สมการเชิงฟังก์ชันค่าเฉลี่ยศูนย์บนรูป n เหลี่ยมในระนาบ

นางสาวจันทร์นภา อุตตะมะ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ประยุกต์และวิทยาการคณนา
ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2555

ZERO-MEAN FUNCTIONAL EQUATION ON PLANAR N-GONS

Miss Jannapa Uttama

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Applied Mathematics and
Computational Science

Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University

Academic Year 2012

Thesis Title	Zero-Mean Functional Equation On Planar N-gons
By	Miss Jannapa Uttama
Field of Study	Applied Mathematics and Computational Science
Thesis Advisor	Associate Professor Paisan Nakmahachalasint, Ph.D.
Thesis Co-advisor	Assistant Professor Nataphan Kitisin, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

....Dean of the Faculty of Science
(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

....Chairman
(Khamron Mekchay, Ph.D.)

....Thesis Advisor
(Associate Professor Paisan Nakmahachalasint, Ph.D.)

....Thesis Co-advisor
(Assistant Professor Nataphan Kitisin, Ph.D.)

....Examiner
(Associate Professor Patanee Udomkavanich, Ph.D.)

....External Examiner
(Charinthip Hengkrawit, Ph.D.)

จันทร์นภา อุตตะมะ : สมการเชิงฟังก์ชันค่าเฉลี่ยศูนย์บนรูป n เหลี่ยมในระนาบ. (ZERO-MEAN FUNCTIONAL EQUATION ON PLANAR N-GONS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก: รศ.ดร.ไพศาล นาคมหาชลาสินธุ์, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม: ผศ.ดร.ณัฐพันธ์ กิติสิน, 21 หน้า.

ในงานวิจัยนี้ผู้เขียนจะทำการหาผลเฉลยทั่วไปของสมการเชิงฟังก์ชันค่าเฉลี่ยศูนย์บนรูป n เหลี่ยมในระนาบ โดยมีข้อกำหนดว่า แต่ละรูป n เหลี่ยมซึ่งเกิดจากการเลื่อน หรือขยายจากรูป n เหลี่ยมที่ถูกกำหนดมาแล้วรูปหนึ่ง ค่าฟังก์ชันที่จุดยอดของรูป n เหลี่ยมรวมกันมีค่าเท่ากับศูนย์ สมการเชิงฟังก์ชันดังกล่าวจะอยู่ในรูป

$$\sum_{j=1}^{n} f(z+\lambda a_j)=0,$$

สำหรับทุก $z\in\mathbb{C}$ และ $\lambda\in\mathbb{R},\ \lambda>0$ เมื่อ $a_j's, 1\le j\le n$ เป็นจุดคงที่บนระนาบ

ภาควิชา	คณิตศาสตร์และวิทยาการคอมพิวเตอร์	ลายมือชื่อนิสิต..
		ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก......................
สาขาวิชา	คณิตศาสตร์ประยุกต์และวิทยาการคณนา	ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์ร่วม.......................
ปีการศึกษา	2555	

5471926623 : MAJOR APPLIED MATHEMATICS AND COMPUTATIONAL SCIENCE
KEYWORDS : FUNCTIONAL EQUATION / MEAN FUNCTIONAL EQUATION

JANNAPA UTTAMA : ZERO-MEAN FUNCTIONAL EQUATION ON PLANAR N-GONS. ADVISOR : ASSOC.PROF. PAISAN NAKMAHACHALASINT, Ph.D. CO-ADVISOR : ASST.PROF. NATAPHAN KITISIN, Ph.D., 21 pp.

In this thesis, we establish the general solutions to the zero-mean functional equation on planar n-gon with the constraint that for each planar n-gon obtained by translations and dilations of an arbitrary fixed n-gon, the sum of the values of f at all vertices is zero. In particular, we will determine the general solution $f : \mathbb{C} \to \mathbb{C}$ to the zero-mean functional equation

$$\sum_{j=1}^{n} f(z + \lambda a_j) = 0,$$

for all $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}, \lambda > 0$ where $a_j' s, 1 \leq j \leq n$ are fixed points in $\mathbb{C}$.

Department :	..Mathematics......................	Student's Signature :	....
	..and Computer Science.......	Advisor's Signature :	....
Field of Study :	..Applied Mathematics and..	Co-advisor's Signature :	....
	..Computational Science.......		
Academic Year :	..................2012..................		

ACKNOWLEDGEMENTS

This thesis would not be possible without the guidance and the help of several people who in one way or another made contributions and extended their valuable assistance in the completion of this work. First and foremost, my utmost gratitude to my thesis advisor, Associate Professor Paisan Nakmahachalasint and my co-advisor, Assistant Professor Nataphan Kitisin for their continuous support of my graduate study and research, for their patience, motivation and immense knowledge in the field of Functional Equation. Their guidance helped me in all the time of research and writing of this thesis. I would like to thank my thesis committee members, Dr. Khamron Mekchay, Associate Professor Patanee Udomkavanich, and Dr. Charinthip Hengkrawit for their encouragement and insightful comments. Completing this work would have been all the more difficult were it not for the helps of so many friends, Rittigrai, Chudej, Woravit to name just a few. I am indebted to them all. Last but not the least, I would like to thank my family, my parents and my younger sister. They were all supporting me, encouraging me with their best wishes and standing by me through the good times and bad.

CONTENTS

CHAPTER I
INTRODUCTION

1.1 Functional Equation

Functional equations are equations such that the unknowns are functions rather than real numbers. Some examples of functional equations are

$$\begin{aligned}
f(x+y) &= f(x)+f(y),\\
f(x+y) &= f(x)\cdot f(y),\\
f(xy) &= f(x)+f(y),\\
f(xy) &= f(x)\cdot f(y),\\
f(x+y)-f(x-y) &= 2f(x)\cdot f(y),\\
f(x+y) &= g(xy)+h(x-y),\\
f(x)-f(y) &= (x-y)\cdot g(x+y),
\end{aligned}$$

and

$$g(f(x)) = g(x)+c.$$

To solve functional equation is to find all functions that satisfy the functional equation. Unfortunately, there is no systematic method or algorithm to solve general functional equations. But the very first step is often involved the suitable substitutions. Once the solution is obtained, it is imperative to verify the solution by substituting back to the original functional equation. For example;

Example 1.1. *Find all function $f : \mathbb{R}\backslash\{0,1\} \to \mathbb{R}$ satisfying the functional equation*

$$f(x)+f(\frac{x-1}{x}) = 1+x. \tag{1.1}$$

Solution. Substitute x by $x = \frac{1}{1-x}$ in (1.1), we get

$$f(\frac{1}{1-x}) + f(x) = 1 + \frac{1}{1-x}. \tag{1.2}$$

Substituting x by $x = \frac{x-1}{x}$ in (1.1), we obtain

$$f(\frac{x-1}{x}) + f(\frac{1}{1-x}) = 1 + \frac{x-1}{x}. \tag{1.3}$$

By adding (1.1), (1.2) and (1.3) and dividing the result by two, we have

$$f(x) + f(\frac{x-1}{x}) + f(\frac{1}{1-x}) = \frac{1}{2}(3 + \frac{1}{1-x} + \frac{x-1}{x} + x). \tag{1.4}$$

By (1.3) and (1.4), we get

$$\begin{aligned} f(x) &= \frac{1}{2}(3 + \frac{1}{1-x} + \frac{x-1}{x} + x) - (1 - \frac{x-1}{x}) \\ &= \frac{1}{2}(1 + \frac{1}{1-x} + x - \frac{x-1}{x}) \\ &= \frac{-x^3 + x^2 + 1}{2x(1-x)}. \end{aligned}$$

Conversely, we can easily verify that $f(x) = \frac{-x^3+x^2+1}{2x(1-x)}$ satisfies the functional equation. Therefore, the function $f(x) = \frac{-x^3+x^2+1}{2x(1-x)}$ for all $x \in \mathbb{R}\backslash\{0,1\}$ is the unique solution of the functional equation. □

In some cases, the solution for the functional equation may not exist.

Example 1.2. *Is there a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation*

$$2f(5-x) + xf(x) = 1, \tag{1.5}$$

for all $x \in \mathbb{R}$?

Solution. We will prove that there is no solution by contradiction. Suppose that there is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.5).
Let $x = 1$ in (1.5), we get

$$2f(4) + f(1) = 1. \tag{1.6}$$

Let $x = 4$ in (1.5), we have

$$2f(1) + 4f(4) = 1. \tag{1.7}$$

It is obvious that there is no $f(1)$ and $f(4)$ satisfying both equations (1.6) and (1.7) simultaneously. Therefore, there is no solution for the functional (1.5). □

The functional equations may involve more than one variable such as in the following example.

Example 1.3. *Find all function $f : \mathbb{R} \to \mathbb{R}$ satisfying*

$$f(x^2 + f(y)) = y + xf(x), \tag{1.8}$$

for all $x, y \in \mathbb{R}$.

Solution. Putting $x = 0$, then $f(f(y)) = y$. Therefore, we get

$$f(y + xf(x)) = f(f(x^2 + f(y))) = x^2 + f(y). \tag{1.9}$$

Now substituting x by $f(x)$, we have

$$f(y + f(x)f(f(x))) = (f(x))^2 + f(y).$$

Since $f(f(y)) = y$, we obtain

$$f(y + xf(x)) = (f(x))^2 + f(y). \tag{1.10}$$

By (1.9) and (1.10), we get

$$(f(x))^2 = x^2. \tag{1.11}$$

Note that $f(0) = 0$.
Now replacing y by $f(y)$ in the original equation, we have

$$f(x^2 + y) = f(y) + xf(x).$$

Squaring both sides, we get

$$\begin{aligned}(x^2+y)^2 &= (f(x^2+y))^2 \\ &= (f(y)+xf(x))^2 \\ &= (f(y))^2 + x^2(f(x))^2 + 2xf(x)\cdot f(y) \\ &= x^4 + y^2 + 2xf(x)\cdot f(y).\end{aligned}$$

From this, we obtain

$$2x^2y = 2xf(x)\cdot f(y)$$

Hence, for $x \neq 0$, we get

$$xy = f(x)\cdot f(y). \tag{1.12}$$

From (1.11), we have $f(x) = x$ or $f(x) = -x$.
If $f(x) = x$, then from (1.12) we have $f(x) = x$ for all $x \in \mathbb{R}$.
If $f(x) = -x$, from (1.12) we get $f(x) = -x$ for all $x \in \mathbb{R}$.
By verifying $f(x) = x$ and $f(x) = -x$, we see that both solutions satisfy the original functional equation. Therefore $f(x) = x$ and $f(x) = -x$ are solution of the functional equation. □

The following example shows that there may be more than one function involved in the function equation.

Example 1.4. *Find all functions $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation*

$$f(x+y) = g(x) + h(y), \tag{1.13}$$

for all $x, y \in \mathbb{R}$.

Solution. Let $x = 0$ in (1.13), we have

$$f(y) = g(0) + h(y). \tag{1.14}$$

Let $y = 0$ in (1.13), we get

$$f(x) = g(x) + h(0). \tag{1.15}$$

From (1.13), (1.14) and (1.15), we obtain

$$f(x + y) + g(0) + h(0) = f(x) + f(y). \tag{1.16}$$

Define $A : \mathbb{R} \to \mathbb{R}$ by $A(x) = f(x) - g(0) - h(0)$ for all $x \in \mathbb{R}$.
Therefore, we can write equation (1.16) as

$$A(x + y) = A(x) + A(y). \tag{1.17}$$

It is well known that the general solution of equation (1.17) is the Cauchy additive function. Hence, we have that

$$f(x) = A(x) + g(0) + h(0),$$

$$g(x) = A(x) + g(0),$$

and

$$h(x) = A(x) + h(0).$$

The functions $f(x), g(x), h(x)$ can be easily verified that they satisfy the original functional equation. $\square$

Next, we give an example pertaining to our work where the domain of f is $\mathbb{R}^2$. Note that there is a parameter $\lambda \in \mathbb{R}\backslash\{0\}$ in the functional equation. Geometrically, let ℓ_{z,z_0} be the line passing through the point z parallel to a fixed vector $z_0 \in \mathbb{R}^2$. The following functional equation implies that the sum of the values of f on any two points on ℓ_{z,z_0} is always equal to zero.

Example 1.5. *Let $z_0 \in \mathbb{R}^2$. Find all functions $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying the functional equation*

$$f(z) + f(z + \lambda z_0) = 0, \tag{1.18}$$

for all $z \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}, \lambda \neq 0$.

Solution. Replacing z by $z + \lambda z_0$ in (1.18), we have

$$f(z + \lambda z_0) + f(z + 2\lambda z_0) = 0. \tag{1.19}$$

By (1.18) and (1.19), we get

$$f(z) = f(z + 2\lambda z_0). \tag{1.20}$$

Replacing λ by $\frac{\lambda}{2}$ in (1.20), we have

$$f(z) = f(z + \lambda z_0). \tag{1.21}$$

By (1.19) and (1.21), we obtain that $2f(z) = 0$ and therefore $f(z) = 0$, for all $z \in \mathbb{R}^2$. Conversely, $f(z) = 0$ obviously satisfies the original functional equation. Hence $f(z) = 0$, for all $z \in \mathbb{R}^2$, is the unique solution of the functional equation (1.18). $\square$

1.2 Literature Review

It is well known that harmonic functions on $\mathbb{R}^2$ can be characterized as those continuous functions which have the mean value property over any circles in $\mathbb{R}^2$. In particular, it is the result of the following theorem.

Theorem 1.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Suppose that $\forall z \in \mathbb{R}^2$ and $\forall r > 0$,*

$$f(z) = \frac{1}{2\pi r} \int_0^{2\pi} f(z + re^{i\theta}) d\theta.$$

Then f is harmonic.

Analogously, functions satisfying the mean value property over other geometric figures in $\mathbb{R}^2$ have also been studied. For examples in 1968, J. Aczél, H. Haruki, M.A. McKiernan and G. N. Sakovič studied the functional equations

$$f(x+u, y+v) + f(x+u, y-v) + f(x-u, y+v) + f(x-u, y-v) = 4f(x, y), \tag{1.22}$$

and

$$f(x+u,y)+f(x-u,y)+f(x,y+v)+f(x,y-v)=4f(x,y). \tag{1.23}$$

Geometrically, the functional equation (1.22) says that the value of f at the center of any rectangle, with its sides parallel to the coordinate axis, equals to the mean value of f taken at all its vertices.

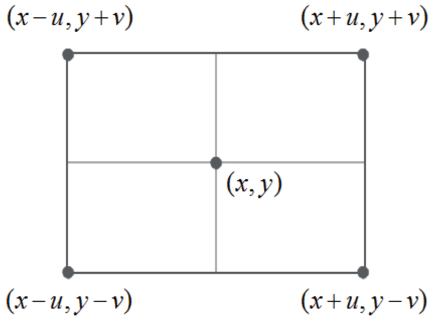

Figure 1.1 : Rectangle

Similarly, we can geometrically interpret functional equation (1.23) as the value of f at the center of the rhombus, whose diagonals are parallel to the coordinate axes, equals to the mean value of f taken at all its vertices.

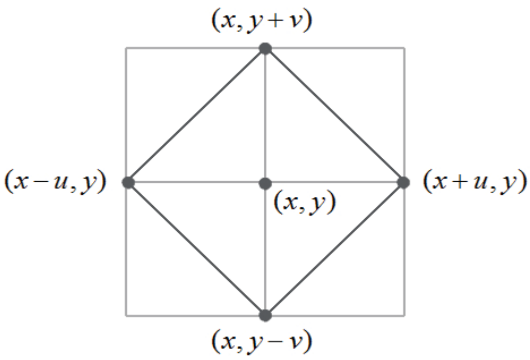

Figure 1.2 : Rhombus

Later on, in 1982, S.Haruki[4] was able to determine the general solution of the functional equation;

$$f(x-t,y-\frac{t}{\sqrt{3}})+f(x+t,y-\frac{t}{\sqrt{3}})+f(x,y+\frac{2t}{\sqrt{3}})=3f(x,y). \tag{1.24}$$

Geometrically, the functional equation (1.24) implies that for each triangle obtained by translations and dilations of an equilateral triangle, with one side parallel to the x-axis, the value of the function at its centroid is the arithmetic mean of its values at the vertices.

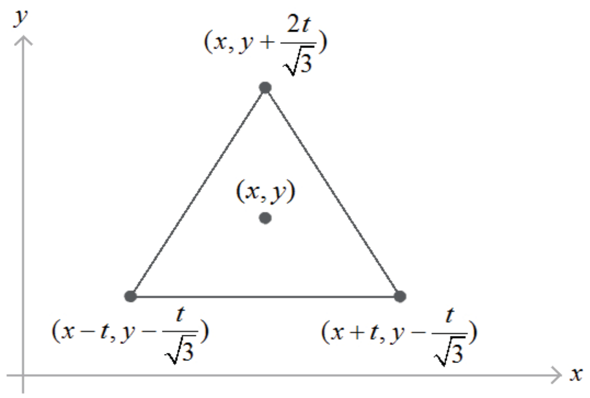

Figure 1.3 : Equilateral triangle with one side parallel to x-axis

With more regularity conditions imposed on f, J. Aczél, et al[1] were able to establish the general solution to the mean value functional equation on regular n-gons;

$$\sum_{k=0}^{n-1} f(z + \lambda\omega^k) = nf(z), \qquad n \geq 3 \tag{1.25}$$

for all n-gons $z + \lambda\omega^k$ where $\omega = e^{2\pi i/n}$. In particular, they have shown that the general solution to (1.25) must be a harmonic polynomial of degree n.
Note that functional equation (1.25) says that the value of the function at the center of homothetic regular n-gons is the arithmetic mean of its values taken at its vertices.

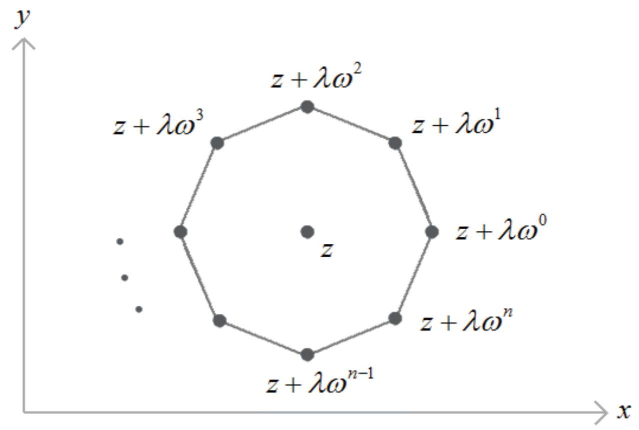

Figure 1.4 : Regular n-gons

In 1995, J.A.Baker[2] studied the triangular mean value functional equation;

$$f(z+e^{it})+f(z+e^{it}\omega)+f(z+e^{it}\bar{\omega})=3f(z), \tag{1.26}$$

for all $z \in \mathbb{C}$, $t \in \mathbb{R}$ and $\omega = e^{2\pi i/3}$. The functional equation (1.26) is different from (1.25) in the sense that it only allows the rotation instead of dilation of an equilateral triangle. The author was able to prove that the general solution must also be a harmonic polynomial, provided f is continuous. Recently, R. Kotnara[3] has studied the case when the right hand sides of equation (1.22)-(1.24) are equal to zero. He was able to determine the general solutions of the so called zero-mean functional equations on arbitrary triangle and quadrilateral as follows.

Theorem 1.7. *Let $a_1, a_2, a_3 \in \mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ satisfies the functional equation*

$$f(z+\lambda a_1)+f(z+\lambda a_2)+f(z+\lambda a_3)=0,$$

for all $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}, \lambda > 0$ if and only if $f(z) = 0$ for all $z \in \mathbb{C}$.

Theorem 1.8. *Let $a_1, a_2, a_3, a_4 \in \mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ satisfies the functional equation*

$$f(z+\lambda a_1)+f(z+\lambda a_2)+f(z+\lambda a_3)+f(z+\lambda a_4)=0,$$

for all $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}, \lambda > 0$ if and only if $f(z) = 0$ for all $z \in \mathbb{C}$.

The geometric meaning of Theorem 1.7 and 1.8 are that if for each triangle (quadrilateral), which is obtained from dilations and translations of a fixed triangle(quadrilateral), the sum of the values of f at its vertices is always equal to zero, then f must be identically zero on $\mathbb{C}$.

1.3 Proposed Work

Our main result will be divided into two parts. First, we will give another proof to the R. Kotnara's result[3] on the zero-mean functional equation on planar quadrilateral. Afterward, we will extend the result by establishing the general solution for the zero-mean functional equation on planar n-gons.

CHAPTER II
ZERO-MEAN FUNCTIONAL EQUATION ON PLANAR QUADRILATERAL

In this chapter, we will give another proof to the following theorem which first appeared in R. Kotnara[3].

Theorem 2.1. *Given $a_1, a_2, a_3, a_4 \in \mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ satisfies the functional equation*

$$f(z + \lambda a_1) + f(z + \lambda a_2) + f(z + \lambda a_3) + f(z + \lambda a_4) = 0, \tag{2.1}$$

for all $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}, \lambda > 0$ if and only if $f(z) = 0$ for all $z \in \mathbb{C}$.

Proof. **Step 1**: We let $\lambda = 1$ in equation (2.1) and have

$$f(z + a_1) + f(z + a_2) + f(z + a_3) + f(z + a_4) = 0. \tag{2.2}$$

We then make the following substitutions.

Substituting z by $z + a_1 + a_2 + a_3$ in equation (2.2), we get

$$\begin{aligned} &f(z + 2a_1 + a_2 + a_3) + f(z + a_1 + 2a_2 + a_3)+ \\ &f(z + a_1 + a_2 + 2a_3) + f(z + a_1 + a_2 + a_3 + a_4) = 0. \end{aligned} \tag{2.3}$$

Substituting z by $z + a_1 + a_2 + a_4$ in equation (2.2), we have

$$\begin{aligned} &f(z + 2a_1 + a_2 + a_4) + f(z + a_1 + 2a_2 + a_4)+ \\ &f(z + a_1 + a_2 + a_3 + a_4) + f(z + a_1 + a_2 + 2a_4) = 0. \end{aligned} \tag{2.4}$$

Substituting z by $z + a_1 + a_3 + a_4$ in equation (2.2), we obtain

$$\begin{aligned} &f(z + 2a_1 + a_3 + a_4) + f(z + a_1 + a_2 + a_3 + a_4)+ \\ &f(z + a_1 + 2a_3 + a_4) + f(z + a_1 + a_3 + 2a_4) = 0. \end{aligned} \tag{2.5}$$

Substituting z by $z + a_2 + a_3 + a_4$ in equation (2.2), we get

$$\begin{aligned} &f(z + a_1 + a_2 + a_3 + a_4) + f(z + 2a_2 + a_3 + a_4)+ \\ &f(z + a_2 + 2a_3 + a_4) + f(z + a_2 + a_3 + 2a_4) = 0. \end{aligned} \tag{2.6}$$

Step 2, We let $\lambda = 2$ in equation (2.1) and have

$$f(z + 2a_1) + f(z + 2a_2) + f(z + 2a_3) + f(z + 2a_4) = 0. \tag{2.7}$$

We then make the following substitutions.

Substituting z by $z + a_1 + a_2$ in equation (2.7), we get

$$\begin{aligned} &f(z + 3a_1 + a_2) + f(z + a_1 + 3a_2)+ \\ &f(z + a_1 + a_2 + 2a_3) + f(z + a_1 + a_2 + 2a_4) = 0. \end{aligned} \tag{2.8}$$

Substituting z by $z + a_1 + a_3$ in equation (2.7), we obtain

$$\begin{aligned} &f(z + 3a_1 + a_3) + f(z + a_1 + a_3 + 2a_2)+ \\ &f(z + a_1 + 3a_3) + f(z + a_1 + a_3 + 2a_4) = 0. \end{aligned} \tag{2.9}$$

Substituting z by $z + a_1 + a_4$ in equation (2.7), we get

$$\begin{aligned} &f(z + 3a_1 + a_4) + f(z + a_1 + a_4 + 2a_2)+ \\ &f(z + a_1 + a_4 + 2a_3) + f(z + a_1 + 3a_4) = 0. \end{aligned} \tag{2.10}$$

Substituting z by $z + a_2 + a_3$ in equation (2.7), we have

$$\begin{aligned} &f(z + a_2 + a_3 + 2a_1) + f(z + 3a_2 + a_3)+ \\ &f(z + a_2 + 3a_3) + f(z + a_2 + a_3 + 2a_4) = 0. \end{aligned} \tag{2.11}$$

Substituting z by $z + a_2 + a_4$ in equation (2.7), we obtain

$$\begin{aligned} &f(z + a_2 + a_4 + 2a_1) + f(z + 3a_2 + a_4)+ \\ &f(z + a_2 + a_4 + 2a_3) + f(z + a_2 + 3a_4) = 0. \end{aligned} \tag{2.12}$$

Substituting z by $z+a_3+a_4$ in equation (2.7), we get

$$\begin{aligned}&f(z+a_3+a_4+2a_1)+f(z+a_3+a_4+2a_2)+\\&f(z+3a_3+a_4)+f(z+a_3+3a_4)=0.\end{aligned} \tag{2.13}$$

Before proceeding to step 3, we make the following computations.
Subtracting the sum of equations (2.8)-(2.13) from the sum of equations (2.3)-(2.6), we have

$$\begin{aligned}&4f(z+a_1+a_2+a_3+a_4)-f(z+3a_1+a_2)-f(z+3a_2+a_1)-\\&f(z+3a_1+a_3)-f(z+3a_3+a_1)-f(z+3a_1+a_4)-f(z+3a_4+a_1)-\\&f(z+3a_2+a_3)-f(z+3a_3+a_2)-f(z+3a_2+a_4)-f(z+3a_4+a_2)-\\&f(z+3a_3+a_4)-f(z+3a_4+a_3)=0.\end{aligned} \tag{2.14}$$

Step 3, We let $\lambda=3$ in equation (2.1) and have

$$f(z+3a_1)+f(z+3a_2)+f(z+3a_3)+f(z+3a_4)=0. \tag{2.15}$$

We then make the following substitutions.

Substituting z by $z+a_1$ in equation (2.15), we get

$$f(z+4a_1)+f(z+a_1+3a_2)+f(z+a_1+3a_3)+f(z+a_1+3a_4)=0. \tag{2.16}$$

Substituting z by $z+a_2$ in equation (2.15), we obtain

$$f(z+a_2+3a_1)+f(z+4a_2)+f(z+a_2+3a_3)+f(z+a_2+3a_4)=0. \tag{2.17}$$

Substituting z by $z+a_3$ in equation (2.15), we get

$$f(z+a_3+3a_1)+f(z+a_3+3a_2)+f(z+4a_3)+f(z+a_3+3a_4)=0. \tag{2.18}$$

Substituting z by $z+a_4$ in equation (2.15), we have

$$f(z+a_4+3a_1)+f(z+a_4+3a_2)+f(z+a_4+3a_3)+f(z+4a_4)=0. \tag{2.19}$$

By adding equations (2.16)-(2.19), we get

$$\begin{aligned}
&f(z+4a_1)+f(z+4a_2)+f(z+4a_3)+f(z+4a_4)+\\
&f(z+a_1+3a_2)+f(z+a_1+3a_3)+f(z+a_1+3a_4)+\\
&f(z+a_2+3a_1)+f(z+a_2+3a_3)+f(z+a_2+3a_4)+\\
&f(z+a_3+3a_1)+f(z+a_3+3a_2)+f(z+a_3+3a_4)+\\
&f(z+a_4+3a_1)+f(z+a_4+3a_2)+f(z+a_4+3a_3)=0.
\end{aligned} \tag{2.20}$$

We add equation (2.14) with equation (2.20) and obtain

$$\begin{aligned}
&4f(z+a_1+a_2+a_3+a_4)+f(z+4a_1)+\\
&f(z+4a_2)+f(z+4a_3)+f(z+4a_4)=0.
\end{aligned} \tag{2.21}$$

Step 4, We let $\lambda = 4$ in equation (2.2) and get

$$f(z+4a_1)+f(z+4a_2)+f(z+4a_3)+f(z+4a_4)=0. \tag{2.22}$$

From equations (2.21) and (2.22), we finally obtain

$$4f(z+a_1+a_2+a_3+a_4)=0. \tag{2.23}$$

But equation (2.23) is true for all $z \in \mathbb{C}$. Therefore we must have $f(z) = 0$, for all $z \in \mathbb{C}$.

$\square$

CHAPTER III

ZERO-MEAN FUNCTIONAL EQUATION ON PLANAR N-GONS

In this chapter, we will extend the previous result to any planar n-gons. In particular, we will show that the only solution to the zero-mean functional equation on planar n-gons is the identically zero function.

First, since we are dealing with an arbitrary n-gon, let us introduce some notations.

Given integer $n \in \mathbb{N}$ and $a_1, a_2, ..., a_n \in \mathbb{C}$.
Let $\alpha = (i_1, i_2, ..., i_k)$, where $k \in \mathbb{N}, i_k \in \mathbb{N}$ and $1 \leq i_k \leq n$, be a multi-index.
We define $A_\alpha = a_{i_1} + a_{i_2} + ... + a_{i_k}$ and

$$\sum_{1 \leq i_1 < i_2 < i_3 < ... < i_k \leq n} A_\alpha$$

be the sum of all A_α where α is subject to the condition $1 \leq i_1 < i_2 < i_3 < ... < i_k \leq n$.

Example 3.1. *For $n = 4$, $\alpha = (i_1, i_2, i_3)$ we have $A_{(1,2,4)} = a_1 + a_2 + a_4$ and*

$$\sum_{1 \leq i_1 < i_2 < i_3 \leq 4}^{4} A_\alpha = A_{(1,2,3)} + A_{(1,2,4)} + A_{(1,3,4)} + A_{(2,3,4)}.$$

Our main result is to establish the general solution to the zero-mean functional equation on planar n-gons. Geometrically, the functional equation implies that for each n-gon obtained by translation and dilation of an arbitrary fixed n-gon determined by $a_1, a_2, ..., a_n$, the sum of values of f at all its vertices equals to zero.

Theorem 3.2. *Given $n \in \mathbb{N}$ and $a_1, a_2, ..., a_n \in \mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ satisfies the zero-mean functional equation;*

$$\sum_{j=1}^{n} f(z + \lambda a_j) = 0, \tag{3.1}$$

for all $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}, \lambda > 0$ if and only if $f(z) = 0$, for all $z \in \mathbb{C}$.

Proof. We make various substitutions in steps as follows:

Step 1: First, we let $\lambda = 1$ in (3.1) and get

$$\sum_{j=1}^{n} f(z + a_j) = 0. \tag{3.2}$$

We then make $\binom{n}{n-1}$ substitutions by replacing z by $z + A_\alpha$ in (3.2),
where $\alpha = (i_1, i_2, ..., i_{n-1})$ and $1 \leq i_1 < i_2 < i_3 < ... < i_{n-1} \leq n$.
Observe that when we substitute z with $z + A_\alpha$ in each terms of (3.2),
it will result in either kinds of these terms.

If $a_j = a_{i_k}$ in $\alpha = (i_1, i_2, ..., i_{n-1})$ for some k, $1 \leq k \leq n-1$,
then $f(z + A_\alpha + a_j) = f(z + 2a_j + A_\beta)$ where $\beta = (i_1, i_2, ..., i_{n-2})$ and $i_k \neq j$
for all $1 \leq k \leq n-2$ in β.

If $a_j \neq a_{i_k}$ in $\alpha = (i_1, i_2, ..., i_{n-1})$ for all k, $1 \leq k \leq n-1$,
then $f(z + A_\alpha + a_j) = f(z + \sum\limits_{\ell=1}^{n} a_\ell)$.

Note that we will have new $\binom{n}{n-1}$ equations after all the substitutions.
Adding all these new $\binom{n}{n-1}$ equations up and denoting the sum by I_1, we therefore get

$$I_1 = \left(\sum_{j=1}^{n} \sum_{\substack{\beta=(i_1,i_2,...,i_{n-2}) \\ 1\leq i_1<i_2<...<i_{n-2}\leq n \\ i_k \neq j}} f(z + 2a_j + A_\beta) \right) + \left(nf(z + \sum_{j=1}^{n} a_j \right) = 0.$$

Step 2: We let $\lambda = 2$ in (3.1) and get

$$\sum_{j=1}^{n} f(z + 2a_j) = 0. \tag{3.3}$$

We then make $\binom{n}{n-2}$ substitutions by replacing by $z + A_\alpha$ in (3.3), where $\alpha = (i_1, i_2, ..., i_{n-2})$ and $1 \leq i_1 < i_2 < i_3 < ... < i_{n-2} \leq n$.
Observe that when we substitute z with $z + A_\alpha$ in each terms of (3.3), it will result in either of these terms.

If $a_j = a_{i_k}$ in $\alpha = (i_1, i_2, ..., i_{n-2})$ for some k, $1 \leq k \leq n-2$,
then $f(z + A_\alpha + 2a_j) = f(z + 3a_j + A_\beta)$ where $\beta = (i_1, i_2, ..., i_{n-3})$ and $i_k \neq j$ for all $1 \leq k \leq n-3$ in β.

If $a_j \neq a_{i_k}$ in $\alpha = (i_1, i_2, ..., i_{n-2})$ for all k, $1 \leq k \leq n-2$,
then $f(z + A_\alpha + 2a_j) = f(z + 2a_j + A_\alpha)$ where $\alpha = (i_1, i_2, ..., i_{n-2})$ and $i_k \neq j$ in α.

Note that we will have new $\binom{n}{n-2}$ equations after all the substitution. Adding all these new $\binom{n}{n-2}$ equations up and denoting the sum by I_2, we therefore get

$$I_2 = \left(\sum_{j=1}^{n} \sum_{\substack{\beta=(i_1,i_2,...,i_{n-3}) \\ 1 \leq i_1 < i_2 < ... < i_{n-3} \leq n \\ i_k \neq j}} f(z + 3a_j + A_\beta) \right) + \left(\sum_{j=1}^{n} \sum_{\substack{\beta=(i_1,i_2,...,i_{n-2}) \\ 1 \leq i_1 < i_2 < ... < i_{n-2} \leq n \\ i_k \neq j}} f(z + 2a_j + A_\beta) \right) = 0.$$

We continue this process similarly for each step. The following is the p^{th} step where $1 \leq p \leq n-1$.

Step p: We let $\lambda = p$ in (3.1) and get

$$\sum_{j=1}^{n} f(z + pa_j) = 0. \tag{3.4}$$

We then make $\binom{n}{n-p}$ substitutions by replacing z by $z + A_\alpha$ in (3.4), where $\alpha = (i_1, i_2, ..., i_{n-p})$ and $1 \leq i_1 < i_2 < i_3 < ... < i_{n-p} \leq n$.
Observe that when we substitute z with $z + A_\alpha$ in each term of(3.4), it will result in either kinds of these terms.

If $a_j = a_{i_k}$ in $\alpha = (i_1, i_2, ..., i_{n-p})$ for some k, $1 \leq k \leq n - p$,
then $f(z + A_\alpha + pa_j) = f(z + (p+1)a_j + A_\beta)$, where $\beta = (i_1, i_2, ..., i_{n-p-1})$ and $i_k \neq j$ in β.

If $a_j \neq a_{i_k}$ in $\alpha = (i_1, i_2, ..., i_{n-p})$ for all k, $1 \leq k \leq n - p$,
then $f(z + A_\alpha + pa_j) = f(z + pa_j + A_\alpha)$, where $\alpha = (i_1, i_2, ..., i_{n-p})$ and $i_k \neq j$ in α.

Note that we will have new $\binom{n}{n-p}$ equations after all the substitutions. Adding all these new $\binom{n}{n-p}$ equations up and denoting the sum by I_p, we therefore get

$$I_p = \left(\sum_{j=1}^{n} \sum_{\substack{\beta=(i_1,i_2,...,i_{n-p-1}) \\ 1\leq i_1<i_2<...<i_{n-p-1}\leq n \\ i_k \neq j}} f(z + (p+1)a_j + A_\beta) \right)$$
$$+ \left(\sum_{j=1}^{n} \sum_{\substack{\beta=(i_1,i_2,...,i_{n-p}) \\ 1\leq i_1<i_2<...<i_{n-p}\leq n \\ i_k \neq j}} f(z + pa_j + A_\beta) \right) = 0.$$

Finally, we make the last substitution by letting $\lambda = n$ in (3.1) and get

$$I_n = \sum_{j=1}^{n} f(z + na_j) = 0.$$

Since $I_\ell = 0$ for all ℓ, $1 \leq \ell \leq n$, we clearly have

$$\sum_{\ell=1}^{n} (-1)^{\ell+1} I_\ell = 0.$$

By computing the telescoping sum above, we get

$$n \cdot f(z + \sum_{j=1}^{n} a_j) = 0, \tag{3.5}$$

for all $z \in \mathbb{C}$.

For any $y \in \mathbb{C}$, we have

$$f(y) = f((y - \sum_{j=1}^{n} a_j) + \sum_{j=1}^{n} a_j) = 0$$

by letting $z = y - \sum_{j=1}^{n} a_j$ in (3.5). Since y is arbitrary, we therefore get $f(z) = 0$, for all $z \in \mathbb{C}$.

Since $f(z) = 0$ for all $z \in \mathbb{C}$ is obviously the solution of (3.1). Hence, the only solution to the functional equation (3.1) is $f(z) = 0$, for all $z \in \mathbb{C}$. $\square$

REFERENCES

[1] J.Aczél, H.Haruki, M.A.Mckiernan, and G.N.Sakovič *General and Regular Solutions of Functional Equations Characterizing Harmonic Polynomials.* Aequationes Math. **1**, 37-53 (1968).

[2] J.A.Baker, *Triangular mean value theorems and Frchet's equation.* Acta Math. Hungar. **25**, 111-126 (1995).

[3] R. Kotnara, Functional Equation on Planar Quadrilaterals, M.Sc Thesis, Chulalongkorn University, 2011.

[4] S.Haruki, *On the general solution of the triangle mean value equation.* Aequationes Math. **25**, 209-215 (1982).

VITA

Name	Miss Jannapa Uttama
Date of Birth	16 October 1988
Place of Birth	Ratchaburi, Thailand
Education	B.Sc. (Mathematics),
	Chulalongkorn University, 2011

PUBLICATION

J. Uttama, N. Kitisin, P. Nakmahachalasint, *Zero-Mean Functional Equation On Planar N-gons*, in:Proc. 18th Annual Meeting in Mathematics, Krabi (2013).