(z)|}{\|K_w\|_\psi} &= \frac{|K(z,w)|}{\|K_w\|_\psi} \\
&= \Big(\frac{c}{4\pi}\Big)^{1/2}\exp\left[\frac{1}{2}\Big(-\frac{c}{4}|z-w|^2+\psi(z)\Big)\right].
\end{aligned}$$

□

Lemma 3.4. *Define a positive measure* μ *by*

$$\mu(E) = \int_{\varphi^{-1}(E)} |u(z)|^2 e^{-\psi(z)}\, dA(z),$$

where E *is a Borel subset of* $\mathbb{C}$. *Then*

$$\int_{D(w,1)} e^{\psi(z)} d\mu(z) \le \frac{4\pi}{c} e^{\frac{c}{4}}\, B_\varphi(|u|^2)(w)$$

for all $w \in \mathbb{C}$.

Proof. For each $z \in D(w,1)$, by Lemma 3.3, we have

$$|k_w(z)|^2 = \frac{c}{4\pi} \exp\Big[-\frac{c}{4}|z-w|^2 + \psi(z)\Big].$$

Since $|z-w| < 1$, we obtain an inequality

$$|k_w(z)|^2 \ge \frac{c}{4\pi} \exp\Big[-\frac{c}{4} + \psi(z)\Big].$$

Hence

$$\frac{c}{4\pi} e^{-\frac{c}{4}} \int_{D(w,1)} e^{\psi(z)}\, d\mu(z) \le \int_{D(w,1)} |k_w(z)|^2 d\,\mu(z) \le \int_{\mathbb{C}} |k_w(z)|^2\, d\mu(z).$$

By the definitions of measure μ and the integral operator $B_\varphi(|u|^2)$, we see that

$$\begin{aligned}\int_{\mathbb{C}} |k_w(z)|^2\, d\mu(z) &= \int_{\varphi^{-1}(\mathbb{C})} |u(z)|^2 |k_w \circ \varphi(z)|^2\, e^{-\psi(z)}\, dA(z) \\ &\le \int_{\mathbb{C}} |u(z)|^2 |k_w \circ \varphi(z)|^2\, e^{-\psi(z)}\, dA(z) \\ &= B_\varphi(|u|^2)(w).\end{aligned}$$

Thus, we obtain the desired inequality:

$$\int_{D(w,1)} e^{\psi(z)}\, d\mu(z) \le \frac{4\pi}{c} e^{\frac{c}{4}}\, B_\varphi(|u|^2)(w).$$

□

Theorem 3.5. *Let φ and u be entire functions on $\mathbb{C}$. Then uC_φ is a bounded linear operator on $\mathcal{H}L^2(\mathbb{C}, e^{-\psi})$ if and only if $B_\varphi(|u|^2) \in L^\infty(\mathbb{C})$.*

Proof. First, suppose that uC_φ is bounded on $\mathcal{H}L^2(\mathbb{C}, e^{-\psi})$. Then

$$\|uC_\varphi(k_w)\|_\psi^2 \leq L\|k_w\|_\psi^2 = L$$

for some constant $L > 0$ and for all $w \in \mathbb{C}$. On the other hand,

$$\begin{aligned}
\|uC_\varphi(k_w)\|_\psi^2 &= \int_{\mathbb{C}} |u(z)|^2 |k_w(\varphi(z))|^2 e^{-\psi(z)}\, dA(z) \\
&= \int_{\mathbb{C}} |u(z)|^2 \exp\Big(- \frac{c}{4}|\varphi(z) - w|^2 + \psi(\varphi(z))\Big) e^{-\psi(z)}\, dA(z) \\
&= \int_{\mathbb{C}} |u(z)|^2 \exp\Big(- \frac{c}{4}|\varphi(z) - w|^2 + \psi(\varphi(z)) - \psi(z)\Big)\, dA(z) \\
&= B_\varphi(|u|^2)(w).
\end{aligned}$$

Thus, $B_\varphi(|u|^2)(w) \leq L$ for all $w \in \mathbb{C}$. This implies that $B_\varphi(|u|^2) \in L^\infty(\mathbb{C})$.

Conversely, by the definition of measure μ, we obtain

$$\|uC_\varphi f\|_\psi^2 = \int_{\mathbb{C}} |u(z)|^2 |f(\varphi(z))|^2 e^{-\psi(z)}\, dA(z) = \int_{\mathbb{C}} |f(z)|^2\, d\mu(z).$$

It follows from Lemma 3.2 that

$$\begin{aligned}
\|uC_\varphi f\|_\psi^2 &= \int_{\mathbb{C}} |f(z)|^2\, d\mu(z) \\
&\leq \int_{\mathbb{C}} M\, e^{\psi(z)} \int_{D(z,1)} |f(w)|^2 e^{-\psi(w)}\, dA(w)\, d\mu(z) \\
&= M \int_{\mathbb{C}} \int_{\mathbb{C}} e^{\psi(z)}\, \chi_{D(z,1)}(w)\, |f(w)|^2 e^{-\psi(w)}\, dA(w)\, d\mu(z)
\end{aligned}$$

where $\chi_{D(z,1)}$ is the characteristic function of $D(z, 1)$. Since $\chi_{D(z,1)}(w) = \chi_{D(w,1)}(z)$, we have

$$
\begin{aligned}
\|uC_\varphi f\|_\psi^2 &\le M \int_{\mathbb{C}}\int_{\mathbb{C}} e^{\psi(z)}\, \chi_{D(z,1)}(w)\, |f(w)|^2 e^{-\psi(w)}\, dA(w)\, d\mu(z) \\
&= M \int_{\mathbb{C}}\int_{\mathbb{C}} e^{\psi(z)}\, \chi_{D(w,1)}(z)\, |f(w)|^2 e^{-\psi(w)}\, dA(w)\, d\mu(z) \\
&= M \int_{\mathbb{C}}\int_{\mathbb{C}} e^{\psi(z)}\, \chi_{D(w,1)}(z)\, |f(w)|^2 e^{-\psi(w)}\, d\mu(z)\, dA(w) \qquad (3.2) \\
&= M \int_{\mathbb{C}} |f(w)|^2 e^{-\psi(w)} \left[\int_{\mathbb{C}} \chi_{D(w,1)}(z)\, e^{\psi(z)}\, d\mu(z)\right] dA(w) \\
&= M \int_{\mathbb{C}} |f(w)|^2 e^{-\psi(w)} \left[\int_{D(w,1)} e^{\psi(z)}\, d\mu(z)\right] dA(w) \\
&\le M \int_{\mathbb{C}} |f(w)|^2 e^{-\psi(w)} \left[\frac{4\pi}{c} e^{\frac{c}{4}} B_\varphi(|u|^2)(w)\right] dA(w) \qquad (3.3) \\
&= \frac{4\pi M}{c} e^{\frac{c}{4}} \int_{\mathbb{C}} |f(w)|^2 e^{-\psi(w)}\, B_\varphi(|u|^2)(w)\, dA(w) \\
&\le \frac{4\pi M}{c} e^{\frac{c}{4}}\, \|B_\varphi(|u|^2)\|_\infty \int_{\mathbb{C}} |f(w)|^2\, e^{-\psi(w)}\, dA(w) \qquad (3.4) \\
&\le \frac{4\pi M}{c} e^{\frac{c}{4}}\, \|B_\varphi(|u|^2)\|_\infty\, \|f\|_\psi^2.
\end{aligned}
$$

Using Fubini's theorem allows one to interchange order of the integration in (3.2). We also use Lemma 3.4 in (3.3). Moreover, (3.4) follows from $B_\varphi(|u|^2) \in L^\infty(\mathbb{C})$. Hence uC_φ is a bounded linear operator on $\mathcal{H}L^2(\mathbb{C}, e^{-\psi})$. □

In case $\varphi(z) = z$ the operator uC_φ reduces to the *multiplication operator*, M_u. We obtain the following corollary:

Corollary 3.6. *Let u be an entire function on $\mathbb{C}$. Then M_u is a bounded linear operator on $\mathcal{H}L^2(\mathbb{C}, e^{-\psi})$, if and only if $B_z(|u|^2) \in L^\infty(\mathbb{C})$, where*

$$B_z(|u|^2)(w) = \int_{\mathbb{C}} |u(w)|^2\, e^{-\frac{c}{4}|z-w|^2}\, dA(z).$$

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