CHAPTER I

INTRODUCTION AND PRELIMINARIES

A *congruence* on a semigroup S is an equivalence relation ρ on S such that for all $x, y, z \in S$, $x \,\rho\, y$ implies $xz \,\rho\, yz$ and $zx \,\rho\, zy$. If ρ is a congruence on S, then S/ρ is a semigroup under the multiplication defined by $(x\rho)(y\rho) = (xy)\rho$ for all $x, y \in S$ which is called the *quotient semigroup of S relative to ρ*.

Let $\mathbb{N}, \mathbb{Z}$ and $\mathbb{R}$ denote respectively the set of natural numbers (positive integers), the set of integers and the set of real numbers.

For nonempty subsets X and Y of a ring A, let $\mathbb{Z}X$ and XY denote respectively the set of all finite sums of the form $\sum k_i x_i$ and the set of all finite sums of the form $\sum x_i y_i$ where $k_i \in \mathbb{Z}, x_i \in X$ and $y_i \in Y$. If X consists of a single element x, we write $\mathbb{Z}x$ and xY for $\mathbb{Z}X$ and XY, respectively. Similarly, if $Y = \{y\}$, we write Xy for XY. A *quasi-ideal* of a ring A is a subring Q of A such that $AQ \cap QA \subseteq Q$, and by a *bi-ideal* of A we mean a subring B of A such that $BAB \subseteq B$. Then a nonempty subset Q of $(A, +)$ is a quasi-ideal of A if Q is a subgroup of $(A, +)$ and $AQ \cap QA \subseteq Q$. Every one-sided ideal of a ring A is clearly a quasi-ideal and every quasi-ideal of A is a bi-ideal. The notion of quasi-ideal in rings was introduced by O. Steinfeld [9] in 1953 while the notion of bi-ideal was introduced much later. It was actually introduced by S. Lajos and F. Szász [6] in 1971. Note that if A is commutative, then the quasi-ideals and the ideals of A coincide.

Example 1.1. Let F be a field, $n \in \mathbb{N}$ and $M_n(F)$ the ring of all $n \times n$ matrices over F under the usual addition and multiplication of matrices. For $C \in M_n(F)$, let C_{ij} denote the entry of C in the i^{th} row and the j^{th} column. For $k, l \in \{1, 2, \ldots, n\}$, let $Q_n^{kl}(F)$ consist of all matrices $C \in M_n(F)$ such that

$$C_{ij} = 0 \quad \text{if} \quad i \neq k \quad \text{or} \quad j \neq l.$$

Then for $k, l \in \{1, 2, \ldots, n\}$, $Q_n^{kl}(F)$ is a subring of $M_n(F)$,

$$M_n(F)Q_n^{kl}(F) = \left\{ \begin{array}{c} l \\ \downarrow \\ \begin{bmatrix} 0 & \ldots & 0 & x_1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & x_2 & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & x_n & 0 & \ldots & 0 \end{bmatrix} \end{array} \;\middle|\; x_1, x_2, \ldots, x_n \in F \right\}$$

and

$$Q_n^{kl}(F)M_n(F) = \left\{ \begin{array}{c} \\ \\ \\ k \rightarrow \\ \\ \\ \\ \end{array} \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \\ x_1 & x_2 & \ldots & x_n \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \;\middle|\; x_1, x_2, \ldots, x_n \in F \right\}$$

which implies that $M_n(F)Q_n^{kl}(F) \cap Q_n^{kl}(F)M_n(F) = Q_n^{kl}(F)$, so $Q_n^{kl}(F)$ is a quasi-ideal of $M_n(F)$. Moreover, if $n > 1$, then for all $k, l \in \{1, 2, \ldots, n\}$, $Q_n^{kl}(F)$ is neither a left ideal nor a right ideal of $M_n(F)$.

Example 1.2. Let F be a field, $n \in \mathbb{N}, n \geq 4$ and $SU_n(F)$ the ring of all strictly upper triangular matrices over F under the usual addition and multiplication of matrices. Let

$$B = \left\{ \begin{bmatrix} 0 & \dots & 0 & x & 0 \\ 0 & \dots & 0 & 0 & y \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \;\middle|\; x,\ y \in F \right\}.$$

Then $B^2 = \{0\}$, so B is a subring of $SU_n(F)$. Moreover, $BSU_n(F)B = \{0\} \subseteq B$. But

$$\begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

$$\in (SU_n(F)B \cap BSU_n(F)) \setminus B,$$

so B is a bi-ideal but not a quasi-ideal of $SU_n(F)$.

Example 1.1 shows that quasi-ideals of rings are a generalization of one-sided ideals. It is shown in Example 1.2 that bi-ideals of rings generalize quasi-ideals.

It is well-known in rings that if A is not a zero ring, then A is a division ring if and only if A and $\{0\}$ are only left [right] ideals of A. This is also true if "left [right] ideals" is replaced by "quasi-ideals".

Proposition 1.3. ([10], page 6). *Let A be a ring such that $A^2 \neq \{0\}$. Then A is a division ring if and only if A and $\{0\}$ are the only quasi-ideals of A.*

We also have

Proposition 1.4. ([10], page 10 and 12). *Let A be a ring. Then:*

(i) *The intersection of a set of quasi-ideals of A is a quasi-ideal of A.*

(ii) *The intersection of a set of bi-ideals of A is a bi-ideal of A.*

For a subset X of a ring A, let $(X)_q$ and $(X)_b$ denote the intersection of all quasi-ideals of A containing X and the intersection of all bi-ideals of A containing X, respectively. Then for $X \subseteq A$, $(X)_q$ $[(X)_b]$ is the smallest quasi-ideal [bi-ideal] of A containing X. Since every quasi-ideal of A is a bi-ideal, we have $(X)_b \subseteq (X)_q$ for every subset X of A.

Proposition 1.5. (H. J. Wilnert [12]). *For a nonempty subset X of a ring A,*

$$(X)_q = \mathbb{Z}X + (AX \cap XA).$$

Proposition 1.6. (S. Lajos and F. Szász [6]). *For a nonempty subset X of a ring A,*

$$(X)_b = \mathbb{Z}X + \mathbb{Z}X^2 + XAX.$$

A ring A is said to be a (Von Neumann) *regular ring* if for every $x \in A, x = xyx$ for some $y \in A$. These two facts are known.

Proposition 1.7. ([10], page 69). *A ring A is regular if and only if $QAQ = Q$ for every quasi-ideal Q of A.*

It is clearly seen that the intersection of a left ideal and a right ideal of a ring A is a quasi-ideal. However, a quasi-ideal of A may not be obtained in this way. See [10], page 8, [7] and [3], for examples. A quasi-ideal Q of A is said to have *the intersection property* if $Q = L \cap R$ for some left ideal L and right ideal R of A, and we say that A *has the intersection property of quasi-ideals* if every quasi-ideal of A has the intersection property. It is known that a ring with a one-sided identity has the intersection property of quasi-ideals. This is a special case of the following proposition.

Proposition 1.8. ([10], page 9) *Let Q be a quasi-ideal of a ring A. If $Q \subseteq AQ$ or $Q \subseteq QA$, then*

$$Q = (Q + AQ) \cap (Q + QA).$$

In this case, Q has the intersection property (since $Q + AQ$ and $Q + QA$ are a left ideal and a right ideal of A, respectively.)

H. J. Wilnert [12] and Z. Moucheng and etc. [7] characterized quasi-ideals in rings having the intersection property as follows:

Proposition 1.9. (H. J. Wilnert [12]). *Let Q be a quasi-ideal of a ring A. Then the following statements are equivalent.*

(i) *Q has the intersection property.*

(ii) $(Q + AQ) \cap (Q + QA) = Q$.

(iii) $AQ \cap (Q + QA) \subseteq Q$.

(iv) $QA \cap (Q + AQ) \subseteq Q$.

Proposition 1.10. (Z. Moucheng and etc. [7]). *Let X be a nonempty subset of a ring A. Then the following statements are equivalent.*

(i) *$(X)_q$ has the intersection property.*

(ii) $(\mathbb{Z}X + AX) \cap (\mathbb{Z}X + XA) = (X)_q$.

(iii) $AX \cap (\mathbb{Z}X + XA) \subseteq (X)_q$.

(iv) $XA \cap (\mathbb{Z}X + AX) \subseteq (X)_q$.

Z. Moucheng and etc. [7] also characterized rings having the intersection property of quasi-ideals.

Proposition 1.11. (Z. Moucheng and etc. [7]). *The following statements for a ring A are equivalent.*

(i) *A has the intersection property of quasi-ideals.*

(ii) *For any finite nonempty subset X of A,*

$$AX \cap (\mathbb{Z}X + XA) \subseteq \mathbb{Z}X + (AX \cap XA)\,(= (X)_q)\,.$$

(iii) *For any finite subset $X = \{x_1, x_2, \ldots, x_n\}$ of A and $a_1, a_2, \ldots, a_n \in A$, if*

$$\sum_{i=1}^{n}(a_i x_i + k_i x_i + x_i a_i') = 0,$$

for some $a_i' \in A$ and $k_i \in \mathbb{Z}$, then $\sum_{i=1}^{n} a_i x_i \in (X)_q$.

K. M. Kapp gave a nice proof of the following result in [2].

Proposition 1.12. (K. M. Kapp [2]). *Let B be a bi-ideal of a ring A. If every element of B is regular in A, then B is a quasi-ideal of A.*

To be convenient, let us call a ring A a *BQ-ring* if the bi-ideals and the quasi-ideals of A coincide, that is, every bi-ideal of A is a quasi-ideal. Then from Proposition 1.12, we have

Proposition 1.13. *Every regular ring is a BQ-ring.*

The next proposition gives a necessary and sufficient condition of a ring to be a BQ-ring.

Proposition 1.14. ([10], page 77). *A ring A is a BQ-ring if and only if for every finite subset X of A, $(X)_b = (X)_q$.*

We call a nonzero quasi-ideal Q of a ring A a *minimal quasi-ideal* of A if Q does not properly contain a nonzero quasi-ideal of A. The following fact is clearly true.

Proposition 1.15. *A nonzero quasi-ideal Q of a ring A is a minimal quasi-ideal of A if and only if $(x)_q = Q$ for all $x \in Q\backslash\{0\}$.*

A *minimal left [right] ideal* of a ring A is a nonzero left [right] ideal of A which does not properly contain a nonzero left [right] ideal of A. There is a relation among minimal quasi-ideals, minimal left ideals and minimal right ideals as follows:

Proposition 1.16. ([10], page 34). *If L and R are a minimal left ideal and a minimal right ideal of a ring A, respectively, then either $L \cap R = \{0\}$ or $L \cap R$ is a minimal quasi-ideal of A.*

Necessary conditions and a partial converse for a quasi-ideal of a ring A to be minimal are as follows:

Proposition 1.17. ([10], pages 35 and 37). *Let Q be a quasi-ideal of a ring A.*

(i) *If Q is a minimal quasi-ideal of A, then Q is either a zero ring or a division subring of A. In the second case, $Q = eAe = Ae \cap eA$ where e is the identity of Q.*

(ii) *If Q is a division subring of A, then Q is a minimal quasi-ideal of A.*

Recall that for an element x of a ring A, the principal left [right] ideal of A generated by x is $\mathbb{Z}x + Ax$ $[\mathbb{Z}x + xA]$ which is denoted by $(x)_l$ $[(x)_r]$. P. N. Stewart [11] gave a necessary and sufficient condition for a quasi-ideal of a ring to be minimal in terms of principal left ideals and principal right ideals as follows:

Proposition 1.18. (P. N. Stewart [11]). *A quasi-ideal Q of a ring A is a minimal quasi-ideal of A if and only if for any two nonzero elements x, y in Q,*

$$(x)_l = (y)_l \quad \textit{and} \quad (x)_r = (y)_r.$$

Next, we shall give the definitions of Krasner hyperrings and their left [right] hyperideals, quasi-hyperideals, bi-hyperideals, etc. accordingly as in rings.

For a set X, let $P(X)$ denote the power set of X and let $P^*(X) = P(X)\backslash\{\varnothing\}$.

A *hyperoperation* on a nonempty set H is a mapping of $H \times H$ into $P^*(H)$. A *hypergroupoid* is a system $(H, \circ)$ consisting of a nonempty set H and a hyperoperation $\circ$ on H.

Let $(H, \circ)$ be a hypergroupoid. For nonempty subsets X, Y of H, let

$$X \circ Y = \bigcup_{\substack{x \in X \\ y \in Y}} (x \circ y)$$

and let $X \circ y = X \circ \{y\}$ and $y \circ X = \{y\} \circ X$ for all $y \in H$. An element e of H is called an *identity* of $(H, \circ)$ if $x \in (x \circ e) \cap (e \circ x)$ for all $x \in H$. An element e of H is called a *scalar identity* of $(H, \circ)$ if $x \circ e = e \circ x = \{x\}$ for all $x \in H$. If e is a scalar identity of $(H, \circ)$, then e is a unique identity of $(H, \circ)$.

A hypergroupoid $(H, \circ)$ is said to be *commutative* if $x \circ y = y \circ x$ for all $x, y \in H$.

A *semihypergroup* is a hypergroupoid $(H, \circ)$ such that $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$. A *hypergroup* is a semihypergroup $(H, \circ)$ such that $x \circ H = H \circ x = H$ for all $x \in H$.

An element x in a semihypergroup $(H, \circ)$ is said to be an *inverse* of an element y in $(H, \circ)$ if there exists an identity e of $(H, \circ)$ such that $e \in (x \circ y) \cap (y \circ x)$, that is, $(x \circ y) \cap (y \circ x)$ contains at least one identity of $(H, \circ)$.

A hypergroup $(H, \circ)$ is called *regular* if every element of H has at least one inverse in $(H, \circ)$. A regular hypergroup $(H, \circ)$ is said to be *reversible* if for

$x, y, z \in H$, $x \in y \circ z$ implies $z \in u \circ x$ and $y \in x \circ v$ for some inverse u of y and some inverse v of z.

A *canonical hypergroup* is a commutative reversible hypegroup $(H, \circ)$ such that $(H, \circ)$ has a scalar identity and every element of H has a unique inverse in $(H, \circ)$. Hence a hypergroup $(H, \circ)$ is a canonical hypergroup if and only if

1. $(H, \circ)$ is commutative,
2. $(H, \circ)$ has a scalar identity,
3. every element x of H has a unique inverse x^{-1} in $(H, \circ)$ and
4. for $x, y, z \in H$, $x \in y \circ z$ implies $z \in y^{-1} \circ x$.

For a nonempty subset X of H, let X^{-1} denote the set $\{x^{-1} \mid x \in X\}$ in the canonical hypergroup $(H, \circ)$. The following proposition shows an analogous property between canonical hypergroups and abelian groups.

Proposition 1.19. ([1], page 98). *If $(H, \circ)$ is a canonical hypergroup and $x_1, x_2, \ldots, x_n \in H$, then $(x_1 \circ x_2 \circ \cdots \circ x_n)^{-1} = x_1^{-1} \circ x_2^{-1} \circ \cdots \circ x_n^{-1}$.*

By a *canonical subhypergroup* of a canonical hypergroup $(H, \circ)$ we mean a subset H_1 which is a canonical hypergroup under the hyperoperation $\circ$ of H restricted to H_1. By the definition of canonical hypergroups, we clearly have

Proposition 1.20. *A nonempty subset H_1 of a canonical hypergroup $(H, \circ)$ is a canonical subhypergroup of $(H, \circ)$ if and only if*

1. *$x \circ y \subseteq H_1$ for all $x, y \in H_1$,*
2. *$e \in H_1$ where e is the scalar identity of $(H, \circ)$ and*
3. *for every $x \in H_1$, $x^{-1} \in H_1$.*

Note that if H_1 satisfies 1.– 3. of Proposition 1.20, then for every $x, y \in H_1$, $y \in e \circ y \subseteq x \circ x^{-1} \circ y = x \circ (x^{-1} \circ y) \subseteq x \circ H_1$ which implies that $x \circ H_1 = H_1$

for all $x \in H_1$. As a consequence of Proposition 1.20, the following proposition is clearly obtained.

Proposition 1.21. *The intersection of a collection of canonical subhypergroups of a hypergroup* $(H, \circ)$ *is a canonical subhypergroup of* $(H, \circ)$.

By a *Krasner hyperring* we mean a system $(A, +, \cdot)$ where

1. $(A, +)$ is a canonical hypergroup,
2. $(A, \cdot)$ is a semigroup with zero 0 where 0 is the scalar identity of $(A, +)$ and
3. $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$ for all $x, y, z \in A$.

The hyperoperation $+$ and the operation $\cdot$ of a hyperring $(A, +, \cdot)$ are called the *addition* and the *multiplication* of A, respectively. We shall write A instead of $(A, +, \cdot)$ when there is no danger of ambiguity. Hence every ring is a Krasner hyperring. It can be seen later that there are many Krasner hyperrings which are not rings. Hence Krasner hyperrings are a generalization of rings.

A Krasner hyperring $(A, +, \cdot)$ is said to be *commutative* if $(A, \cdot)$ is a commutative semigroup. An element $1 \in A$ is called an *identity* of $(A, +, \cdot)$ if 1 is the identity of the semigroup $(A, \cdot)$. A Krasner hyperring A is called a *zero Krasner hyperring* if $x \cdot y = 0$ for all $x, y \in A$.

Let $(A, +, \cdot)$ be a Krasner hyperring. The scalar identity of the canonical hypergroup $(A, +)$ which is the zero of the semigroup $(A, \cdot)$ is called the *zero* of $(A, +, \cdot)$ and it is usually denoted by 0. For $x, y \in A$, let $-x$ denote the unique inverse of x in the canonical hypergroup $(A, +)$ which is called the *additive inverse* of x in $(A, +, \cdot)$ and let xy and $x - y$ denote $x \cdot y$ and $x + (-y)$, respectively. Due to the definition of a Krasner hyperring above, $x \cdot (y + z)$ and $(y + z) \cdot x$ in 3. mean $\{x \cdot t \mid t \in y + z\}$ and $\{t \cdot x \mid t \in y + z\}$, respectively. Moreover, for $x, y, u, v \in A$, one uses $(x + y) \cdot (u + v)$ to denote the set $\{s \cdot t \mid s \in x + y \text{ and } t \in u + v\}$. Because of these facts, we cannot use the notation XY in the similar meaning as in rings where X and Y are nonempty subsets of a Krasner hyperring A. To distinguish between the following two subsets of a Krasner hyperring A: $\{xy \mid x \in X \text{ and } y \in Y\}$ and the union of all finite sums of the form $\sum x_i y_i$ where $x_i \in X$ and $y_i \in Y$ for all nonempty subsets X, Y of A, we shall let XY and $< XY >$ denote the first set and the second one, respectively, that is,

$$\begin{aligned} XY &= \{xy \mid x \in X \text{ and } y \in Y\}, \\ <XY> &= \bigcup_{\substack{x_i \in X, y_i \in Y \\ n \in \mathbb{N}}} (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n). \end{aligned}$$

Due to our proofs in some theorems, we generally let $< X_1 X_2 \ldots X_n >$ denote the union of all finite sum of the form $\sum x_i^{(1)} x_i^{(2)} \ldots x_i^{(n)}$ where $x_i^{(1)} \in X_1, x_i^{(2)} \in X_2, \ldots, x_i^{(n)} \in X_n$ for any nonempty subsets $X_1, X_2, \ldots, X_n$ of A where $n \in \mathbb{N}$. Because of the given definition and the associative law of the multiplication, the notation $< X_1 X_2 \ldots X_n >$ is well-defined, not depending on parenthesis. For $n \in \mathbb{Z}$ and $x \in A$, nx is defined as follows:

$$nx = \begin{cases} x + x + \cdots + x \ (n \text{ copies}) & \text{if } n > 0, \\ \{0\} & \text{if } n = 0, \\ (-x) + (-x) + \cdots + (-x) \ (-n \text{ copies}) & \text{if } n < 0. \end{cases}$$

For a nonempty subset X of A, let $\mathbb{Z}X$ denote the union of all finite sums of the form $\sum n_i x_i$ where $n_i \in \mathbb{Z}$ and $x_i \in X$. For $a \in A$ and $\varnothing \neq X \subseteq A$, let Xa, aX and $\mathbb{Z}a$ denote $X\{a\}, \{a\}X$ and $\mathbb{Z}\{a\}$, respectively. Clearly, for $a, b \in A$,

$$< Aa >= \{xa \mid x \in A\} = Aa, \quad < aA >= \{ax \mid x \in A\} = aA,$$

$$< aAb >= \{axb \mid x \in A\} = aAb.$$

From the notations defined above, we easily obtain the following proposition.

Proposition 1.22. *In a Krasner hyperring* $(A, +, \cdot)$, *the following statements hold.*

1. $-0 = 0$.
2. $-(-x) = x$ *for all* $x \in A$.
3. $(-x)y = -(xy) = x(-y)$ *for all* $x, y \in A$ (from [1], page 167).
4. $(-x)(-y) = xy$ *for all* $x, y \in A$.
5. $(x + y)(z + v) \subseteq xz + xv + yz + yv$ *for all* $x, y, z, v \in A$ (from [1], page 167).
6. *For* $x, y, z \in A, x \in y + z \Rightarrow -x \in -y - z$ (from Proposition 1.19).
7. *For* $x \in A$ *and* $n \in \mathbb{Z}$, $-(nx) = (-n)x = n(-x)$ (from (2) and Proposition 1.19).
8. *For* $x, y \in A$ *and* $n \in \mathbb{Z}$, $x(ny) = n(xy) = (nx)y$ (from (3)).
9. *For nonempty subsets* X, Y, Z *of* A,

$$<< X >>=< X >,$$

$$< XY >=<< X >< Y >>,$$

$$< X < YZ >>=< XYZ >=<< XY > Z >,$$

$$< X > + < X >=< X > \quad \textit{if} \quad 0 \in X.$$

10. *For nonempty subsets* X, Y, Z, V *of* A, $< X + Y >< Z + V > \subseteq < XZ > + < XV > + < YZ > + < YV >$.
11. *For nonempty subsets* X, Y *of* A, $< X(\mathbb{Z}Y) >= \mathbb{Z}(XY) =< (\mathbb{Z}X)Y >$.

Some examples of Krasner hyperrings which are not rings are as follows:

Example 1.23. (Y. Punkla [8]). Define the hyperoperation $\oplus$ on $\mathbb{Z}_3$ as follows:

$\oplus$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{1\}$	$\mathbb{Z}_3$
2	$\{2\}$	$\mathbb{Z}_3$	$\{2\}$

Then $(\mathbb{Z}_3, \oplus, \cdot)$ is a Krasner hyperring where $\cdot$ is the usual multiplication on $\mathbb{Z}_3$. Observe that 0 is its zero and 1 is the additive inverse of 2 in this hyperring.

Example 1.24. (Y. Kemprasit [3]). Let $a \in \mathbb{R}$ be such that $0 < a \leq 1$ and $A = [0, a]$ or $[0, a)$. Define a hyperoperation $\oplus$ on A by

$$x \oplus y = \begin{cases} \{\max\{x, y\}\} & \text{if } x \neq y, \\ [0, x] & \text{if } x = y. \end{cases}$$

Then $(A, \oplus, \cdot)$ is a Krasner hyperring where $\cdot$ is the usual multiplication on A. In this Krasner hyperring, 0 is the zero and the additive inverse of $x \in A$ is x itself.

Example 1.25. (Y. Kemprasit [3]). Let $a \in \mathbb{R}$ be such that $a \geq 1$ and $A = [a, \infty) \cup \{0\}$ or $(a, \infty) \cup \{0\}$. Define a hyperoperation $\oplus$ on A by

$$\begin{aligned} &x \oplus 0 = 0 \oplus x = \{x\} && \text{for all } x \in A, \\ &x \oplus x = [x, \infty) \cup \{0\} && \text{for all } x \in A \backslash \{0\} \text{ and} \\ &x \oplus y = \{\min\{x, y\}\} && \text{for all } x, y \in A \backslash \{0\} \text{ with } x \neq y. \end{aligned}$$

Then $(A, \oplus, \cdot)$ is a Krasner hyperring where $\cdot$ is the usual multiplication on A. Note that in this Krasner hyperring, 0 is the zero and the additive inverse of $x \in A$ is x itself.

Example 1.26. (Y. Kemprasit [3]). Let $a \in \mathbb{R}$ be such that $0 < a \leq 1$ and $A = [-a, a]$ or $(-a, a)$. Define a hyperoperation $\oplus$ on A by

$$\begin{aligned} &x \oplus x = \{x\} && \text{for all } x \in A, \\ &x \oplus y = y \oplus x = \{x\} && \text{for all } x, y \in A \text{ with } |y| < |x| \text{ and} \\ &x \oplus (-x) = [-|x|, |x|] && \text{for all } x \in A. \end{aligned}$$

Then $(A, \oplus, \cdot)$ is a Krasner hyperring where $\cdot$ is the usual multiplication on A. Note that in this Krasner hyperring, 0 is its zero and the additive inverse of $x \in A$ is $-x$.

For a semigroup S, let S^0 be S if S has a zero and S contains more than one element, otherwise, let S^0 be the semigroup $S \cup \{0\}$ where $0 \notin S$ with the operation extended from S by defining $x0 = 0x = 0$ for all $x \in S \cup \{0\}$.

Example 1.27. ([1], page 170 and Y. Punkla [8]). Let G be a group and define a hyperoperation $+$ on G^0 by

$$
\begin{aligned}
x + 0 = 0 + x &= \{x\} && \text{for all } x \in G^0, \\
x + x &= G^0 \smallsetminus \{x\} && \text{for all } x \in G \text{ and} \\
x + y &= \{x, y\} && \text{for all } x, y \in G \text{ with } x \neq y.
\end{aligned}
$$

It is given in [1], page 170, that if G is an abelian group, then $(G^0, +, \cdot)$ is a hyperfield where $\cdot$ is the operation on G^0. A *hyperfield* is defined naturally to be a Krasner hyperring $(A, +, \cdot)$ such that $\big(A\backslash\{0\}, \cdot\big)$ is an abelian group. In fact, it was proved by Y. Punkla [8] that $(G^0, +, \cdot)$ is a Krasner hyperring without assuming the commutativity of the group G. In this case we also have that $(G^0, +, \cdot)$ is a division hyperring. By a *division hyperring* we mean a Krasner hyperring $(A, +, \cdot)$ such that $\big(A\backslash\{0\}, \cdot\big)$ is a group. In this hyperring, 0 is the zero and the additive inverse of $x \in G^0$ is x itself.

Example 1.28. ([1], page 170). Let $(A, +, \cdot)$ be a division ring and N a normal subgroup of $(A\backslash\{0\}, \cdot)$. Set

$$A_N = \{\ xN \mid x \in A\ \}.$$

Define a hyperoperation $\oplus$ and an operation $\circ$ on A_N by

$$xN \oplus yN = \{ \ tN \mid t \in xN + yN \ \},$$

$$xN \circ yN = xyN \text{ for all } x, y \in A.$$

Then $(A_N, \oplus, \circ)$ is a Krasner hyperring in which $0N(= \{0\})$ is the zero, N is the identity and the additive inverse of xN is $(-x)N$ where $-x$ is the additive inverse of x in the ring A. In fact, it is a division hyperring with $(xN)^{-1} = x^{-1}N$ for all $x \in A \backslash \{0\}$.

For a nonempty subset S of a Krasner hyperring $A = (A, +, \cdot)$, one says that S is a *subhyperring* of A if $(S, +)$ is a canonical subhypergroup of $(A, +)$ and $(S, \cdot)$ is a subsemigroup of $(A, \cdot)$. The following proposition is clearly obtained.

Proposition 1.29. *If S is a subhyperring of a Krasner hyperring A, then $\mathbb{Z}S = S$.*

By a *left* [*right*] *hyperideal* of a Krasner hyperring $A = (A, +, \cdot)$ we mean a subhyperring S of A such that $AS \subseteq S$ [$SA \subseteq S$]. If S is both a left and a right hyperideal of A, then it is called a *(two-sided)hyperideal* of A. A subhyperring Q of A is called a *quasi-hyperideal* of A if $< AQ > \cap < QA > \subseteq Q$. Then a nonempty subset Q of A is a quasi-hyperideal of A if Q is a canonical subhypergroup of $(A, +)$ and $< AQ > \cap < QA > \subseteq Q$. By a *bi-hyperideal* of A we mean a subhyperring B of A such that $< BAB > \subseteq B$. It then follows that a subhyperring B of A is a bi-hyperideal of A if and only if $BAB \subseteq B$. We also have quasi-hyperideals are a generalization of left hyperideals and right hyperideals and bi-hyperideals generalize quasi-hyperideals (see Example 1.1 and Example 1.2). Especially, quasi-hyperideals in Krasner hyperrings generalize quasi-ideals in rings. Note that if A is commutative, then the quasi-hyperideals and the hyperideals of A coincide. Similarly as in rings, an element $a \in A$ is said to be *regular* if $a = axa$ for some $x \in A$ and we call A a *regular hyperring* if every element of A is regular.

The intersection of a left hyperideal and a right hyperideal is clearly a quasi-hyperideal. As mentioned on page 4, it follows that a quasi-hyperideal may not be obtained in this way. A quasi-hyperideal of a Krasner hyperring A is said to have *the intersection property* if it is an intersection of a left-hyperideal and a right hyperideal of A and we say that A *has the intersection property of quasi-hyperideals* if every quasi-hyperideal of A has the intersection property.

Let $(A, +, \cdot)$ be a Krasner hyperring. A nonzero left hyperideal L of A is said to be *minimal* if L does not contain nonzero proper left hyperideals of A. A minimal right [two-sided, quasi-] hyperideal of A is defined similarly.

In Chapter II, Proposition 1.3 to Proposition 1.7 are generalized to quasi-hyperideals in Krasner hyperrings. All the qausi-hyperideals of the Krasner hyperrings in Example 1.23–Example 1.28 are also determined in this chapter. Observe that the given examples of Krasner hyperrings are either commutative Krasner hyperrings or division hyperrings. An example of a noncommutative Krasner hyperring which is not a division hyperring is given in this chapter and its quasi-hyperideals are also determined.

We generalize Proposition 1.8 – Proposition 1.11 to quasi-hyperideals in Krasner hyperrings in Chapter III. Krasner hyperrings whose bi-hyperideals and quasi-hyperideals coincide are studied in Chapter IV in order to generalize Proposition 1.12 – Proposition 1.14.

Finally, minimal quasi-hyperideals of Krasner hyperrings are studied in the last chapter. Our purpose of this chapter is to provide some generalizations of Proposition 1.15 – Proposition 1.18.

To be convenient, Krasner hyperrings are simply called *hyperrings* in the remainder of this thesis.