CHAPTER II

GENERAL PROPERTIES AND EXAMPLES

In this chapter, we first generalize Proposition 1.3 – Proposition 1.7 of Chapter I. After that all the quasi-hyperideals of the hyperrings in Example 1.23 – Example 1.28 are characterized. Finally, an example of a noncommutative hyperring which is not a division hyperring is provided. Moreover, all of its quasi-hyperideals are investigated.

To generalize Proposition 1.3, the following three lemmas are required. The first one is clearly true in semigroups.

Lemma 2.1. *Let S be a semigroup with zero 0. Then S is a group with zero (that is, $S\backslash\{0\}$ is a group under the operation of S) if and only if $Sx = S = xS$ for all $x \in S\backslash\{0\}$.*

Lemma 2.2. *If H_1 and H_2 are canonical subhypergroups of a hypergroup $(H, \circ)$ then $H_1 \circ H_2$ is a canonical subhypergroup of H.*

Proof. Let e be the scalar identity of $(H, \circ)$. Then $e \in H_1$ and $e \in H_2$, so $e \in e \circ e \subseteq H_1 \circ H_2$. Let $x, y \in H_1 \circ H_2$. Then $x \in a_1 \circ a_2$ and $y \in b_1 \circ b_2$ for some $a_1, b_1 \in H_1$ and $a_2, b_2 \in H_2$. Since $(H, \circ)$ is commutative, we have

$$x \circ y \subseteq (a_1 \circ a_2) \circ (b_1 \circ b_2) = (a_1 \circ b_1) \circ (a_2 \circ b_2) \subseteq H_1 \circ H_2.$$

Since $x \in a_1 \circ a_2$, by Proposition 1.19, $x^{-1} \in a_1^{-1} \circ a_2^{-1} \subseteq H_1 \circ H_2$. Thus $H_1 \circ H_2$ is a canonical subhypergroup of $(H, \circ)$ by Proposition 1.20. $\square$

Lemma 2.3. *Let $(A,+,\cdot)$ be a hyperring and X and Y nonempty subsets of A. Then the following statements hold.*

(i) *$\mathbb{Z}X$ is a canonical subhypergroup of $(A,+)$ containing X.*

(ii) *$<AX>$ and $<XA>$ are a left hyperideal and a right hyperideal of A, respectively.*

(iii) *$<XAY>$ is a subhyperring of A.*

(iv) *$\mathbb{Z}X+<AX>$ and $\mathbb{Z}X+<XA>$ are respectively a left hyperideal and a right hyperideal of A containing X.*

Proof. (i) It is easy to see from the definition of $\mathbb{Z}X$ on page 11 that $X \subseteq \mathbb{Z}X$, $\mathbb{Z}X+\mathbb{Z}X \subseteq \mathbb{Z}X$ and $0 \in \mathbb{Z}X$. Let $a \in \mathbb{Z}X$. Then $a \in \sum_{i=1}^{k} n_i x_i$ for some $n_1,\ldots,n_k \in \mathbb{Z}$ and $x_1,\ldots,x_k \in X$. By Proposition 1.19 and Proposition 1.22(7), $-a \in \sum_{i=1}^{k} -(n_i x_i) = \sum_{i=1}^{k}(-n_i)x_i \in \mathbb{Z}X$. By Proposition 1.20, (i) holds.

(ii) Clearly, $<AX>+<AX>\subseteq<AX>$. By Proposition 1.22(9), $<<AX><AX>>=<AXAX>$, so $<AX><AX>\subseteq<AXAX>=<(AXA)X>\subseteq<AX>$. Since $\{0\}=0X\subseteq<AX>, 0\in<AX>$. Next, let $b\in<AX>$. Then $b\in\sum_{i=1}^{n} a_i x_i$ for some $a_1,\ldots,a_n\in A, x_1,\ldots,x_n\in X$. By Proposition 1.19 and Proposition 1.22(3), we have

$$-b \in \sum_{i=1}^{n} -(a_i x_i) = \sum_{i=1}^{n}(-a_i)x_i \in <AX> .$$

By Proposition 1.20, $(<AX>,+)$ is a canonical subhypergroup of $(A,+)$. We also have $(<AX>,\cdot)$ is a subsemigroup of $(A,\cdot)$. Hence $<AX>$ is a subhyperring of $(A,+,\cdot)$. Since $A<AX>\subseteq<A<AX>>\subseteq<AAX>\subseteq<AX>$ by Proposition 1.22(9), $<AX>$ is a left hyperideal of $(A,+,\cdot)$.

Similarly, $<XA>$ is a right hyperideal of $(A,+,\cdot)$.

(iii) We have $(<XAY>,\cdot)$ is a subsemigroup of $(A,\cdot)$ since $<XAY><XAY> \subseteq<XAYXAY>=<X(AYXA)Y>\subseteq<XAY>$ by Proposition 1.22(9). Since

$0 \in A$, we have $0 \in < XAY >$. Since any element of $< XAY >$ is a member of a set of the form $\sum x_i a_i y_i$ where $x_i \in X, y_i \in Y$ and $a_i \in A$, we can conclude that $< XAY > + < XAY > \subseteq < XAY >$. Next, let $b \in < XAY >$. Then $b \in \sum_{i=1}^{n} x_i a_i y_i$ for some $x_i \in X, a_i \in A, y_i \in Y$ and $n \in \mathbb{N}$. By Proposition 1.19 and Proposition 1.22(3),

$$-b \in \sum_{i=1}^{n} -(x_i a_i y_i) = \sum_{i=1}^{n} x_i(-a_i)y_i \subseteq < XAY > .$$

By Proposition 1.20, $(< XAY >, +)$ is a canonical subhypergroup of $(A, +)$. Hence $< XAY >$ is a subhyperring of A.

(iv) Since $0 \in < AX >$ and $0 \in < XA >$, $X \subseteq \mathbb{Z}X + < AX >$ and $X \subseteq \mathbb{Z}X + < XA >$. By (i), (ii) and Lemma 2.2, $\mathbb{Z}X + < AX >$ and $\mathbb{Z}X + < XA >$ are canonical subhypergroups of $(A, +)$. Since

$$\begin{aligned} A(\mathbb{Z}X + < AX >) &\subseteq A(\mathbb{Z}X) + A < AX > \\ &\subseteq \mathbb{Z}(AX) + < AX > \quad \text{by Proposition 1.22(11) and (ii)} \\ &\subseteq \mathbb{Z} < AX > + < AX > \\ &= < AX > + < AX > \quad \text{by Proposition 1.29 and (ii)} \\ &= < AX > \ \subseteq \ \mathbb{Z}X + < AX > \end{aligned}$$

and

$$\begin{aligned} (\mathbb{Z}X + < XA >)A &\subseteq (\mathbb{Z}X)A + < XA > A \\ &\subseteq \mathbb{Z}(XA) + < XA > \quad \text{by Proposition 1.22(11) and (ii)} \\ &\subseteq \mathbb{Z} < XA > + < XA > \\ &= < XA > + < XA > \quad \text{by Proposition 1.29 and (ii)} \\ &= < XA > \ \subseteq \ \mathbb{Z}X + < XA >, \end{aligned}$$

it follows that $\mathbb{Z}X + < AX >$ and $\mathbb{Z}X + < XA >$ are a left hyperideal and a right hyperideal of A, respectively. □

Let A be a hyperring. Then by Proposition 1.21, the intersection of a collection of left [right] hyperideals of A is also a left [right] hyperideal of A.

For $\varnothing \neq X \subseteq A$, let $(X)_l[(X)_r]$ denote the intersection of all left [right] hyperideals of A containing X. Therefore $(X)_l[(X)_r]$ is the smallest left [right] hyperideal of A containing X and it is called the *left [right] hyperideal of A generated by X*. For $a \in A$, let $(a)_l[(a)_r]$ denote $(\{a\})_l[(\{a\})_r]$ and it is called the *principal left [right] hyperideal of A generated by a.*

Lemma 2.4. *For any nonempty subset X of a hyperring A,*

$$(X)_l = \mathbb{Z}X + < AX > \quad \textit{and} \quad (X)_r = \mathbb{Z}X + < XA > .$$

In particular, for $a \in A$,

$$(a)_l = \mathbb{Z}a + Aa \quad \textit{and} \quad (a)_r = \mathbb{Z}a + aA.$$

Proof. From Lemma 2.3 (iv), $(X)_l \subseteq \mathbb{Z}X + < AX >$. Since $(X)_l$ is a canonical subhypergroup of $(A, +)$ containing X, we have that $\mathbb{Z}X \subseteq (X)_l$. Also $< AX > \subseteq < A(X)_l > \subseteq (X)_l$ because $(X)_l$ is a left hyperideal of A containing X. Hence $\mathbb{Z}X + < AX > \subseteq (X)_l$. Therefore we deduce that $(X)_l = \mathbb{Z}X + < AX >$. We can show similarly that $(X)_r = \mathbb{Z}X + < XA >$. Since for $a \in A$, $< Aa > = Aa$ and $< aA > = aA$, it follows that the last two equalities hold. □

Theorem 2.5. *If A is a hyperring such that $A^2 \neq \{0\}$, then A has no proper nonzero quasi-hyperideals if and only if A is a division hyperring.*

Proof. Assume that A has no proper nonzero quasi-hyperideals. Let $a \in A \setminus \{0\}$. Since $(a)_l$ is a quasi-hyperideal of A containing $a \neq 0$, by the assumption, $(a)_l = A$.

Moreover,

$$
\begin{aligned}
Aa \subseteq\, < A(a)_l > &=\, < A(\mathbb{Z}a + Aa) > && \text{by Lemma 2.4} \\
&\subseteq\, < A(\mathbb{Z}a) + AAa > && \\
&\subseteq\, < \mathbb{Z}(Aa) + Aa > && \text{by Proposition 1.22(11)} \\
&=\, < Aa + Aa > && \text{by Proposition 1.29 and Lemma 2.3(ii)} \\
&=\, < Aa > && \text{by Lemma 2.3(ii)} \\
&= Aa,
\end{aligned}
$$

so $< A(a)_l >= Aa$. Since $A^2 \neq \{0\}$, $Aa =< A(a)_l >=< A^2 >\neq \{0\}$. Since Aa is a quasi-hyperideal such that $Aa \neq \{0\}$, $Aa = A$. Similarly, we obtain that $aA = A$. Then $Aa = A = aA$, so by Lemma 2.1, $(A\backslash\{0\}, \cdot)$ is a group. Hence A is a division hyperring.

Conversely, assume that A is a division hyperring. Then $(A\backslash\{0\}, \cdot)$ is a group, so by Lemma 2.1, $Aa = A = aA$ for all $a \in A\backslash\{0\}$. Let Q be a nonzero quasi-hyperideal of A. Then there exists $q \in Q$ such that $q \neq 0$. Thus $Aq = A = qA$ which implies that $A = Aq \cap qA \subseteq< AQ > \cap < QA >\subseteq Q$. Hence $Q = A$. Therefore A has no proper nonzero quasi-hyperideals. □

Proposition 1.3 becomes a corollary of Theorem 2.5.

Corollary 2.6. *Let A be a ring such that $A^2 \neq \{0\}$. Then A is a division ring if and only if A and $\{0\}$ are the only quasi-ideals of A.*

We also have the following facts in hyperrings.

Theorem 2.7. *Let A be a hyperring. Then:*

(i) *The intersection of a set of quasi-hyperideals of A is a quasi-hyperideal of A.*

(ii) *The intersection of a set of bi-hyperideals of A is a bi-hyperideal of A.*

Proof. (i) Let $\{Q_\alpha \mid \alpha \in \Lambda\}$ be a set of quasi-hyperideals of A. By Proposition 1.21, $(\bigcap_{\alpha\in\Lambda} Q_\alpha, +)$ is a canonical subhypergroup of $(A, +)$. Since each Q_α is a quasi-hyperideal of A, we have that for every $\beta \in \Lambda$,

$$< A(\bigcap_{\alpha\in\Lambda} Q_\alpha) > \cap < (\bigcap_{\alpha\in\Lambda} Q_\alpha)A > \subseteq < AQ_\beta > \cap < Q_\beta A > \subseteq Q_\beta.$$

Consequently, $< A(\bigcap_{\alpha\in\Lambda} Q_\alpha) > \cap < (\bigcap_{\alpha\in\Lambda} Q_\alpha)A > \subseteq \bigcap_{\alpha\in\Lambda} Q_\alpha$. Hence $\bigcap_{\alpha\in\Lambda} Q_\alpha$ is a quasi-hyperideal of A.

(ii) First, we note that an arbitrary nonempty intersection of subsemigroups of a semigroup S is a subsemigroup of S. Let $\{B_\alpha \mid \alpha \in \Lambda\}$ be a set of bi-hyperideals of A. From Proposition 1.21, $(\bigcap_{\alpha\in\Lambda} B_\alpha, +)$ is a canonical subhypergroup of $(A, +)$. Then $\bigcap_{\alpha\in\Lambda} B_\alpha$ is a subhyperring of A. Since each B_α is a bi-hyperideal of A, we have that for every $\beta \in \Lambda$,

$$< (\bigcap_{\alpha\in\Lambda} B_\alpha)A(\bigcap_{\alpha\in\Lambda} B_\alpha) > \subseteq < B_\beta A B_\beta > \subseteq B_\beta.$$

It then follows that $< (\bigcap_{\alpha\in\Lambda} B_\alpha)A(\bigcap_{\alpha\in\Lambda} B_\alpha) > \subseteq \bigcap_{\alpha\in\Lambda} B_\alpha$. Hence $\bigcap_{\alpha\in\Lambda} B_\alpha$ is a bi-hyperideal of A. □

Proposition 1.4 is immediately a consequence of the above theorem.

Corollary 2.8. *Let A be a ring. Then:*

(i) *The intersection of a set of quasi-ideals of A is a quasi-ideal of A.*

(ii) *The intersection of a set of bi-ideals of A is a bi-ideal of A.*

Let A be a hyperring. For $\varnothing \neq X \subseteq A$, the *quasi-hyperideal of A generated by X* is the intersection of all quasi-hyperideals of A containing X which is denoted by $(X)_q$. The *bi-hyperideal of A generated by* $X \subseteq A$ with $X \neq \varnothing$ is defined similarly and it is denoted by $(X)_b$. Then for $X \subseteq A$, $(X)_q[(X)_b]$ is the smallest quasi-hyperideal [bi-hyperideal] of A containing X. For $a \in A$, let $(a)_q$

denote $(\{a\})_q$ and it is called the *principal quasi-hyperideal of A generated by a*. Since every quasi-hyperideal of A is a bi-hyperideal of A, $(X)_b \subseteq (X)_q$ for every nonempty subset X of A.

Theorem 2.9. *For a nonempty subset X of a hyperring A,*

$$(X)_q = \mathbb{Z}X + (< AX > \cap < XA >).$$

In particular, for $a \in A$,

$$(a)_q = \mathbb{Z}a + (Aa \cap aA).$$

Proof. First, we show that $\mathbb{Z}X + (< AX > \cap < XA >)$ is a quasi-hyperideal containing X. Since $X \subseteq \mathbb{Z}X$ and $0 \in < AX > \cap < XA >$, $X \subseteq \mathbb{Z}X + (< AX > \cap < XA >)$. We know from Lemma 2.3 that $\mathbb{Z}X, < AX >$ and $< XA >$ are canonical subhypergroups of $(A, +)$. By Proposition 1.21 and Lemma 2.2, we have $\mathbb{Z}X + (< AX > \cap < XA >)$ is a canonical subhypergroup of $(A, +)$. We also have

$$\begin{aligned}
< A(\mathbb{Z}X + (< AX > \cap < XA >)) > &\subseteq < A(\mathbb{Z}X+ < AX >) > \\
&\subseteq < A(\mathbb{Z}X) + A < AX >> \\
&\subseteq < \mathbb{Z}(AX)+ < AX >> \\
&\qquad \text{by Proposition 1.22(11) and Lemma 2.3(ii)} \\
&\subseteq < \mathbb{Z} < AX > + < AX >> \\
&\subseteq << AX > + < AX >> \\
&\qquad \text{by Proposition 1.29 and Lemma 2.3(ii)} \\
&= << AX >> \quad \text{by Proposition 1.22(9)} \\
&= < AX > \quad \text{by Proposition 1.22(9)}
\end{aligned}$$

and

$$
\begin{aligned}
< (\mathbb{Z}X + (< AX > \cap < XA >))\, A > &\subseteq < (\mathbb{Z}X + < XA >)A > \\
&\subseteq < (\mathbb{Z}X)A + < XA > A > \\
&\subseteq < \mathbb{Z}(XA) + < XA >> \\
&\subseteq < \mathbb{Z} < XA > + < XA >> \\
&\subseteq << XA > + < XA >> \\
&= << XA >> \\
&= < XA > .
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
< A\,(\mathbb{Z}X + (< AX > \cap < XA >)) > \cap < (\mathbb{Z}X + (< AX > \cap < XA >))\, A > \\
\subseteq < AX > \cap < XA > \\
\subseteq \mathbb{Z}X + (< AX > \cap < XA >).
\end{aligned}
$$

Hence $\mathbb{Z}X + (< AX > \cap < XA >)$ is a quasi-hyperideal of A containing X. Then $(X)_q \subseteq \mathbb{Z}X + (< AX > \cap < XA >)$.

Since $((X)_q, +)$ is a canonical subhypergroup of $(A, +)$, $\mathbb{Z}X \subseteq (X)_q$. We also have $< AX > \cap < XA > \subseteq < A(X)_q > \cap < (X)_q A > \subseteq (X)_q$ which implies that $\mathbb{Z}X + (< AX > \cap < XA >) \subseteq (X)_q$. Therefore $(X)_q = \mathbb{Z}X + (< AX > \cap < XA >)$. □

Theorem 2.10. *For a nonempty subset X of a hyperring A,*

$$(X)_b = \mathbb{Z}X + \mathbb{Z}X^2 + < XAX > .$$

Proof. By Lemma 2.2 and Lemma 2.3 ((i) and (iii)), $\mathbb{Z}X + \mathbb{Z}X^2 + < XAX >$ is a canonical subhypergroup of (A, +). Since

$$
\begin{aligned}
&(\mathbb{Z}X + \mathbb{Z}X^2 + <XAX>)^2 \\
&\quad \subseteq (\mathbb{Z}X)(\mathbb{Z}X) + (\mathbb{Z}X)(\mathbb{Z}X^2) + (\mathbb{Z}X) <XAX> + (\mathbb{Z}X^2)(\mathbb{Z}X) + (\mathbb{Z}X^2)(\mathbb{Z}X^2) \\
&\qquad + (\mathbb{Z}X^2) <XAX> + <XAX> (\mathbb{Z}X) + <XAX> (\mathbb{Z}X^2) \\
&\qquad + <XAX><XAX> \qquad \text{from Proposition 1.22(5)} \\
&\quad \subseteq \mathbb{Z}X^2 + \mathbb{Z}X^3 + \mathbb{Z}(X <XAX>) + \mathbb{Z}X^3 + \mathbb{Z}X^4 + \mathbb{Z}(X^2 <XAX>) \\
&\qquad + \mathbb{Z}(<XAX> X) + \mathbb{Z}(<XAX> X^2) + <<XAX><XAX>> \\
&\qquad\qquad \text{from Proposition 1.22(11)} \\
&\quad \subseteq \mathbb{Z}X^2 + \mathbb{Z}X^3 + \mathbb{Z} <X^2AX> + \mathbb{Z}X^3 + \mathbb{Z}X^4 + \mathbb{Z} <X^3AX> + \mathbb{Z} <XAX^2> \\
&\qquad + \mathbb{Z} <XAX^3> + <(XAX)^2> \qquad \text{from Proposition 1.22(9)} \\
&\quad \subseteq \mathbb{Z}X^2 + \mathbb{Z} <XAX> + \mathbb{Z} <XAX> + \mathbb{Z} <XAX> + \mathbb{Z} <XAX> \\
&\qquad + \mathbb{Z} <XAX> + \mathbb{Z} <XAX> + \mathbb{Z} <XAX> + <XAX> \\
&\quad \subseteq \mathbb{Z}X^2 + <XAX> \qquad \text{from Proposition 1.29 and Lemma 2.3(iii)} \\
&\quad \subseteq \mathbb{Z}X + \mathbb{Z}X^2 + <XAX>,
\end{aligned}
$$

we deduce that $\mathbb{Z}X + \mathbb{Z}X^2 + <XAX>$ is a subhyperring of A. It is clear that $<XAX> A <XAX> \subseteq <XAX>$. Since

$$
\begin{aligned}
&(\mathbb{Z}X + \mathbb{Z}X^2 + <XAX>)A(\mathbb{Z}X + \mathbb{Z}X^2 + <XAX>) \\
&\quad \subseteq (\mathbb{Z}X + \mathbb{Z}X^2 + <XAX>)(\mathbb{Z}AX + \mathbb{Z}AX^2 + A <XAX>) \\
&\qquad\qquad \text{from Proposition 1.22(11)} \\
&\quad \subseteq \mathbb{Z}(XAX) + \mathbb{Z}(X^2AX) + \mathbb{Z}(<XAX> (AX)) + \mathbb{Z}(XAX^2) \\
&\qquad + \mathbb{Z}(X^2AX^2) + \mathbb{Z}(<XAX> AX^2) + \mathbb{Z}(XA <XAX>)
\end{aligned}
$$

$$+\mathbb{Z}(X^2A < XAX >)+ < XAX > A < XAX >$$

from Proposition 1.22((5) and (11))

$$\subseteq \mathbb{Z} < XAX > +\mathbb{Z} < XAX > +\mathbb{Z} < XAX > +\mathbb{Z} < XAX >$$

$$+\mathbb{Z} < XAX > +\mathbb{Z} < XAX > +\mathbb{Z} < XAX > +\mathbb{Z} < XAX >$$

$$+ < XAX > \quad \text{from Proposition 1.22(9)}$$

$$\subseteq < XAX > \quad \text{from Proposition 1.29 and Lemma 2.3(iii)}$$

$$\subseteq \mathbb{Z}X + \mathbb{Z}X^2 + < XAX >$$

and $X \subseteq \mathbb{Z}X + \mathbb{Z}X^2 + < XAX >$, we conclude that $\mathbb{Z}X + \mathbb{Z}X^2 + < XAX >$ is a bi-hyperideal of A containing X. Hence $(X)_b \subseteq \mathbb{Z}X + \mathbb{Z}X^2 + < XAX >$.

Since $(X)_b$ is a subhyperring of A containing X, it follows that $\mathbb{Z}X \subseteq (X)_b$ and $\mathbb{Z}X^2 \subseteq (X)_b$. But $(X)_b$ is a bi-hyperideal of A containing X, so $< XAX > \subseteq < (X)_b A (X)_b > \subseteq (X)_b$. Consequently, $\mathbb{Z}X + \mathbb{Z}X^2 + < XAX > \subseteq (X)_b$. Hence the theorem is proved. □

Proposition 1.5 and Proposition 1.6 are special cases of Theorem 2.9 and Theorem 2.10, respectively.

Corollary 2.11. *For a nonempty subset X of a ring A,*

$$(X)_q = \mathbb{Z}X + (AX \cap XA).$$

Corollary 2.12. *For a nonempty subset X of a ring A,*

$$(X)_b = \mathbb{Z}X + \mathbb{Z}X^2 + XAX.$$

Theorem 2.13. *A hyperring A is a regular if and only if $< QAQ >= Q$ for every quasi-hyperideal Q of A.*

Proof. Assume that A is a regular hyperring and let Q be a quasi-hyperideal of A. Since A is regular, for every $x \in Q$, $x = xyx$ for some $y \in A$. Thus $x \in QAQ$

for all $x \in Q$, that is, $Q \subseteq < QAQ >$. Since $Q \subseteq < QAQ > \subseteq < AQ >$ and $Q \subseteq < QAQ > \subseteq < QA >$, we obtain that $Q \subseteq < QAQ > \subseteq < AQ > \cap < QA > \subseteq Q$. Hence $Q =< QAQ >$.

Conversely, assume that $< QAQ >= Q$ for every quasi-hyperideal Q of A. To show that A is a regular hyperring, let $a \in A\backslash\{0\}$. Then $(a)_l \cap (a)_r$ is a quasi-hyperideal of A (see page 15). By the assumption, $< ((a)_l \cap (a)_r)\, A\, ((a)_l \cap (a)_r) >= (a)_l \cap (a)_r$. Since $a \in (a)_l \cap (a)_r$, $a \in < ((a)_l \cap (a)_r)\, A\, ((a)_l \cap (a)_r) > \subseteq < (a)_r A (a)_l >$. By Lemma 2.4, $(a)_r = \mathbb{Z}a + aA$ and $(a)_l = \mathbb{Z}a + Aa$. Then

$$
\begin{aligned}
a \in < (a)_r A (a)_l > &=< (\mathbb{Z}a + aA)A(\mathbb{Z}a + Aa) > \\
&\subseteq < \mathbb{Z}(aAa) + \mathbb{Z}(aAa) + \mathbb{Z}(aA^2a + aA^3a > \\
&\qquad \text{from Proposition 1.22(11)} \\
&\subseteq < aAa + aAa + aAa + aAa > \\
&\qquad \text{from Proposition 1.29 and Lemma 2.3(iii)} \\
&\subseteq < aAa >= aAa
\end{aligned}
$$

which implies that a is a regular element of A. □

Proposition 1.7 becomes a corollary of Theorem 2.13.

Corollary 2.14. *A ring A is regular if and only if $QAQ = Q$ for every quasi-ideal Q of A.*

Next, we shall determine all quasi-hyperideals of the hyperrings in Example 1.23 – Example 1.28. By Theorem 2.5, we obtain that all quasi-hyperideals of the hyperrings in Example 1.23, Example 1.27 and Example 1.28 are the only $\{0\}$ and itself. The next three propositions determine all quasi-hyperideals (hyperideals) of the hyperrings in Example 1.24, Example 1.25 and Example 1.26, respectively.

Proposition 2.15. *Let* $(A, \oplus, \cdot)$ *be a hyperring where* $A = [0, a]$ *or* $[0, a)$, $0 < a \leq 1$ *and*

$$x \oplus y = \begin{cases} \{\max\{x, y\}\} & \text{if } x \neq y, \\ [0, x] & \text{if } x = y. \end{cases}$$

Then

$$\{[0, b] \mid b \in A\} \cup \{[0, b) \mid b \in A \backslash \{0\}\}$$

is the set of all quasi-hyperideals of $(A, \oplus, \cdot)$.

Proof. Trivially, $\{0\}$ is a quasi-hyperideal of $(A, \oplus, \cdot)$ and it is clear that for every $b \in A \backslash \{0\}$, $[0, b]$ and $[0, b)$ are subhyperrings of $(A, \oplus, \cdot)$. Since $0 < a \leq 1$, we have $A[0, b] \subseteq [0, 1][0, b] = [0, b]$ and $A[0, b) \subseteq [0, 1][0, b) = [0, b)$ for all $b \in A \backslash \{0\}$. Hence $[0, b]$ and $[0, b)$ are quasi-hyperideals of $(A, \oplus, \cdot)$ for every $b \in A \backslash \{0\}$.

For the converse, let Q be a nonzero quasi-hyperideal of $(A, \oplus, \cdot)$. Since $Q \subseteq A \subseteq [0, 1]$, sup Q exists in $\mathbb{R}$, say b.

Case 1: $b \in Q$. Then $Q \subseteq [0, b]$. But $b \oplus b = [0, b] \subseteq Q$, so $Q = [0, b]$.

Case 2: $b \notin Q$. Then $Q \subseteq [0, b)$. Let $c \in [0, b)$. Then $c < b$. But $b =$ sup Q, so there exists $d \in Q$ such that $c < d < b$. Then $c \in [0, d] = d \oplus d \subseteq Q$. Hence $Q = [0, b)$. □

Proposition 2.16. *Let* $(A, \oplus, \cdot)$ *be a hyperring where* $A = [a, \infty) \cup \{0\}$ *or* $(a, \infty) \cup \{0\}, a \geq 1$ *and*

$$x \oplus 0 = 0 \oplus x = \{x\} \quad \text{for all } x \in A,$$

$$x \oplus x = [x, \infty) \cup \{0\} \quad \text{for all } x \in A \backslash \{0\} \text{ and}$$

$$x \oplus y = \{\min\{x, y\}\} \text{ for all } x, y \in A \backslash \{0\} \text{ with } x \neq y.$$

Then

$$\{\{0\}\} \cup \{[b, \infty) \cup \{0\} \mid b \in A \backslash \{0\}\} \cup \{(b, \infty) \cup \{0\} \mid b \in A \backslash \{0\}\}$$

is the set of all quasi-hyperideals of $(A, \oplus, \cdot)$.

Proof. Clearly, $\{0\}, [b, \infty) \cup \{0\}$ and $(b, \infty) \cup \{0\}$ are subhyperrings of A for all $b \in A \backslash \{0\}$. Since $a \geq 1$, we have $A[b, \infty) \subseteq ([1, \infty) \cup \{0\}) [b, \infty) = [b, \infty) \cup \{0\}$ and $A(b, \infty) \subseteq ([1, \infty) \cup \{0\}) (b, \infty) = (b, \infty) \cup \{0\}$ for all $b \in A \backslash \{0\}$. We also have $A\{0\} = \{0\}$. Hence $\{0\}, [b, \infty) \cup \{0\}$ and $(b, \infty) \cup \{0\}$ are quasi-hyperideals of $(A, \oplus, \cdot)$ for all $b \in A \backslash \{0\}$.

For the reverse inclusion, let Q be a nonzero quasi-hyperideal of $(A, \oplus, \cdot)$. Then $0 \in Q$. Since $Q \subseteq A \subseteq [1, \infty) \cup \{0\}$, inf $(Q \backslash \{0\})$ exists in $\mathbb{R}$, say b.

Case 1: $b \in Q$. Then $Q \subseteq [b, \infty) \cup \{0\}$. But $b \oplus b = [b, \infty) \cup \{0\} \subseteq Q$, so $Q = [b, \infty) \cup \{0\}$.

Case 2: $b \notin Q$. Then $Q \subseteq (b, \infty) \cup \{0\}$. Let $c \in (b, \infty)$. Then $b < c$. But $b = \inf\, (Q \backslash \{0\})$, so there exists $d \in Q \backslash \{0\}$ such that $b < d < c$. Thus $c \in [d, \infty) \subseteq [d, \infty) \cup \{0\} = d \oplus d \subseteq Q$. Hence $Q = (b, \infty) \cup \{0\}$. □

Proposition 2.17. *Let* $(A, \oplus, \cdot)$ *be a hyperring where* $A = [-a, a]$ *or* $(-a, a)$, $0 < a \leq 1$ *and*

$$x \oplus x = \{x\} \qquad \text{for all } x \in A,$$

$$x \oplus y = y \oplus x = \{x\} \quad \text{for all } x, y \in A \text{ with } |y| < |x| \text{ and}$$

$$x \oplus (-x) = [-|x|, |x|] \quad \text{for all } x \in A.$$

Then

$$\{[-b, b] \mid b \in A \text{ and } b \geq 0\} \cup \{(-b, b) \mid b \in A \text{ and } b > 0\}$$

is the set of all quasi-hyperideals of $(A, \oplus, \cdot)$.

Proof. Clearly, $[-b, b]$ and $(-b, b)$ are subhyperrings of $(A, \oplus, \cdot)$ for every $b \in A$ such that $b > 0$. Since $0 < a \leq 1$, we have $A[-b, b] \subseteq [-1, 1][-b, b] = [-b, b]$

and $A(-b,b) \subseteq [-1,1](-b,b) = (-b,b)$ for every $b \in A$ with $b > 0$. We also have $A\{0\} = \{0\}$. Hence $\{0\}, [-b,b]$ and $(-b,b)$ are quasi-hyperideals of $(A, \oplus, \cdot)$ for all $b \in A$ with $b > 0$.

Conversely, let Q be a nonzero quasi-hyperideal of $(A, \oplus, \cdot)$. Since $Q \subseteq A \subseteq [-1,1]$, sup Q exists in $\mathbb{R}$, say b. Since for every $x \in A$, $-x$ is the inverse of x in $(A, \oplus)$ and Q is a canonical subhypergroup of $(A, \oplus)$, we deduce that

$$\text{for } x \in A,\ x \in Q \Leftrightarrow -x \in Q. \tag{1}$$

From (1), we have

$$b > 0,\ \ b \in Q \Rightarrow Q \subseteq [-b,b] \quad \text{and} \quad b \notin Q \Rightarrow Q \subseteq (-b,b). \tag{2}$$

Case 1: $b \in Q$. From (1) and (2), $-b \in Q$ and $Q \subseteq [-b,b]$, respectively. But $b \oplus (-b) = [-b,b]$, so we have $[-b,b] \subseteq Q$. Hence $Q = [-b,b]$.

Case 2: $b \notin Q$. By (2), $Q \subseteq (-b,b)$. Let $c \in (-b,b)$. Then there exists $x \in Q$ such that $|c| < x < b$ since $b = \sup Q$. Thus $-x \in Q$ and $c \in [-x,x] = x \oplus (-x) \subseteq Q$. Hence we have $Q = (-b,b)$. □

The following theorem shows that there are noncommutative hyperrings which are not division hyperrings and its quasi-hyperideals are also determined later.

Theorem 2.18. *Let $(A, +, \cdot)$ be a ring. Define a relation ρ on $(A, \cdot)$ by*

$$x\ \rho\ y \ \Leftrightarrow\ y = x \text{ or } y = -x \quad \text{for all } x, y \in A.$$

Then ρ is a congruence on $(A, \cdot)$. Define a hyperoperation $\oplus$ on A/ρ by

$$x\rho \oplus y\rho = \{(x+y)\rho, (x-y)\rho\} \quad \text{for all } x, y \in A.$$

Then $(A/\rho, \oplus, \circ)$ is a hyperring where $\circ$ is the usual multiplication on A/ρ.

Proof. Clearly, ρ is an equivalence relation on A. First, we shall show that $(A/\rho, \oplus)$ is a canonical hypergroup. By [1], page 11, we have that $(A/\rho, \oplus)$ is a hypergroup. Since $(A, +)$ is an abelian group, $(A/\rho, \oplus)$ is commutative. Let $x, y \in A$. Then we have

$$0\rho \oplus x\rho = \{x\rho, (-x)\rho\} = \{x\rho\},$$

$$0\rho \in \{(2x)\rho, 0\rho\} = \{(x+x)\rho, (x-x)\rho\} = x\rho \oplus x\rho.$$

Assume that $0\rho \in x\rho \oplus y\rho$. But $x\rho \oplus y\rho = \{(x+y)\rho, (x-y)\rho\}$, thus $0\rho = (x+y)\rho$ or $0\rho = (x-y)\rho$. This implies that

$$0 = x + y,\ 0 = -(x+y),\ 0 = x - y \ \text{ or } \ 0 = -(x-y).$$

and hence $y = x$ or $y = -x$ which yields $y\rho = x\rho$. These prove that 0ρ is the scalar identity of $(A/\rho, \oplus)$ and for every $x \in A, x\rho$ is the unique inverse of $x\rho$.

To show that $(A/\rho, \oplus)$ is reversible, let $x, y, z \in A$ be such that $x\rho \in y\rho \oplus z\rho$. Then $x\rho \in \{(y+z)\rho, (y-z)\rho\}$, so we have $x \in \{y+z, -(y+z), y-z, -(y-z)\}$. Consequently, $y \in \{x-z, -(x+z), x+z, -(x-z)\}$. Since $a\rho = (-a)\rho$ for all $a \in A$, we obtain

$$y\rho \in \{(x-z)\rho, (-(x+z))\rho, (x+z)\rho, (-(x-z))\rho\}$$

$$= \{(x-z)\rho, (x+z)\rho\} = x\rho \oplus z\rho.$$

If $x, y, z \in A$ are such that $x\ \rho\ y$, then $x = y$ or $x = -y$ which implies that (i) $zx = zy$ or $zx = -zy$ and (ii) $xz = yz$ or $xz = -yz$, $zx\ \rho\ zy$ and $xz\ \rho\ yz$. Thus ρ is a congruence on $(A, \cdot)$.

Since 0 is the zero of $(A, \cdot)$, 0ρ is the zero of $(A/\rho, \circ)$ where $\circ$ is the operation on the quotient semigroup of $(A, \cdot)$ relative to ρ.

Finally, to show that $\circ$ is distributive over $\oplus$, let $x, y, z \in A$. Then

$$\begin{aligned}(x\rho \oplus y\rho) \circ z\rho &= \{(x+y)\rho, (x-y)\rho\} \circ z\rho \\ &= \{(x+y)\rho \circ z\rho, (x-y)\rho \circ z\rho\} \\ &= \{((x+y)z)\rho, ((x-y)z)\rho\} \\ &= \{(xz+yz)\rho, (xz-yz)\rho\} \\ &= (xz)\rho \oplus (yz)\rho \\ &= (x\rho \circ z\rho) \oplus (y\rho \circ z\rho).\end{aligned}$$

Hence $(A/\rho, \oplus, \circ)$ is a hyperring, as required. □

Example 2.19. Let $n \in \mathbb{N}\backslash\{1\}, A$ a ring with identity $1 \neq 0$ and $M_n(A)$ the ring of all $n \times n$ matrices over A with the usual addition and multiplication of matrices. Define the equivalence relation ρ on $M_n(A)$ by

$$C \ \rho \ D \Leftrightarrow C = D \text{ or } C = -D.$$

By Proposition 2.18, $(M_n(A)/\rho, \oplus, \circ)$ is a hyperring where

$$\begin{aligned}C\rho \oplus D\rho &= \{(C+D)\rho, (C-D)\rho\}, \\ C\rho \circ D\rho &= (CD)\rho \quad \text{for all } C, D \in M_n(A).\end{aligned}$$

Let $E, F \in M_n(A)$ be defined by

$$E = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

Then $(E\rho) \circ (F\rho) = (EF)\rho = F\rho \neq [0]\rho = (FE)\rho = (F\rho) \circ (E\rho)$ where $[0]$ denotes the zero matrix in $M_n(A)$. This shows that $(M_n(A)/\rho, \oplus, \circ)$ is neither a commutative hyperring nor a division hyperring.

Next, we consider a relation between quasi-ideals in any ring $(A, +, \cdot)$ and quasi-hyperideals in the hyperring $(A/\rho, \oplus, \circ)$ defined as above. We first prove the following lemma.

Lemma 2.20. *Let $(A/\rho, \oplus, \circ)$ be the hyperring defined from a ring $(A, +, \cdot)$ as in Theorem 2.18. If S is a subring of $(A, +, \cdot)$, then $\{x\rho \mid x \in S\}$ is a subhyperring of the hyperring $(A/\rho, \oplus, \circ)$.*

Proof. Recall that

$$x\rho = \{x, -x\},$$

$$x\rho \oplus y\rho = \{(x+y)\rho, (x-y)\rho\},$$

$$x\rho \circ y\rho = (xy)\rho \quad \text{for all} \quad x, y \in A.$$

Let S be a subring of $(A, +, \cdot)$ and let $S' = \{x\rho \mid x \in S\}$. Since for $x, y \in S$, $x+y$ and $x-y$ are in S, it follows that $x\rho \oplus y\rho = \{(x+y)\rho, (x-y)\rho\} \subseteq S'$. Also, $0\rho \in S'$ since $0 \in S$ and for every $x \in A$, $x\rho$ is the inverse of $x\rho$ in $(A/\rho, \oplus)$. By Propositon 1.20, S' is a canonical subhypergroup of $(A/\rho, \oplus)$. Since $xy \in S$ for all $x, y \in S$, we have $x\rho \circ y\rho = (xy)\rho \in S'$ for all $x, y \in S$.

This proves that S' is a subhyperring of $(A/\rho, \oplus, \circ)$, as desired. □

Theorem 2.21. *Let $(A/\rho, \oplus, \circ)$ be a hyperring in Theorem 2.18. For each quasi-ideal Q of A, let $Q' = \{x\rho \mid x \in Q\}$. Then the map $Q \mapsto Q'$ is a bijection from the set of all quasi-ideals in $(A, +, \cdot)$ onto the set of all quasi-hyperideals in $(A/\rho, \oplus, \circ)$.*

Proof. Let Q be a quasi-ideal of A. We shall show that Q' is a quasi-hyperideal of A/ρ. By Lemma 2.20, Q' is a subhyperring of $(A/\rho, \oplus, \circ)$. To show that

$< (A/\rho) \circ Q' > \cap < Q' \circ (A/\rho) > \subseteq Q'$, let $x \in A$ be such that $x\rho \in < (A/\rho) \circ Q' >$ $\cap < Q' \circ (A/\rho) >$. Then

$$x\rho \in (x_1\rho) \circ (q_1\rho) \oplus (x_2\rho) \circ (q_2\rho) \oplus \cdots \oplus (x_n\rho) \circ (q_n\rho)$$

and

$$x\rho \in (p_1\rho) \circ (y_1\rho) \oplus (p_2\rho) \circ (y_2\rho) \oplus \cdots \oplus (p_m\rho) \circ (y_m\rho)$$

for some $x_1, \ldots, x_n, y_1, \ldots, y_m \in A$ and $q_1, \ldots, q_n, p_1, \ldots, p_m \in Q$. Then we have

$$x\rho \in (x_1q_1)\rho \oplus (x_2q_2)\rho \oplus \cdots \oplus (x_nq_n)\rho$$

and

$$x\rho \in (p_1y_1)\rho \oplus (p_2y_2)\rho \oplus \cdots \oplus (p_my_m)\rho.$$

Hence

$$x\rho = (x_1'q_1 + x_2'q_2 + \cdots + x_n'q_n)\rho = (p_1y_1' + p_2y_2' + \cdots + p_my_m')\rho$$

for some $x_i' \in \{x_i, -x_i\}$ and $y_j' \in \{y_j, -y_j\}$ for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$. These imply that

$$\begin{aligned}
x &\in \{x_1'q_1 + x_2'q_2 + \cdots + x_n'q_n, -(x_1'q_1 + x_2'q_2 + \cdots + x_n'q_n)\} \cap \\
&\quad \{p_1y_1' + p_2y_2' + \cdots + p_my_m', -(p_1y_1' + p_2y_2' + \cdots + p_my_m')\} \\
&= \{x_1'q_1 + x_2'q_2 + \cdots + x_n'q_n, (-x_1')q_1 + (-x_2')q_2 + \cdots + (-x_n')q_n\} \cap \\
&\quad \{p_1y_1' + p_2y_2' + \cdots + p_my_m', p_1(-y_1') + p_2(-y_2') + \cdots + p_m(-y_m')\} \\
&\subseteq < AQ > \cap < QA > \subseteq Q.
\end{aligned}$$

Hence $x\rho \in Q'$. This proves that Q' is a quasi-hyperideal of A/ρ.

Let Q_1 and Q_2 be quasi-ideals of $(A, +, \cdot)$ such that $Q_1' = Q_2'$. Then $\{x\rho \mid x \in Q_1\}$ $= \{x\rho \mid x \in Q_2\}$. To show that $Q_1 = Q_2$, let $a \in Q_1$. Then $a\rho \in Q_1' = Q_2'$, so $a\rho = b\rho$ for some $b \in Q_2$. Hence $a \in a\rho = b\rho = \{b, -b\} \subseteq Q_2$ since $(Q_2, +)$

is a subgroup of $(A,+)$ and $b \in Q_2$. Therefore $Q_1 \subseteq Q_2$. We obtain $Q_2 \subseteq Q_1$ similarly. Hence $Q_1 = Q_2$. Therefore the given map is one-to-one.

To show that the map is onto, let P be a quasi-hyperideal of $(A/\rho, \oplus, \circ)$. Let $Q = \{x \in A \mid x\rho \in P\}$. If $x, y \in Q$, then $x\rho, y\rho \in P$ and hence $(x-y)\rho \in x\rho \oplus y\rho \subseteq P$ which implies $x - y \in Q$. Hence Q is a subgroup of $(A,+)$. Next, let $x \in < AQ > \cap < QA >$. Then $x = x_1q_1 + x_2q_2 + \cdots + x_nq_n = p_1y_1 + p_2y_2 + \cdots + p_my_m$ for some $x_1, \ldots, x_n, y_1, \ldots, y_m \in A$ and $q_1, \ldots, q_n, p_1, \ldots, p_m \in Q$. Therefore $q_i\rho, p_j\rho \in P$ for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$, so

$$\begin{aligned}
x\rho &= (x_1q_1 + x_2q_2 + \cdots + x_nq_n)\rho = (p_1y_1 + p_2y_2 + \cdots + p_my_m)\rho \\
&\in \big((x_1q_1)\rho \oplus (x_2q_2)\rho \oplus \cdots \oplus (x_nq_n)\rho\big) \cap \\
&\quad \big((p_1y_1)\rho \oplus (p_2y_2)\rho \oplus \cdots \oplus (p_my_m)\rho\big) \\
&= \big((x_1\rho) \circ (q_1\rho) \oplus (x_2\rho) \circ (q_2\rho) \oplus \cdots \oplus (x_n\rho) \circ (q_n\rho)\big) \cap \\
&\quad \big((p_1\rho) \circ (y_1\rho) \oplus (p_2\rho) \circ (y_2\rho) \oplus \cdots \oplus (p_m\rho) \circ (y_m\rho)\big) \\
&\subseteq < (A/\rho) \circ P > \cap < P \circ (A/\rho) > \subseteq P
\end{aligned}$$

which implies that $x \in Q$. Hence Q is a quasi-ideal of A. Clearly, $Q' = P$.

Hence the theorem is proved. □